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Finite time future singularities in the interacting dark sector

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We construct a piecewise model that gives a physical viable realization of finite-time future singularity for a spatially flat Friedmann-Robertson-Walker universe within the interacting dark matter–dark energy framework, with the latter in the form of a variable vacuum energy. The scale factor solutions provided by the model are accommodated in several branches defined in four regions delimited by the scale factor and the effective energy density. A branch starts from a big bang singularity and describes an expanding matter-dominated universe until the sudden future singularity occurs. Then, an expanding branch emerges from a past singularity, reaches a maximum, reverses its expansion, and possibly collapses into itself while another expanding branch emerges from the latter singularity and has a de Sitter phase which is intrinsically stable. We obtain a different piecewise scale factor which describes a contracting de Sitter universe in the distant past until the finite-time future singularity happens. It emerges and continues in a contracting phase, bounces at the minimum, reverses, and enters into a stable de Sitter phase without a dramatic final. Also, we explore the aforesaid cosmic scenarios by focusing on the leading contributions of some physical quantities near the sudden future singularity and applying the geometric Tipler and Królak criteria in order to inspect the behavior of timelike geodesic curves around such singularity.

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I. INTRODUCTION

Despite the overwhelming observational evidence supporting the current cosmic acceleration of the Universe coming from supernovae data, the cosmic microwave background anisotropies, and a brand-new type of launched satellite for exploring the dark side of gravity [1,2], quite little is known about the nature of dark energy (DE). More precisely, there is no a fundamental theory at the microscopic level for explaining the origin of DE; only some properties are known. For instance, astrophysical observations suggest that DE is dominated by a strongly negative pressure acting as a repulsive force [1]. Another side of this puzzle refers to what will be the ultimate fate of the Universe [3]: could DE make the Universe undergo an extremely violent final event as a big rip singularity [4,5,6]. This kind of singularity happens at a finite cosmic time where the scale factor, the Hubble parameter, and its cosmic time derivative diverge [5].

Sudden future singularity is a one-of-a-kind future scenario that has gained great interest recently [7,8] because it offers an alternative and smooth "ultimate" fate for the Universe, opening the possibility for a noncatastrophic transition which can lead to a new phase in the Universe's evolution. The latter type of doomsday happens at a finite cosmic time where the scale factor and the Hubble parameter remain finite but the cosmic time derivative diverges [7]. Indeed, the scale factor along with Hubble parameter both remain bounded which implies the Christoffel symbols are regular at this singularity. Hence, the geodesics are well behaved and they can cross the singularity [9]. Moreover, it was found that a sudden future singularity does not experience geodesic incompleteness [10] and demonstrated that the Tipler and Królak conditions do not hold [11,12]; as a result of that, finite objects (string or membrane) are not crushed when crossing this singularity and, therefore, it can be classified as a nonharmful transversable singularity. Also, it was argued that the particles crossing the singularity will generate the new geometry of the spacetime, providing in such a way a"soft rebirth" of the Universe after the singularity crossing [13]. Relaxing one of the conditions that characterizes a typical sudden future singularity, another interesting realization of sudden future singularity, dubbed "big brake" singularity [14], was discovered that depends upon the Hubble parameter vanishing at a finite cosmic time. The compatibility of this doomsday with supernovae data was addressed in [15], and an explicit example of the crossing of this singularity was described in [13], where a tachyon field passes through the singularity, continuing its evolution until the Universe recollapses due to the existence of final big crunch singularity.

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Some appealing models based on a mixture of anti-Chaplygin gas plus dustlike matter exhibiting a soft (sudden/big brake) future singularity were reported in the literature [16]. In fact, the distributional version of anti-Chaplygin gas was presented in Ref. [16], focusing in the role played by these distributional quantities together with the junction conditions at the singularity. Recently, the issue of describing a big brake singularity in terms of a scalar field model was analyzed in great detail in [17]. It turned out that a big brake future singularity can be obtained by a modified Chaplygin gas equation of state using a unified (effective) scheme. Furthermore, this singularity was accommodated in terms of an exotic quintessence model and a full perturbation analysis was carrried out near such event [17].

One of the aims of the present paper is to study some reasonable cosmological models with a finite-time future singularity when the Universe is filled with dark matter (DM) and variable vacuum energy (VVE) accommodated as dark energy. Contrary to the usual picture where the aforesaid components are decoupled, we propose that both fluids exchange energy [18–21], being the interaction a nonlinear combination of the total density and its first derivative. A physical motivation for selecting this kind of nonlinear interaction is that the two interacting fluid model can be mapped into an effective unified model characterized by the relaxed version of the Chaplygin gas equation of state introduced in [22]. This equation of state produces an energy density that can be separated into three branches, one of them gives rise to a scale factor that interpolates between a matter-dominated universe at early time and a de Sitter phase dominated by VVE at late times while the other two branches produce and energy density that vanishes o diverges at a finite scale factor value a_s. These are attractive characteristics to investigate finite-time future singularities whenever $a_s = a(t_s)$ is finite for a finite cosmic time t_s. In doing so, we can provide a physical description of the behavior of the interacting components near the singularity but also we can compare with the behavior of effective quantities such as total energy and total pressure. In this way, we will be able to establish a physical correspondence between the critical behavior of interaction and the effective equation of state. Having mentioned the physical motivation of the nonlinear interaction, we are going to use the so called "source equation" to determine the effective density, total pressure, and partial densities as well. Thus, We will reconstruct the explicit dependence of the interaction, total density, total pressure and partial densities in terms of the scale factor. Such procedure has the virtue of keeping the analysis simple but also it allows us to demonstrate the critical behavior of the total pressure, the interaction and the partial densities around the future singular event [23]. A remarkable fact is that the appearance of finite-time future singularity will be deeply connected with the critical behavior of the dark components near such event; that is, the total energy density remains finite whereas the dark densities along with the total presssure will grow without limit.

The layout of the paper is as follows. In Sec. II, we introduce in detail the model of an interacting dark sector composed of DM and VVE. In doing so, we solve the source equation, reconstruct all the geometrical, source variables, and provide a classification of the different interaction pieces in terms of the scale factor and the energy density in the (a, ρ) plane. This analysis is followed in Sec. III by solving the Friedmann equation and exploring all the possible types of cosmic scenarios such as big bang singularity, bounce, big crunch, sudden singularity, de Sitter, and anti-de Sitter phases. Also, we study the behavior of dark energy components: the energy density, the equation of state and the stability of the de Sitter phases. In Sec. IV, we present two piecewise nonlinear models obtained after matching two interacting dark sector models at the finite-time future singularity, which are characterized by a relaxed Chaplygin gas equation of state, and describe the Universe's evolution identified with both models. In Sec. V, we examine those singular events in terms of comoving observers approaching to them by using the criteria of Królak and Tippler. Section VI is devoted to present a summary of our main results.

II. NONLINEAR INTERACTION AND FINITE-TIME FUTURE SINGULARITIES IN THE FRW UNIVERSE

The purpose of our paper is to examine the existence of finite-time future singularities from the interaction point of view. In doing so, we investigate a two fluid model for a spatially flat FRW metric, where the Universe is filled with DM and an unknown component that we choose as VVE. The former one has a linear equations of state $p_m = (\gamma_m - 1)\rho_m$, where the barotropic index γ_m is assumed to be a constant such that $\gamma_m \simeq 1$, while the latter one has a vacuum equation of state $p_x = -\rho_x$ with $\gamma_x = 0$. The energy density and the conservation equation of this system are given by,

$$\rho = \rho_m + \rho_x, \tag{1}$$

$$\rho' = -\gamma_m \rho_m, \tag{2}$$

while the pressure of the whole system is

$$p = -\rho - \rho', \tag{3}$$

and can be recast as $p = \gamma_m \rho_m - \rho$. The prime stands for derivatives with respect to the variable η defined as $' \equiv d/d\eta = d/3Hdt = d/d\ln(a/a_0)^3$, being a_0 some value of reference for the scale factor a. $H = \dot{a}/a$ is the expansion rate and the dot means $\dot{} \equiv d/dt$. Solving the linear algebraic system of Eqs. (2), one finds ρ_m and ρ_x as functions of ρ and its derivative ρ':

$$\rho_m = -\frac{\rho'}{\gamma_m}, \tag{4}$$

$$\rho_x = \rho + \frac{\rho'}{\gamma_m}. \tag{5}$$

We build a model of interacting DM and DE by splitting the conservation equation (2) into two balance equations and introducing the interaction term Q with a factorized dependence as $3HQ(\eta, \rho, \rho')$,

$$\rho_m' + \gamma_m \rho_m = -Q, \tag{6}$$

$$\rho_x' = Q, \tag{7}$$

so there is an exchange of energy between DM and VVE components. From Eqs. (4) and (7), we obtain the "source equation" [22] for the total energy density

$$\rho'' + \gamma_m \rho' = \gamma_m Q. \tag{8}$$

Knowledge of the evolution of ρ requires to solve the source equation (8) for a given Q. After having obtained ρ, we are in a position to determine the energy densities of DM and VVE components from Eqs. (4) and (5).

With the purpose of getting a solvable model and showing the existence of finite-time future singularities in the interacting dark sector, we present an interacting scenario generated by a nonlinear interaction in the form

$$Q(\rho, \rho') = -n\left[\rho' + \frac{\rho'^2}{\gamma_m(\rho - \rho_s)}\right], \tag{9}$$

where ρ_s is a constant positive energy density and n is the coupling constant. Note that Q diverges in the limit $\rho \to \rho_s$. Let us mention which are the physical motivations for selecting this ansatz over other choices. Cosmological scenarios where dark matter exchanges energy with a modified holographic Ricci dark energy [18,19], can be accommodated in terms of a nonlinear interaction, as that represented by Eq. (9). In dealing with that interacting framework, one finds that the effective one-fluid obeys the equation of state of a relaxed Chaplygin gas. Therefore, the Universe is dominated by pressureless dark matter at early times and undergoes an accelerated expansion in the far future driven by a strong negative pressure [18]. Another interesting case where the nonlinear interaction takes place is related with the evolution of a Universe that has an interacting dark matter, a modified holographic Ricci dark energy, plus a decoupled radiation term. The aforesaid model seems to be consistent with the Hubble data, the constraints coming from the amount of dark energy during the recombination along with the abundance of light elements obtained with the big bang nucleosynthesis (BBN) data [19].

Combining Eqs. (8) and (9), we obtain the equation that governs the dynamics of the energy density,

$$xx'' + \gamma_m(n+1)xx' + nx'^2 = 0, \tag{10}$$

where a new dimensionless variable is defined as

$$x = \frac{\rho - \rho_s}{\beta \rho_s}, \tag{11}$$

where this positive variable is such that $x_s = x(\rho_s) = 0$. Here the role played by the parameter β is the following: in the case $\beta = -1$ the energy density and the variable x range between $0 \le \rho < \rho_s$ and $0 < x \le 1$, while for $\beta = 1$ they define another physically admissible region given by $\rho > \rho_s$ and $x > 0$. Then, the energy density and its derivative can be written as follows

$$\rho = \rho_s(1 + \beta x), \qquad \rho' = \beta \rho_s x'. \tag{12}$$

Once the function $x = x(a)$ is known, we calculate the interaction (9), the energy density and its derivative (12), the equation of state (3) and the DM and VVE densities (4) and (5) as functions of the scale factor. To this end, we find the first integral of the source equation (10) and its general solution which read,

$$x' = -\gamma_m[x + \alpha x^{-n}], \tag{13}$$

$$x = \left\{\alpha\left[\left(\frac{a_s}{a}\right)^{3\gamma_m(n+1)} - 1\right]\right\}^{\frac{1}{n+1}}, \quad n > -1, \tag{14}$$

where α is an integration constant. The integration constant appearing in the general solution (14), after integrating the first integral (13), was chosen so that $x_s = x(a_s) = x(\rho_s) = 0$ for a finite value of the scale factor $a_s \neq 0$, which is equivalent to demand the existence of finite density at a_s, namely $\rho_s = \rho(x_s) = \rho(a_s)$. Such demand is well justified provided we are seeking for a realization of a future singularity and it must have a nonvanishing Hubble function at finite time t_s as we will see later on. Besides, we must combine $\alpha > 0$ with $a < a_s$ or $\alpha < 0$ with $a > a_s$ in order to ensure that the square bracket in Eq. (14) is defined positive. Finally, if Eqs. (13) and (14) are combined, we can obtain how the first integral behaves with the scale factor, $x' = -\alpha\gamma_m x^{-n}(a_s/a)^{3\gamma_m(n+1)}$.

In the particular case of a vanishing integration constant α, the source equation (10) of the interacting dark sector in presence of different forms of dark matter components includes the ΛCDM model of the General Relativity Theory. In fact, solving Eq. (13) and inserting the solution into Eq. (12), we obtain the energy density

$$\rho_{(\alpha=0)} = \rho_s\left\{1 \pm \left(\frac{a_0}{a}\right)^{3\gamma_m}\right\}, \tag{15}$$

with a_0 an integration constant. From this equation, we get $\rho'_{(\alpha=0)} = -\gamma_m(\rho_{(\alpha=0)} - \rho_s)$, so that inserting it into Eqs. (3), (4), (5), and (9), we have the equation of state $p_{(\alpha=0)} = -\gamma_m \rho_s + (\gamma_m - 1)\rho_{(\alpha=0)}$, which reduces to $p_{(\alpha=0)} = -\rho_s$ for cold dark matter(CDM) when $\gamma_m = 1$, the dark energy densities $\rho_m = \rho_{(\alpha=0)} - \rho_s$, $\rho_x = \rho_s$ and a vanishing interaction term $Q_{(\alpha=0)} = 0$. The upper sign in Eq. (15) corresponds to the ΛCDM model for $\gamma_m = 1$ and the lower one to a nonsingular bouncing model.

Equations (10)–(13) also admit a simple constant solution $\rho_{\rm dS} = \rho_s[1 + \beta(-\alpha)^{1/(n+1)}]$, which after having used Eqs. (4)–(5) they lead to $\rho_m = 0$ and $\rho_x = \rho_{\rm dS}$. Below, we will return to this solution that is related with the branches of solutions which have an initial contracting or a final expanding de Sitter phase. These stages are in turn related with an asymptotically vanishing interaction.

In order to study the most relevant outcome of our model, we collect all the meaningful quantities which will be used later on. From Eqs. (3)–(5), (9), and (12)–(14), we find that

$$Q(a) = -n\alpha^2\beta\rho_s\gamma_m\left(\frac{a_s}{a}\right)^{3\gamma_m(n+1)} \times\left\{\alpha\left[\left(\frac{a_s}{a}\right)^{3\gamma_m(n+1)} - 1\right]\right\}^{-\frac{2n+1}{n+1}}, \quad (16)$$

$$\rho = \rho_s\left\{1 + \beta\left\{\alpha\left[\left(\frac{a_s}{a}\right)^{3\gamma_m(n+1)} - 1\right]\right\}^{\frac{1}{n+1}}\right\}, \quad (17)$$

$$\rho' = -\alpha\beta\rho_s\gamma_m\left(\frac{a_s}{a}\right)^{3\gamma_m(n+1)}\left\{\alpha\left[\left(\frac{a_s}{a}\right)^{3\gamma_m(n+1)} - 1\right]\right\}^{\frac{-n}{n+1}}, \quad (18)$$

$$p = -\rho_s + (\gamma_m - 1)(\rho - \rho_s) + \alpha\beta\rho_s\gamma_m\left(\frac{\beta\rho_s}{\rho - \rho_s}\right)^n, \quad (19)$$

$$\rho_m = \alpha\beta\rho_s\left(\frac{a_s}{a}\right)^{3\gamma_m(n+1)}\left\{\alpha\left[\left(\frac{a_s}{a}\right)^{3\gamma_m(n+1)} - 1\right]\right\}^{\frac{-n}{n+1}}, \quad (20)$$

$$\rho_x = \rho_s\left\{1 - \alpha\beta\left\{\alpha\left[\left(\frac{a_s}{a}\right)^{3\gamma_m(n+1)} - 1\right]\right\}^{\frac{-n}{n+1}}\right\}. \quad (21)$$

Now, we will focus our investigation in the family of finite-time future singularities such that the scale factor and the Hubble parameter remain finite at some finite cosmic time called t_s but the pressure diverges at this time. Meaning that the scale factor $a_s = a(t_s)$, its first time derivative $\dot{a}_s = \dot{a}(t_s)$ are finite but its second derivative

$$\ddot{a}_s = \ddot{a}(t \to t_s) = \pm\infty, \quad (22)$$

diverging in the limit $t \to t_s$. As both the scale factor and Hubble parameter remain bounded at $t = t_s$, the Christoffel symbols are regular at this singularity and the geodesics are well behaved and they can cross the singularity [8]. The existence of this kind of future singularity in the interacting dark sector requires that the pressure (19), relaxed version of the well-known Chaplygin gas, diverges in the limit $\rho \to \rho_s$. Bearing this in mind and taking into account Eq. (19), we restrict the coupling constant to be positive $n > 0$, so that

$$p_s = p(\rho \to \rho_s) = \pm\infty, \quad (23)$$

as $\rho \to \rho_s$.Thus, $p \to -2\dot{H} \to -2\ddot{a}/a_s$ and the cosmic acceleration diverges $\ddot{a} \to -a_s p_s/2 \to \mp\infty$ when $\rho \to \rho_s$. Also, the interaction Q, the η-derivative of the energy density ρ', the DM and VVE densities ρ_m and ρ_x diverge in the limit $a \to a_s$ whereas their sign near a_s is determined by the sign of the product $\alpha\beta$. In this scenario, the pressure $p = \gamma_m\rho_m - \rho$ diverges as $a \to a_s$ provided the DM and VVE densities $\rho_m \to \alpha\beta\infty$ and $\rho_x \to -\alpha\beta\infty$ blows-up at the limit $a \to a_s$ whereas the energy density remains finite, namely $\rho = \rho_m + \rho_x \to \rho_s$. Our conclusion is that the appearance of a finite-time future singularity is directly linked with the behavior of the dark energy densities near the singularity given that they diverge at a_s together with the pressure as well. Hence, the interaction between the dark components plays an important role in the occurrence of the future singularity. This simple fact shows a close relation between a divergent interaction term and divergent dark energy components at a_s.

One way to address the full analysis of this type of singular event is to show more explicitly how it can be crossed and by doing so one necessarily emerges in another kind of Universe. In order to show how the matching of two nonlinear interacting dark sector can be obtained, it is useful to start by classifying all the physical distinctive dynamical regions of the model. Such a task is tackle by considering a generic point (a_s, ρ_s) in the plane (a, ρ), where Q, ρ', p, ρ_m, ρ_x and the effective barotropic index $\gamma = -\rho'/\rho$ diverge. The aforesaid analysis points out that there are four regions in which the interaction term and the others quantities change their specific forms (see for example Fig. 1 for the energy density). These regions are uniquely identified with the signs of the parameters α and β [see Eqs. (11) and (14)]. Clearly, the identification of these four interaction pieces is characterized by the signs of α and β, so that we are going to define a new symbol $Q^{(\mathrm{sign}(\alpha),\mathrm{sign}(\beta))}$ for that purpose. The four kinds of interactions can be classified in the following way:

$$Q^{(+,+)}(a) = Q(\alpha > 0, \beta = 1, a < a_s, \rho > \rho_s), \quad (24)$$

$$Q^{(-,-)}(a) = Q(\alpha < 0, \beta = -1, a > a_s, \rho < \rho_s), \quad (25)$$

$$Q^{(-,+)}(a) = Q(\alpha < 0, \beta = 1, a > a_s, \rho > \rho_s), \quad (26)$$

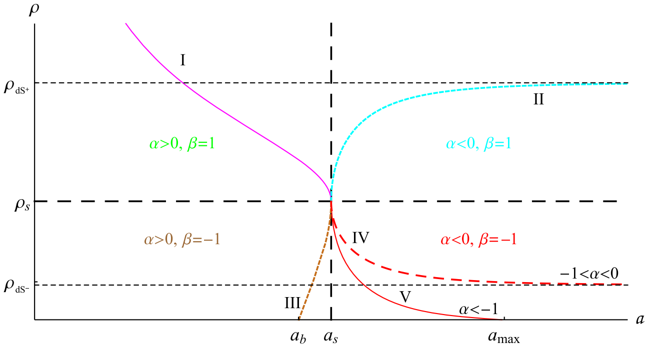

FIG. 1. Figure shows the energy density as a function of the scale factor for the four regions characterized by the signs of α and β parameters. Solutions II and IV tend to constant values ρ_{dS^+} and ρ_{dS^-}, while solutions III and V reach a bounce at a_b and a maximum at a_{max}. In Sec. IV, we describe three different piecewise models constructed by matching solutions I with IV, I with V, and II with III at $a = a_s$.

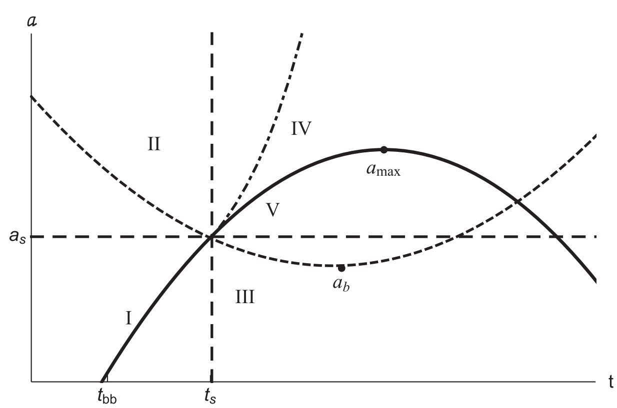

FIG. 2. Figure shows the qualitative behavior of the scale factor as a function of the cosmic time obtained from the five energy densities of the Fig. 1. The solid line represents the scale factor of the piecewise model corresponding to energy densities I and V, the dashed line to II and III, and both the solid and the dot-dashed lines to I and IV.

$$Q^{(+,-)}(a) = Q(\alpha > 0, \beta = -1, a < a_s, \rho < \rho_s). \quad (27)$$

In the forthcoming sections, we are going to explore with enough detail two piecewise models, one will be driven by interaction pieces (24)–(25) and the other will be associated with (26) and (27), respectively.

III. SOLUTIONS FOR THE SCALE FACTOR

From the conclusions we have reached in Sec. II, we are in a position to assert that the Universe may evolve to a finite-time future singularity. We are going to take a closer look at this possibility by inspecting the physical mechanism behind the occurrence of such kind of event within the interacting dark sector model. One way to achieve such a goal is examining not only the source variables as energy density and pressure as functions of the scale factor but also studying some of these quantities, for example, the scale factor and the energy density as functions of the cosmic time and its subsequent time derivatives provided both variables are related through the Einstein's field equation. In this direction, we will focus on the geometric part of the interacting dark sector model and use the Friedmann equation, $3H^2 = \rho$, to describe the time evolution of the scale factor along with the energy density. For that purpose, we rewrite the Friedmann equation in terms of the dimensionless variable x as

$$\dot{x}^2 = 3\rho_s(1 + \beta x)x'^2. \quad (28)$$

Equation (28) suggests that we can find the approximate solution for the scale factor along with the related quantities in certain limiting cases as a useful manner to find some connection between the interacting dark sector and the existence of finite-time future singularities. We will examine how this procedure can be implemented in a consistent manner in the forthcoming subsections.

A. Scale factor solution near the sudden future singularity

Near the singularity at $a = a_s$, where the interaction term (9) diverges, $\rho \to \rho_s$ and $x \to 0$, we consider only the contribution of the term, αx^{-n}, in the first integral (13) of the source equation (10) provided it is the leading term, so that the first integral reduces to the following one:

$$x' \approx -\alpha\gamma_m x^{-n}. \quad (29)$$

Combining Eqs. (28) and (29), we obtain the Friedmann equation near a_s,

$$\dot{x} \approx -3\alpha\gamma_m H_s x^{-n}\left[1 + \frac{\beta}{2}x\right], \quad (30)$$

where the expansion rate $H_s = \sqrt{\rho_s/3}$ and whose approximate implicit solution is given by

$$\frac{x^{n+1}}{n+1} - \frac{\beta x^{n+2}}{2(n+2)} \approx -3\alpha\gamma_m H_s(t - t_s). \quad (31)$$

Solving iteratively the last equation, we get the approximate time dependence of the scale factor, its first time derivative, and the cosmic acceleration near the singularity at $t = t_s$; these are

$$a(t) \simeq a_s\left[1 + H_s\Delta t - \frac{\beta[-3\alpha\gamma_m(n+1)H_s\Delta t]^{\frac{n+2}{n+1}}}{6\alpha\gamma_m(n+2)} + \cdots\right], \quad (32)$$

$$\dot{a}(t) \simeq a_s H_s\left[1 + \frac{\beta}{2}[-3\alpha\gamma_m(n+1)H_s\Delta t)]^{\frac{1}{n+1}} + \cdots\right], \quad (33)$$

$$\ddot{a}(t) \simeq -\frac{3\alpha\beta\gamma_m a_s H_s^2}{2}[-3\alpha\gamma_m(n+1)H_s\Delta t)]^{\frac{-n}{n+1}} + \cdots, \tag{34}$$

where $\Delta t = t - t_s$. For $n > 0$, we have that the scale factor $a(t_s) = a_s$ and its first time derivative $\dot{a}(t_s) = a_s H_s$ are both finite but its second time derivative $\ddot{a} \to -\alpha\beta\infty$ and the subsequent time derivatives diverge in the limit $t \to t_s$. When the parameters α and β involved in the interacting piecewise model have the same signs a sudden future singularity will occur at the cosmic time t_s.

For the sake of completeness, we integrate the approximate first integral (29) and compose the solution with the scale factor (32), in order to find a power expansion of the energy density (17) as well as its time derivative near the singularity at t_s,

$$\rho \approx \rho_s \left\{1 + \beta[-3\alpha\gamma_m(n+1)H_s\Delta t]^{\frac{1}{n+1}} + \frac{\beta^2[-3\alpha\gamma_m(n+1)H_s\Delta t]^{\frac{2}{n+1}}}{2(n+2)} + \cdots\right\}, \tag{35}$$

$$\dot{\rho} \approx -3\alpha\beta\rho_s\gamma_m H_s[-3\alpha\gamma_m(n+1)H_s\Delta t]^{\frac{-n}{n+1}} + \cdots. \tag{36}$$

Combining the cosmic acceleration (34) with the first time derivative of the energy density (36), we obtain a simple relation between them:

$$\dot{\rho} \approx 6H_s \frac{\ddot{a}}{a_s}. \tag{37}$$

From Eqs. (35)–(37), we have explicitly a finite energy density $\rho(t_s) = \rho_s$ at the singularity while its time derivative $\dot{\rho} \to -\alpha\beta H_s\infty$ in the limit $t \to t_s$ as long as the coupling constant remains positive ($n > 0$). In this case, a sudden future singularity happens for an expanding universe, $H_s > 0$. In addition, from Eqs. (4) and (5), we find that the dark energy densities and the barotropic index $\rho_m \to \pm\infty$, $\rho_x \to \mp\infty$ and $\gamma = -\dot{\rho}/3H\rho \to \mp\infty$ diverge in the limit $t \to t_s$. At the same time, the interaction term Q diverges at the cosmic time t_s.

B. Big bang and big crunch singularities

The interacting dark sector model has an initial singularity at $t = t_{\mathrm{BB}}$ when $\beta = 1$, $\alpha > 0$ and $n > -1$. In this case, the leading term in the energy density (17) is given by $\rho_{\mathrm{BB}} \approx \rho_s\alpha^{1/(n+1)}(a_s/a_{\mathrm{BB}})^{3\gamma_m}$., since near the initial singularity $a_{\mathrm{BB}} \to 0$ and the energy density blows up. Then, inserting ρ_{BB} into the Friedmann equation (28), we find that the dominant term of the approximated solution is the power law scale factor,

$$a_{\mathrm{BB}}(t) \simeq a_s\left[\frac{3}{2}\gamma_m H_s\alpha^{1/2(n+1)}(t - t_{\mathrm{BB}})\right]^{\frac{2}{3\gamma_m}} + \cdots, \tag{38}$$

with $t > t_{\mathrm{BB}}$ and $H_s > 0$ for an expanding universe. We have that $\rho \to \rho_{\mathrm{BB}} \to \infty$ and $a \to 0$ when $t \to t_{\mathrm{BB}}$ while the equation of state (19) becomes linear $p_{\mathrm{BB}} \approx (\gamma_m - 1)\rho_{\mathrm{BB}}$ and the pressure diverges in the same limit. Then, the big bang singularity occurs at $t = t_{\mathrm{BB}}$, where the energy density ρ, the pressure p, $\dot{\rho}$, $\dot{p}$ and their higher derivatives diverge. Near this primordial singularity $\rho_m \to \infty$, $\rho_x \to \rho_s$, the interaction term (16) has a vanishing limit and dark components decouple in the limit $t \to t_{\mathrm{BB}}$. So the appearance of the initial singularity is essentially caused by the growing without limit of the DM energy density. In contrast, at the sudden future singularity, the interaction Q diverges and the dark components become strongly coupled. This is a very significant result because the appearance of a finite-time future singularity appears to be strongly linked with the divergence of the interaction term at the limit $t \to t_s$. In the following, we will reinforce such reciprocity.

Also there exists a possibility that a big crunch singularity occurs at cosmic time $t_{\mathrm{BC}} = t_{\mathrm{BB}}$ when the entire Universe contracts from the past and collapses in the limit $t \to t_{\mathrm{BC}}$, such that $t < t_{\mathrm{BC}}$ and $H_s < 0$, under its own gravity until all known structures are concentrated at one point (see Fig. 2).

C. Bouncing scenario

For $\alpha > 0$, $\beta = -1$ and $\rho \le \rho_s$ there exists a scale factor value a_e,

$$a_e = a_s[1 + \alpha^{-1}]^{\frac{-1}{3\gamma_m(n+1)}}, \tag{39}$$

such that the energy density (17) vanishes ($\rho_e = \rho(a_e) = 0$) and the scale factor has an extremum. To explore the behavior of physical quantities near a_e, it is useful to define a small departure δ from a_e as follows

$$\delta = \frac{a - a_e}{a_e}, \qquad |\delta| \ll 1. \tag{40}$$

Taking into account that $\rho_e = 0$, we make a first order Taylor expansion of the energy density (17) around a_e,

$$\rho(\eta) \approx \rho'(\eta_e)(\eta - \eta_e) + \cdots, \tag{41}$$

$$\rho'_e = \rho'(\eta_e) = (1 + \alpha)\rho_s\gamma_m, \tag{42}$$

where $\rho'(\eta_e)$ is positive definite, $\eta_e = 3\ln a_e$ and $\eta - \eta_e = 3\ln a/a_e = 3\ln(1 + \delta) \approx 3\delta$. The integration of the approximate Friedmann equation, $\dot{\delta} \approx \sqrt{\rho'(a_e)\delta}$, leads to the scale factor in terms of the cosmic time, and it reads

$$a(t) \simeq a_e\left[1 + \frac{\rho_e'}{4}(t - t_e)^2 + \cdots\right]. \qquad (43)$$

Then, a_e represents a minimum of the scale factor where it bounces at $t_b = t_e$ and $a_b = a(t_b) = a_e$.

From Eqs. (16)–(21), we have that the interaction term (16) becomes constant at the bounce $Q(a_b) = -n\alpha\beta(1+\alpha)\rho_s\gamma_m$, a similar result can be noticed for the DM and VVE densities $\rho_{mb} = \rho_m(a_b) = -(1+\alpha)\rho_s = -\rho_{xb} > 0$ and the pressure $p_b = p(a_b) = -\rho_b' = -(1+\alpha)\gamma_m\rho_s$ turns negative. In short, all the source variables have a constant nonvanishing value at a_b. Such signature can be used as a physical property to distinguish a finite-time singularity from a bounce event.

On the other hand, the energy density also vanishes if $\alpha < -1$, $\beta = -1$, and $\rho \le \rho_s$. The particular value $\alpha = -1$ is excluded from the analysis because the energy density vanishes in the limit $a \to \infty$. For the remaining values of α, the extremum a_e represents a maximum of the scale factor, meaning that the scale factor expands until it reaches the maximum value $a_{\max} = a_e$ and then begins to contract (see Fig. 2).

D. de Sitter phase

In Sec. II, we have obtained the simple constant solution,

$$\rho_{\rm dS} = \rho_s[1 + \beta(-\alpha)^{1/(n+1)}], \qquad H_{\rm dS} = \sqrt{\frac{\rho_{\rm dS}}{3}}, \qquad (44)$$

which includes a contracting de Sitter solutions for $\alpha < 0$, $\beta = 1$ and an expanding one for $-1 < \alpha < 0$, $\beta = -1$. For these solutions, we find that the dark energy densities are $\rho_{m_{\rm dS}} = 0$, and $\rho_{x_{\rm dS}} = \rho_{\rm dS}$ with $p_{\rm dS} = -\rho_{\rm dS}$. In these cases, the interaction term (9) vanishes and the model becomes decoupled. We investigate the behavior of solutions $x = x_{\rm dS}(1 + \epsilon)$ of Eq. (13) near to $x_{\rm dS} = \rho_s(1 + \beta\rho_{\rm dS})$ with $\epsilon \ll 1$ by expanding the Eq. (13) to first order in ϵ,

$$\dot\epsilon = -3(1 + n)\gamma_m H_{\rm dS}\epsilon. \qquad (45)$$

This shows that in the regions defined by $\alpha < 0$ and $\beta = \pm 1$, the contracting de Sitter phase (anti-de Sitter) $a \propto \exp(-H_{\rm dS}t)$ is unstable for $n > -1$ and the expanding one $a \propto \exp(H_{\rm dS}t)$ is stable. Then, the solution of the source equation (10) approach to the stable de Sitter solution (44) in the limit $t \to +\infty$.

E. Comments on big brake and sudden future singularities

In the last part of this section, we will make some comments relative to the effective equation of state in the interacting dark sector and put in evidence of the main features of the big brake and the sudden future singularities when they occur as a product of nonlinear interacting processes. In particular, we will refer to two different cases which might seem to share some similarities but they are really different in nature as we will show later. In Ref. [23] we have investigated the occurrence of the big brake singularity when the dark sector includes two components, dark matter and VVE coupled with an interaction term having the form of Eq. (9) with $\rho_s = 0$. On the other hand, we have the sudden future singularity examined in the present interacting model, where the nonlinear interaction term is given by (9) with $\rho_s \neq 0$. In the former case, the effective equation of state has the form

$$p_b = (\gamma_m - 1)\rho + \alpha\gamma_m\rho^{-n}, \qquad (46)$$

of a modified Chaplygin gas while in the latter case the effective equation of state is given by Eq. (19). Now, let us explore the essential differences between both equation of states which give rise to the big brake and the sudden future singularities. To this end, we make use of the energy density expansion (35) near the future singularity at the cosmic time t_s

$$\rho \approx \rho_s\{1 + \beta[-3\alpha\gamma_m(n+1)H_s\Delta t]^{\frac{1}{n+1}} + \cdots\}, \qquad (47)$$

where without loss of generality, we have taken into account only the main contribution of the expansion (35) in power of Δt, and inserted it into the effective equation of state (19),

$$p \approx -\rho_s + \beta\rho_s(\gamma_m - 1)[-3\alpha\gamma_m(n+1)H_s\Delta t]^{\frac{1}{n+1}} + \alpha\beta\rho_s\gamma_m[-3\alpha\gamma_m(n+1)H_s\Delta t]^{\frac{-n}{n+1}} + \cdots. \qquad (48)$$

Surprisingly, the pressure $p \to 0$ in the limit $\rho_s \to 0$; hence, it does not reduce to equation of state (46) that characterizes the big brake singularity. However, in the case that $\rho_s \neq 0$, we have a sudden future singularity provided $1/(n+1) > 0$ and $-n/(n+1) < 0$, meaning that $n > 0$. In fact, when $t \to t_s$ in Eqs. (32)–(34), we obtain that the scale factor $a \to a_s$ and its first time derivative $\dot a \to a_s H_s$ are finite while the acceleration $\ddot a \to \pm\infty$ diverges. Also, taking into consideration Eqs. (47) and (48) in the same limit, we have that the energy density $\rho \to \rho_s$ is finite but the pressure $p \to \pm\infty$ diverges. In conclusion, the big brake singularity cannot be obtained by taking the limit $\rho_s \to 0$ in the present model driven by the nonlinear interaction (9). Therefore we need to investigate the big brake and the sudden future singularities separately, being the former one identified with the nonlinear interaction (9) with $\rho_s = 0$ while the latter one must have $\rho_s \neq 0$ due the physical definition of a sudden singularity. The aforesaid results are indicating that the parameter ρ_s plays a special role in describing a Universe with a sudden singularity, namely ρ_s introduces a discontinuity in the limit $\rho_s \to 0$ for the effective equation of state (9) which is linked with the own nature of sudden singularity. In another words, the interacting model associated with the sudden singularity cannot lead continuously to the other interacting model related with the big brake singularity provided the own different physical nature of both singularities, so both scenarios deserve to be explored on their own merits. Indeed, new physical insights can be gained by exploring this cosmological scenario and because of that many authors devoted several efforts to explore the emergence of the sudden singularity on general grounds or alternative frameworks [7,8,10]. Here, we explored other appealing aspects about the risen of sudden singularity along with the possibility of crossing this soft singularity within the interacting dark energy framework. Regarding the second topic, we will examine the joining of two universes connected by a sudden singularity in the next section as a way to show the richness of the nonlinear interaction chosen in this work.

IV. PIECEWISE NONLINEAR MODEL IN THE DARK SECTOR

So far we have presented an interacting dark-sector model in which DM interacts with VVE. The interaction has been divided in four different pieces (24)–(25) according to the four regions in the plane (a, ρ) delimited by the sign of the parameters α and β, where each one of these pieces has a physcial meaning associated with a particular region of that plane. In this section, we will extend our previous analysis on the cosmological model generated by those interaction pieces, in particular, we will match two different kinds of interaction pieces in order to show a possible extension of the Universe through the finite-time future singularity at $t = t_s$. In doing so, we will examine the interaction pieces near the future singularity, explore the behavior of the energy density, its η derivative, the DM, and VVE densities along with the equation of state of the content of the Universe. In a way, we will offer a connection between two universes (one of these prior to the singularity event) and another universe (posterior to that event) emerging from the future singularity, taking into account that in both cases the content of the Universe includes interacting DM and VVE. Also, the interacting model will produce several types of singularity events and different kinds of cosmological scenarios that we will investigate in detail.

A. Cosmological model driven by $Q^{(+,+)}$ and $Q^{(-,-)}$

As we mentioned earlier, the present interacting cosmological model has two parts; one is driven by $Q^{(+,+)}(a)$, which is defined by $\alpha > 0$, $\beta = 1$ and turns out to be restricted to the region $a < a_s$ and $\rho > \rho_s$ in the plane (a, ρ). However, there is another part driven by $Q^{(-,-)}(a)$, which is specified by $\alpha < 0, \beta = -1$, so it is associated with the region $a > a_s$ and $\rho < \rho_s$. The explicit dependence on the scale factor for $Q^{(+,+)}$ and $Q^{(-,-)}$ together with other useful quantities of the model are listed in Eqs. (16)–(27).

For $n > 0$, the interaction piece $Q^{(+,+)} < 0$ and the Universe starts from a big bang singularity at $t = t_{\rm BB}$, where $Q^{(+,+)} \to 0$, $\rho \to \infty$, $\rho' \to -\infty$, the pressure $p \to \infty$, the DM energy density $\rho_m \to \infty$, and the VVE density $\rho_x \to \rho_s$ in the limit $a \to 0$. Later, the model describes a matter-dominated universe with a power-law scale factor. Interestingly enough, the DM density decreases until it reaches its minimum value at $\rho_{m,\min} = \rho_s(1 + n^{-1})(\alpha n)^{1/(n+1)}$, where $\rho'_m = 0$ and ρ has an inflection point. Then, the DM density begins to increase and, finally, $\rho_m \to +\infty$ when $a \to a_s$. However, the VVE density at the initial singularity is $\rho_x = \rho_s$, later decreases with the scale factor provided $\rho'_x = Q^{(+,+)} < 0$, so then vanishes and finally $\rho_x \to -\infty$ as $a \to a_s$ while $\rho = \rho_m + \rho_x \to \rho_s$ remains finite.

Near the cosmic time t_s, the approximate scale factor (32) takes the following form:

$$a^{(+,+)} \simeq a_s\left[1 + H_s\Delta t - \frac{[-3\alpha\gamma_m(n+1)H_s\Delta t]^{\frac{n+2}{n+1}}}{6\alpha\gamma_m(n+2)}\cdots\right],$$
$$\alpha > 0, \quad \beta > 0, \quad a < a_s,$$
$$\rho > \rho_s, \quad H_s > 0, \quad t < t_s. \tag{49}$$

The sudden future singularity occurs at the finite time t_s where the scale factor $a_s = a(t_s)$ and its first time derivative $\dot{a}_s = \dot{a}(t_s) = a_s H_s$ are both finite but its second time derivative $\ddot{a} \to -\infty$ for $t \to t_s$. The Hubble expansion rate $H_s = H(t_s)$ and the energy density (35), $\rho_s = \rho(t_s)$, has a finite value at the singularity; however, the pressure $p = \gamma_m\rho_m - \rho \to +\infty$ diverges in the limit $t \to t_s$.

In the other part of the piecewise model driven by the positive interaction piece $Q^{(-,-)} > 0$, the scale factor (32) near a_s can be recast as

$$a^{(-,-)} \simeq a_s\left[1 + H_s\Delta t + \frac{[-3\alpha\gamma_m(n+1)H_s\Delta t]^{\frac{n+2}{n+1}}}{6\alpha\gamma_m(n+2)}\cdots\right],$$
$$\alpha < 0, \quad \beta < 0, \quad a > a_s,$$
$$\rho < \rho_s, \quad H_s > 0, \quad t > t_s. \tag{50}$$

The DM and VVE densities (20) and (21) have the limits $\rho_m \to +\infty$ and $\rho_x \to -\infty$, respectively, when $a \to a_s$. However, the energy density $\rho = \rho_s$ remains finite although the pressure $p = \gamma_m\rho_m - \rho \to +\infty$ diverges. In addition, we have that the scale factor and its first time derivative a_s and $\dot{a}_s = a_s H_s$ are finite whereas the cosmic acceleration $\ddot{a} \to -\infty$ in the limit $t \to t_s$.

Let us make a comment about the final piecewise model based on the process of matching two interacting universes. The scale factor and expansion rate (geometrical variable) along with the DM and VVE densities, the energy density, its time derivative, and pressure of the effective fluid (source variables) have the same limits in the limit $t \to t_s$ independently whether the future singularity is reached by the first part of the model which we have identified with

$Q^{(+,+)}$ or by the second part of the model identified with $Q^{(-,-)}$. So that, the evolution across the singularity at t_s is completely regular which in turn means that comoving observers can transit from an expanding universe driven by $Q^{(+,+)}$, passing through the sudden future singularity, and emerging in an expanding universe driven by $Q^{(-,-)}$ where the singularity now is located in its distant past.

Far away from the singularity, the solutions of the model driven by $Q^{(-,-)}$ evolve in two different forms according to the values assigned to the integration constant α. For instance, when $\alpha < -1$ there is a set of scale factor solutions for which the energy density vanishes $\rho_0 = \rho(a_{\max}) = 0$ at a specific value of the scale factor,

$$a_{\max} = a_s[1 + \alpha^{-1}]^{\frac{-1}{3\gamma_m(n+1)}}, \tag{51}$$

which is the maximum value reached by the scale factor (43) belonging to that set of solutions, and its expression near $a_{\max} = a(t_{\max})$ is given by

$$a(t) \simeq a_{\max}\left[1 + \frac{(1+\alpha)\rho_s\gamma_m}{4}(t - t_{\max})^2 + \cdots\right]. \tag{52}$$

After that, the Universe reverses its expansion ($H_s < 0$) and begins to collapse, possibly into a big crunch. In this case, it seems that the catastrophic final of the Universe identified with the present piecewise model would be unavoidable anyway.

For the other set of solutions, for which the integration constant ranges between $-1 < \alpha < 0$, the energy density always decreases and varies between $\rho_{\rm dS} < \rho < \rho_s$. In this case, this expanding branch of solutions emerge from a singularity in the past and the interaction piece $Q^{(-,-)}$ accelerates the expansion, driving the Universe to a final de Sitter stage described by the Eq. (44) with $\beta = -1$. From Eqs. (20) and (21), we have that $\rho_m \to 0$, the VVE density increases asymptotically to achieve the final energy density $\rho_x \to \rho \to \rho_{\rm dS}$ and the asymptotic equation of state turns in $p_{\rm dS} \approx -\rho_{\rm dS}$, being the fuel that controls the final dynamic of the Universe. Consequently, the interaction piece $Q^{(-,-)} \to 0$ at late times and the dark sector becomes decoupled in the remote future.

Summarizing, we have constructed a piecewise model driven by the interaction pieces $Q^{(+,+)}$ and $Q^{(-,-)}$. This model describes a universe that begins in the far past from a big bang singularity in the region of the plane (a, ρ) identified with $Q^{(+,+)}$. Then it has a matter-dominated era and exhibits an accelerated expansion until it reaches the sudden future singularity at t_s, where the DM and VVE densities $\rho_m \to +\infty$, $\rho_x \to -\infty$ and the pressure $p = \gamma_m\rho_m - \rho$ diverge in the limit $t \to t_s$. This clearly shows that the DM and VVE densities produce the sudden future singularity. In the other region of the piecewise model, the interaction piece $Q^{(-,-)}$ generates two branches of solutions; one expanding branch emerges from a distant past singular event, with the scale factor reaching a maximum from which the Universe reverses and collapses into itself, possibly in a big crunch with a catastrophic finale. However, the other expanding branch ends in a stable de Sitter phase, avoiding a dramatic final fate.

B. Cosmological model driven by $Q^{(-,+)}$ and $Q^{(+,-)}$

We will construct a different piecewise model by gluing two interacting models. One part of the model is based on the options $\alpha < 0$ and $\beta = 1$ so that $a > a_s$ and $\rho > \rho_s$, indicating that the exchange of energy is driven by the interaction piece $Q^{(-,+)}(a)$. On the other hand, the choice $\alpha > 0$ and $\beta = -1$ corresponds to $a < a_s$ and $\rho < \rho_s$ which is related with the interaction piece $Q^{(+,-)}(a)$. The main quantities involved in this piecewise model, as functions of the scale factor, are obtained from Eqs. (16)–(27).

The Universe begins with a contracting de Sitter (inflationary) phase in the far distant past, with a decoupled dark sector, $Q^{(-,+)} \approx 0$, $\rho_m \approx 0$, $\rho \approx \rho_x \approx \rho_{\rm dS}$ and equation of state $p_{\rm dS} \approx -\rho_{\rm dS}$, until it nearly reaches the cosmic time t_s where the approximate scale factor (32) is given by

$$\begin{aligned} a^{(-,+)} &\simeq a_s\left[1 + H_s\Delta t - \frac{[-3\alpha\gamma_m(n+1)H_s\Delta t]^{\frac{n+2}{n+1}}}{6\alpha\gamma_m(n+2)}\cdots\right], \\ &\alpha < 0, \quad \beta > 0, \quad a > a_s, \\ &\rho > \rho_s, \quad H_s < 0, \quad t < t_s. \end{aligned} \tag{53}$$

From this equation, we have that both $a \to a_s$ and $\dot{a} \to \dot{a}_s = a_s H_s < 0$ remain finite at $t = t_s$, while the cosmic acceleration grows without ($\ddot{a} \to +\infty$) at the limit $t \to t_s$. This points out that the piecewise model also has a finite-time future singularity at t_s. Regarding the dark energy components, we find that $\rho_m \to -\infty$, $\rho_x \to +\infty$ while $\rho \to \rho_s$, $\rho' \to +\infty$ and $p \to -\infty$ when $t \to t_s$. In addition, the energy density ranges between $\rho_{\rm dS} \le \rho \le \rho_s$, whereas the scale factor varies from ∞ to a_s.

When the Universe emerges from the finite time future singularity at t_s, the other interaction piece $Q^{(+,-)}$ takes the control of the dynamics, leading to a contracting phase. Consequently, the scale factor (32) is written as

$$\begin{aligned} a^{(+,-)} &\simeq a_s\left[1 + H_s\Delta t + \frac{[-3\alpha\gamma_m(n+1)H_s\Delta t]^{\frac{n+2}{n+1}}}{6\alpha\gamma_m(n+2)}\cdots\right], \\ &\alpha > 0, \quad \beta < 0, \quad a < a_s, \\ &\rho < \rho_s, \quad H_s < 0, \quad t > t_s. \end{aligned} \tag{54}$$

The interaction piece $Q^{(+,-)}$ and the remaining relevant quantities ρ, ρ', p, ρ_m, ρ_x as functions of the scale factor are given by Eqs. (16)–(21) together with (27). In this piece of model, $Q^{(+,-)} \to +\infty$, $\rho(a_s) = \rho_s$, $\rho' \to +\infty$ and the pressure $p \to -\infty$, causing $\ddot{a} \to +\infty$ as $t \to t_s$. Taking into account that $\rho_m < 0$ and $\rho_x > 0$ in the aforesaid scenario, we find that $\rho_m \to -\infty$ and $\rho_x \to +\infty$ while the sum of the partial densities remains bounded $\rho = \rho_m + \rho_x = \rho_s$ in the limit $t \to t_s$. These outcomes are in agreement with our previous results obtained for *geometrical and source variables* which characterize a finite-time future singularity.

The Universe continues as before until bounces at the scale factor value a_b, given by

$$a_b = a_s[1 + \alpha^{-1}]^{\frac{-1}{3\gamma_m(n+1)}}, \tag{55}$$

for which $\rho(a_b) = 0$, $a_b \le a \le a_s$ [see Eq. (39)] and the approximated scale factor (43) near the bounce at $t = t_b$ takes the next form

$$a(t) \simeq a_b\left[1 + \frac{(1+\alpha)\rho_s\gamma_m}{4}(t - t_b)^2 + \cdots\right]. \tag{56}$$

The Universe reverses its contraction at $t = t_b$, begins to expand ($H_s > 0$) and possible will enter into a stable de Sitter phase driven by the fuel provided by the VVE density which dominates over the DM one.

Summing up, the piecewise model driven by the interaction pieces $Q^{(-,+)}$ and $Q^{(+,-)}$ gives rise to a universe which begins in the distant past in a contracting de Sitter phase, then one can extend the evolution of the Universe through the finite-time future singularity, emerges from that singularity and reaches a bounce until reverses it. Then the Universe begins to expand and the aforesaid dynamical mechanism produces a stable de Sitter phase described by the asymptotic equation of state of the dark sector (19), $p_{\rm dS} \approx -\rho_{\rm dS}$, indicating that the VVE component becomes the dominant contribution.

V. TIPLER AND KRÓLAK METHOD

In the previous sections, we extracted quite general conditions for the existence of a finite-time future singularity within the framework of an interacting dark sector based on a given nonlinear interaction and a piecewise nonlinear interaction counterpart. We elaborated some new appealing cosmic scenarios by matching in a smooth way two different interacting sectors with the idea of showing how the finite-time future singularity can be continued, offering a new scheme where the aforesaid singularity is viewed as a two-door way to another phase of the Universe. We have also mentioned why this ansatz is physically relevant provided that the effective model is related with an relaxed version of the Chaplygin gas equation of state. In this section, we will try to focus on a geometrical aspect of the previous findings. Thus, we would like to complete our analysis of the future singularity by taking into account a geometric point of view based on a method developed by Tipler [11] and Królak [12]. To summarize these criteria briefly, we recall that a spacetime is Tipler strong [11] if, as the proper time $t \to t_s$, the integral

$$\mathcal{T}(t) = \int_0^t dt' \int_0^{t'} |\mathcal{R}_{\alpha\beta}u^\alpha u^\beta| dt'' \to \infty. \tag{57}$$

In same manner, a spacetime is Królak strong [12] if, as the proper time $t \to t_s$, the integral

$$\mathcal{K}(t) = \int_0^t |\mathcal{R}_{\alpha\beta}u^\alpha u^\beta| dt \to \infty, \tag{58}$$

where the components of the Ricci tensor are understood to be written in a frame which is transported parallel along the geodesic curves. A singularity or event can be strong by Królak criteria but weak according to Tipler's criteria; however, the reverse situation always holds. Because weak singularities can be extended beyond them, the method developed by Tipler and Królak are useful tools for determining the fate of the Universe in terms of the fate of geodesic curves near potential strong singular point [24]. Let us consider timelike geodesic curves, $x^i = c$ with i spatial index and c a constant [25], associated with a comoving observer; i.e., we take into account a comoving worldline congruence with velocity $u^\alpha = (\partial_t)^\alpha = (1, 0, 0, 0)$ so that the proper time coincides with the coordinate time. The components of the Ricci tensor measured by the observer along this congruence lead to $\mathcal{R}_{\alpha\beta}u^\alpha u^\beta = -3\ddot{a}(t)/a(t)$ [26]. Let us take into account the first piecewise model. Using (49), calculating its second derivative, and replacing in Eq. (58), we can demonstrate that the Królak invariant yields $\mathcal{K}(t) \propto [\kappa(t - t_s)]^m$ with $m > 0$ provided the exponent satisfies the relation $(n+2)/(n+1) > 1$, so the Krolak measure does not diverge as $t \to t_s$; such fact indicates that the sudden event is K-weak. On the other hand, the Tipler measure involves a second integration, giving a $\mathcal{T}(t) \propto [\kappa(t - t_s)]^{m+1}$ regular quantity as $t \to t_s$ because $n > -1$; this shows that the aforesaid event is T-weak also and, therefore, a transversable singularity. The same kind of procedure can be done for the second part of the piecewise model with (50). The regularity of these invariants on the both sides of the singularity ensures the continuity of the limits at the sudden future singularity. This fact is closely related with the regularity of the scale factor at both sides of the sudden future singularity. We obtained similar results for the other piecewise model associated with (53) and (54). Besides, the weakness of sudden future singularity was recently proved [10].

As a short comment, let us mention that at the bounce the Tipler and Królak measures behave in a regular way because $\ddot{a}(\eta)/a(\eta) \propto [1 - p\eta^2]$ with $\eta = t - t_b$, while for the big bang or big crunch singularities, both invariants diverge.

VI. SUMMARY

We have investigated an interacting dark sector model composed of DM and VVE with linear equations of state for a spatially flat FRW spacetime and obtained the source equation associated with this system. We have solved the source equation and obtained the general solution for the dark sector energy density, when the DM and VVE are coupled with an extension of the nonlinear interaction introduced in Ref. [22]. The general solution includes several family of solutions, so with the purpose of constructing a finite-time future singularity at $t = t_s$ within this interacting framework, we have selected the family of solutions which produces a divergence of the nonlinear interaction for $\rho \to \rho_s = \rho(a(t_s))$, where t_s is an specific value of the cosmic time such that the scale factor $a_s = a(t_s)$ and the energy density $\rho_s = \rho(a_s)$ are both finite at the limit $t \to t_s$. For this family of solutions, we have shown that the cosmic acceleration $\ddot{a}/a = \rho/3 + \rho'/2 \to \pm\infty$ because ρ' diverges as $a \to a_s$. Additionally, the pressure of the whole dark sector $p = -\rho - \rho'$, that is related with a relaxed version of the Chaplygin gas equation of state, diverges in the limit $\rho \to \rho_s$ for $t \to t_s$. In this interacting dark sector model, the pressure $p = \gamma_m\rho_m - \rho = (\gamma_m - 1)\rho - \gamma_m\rho_x$, diverges as a consequence that both DM and VVE densities also diverge in the limit $t \to t_s$. In the same manner, the occurrence of a finite-time future singularity is strongly associated with the divergent behavior of ρ_m and ρ_x. It was shown the importance of considering a nonlinear coupling between them in order to show how the behavior of dark energy densities produces that singularity. After simple considerations, we have separated the nonlinear interaction term in four pieces according to the four regions in the plane (a, ρ), defined by the scale factor, the energy density and uniquely identified with the signs of the parameters α and β. In each one of these regions the nonlinear interaction changes its form and has a proper significance. In this context, we have obtained the solution of the Friedmann equation for the scaled factor which describes several types of singularities and different kinds of scenarios which we have investigated in enough detail. In particular, by using the approximate scale factor we have studied its behavior near the finite-time future singularity, big bang and big crunch as well as power law, bounce, and de Sitter scenarios.

We have constructed a piecewise cosmological model driven by the interaction pieces $Q^{(+,+)}$ and $Q^{(-,-)}$, gluing the corresponding scale factors at t_s to extend the Universe through the sudden future singularity located at t_s. We have explored the dynamics of the model by examining the behavior of the energy density, its derivative, the DM and VVE densities along with the equation of state of the Universe. We have described an expanding matter-dominated universe that starts from a big bang singularity driven for $Q^{(+,+)}$ and then continues to a sudden future singularity. We have encountered that the scale factor $a_s = a(t_s)$, its first time derivative $\dot{a}_s = \dot{a}(t_s)$ and the energy density $\rho_s = \rho(a(t_s))$ are finite when they are evaluated at t_s, but the acceleration $\ddot{a}$ diverges for $t \to t_s$. Regarding the dark densities, we have found that $\rho_m \to +\infty$, $\rho_x \to -\infty$ while the total pressure $p = \gamma_m\rho_m - \rho$ diverge in the limit $t \to t_s$. This showed us the strong relation that exists between sudden future singularity at t_s and divergent behavior of DM and VVE densities for $t \to t_s$. After that, the Universe emerged from a singularity in the past driven by $Q^{(-,-)}$ with two possible branches of solutions. The branch $\alpha < -1$, reaches a maximum at $a_{\rm max} = a(t_{\rm max}) > a_s$, then reverses its expansion, collapses and possibly end in a big crunch singularity. Nevertheless, the case with $-1 < \alpha < 0$ represents an expanding scenario and the Universe evolves asymptotically towards an accelerated de sitter phase. These solutions are asymptotically stable and avoid a dramatic final fate.

We have generated a second piecewise model with the interaction pieces $Q^{(-,+)}$ and $Q^{(+,-)}$, and gluing the respective scale factor solutions at t_s. The Universe begins in the distant past in a contracting de Sitter phase, with a nearly vanishing interaction piece $Q^{(-,+)} \approx 0$ and a decoupled dark sector. The DM energy density $\rho_m \approx 0$, the energy density $\rho \approx \rho_x \approx \rho_{\rm dS}$ and the VVE energy has an approximate de Sitter equation of state $p_{\rm dS} \approx -\rho_{\rm dS}$. The Universe contracts and then continues to a finite-time future singularity at $t = t_s$, where the scale factor $a_s = a(t_s)$, its first time derivative $\dot{a}_s = a_s H_s < 0$ and the energy density remain finite at $t = t_s$, while the cosmic acceleration grows without limit ($\ddot{a} \to +\infty$) in the limit $t \to t_s$, so these results indicate a finite-time future singularity at t_s. Then, the Universe emerges driven by the interaction piece $Q^{(+,-)}$, bounces at the scale value $a_b = a(t_b)$, and after that reverses, begins to expand, and subsequently the VVE density begins to dominate over the DM one. Then, the expanding universe has a stable de Sitter phase, described by the asymptotic dark sector equation of state, $p \to p_x \to p_{\rm dS} \approx -\rho_{\rm dS}$. Notice that the approximate scale factor can be extended at the finite-time future singularity with the one obtained from the interaction piece $Q^{(+,-)}$ at $t = t_s$.

Using the approximate scale factor in the vicinity of this singular event along with the geometric method developed by Tipler and Królak for the case of timelike geodesic curves (comoving observer), we have obtained a regular behavior of Tipler and Krolak measures around the sudden future singularity; that is, both measures are well behaved at both sides of the singularity and matched continuously at t_s, which in turn ensures the regularity of these measures for the piecewise model in the whole spacetime. In short, timelike geodesic curves can be extended beyond such singular event, turning into a traversable singularity.

In the near future, we will implement an analysis of the classical stability associated with the sudden future singularity for the piecewise models which will be mainly focused on the scalar modes of the perturbed metric. Such analysis will complement the already known stability studies of sudden future singularity [27].

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[25] In a FRW spacetime, curves of the form $x^i(t) = c$, with t the proper time, fulfill the geodesic equation, $\dot{x}^{\alpha} + \Gamma^{\alpha}{}_{\beta\nu}u^{\beta}u^{\nu} = 0$, provided the only component of 4-velocity is $u^t = 1$, $\dot{u}^t = 0$, and $\Gamma^t{}_{tt} = 0$.
[26] For a FRW metric and the components of the Ricci tensor are $\mathcal{R}_{00} = -3(H^2 + \dot{H})$ and $\mathcal{R}_{ij} = a^3(H^2 + \dot{H})\gamma_{ij}$ with γ_{ij} the spatial metric of the three-dimensional submanifold, whereas the Ricci scalar is $\mathcal{R} = 6(H^2 + \frac{\ddot{a}}{a})$.
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