

## Noncommutative geometry of Zitterbewegung

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(Received 24 December 2016; published 23 March 2017)

Drawing from the advanced mathematics of noncommutative geometry, we model a “classical” Dirac fermion propagating in a curved spacetime. We demonstrate that the inherent causal structure of the model encodes the possibility of Zitterbewegung—the “trembling motion” of the fermion. We recover the well-known frequency of Zitterbewegung as the highest possible speed of change in the fermion’s “internal space.” Furthermore, we show that the bound does not change in the presence of an external electromagnetic field and derive its explicit analogue when the mass parameter is promoted to a Yukawa field. We explain the universal character of the model and discuss a table-top experiment in the domain of quantum simulation to test its predictions.

DOI: 10.1103/PhysRevD.95.061701

### I. INTRODUCTION

The introduction of the Dirac equation [1] was a prodigious step towards the unification of quantum and relativistic principles. Surprisingly enough, the differential operator engaged in the Dirac equation turned out to play a pivotal role in differential geometry [2]. Moreover, it lies at the heart of Connes’ theory of noncommutative geometry [3], which extends such classical notions as differentiation, distance [3], or causality [4] to an abstract algebraic setting. Nowadays, noncommutative geometry centered around the concept of the Dirac operator provides a compelling framework for the study of fundamental interactions [5], yielding concrete testable predictions in the domain of elementary particles [6,7] and gravitational physics [8].

Building upon Connes’ ideas [9], we model a single massive Dirac fermion with the help of an almost commutative spacetime [6]. The latter turns out to provide a geometric explanation of one of the peculiarities of Dirac theory—the Zitterbewegung [10]. We show that the inherent causal structure of the almost commutative spacetime puts an explicit bound on the frequency of the “trembling motion.” We expound the universality of this feature and outline its consequences for quantum gauge theories. Finally, we explain how the concept of quantum simulation [11] can be promoted to emulate almost commutative spacetimes, thus opening the door to a direct experimental test of Connes’ theory.

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### II. ZITTERBEWEGUNG

The Dirac equation,

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0, \quad (1)$$

reveals a number of striking facts about the nature of massive fermions. One of them concerns the velocity operator in the Heisenberg picture,  $\hat{v}_k(t) := \frac{\partial\hat{x}_k(t)}{\partial t}$ , which turns out to have eigenvalues  $\pm c$  for all moments of time [12]. This suggests that the instantaneous velocity of a massive fermion is always  $\pm c$ , which seems paradoxical. A more detailed analysis unveils that both the velocity and the position operators in the Heisenberg picture have a part that oscillates in time, hence the name Zitterbewegung—“the trembling motion” [10].

For an initial state with vanishing average momentum, the expectation value of the position operator oscillates with the period [13–15]

$$T_{\text{ZB}} = \frac{\pi\hbar}{mc^2}. \quad (2)$$

The original explanation of the mechanism behind Zitterbewegung given by Schrödinger [10], and refined by several authors [12,15,16], relates it to the interference between the positive and negative energy parts of the Dirac wave packet. Indeed, if the initial state  $\psi$  is a purely positive (or negative) energy state then the expectation value  $\langle\psi, \hat{x}_k(t)\psi\rangle$  does not exhibit oscillations [10,12].

The tangibility of Zitterbewegung for actual fermions, being *par force* positive-energetic excitations of a quantum field, is generally questioned [17] (see however [18]). On the other hand, the realness of this effect has been

confirmed in various Dirac-like systems [14,19,20] and found applications, most notably, in graphene [21].

There exists an alternative viewpoint on Zitterbewegung (known also under the name of “chiral oscillations” [22,23]) relating it to the spin components of the fermionic wave function [13,24–26]. Any Dirac spinor can be uniquely decomposed as a sum of two Weyl spinors  $\psi = \psi_+ + \psi_-$ , where  $\psi_{\pm} = P_{\pm}\psi$ , with  $P_{\pm} = (1 \pm \gamma^5)/2$  [26]. The Weyl spinors  $\psi_{\pm}$ , as eigenstates of the chirality operator  $\gamma^5$  with  $\gamma^5\psi_{\pm} = \pm\psi_{\pm}$ , have opposite chirality. By acting with the projector  $P_{\pm}$  on Eq. (1) we obtain  $i\hbar\gamma^{\mu}\partial_{\mu}\psi_{\pm} = mc\psi_{\mp}$ , which can be seen as two coupled equations for Weyl spinors  $\psi_{\pm}$ , one acting as a source of the other [26]. Since Weyl spinors are massless, they move with the speed of light and one obtains a “zigzag picture” of a massive fermion (Fig. 25.1 of [25]). In the fermion’s rest frame, the period of oscillations between the two eigenstates of chirality equals precisely (2) [24].

The two points of view on Zitterbewegung are closely related in the Dirac wave-packet formalism [23]. In particular, purely positive/negative energy solutions to the Dirac equation also do not exhibit chiral oscillations [22]. Hestenes contended [13,24] that the “chiral” interpretation of Zitterbewegung is more natural, as the origin of the effect resides in the geometry of spacetime. In the present article we support this claim, although we argue that the very notion of geometry needs to be refined.

### III. NONCOMMUTATIVE GEOMETRY AND CAUSALITY

The basic objects of noncommutative geometry [3] are spectral triples  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  consisting of a (dense subalgebra of a)  $C^*$ -algebra  $\mathcal{A}$ , a Hilbert space  $\mathcal{H}$  with a faithful representation of  $\mathcal{A}$ , and an unbounded self-adjoint operator  $\mathcal{D}$  acting on  $\mathcal{H}$ . The original framework was designed to describe spaces of Euclidean signature and has recently been extended to encompass the Lorentzian ones [4,27]. In the latter case, the main conceptual change consists in endowing the Hilbert space  $\mathcal{H}$  with an indefinite inner product, turning it into a Krein space  $\mathcal{K}$  [28].

**Example 1.** Let  $M$  be a globally hyperbolic spacetime with spin structure; then  $(\mathcal{A}_M, \mathcal{K}_M, \mathcal{D})$  is a Lorentzian spectral triple, with  $\mathcal{A}_M = C_c^{\infty}(M)$ , i.e., the algebra of smooth compactly supported functions on  $M$ ,  $\mathcal{K}_M = L^2(M, S)$ , i.e., the space of square summable sections of the spinor bundle  $S$  over  $M$ , and  $\mathcal{D} = -i\gamma^{\mu}\nabla_{\mu}^S$ , i.e., the (curved) Dirac operator associated with  $S$ . If  $\langle \cdot, \cdot \rangle$  stands for the Hermitian inner product on  $L^2(M, S)$ , then the indefinite inner product on  $\mathcal{K}_M$  can be defined as  $(\cdot, \cdot) = \langle \cdot\gamma^0, \cdot \rangle$  with  $\gamma^0$  being the Hermitian first flat gamma matrix.

Even-dimensional spacetimes induce an additional structure on the associated spectral triple—a chirality operator  $\gamma_M = \gamma^5$ , splitting the space  $\mathcal{K}_M = \mathcal{K}_M^+ \oplus \mathcal{K}_M^-$ , with  $\gamma_M\psi_{\pm} = \pm\psi_{\pm}$  for  $\psi_{\pm} \in \mathcal{K}_M^{\pm}$ .

The most important examples of spectral triples from the viewpoint of physical applications are the almost commutative geometries [6,7].

**Example 2.** Let  $(\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F)$  be a *finite* spectral triple, i.e.,  $\mathcal{A}_F = M_n(\mathbb{C})$ ,  $\mathcal{H}_F = \mathbb{C}^n$ , and  $\mathcal{D}_F = \mathcal{D}_F^{\dagger} \in M_n(\mathbb{C})$  for some  $n \in \mathbb{N}$ , and let  $(\mathcal{A}_M, \mathcal{K}_M, \mathcal{D})$  be as in Ex. 1 with  $\dim M$  being even. Then,  $(\mathcal{A}, \mathcal{K}, \mathcal{D}) = (\mathcal{A}_M \otimes \mathcal{A}_F, \mathcal{K}_M \otimes \mathcal{H}_F, \mathcal{D} \otimes 1 + i\gamma_M \otimes \mathcal{D}_F)$  forms a Lorentzian spectral triple.

Causality is one of the most fundamental principles underlying physical theories. Within Einstein’s theory it is defined as a partial order relation on the set of events:  $p \leq q$  means that  $q$  lies in the future of  $p$ . However, noncommutative spaces typically admit only a global description and the very notion of an event does not make sense. This raises a question: what is the scene for causal relations and what is the operational meaning of a “noncommutative spacetime”?

In [4] we advocated the idea that noncommutative spacetime ought to be understood as the space of (pure) states on a, possibly noncommutative,  $C^*$ -algebra (cf., [29], p. 188). The motivation behind this step is twofold: First, if the algebra at hand is of the form  $\mathcal{A}_M = C_c^{\infty}(M)$  (see Ex. 1), then its pure states  $P(\mathcal{A}_M)$  are in one-to-one correspondence with the events on  $M$  [4]. Secondly,  $C^*$ -algebras provide a unified framework for an operational formulation of both classical and quantum physics [30–32]. In this context, states on a  $C^*$ -algebra of observables can be understood as the actual states of a given physical system. For instance, if  $\mathcal{A}_F = M_n(\mathbb{C})$  (see Ex. 2), then pure states in  $P(\mathcal{A}_F)$  are precisely the  $n$ -qubits, whereas mixed states  $S(\mathcal{A}_F)$  correspond to density matrices [32].

In [4] we have shown that such a noncommutative spacetime admits a sensible notion of a causal structure associated with a Lorentzian spectral triple. To this end, one has to identify a specific subset  $\mathcal{C}$  of “causal elements,” named the “causal cone.” Concretely,  $\mathcal{C}$  is the cone of all Hermitian elements  $a$  of a preferred unitization  $\mathbb{A} \supset \mathcal{A}$  respecting  $\forall \phi \in \mathcal{K}, (\phi, [\mathcal{D}, a]\phi) \geq 0$  (see [4,33,34] for the details).

**Definition 3.** Let  $(\mathcal{A}, \mathcal{K}, \mathcal{D})$  be a Lorentzian spectral triple and let  $\mathcal{C}$  be the causal cone. We say that two states  $\omega, \eta \in S(\mathcal{A})$  are causally related with  $\omega \leq \eta$  iff  $\forall a \in \mathcal{C}, \omega(a) \leq \eta(a)$ .

In the classical case of  $(\mathcal{A}_M, \mathcal{K}_M, \mathcal{D})$  (see Ex. 1) one recovers the usual causal structure on spacetime  $M$ , via the Gel’fand duality  $P(\mathcal{A}_M) \simeq M$  [4]. Note also that Def. 3 determines a rigorous partial order on the full space  $S(\mathcal{A})$  of mixed states. We exploit this fact when discussing the experimental setup.

The space of pure states on an almost commutative geometry, which deserves the name of an “almost commutative spacetime,” has a particularly pellucid physical interpretation: Let  $\mathcal{A} = \mathcal{A}_M \otimes \mathcal{A}_F$  as in Ex. 2; then  $P(\mathcal{A}) \simeq M \times \mathcal{F}$  for some finite space  $\mathcal{F}$ . In other words,

all pure states on  $\mathcal{A}$  are separable [32]. Hence, an almost commutative spacetime is the Cartesian product of the spacetime  $M$  and an “inner” space of states of the model. Definition 3 guarantees that, if  $(p, \xi), (q, \chi) \in P(\mathcal{A})$  are such that  $(p, \xi) \preceq (q, \chi)$ , then  $p \preceq q$  within the spacetime  $M$  [34]. This result attests that Einstein’s causality in the spacetime component is not violated. On the other hand, in [33] and [35] we discovered that the extended causal structure imposes highly nontrivial restrictions on the evolution in the “inner” space of the model. We now apply the mathematical results obtained in [35] to lift the veil on the origin of Zitterbewegung.

#### IV. MODELING A “CLASSICAL” DIRAC FERMION

Let  $M$  be a globally hyperbolic spacetime of dimension 2 or 4. We associate to it a Lorentzian spectral triple  $(\mathcal{A}_M, \mathcal{K}_M, \mathcal{D})$  in a canonical way (see Ex. 1). As a finite spectral triple we take  $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{C}$ ,  $\mathcal{H}_F = \mathbb{C}^2$ , and  $\mathcal{D}_F = \begin{pmatrix} 0 & \mu \\ \mu^* & 0 \end{pmatrix}$ , for some  $\mu \in \mathbb{C} \setminus \{0\}$ . The product triple thus reads  $\mathcal{A} = C_c^\infty(M) \oplus C_c^\infty(M)$ ,  $\mathcal{K} = L^2(M, S) \otimes \mathbb{C}^2$ ,  $\mathcal{D} = \mathcal{D} \otimes 1 + i\gamma_M \otimes \mathcal{D}_F$  (see [35] for the details).

Noncommutative geometries are equipped with a natural fermionic action defined as  $S_F = \langle \psi, \mathcal{D}\psi \rangle$ , for  $\psi \in \mathcal{K}' \subset \mathcal{K}$  [36,37]. If the finite spectral triple is even, which is the case here, one encounters the “fermion doubling problem” [38]. A consistent prescription to avoid the overcounting of fermionic degrees of freedom has been worked out in [36] and consists in projecting the elements of  $\mathcal{K}$  onto the physical subspace  $\mathcal{K}'$ . We have  $\mathcal{K}' = P_+ \mathcal{K}$ , with  $P_+ = \frac{1}{2}(1 + \gamma)$ ,  $\gamma = \gamma_M \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . A vector in  $\mathcal{K}'$  can thus be written as  $\mathcal{K}' \ni \psi = \psi_+ \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_- \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , with  $\psi_\pm$  denoting the chirality eigenstates. The fermionic action of the model therefore reads

$$S_F = \langle \psi_+, \mathcal{D}\psi_+ \rangle + \langle \psi_-, \mathcal{D}\psi_- \rangle - i\mu \langle \psi_+, \psi_- \rangle + i\mu^* \langle \psi_-, \psi_+ \rangle \\ = \int_M [\bar{\psi}_- \mathcal{D}\psi_- + \bar{\psi}_+ \mathcal{D}\psi_+ + m(\bar{\psi}_- \psi_+ + \bar{\psi}_+ \psi_-)], \quad (3)$$

with the choice  $\mu = im \in i\mathbb{R}^+$ . This is indeed the action describing a single Dirac fermion of mass  $m$  propagating in a curved spacetime  $M$ .

With  $\mathcal{A} = \mathcal{A}_M \otimes \mathcal{A}_F$  we have  $P(\mathcal{A}) \simeq M \sqcup M := M \times \{-, +\}$  and the “inner” space of the model consists of just two points. Since the two pure states on  $\mathcal{A}_F$  are precisely the vector states associated with  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{H}_F$  [35], it is justified to identify the two points of the model’s “inner” space as states of definite chirality. We thus arrive at the interpretation of the space of physical states  $P(\mathcal{A})$  as the space of states of a “classical” fermion—the component  $M$  defines its position in spacetime and  $\mathcal{F} = \{-, +\}$  corresponds to its chirality.

Connes first observed [9] that the (Euclidean version of the) above almost commutative model provides a geometric

viewpoint on Zitterbewegung. However, Connes’ remark was only qualitative and focused on regarding the Higgs field as a gauge boson operating on the finite space  $\{-, +\}$  (see also [39]). We discovered that taking into account the Lorentzian aspects of this model reveals a deeper, quantitative, connection between geometry and the “trembling motion” of fermions (cf., [35], Theorem 9).

**Theorem 4.** Let  $\tau(\gamma)$  be the proper time along a causal curve on  $M$ . Two states  $(p, -), (q, +) \in P(\mathcal{A})$  are causally related with  $(p, -) \preceq (q, +)$  if and only if there exists a causal curve  $\gamma$  yielding  $p \preceq q$  on  $M$  and such that  $\tau(\gamma) \geq \pi/(2|\mu|)$ .

Restoring the physical dimensions in the model (which is unambiguous as  $\mathcal{D}$  has the dimension  $L^{-1}$ ; thus  $\mathcal{D}_F$  must have so), we arrive at

$$(p, -) \preceq (q, +) \Leftrightarrow p \preceq q \quad \text{and} \quad \tau(\gamma) \geq \frac{\pi \hbar}{2|\mu|c^2}. \quad (4)$$

The number on the rhs of (4) is precisely the half of the Zitterbewegung period (2) of a Dirac fermion of mass  $|\mu| = m$ .

It is striking to realize that the possibility of Zitterbewegung is encoded in the geometry of a purely classical model. The bound on the frequency of the fermion’s quivering is of kinematic origin—we have invoked the action (3) only to identify the relevant degrees of freedom. The apparent abrupt change of state implied by Th. 4 becomes more transparent when one considers the subspace of mixed states of indefinite chirality. In the space  $M \times [-1, +1] \subset S(\mathcal{A})$  the boundary of the causal cone becomes a continuous surface (Fig. 2 of [35]) permitting a smooth evolution of the expectation value of chirality.

The advantage of the presented model is its general covariance, which guarantees that Th. 4 applies in any globally hyperbolic spacetime  $M$ . But the framework of noncommutative geometry is even more flexible and allows one to accommodate other fields interacting with the fermion via the fluctuations of the Dirac operator [7].

Let  $\mathcal{A}$  and  $\mathcal{K}$  be as they were previously in this section and let  $\mathcal{D}_\mathbb{A} = (\mathcal{D} + A) \otimes 1 + i\gamma_M \otimes \begin{pmatrix} 0 & \Phi \\ \Phi^* & 0 \end{pmatrix}$ , where  $A = \gamma^\mu A_\mu$  with  $A_\mu = A_\mu^* \in \mathcal{A}_M$  and  $\Phi \in \mathcal{A}_M$ ; then  $(\mathcal{A}, \mathcal{K}, \mathcal{D}_\mathbb{A})$  is still a Lorentzian spectral triple [35]. The fermionic action (3) now reads

$$S_F(\mathcal{D}_\mathbb{A}) = \int_M [\bar{\psi}_- (\mathcal{D} + A)\psi_- + \bar{\psi}_+ (\mathcal{D} + A)\psi_+ \\ + \bar{\psi}_- (-i\Phi)\psi_+ + \bar{\psi}_+ (i\Phi^*)\psi_-]. \quad (5)$$

We thus see that  $A_\mu$  is a vector field on  $M$ , for instance the electromagnetic one, minimally coupled to the fermion, whereas  $\Phi$  is a complex scalar field interacting via a Yukawa coupling [7].

The space of pure states of the interacting model is still  $M \times \{-, +\}$  as the algebra remains unaltered. On the other hand, the causal cone (and *a fortiori* the causal structure)

does change when  $\mathcal{D}$  is modified. The analogue of Th. 4 reads ([35], Theorem 16)

**Theorem 5.** Two pure states  $(p, -), (q, +) \in P(\mathcal{A})$  are causally related with  $(p, -) \preceq (q, +)$  if and only if there exists a causal curve  $\gamma$  giving  $p \preceq q$  on  $M$  such that

$$\int_0^1 ds |\Phi(\gamma(s))| \sqrt{-g_{\mu\nu} \dot{\gamma}^\mu(s) \dot{\gamma}^\nu(s)} \geq \frac{\pi}{2}. \quad (6)$$

An immediate consequence of Th. 5 is that there is no impact of the vector field on the causal relations in the almost commutative spacetime at hand. In particular, it implies that the upper bound on the Zitterbewegung frequency is not altered by the presence of an electromagnetic field. On the other hand, the scalar field  $\Phi$  affects causality in a more complicated way—the lhs of inequality (6) can be seen as a weighted proper time. Indeed, if  $\Phi$  is constant and equal to  $\mu$ , formula (6) reduces to (4). Such a field  $\Phi$  could for example be related to the variation of the mass of a pointlike particle in the Einstein frame of a tensor-scalar theory [40]. It is amusing to observe that the impact of  $\Phi$  on the causal structure is equivalent to a conformal rescaling of the metric on  $M$  by  $|\Phi|^{-1}$ . We note that the connection of the Higgs field with conformal transformations of the space(-time) in the context of non-commutative geometry was discussed in [41], but at the level of the action.

For an actual, quantum, fermion neither the concept of localization nor that of the proper time is well defined [42]; hence formulas (4) and (6) cannot be applied directly. Nevertheless, one can elicit some phenomenological consequences from the presented model by exploiting the fact that the causal order extends to the full space of states  $S(\mathcal{A})$ .

## V. SIMULATING ALMOST COMMUTATIVE SPACETIMES

In an (analogue) quantum simulation some aspects of the dynamics of a complicated quantum system are mimicked in a simpler one, which is under control [11]. Given an almost commutative spectral triple  $(\mathcal{A}, \mathcal{K}, \mathcal{D})$ , together with the action (3), one can extend this concept to probe almost commutative spacetimes in table-top experiments. The dynamics of vectors in the physical space  $\mathcal{K}' \subset \mathcal{K}$  governed by the action (3) always takes the form of a Dirac equation with some external fields. The latter can be rewritten as a Schrödinger equation once a suitable frame has been chosen. If one succeeds in finding a quantum system with the analogous dynamic, then one disposes of an invertible map  $f: \mathcal{H}_{\text{sim}} \rightarrow \mathcal{K}$ , which determines the correspondence between the observables and the states of the system [43]. Concretely, any state of the simulator system  $\varphi(t) \in \mathcal{H}_{\text{sim}}$  at time instant  $t$  in the laboratory frame defines a state on the algebra of the emulated almost commutative spectral triple  $\rho_{\varphi(t)} \in S(\mathcal{A})$ ,

$$\rho_{\varphi(t)}(a) = \int_{\Sigma_t} f(\varphi(t))^\dagger(t, x) a(t, x) f(\varphi(t))(t, x) dS_t(x). \quad (7)$$

$\Sigma_t$  is the  $t$ -slice determined by the chosen laboratory frame and  $x$  refers to the continuous degrees of freedom of the simulator system, corresponding to the space variable on  $\Sigma_t \subset M$ .

We claim that, within the domain of applicability of the simulation, the intrinsic geometry of the almost commutative spacetime should manifest itself in the simulator system. In particular, we expect the evolution of states to be causal in the sense of Def. 3, i.e.,

$$\rho_{\varphi(s)} \preceq \rho_{\varphi(t)}, \quad (8)$$

for  $s \leq t$  and any initial state  $\varphi(0) \in \mathcal{H}_{\text{sim}}$ . The details on the application of the abstract Def. 3 in the wave-packet formalism are explained in [44].

The quantum simulation of a single free Dirac fermion in flat two-dimensional spacetime has been successfully accomplished with cold atoms [14], trapped ions [19], and photonic systems [20]. Furthermore, a suitable experimental setup has been proposed using superconductors [45], semiconductors [46], and graphene [47]. In the trapped-ion setting [19,48], the mass of the simulated fermion can be introduced dynamically, which enables a simulation of a Dirac fermion coupled to a (real) scalar field. Moreover, a possibility of studying the impact of an external electromagnetic field on Zitterbewegung using the framework of [19] was suggested in [49].

In the presented almost commutative model, the general formula (8) is less explicit than the ones derived for the pure states: (4) and (6). It reflects the fact that Zitterbewegung in the wave-packet formalism is not a single-frequency oscillation [15]. Nevertheless, given concrete initial and final states of a simulator system one can unravel the consequences of formula (8) drawing from the fact that the causal cone is completely characterized in [35] and suitable computational tools to handle mixed states were devised in [50]. One immediate upshot is that the causal relation (8) does not depend on the electromagnetic field (cf., Th. 5). Note also that the lack of Zitterbewegung for purely positive-energy states is consistent with (8) [51].

## VI. OUTLOOK

We have shown that the possibility of Zitterbewegung is encoded in the geometry of an almost commutative spacetime of a “classical” massive Dirac fermion. The presence of an electromagnetic field and a Yukawa scalar field affects the geometry, with the latter modifying the bound on the frequency of the fermion’s quivering. We argued that the consequences of this model can be tested in a suitable quantum simulation.

For a free electron the period of Zitterbewegung (2) is of the order of  $10^{-20}$ s, which is far beyond the currently

available experimental time resolution. Moreover, to model a genuine electron one would need to employ the quantum field theoretic description, which seems to exclude Zitterbewegung [17], at least in flat spacetimes [18]. The scheme presented in this article suggests, however, that the very foundations of quantum gauge theories might need to be refined. The principle of microcausality—requiring the observables in spacelike separated regions to commute—is at the heart of all axiomatic approaches to quantum field theory [30,52]. If the fields have additional degrees of freedom then their background geometry is that of an almost commutative spacetime. Consequently, when constructing a quantum theory of fields, one should take into account its inherent causal structure, what might lead to a modified algebra of local observables.

The concept of causality in the space of states is at the core of the presented model. When applied to other almost commutative spacetimes (see [33] for another gauge model), it might cast a new light on such perplexing phenomena as quark mixing or neutrino oscillations, which also involve a “motion” in the fermion’s internal space. Finally, one can reach beyond the almost commutative

setting and study the causal structure of genuinely non-commutative spacetimes [53]. This perspective suggests that, contrary to the common belief [54,55], causal structure need not break down at the Planck scale, but the very notion of spacetime geometry needs to be refined.

## ACKNOWLEDGMENTS

We are indebted to Koen van den Dungen, Christoph Stephan, and Walter van Suijlekom for valuable discussions. We also thank André Füzfa and Henryk Arodz for comments on the manuscript. M. E. and N. F. thank the Hausdorff Research Institute for Mathematics for hospitality during the trimester program “Non-commutative Geometry and its Applications.” M. E. was supported by the Foundation for Polish Science under the program START 2016. M. E. acknowledges the support of the Marian Smoluchowski Kraków Research Consortium “Matter-Energy-Future” within the program KNOW. This publication was made possible through the support of a grant from the John Templeton Foundation (No. 60671).

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