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One spatial dimensional finite volume three-body interaction for a short-range potential

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In this work, we use McGuire's model to describe scattering of three spinless identical particles in one spatial dimension; we first present analytic solutions of Faddeev's equation for scattering of three spinless particles in free space. The three particles interaction in finite volume is derived subsequently, and the quantization conditions by matching wave functions in free space and finite volume are presented in terms of two-body scattering phase shifts. The quantization conditions obtained in this work for the short-range interaction are Lüscher's formula-like and consistent with Yang's results [Phys. Rev. Lett. **19**, 1312 (1967)].

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I. INTRODUCTION

Three-particle interaction plays an important role in modern physics. In certain hadronic reaction processes, three-particle dynamics may be a crucial component of reaction. For example, the discrepancy of decay width of $\eta \to 3\pi$ between experimental measurement [1] and χPT calculations [2–5] can only be well understood when three-body dynamics are properly considered [6–12]. Three-particle or many-particle dynamics have also been proven essential to illustrate or understand some important effects in nuclear and atomic physics, such as precise knowledge of nucleon interaction [13–16] and Efimov effect [17–22]. In the past, many different approaches have been developed to describe three-body dynamics, for instance, quantum field theory based relativistic Bethe-Salpeter equations [23–25], nonrelativistic Faddeev's equation [26–31], and dispersion relation oriented Khuri-Treiman equation [32–42]. Unfortunately, either approach provides a non-expert friendly framework due to sophistication of three-body dynamics. In recent years, three-body dynamics started regaining some popularities in the hadron physics community for many reasons. For examples, precision theoretical hadron-interaction framework is urgently needed for data analysis when high statistic data become available, and a sensible finite volume theory of three-body interaction is also currently demanded by the lattice QCD community.

In the lattice QCD calculation, because computation is performed in Euclidean space, we do not have direct access to scattering amplitudes [43]. Fortunately, taking advantage of the periodic boundary condition, a relation between the energy spectrum extracted from lattice QCD computation and two-body scattering amplitudes in the elastic region is established [44], which is usually referred to as Lüscher's formula. The extension of the framework to moving frames and to inelastic channels has also been developed by many authors [45–56]. The finite volume scattering formalism has been proven valid and effective in the lattice community for extracting hadron-hadron two-body scattering information [57–68].

There have been some attempts on finite volume three-body interactions in recent years [69–76]. These recent developments for the finite volume three-body scattering problem [69–76] are typically momentum representation of quantum field theory approaches, diagrammatic approaches or the Faddeev equation based method. Most of these developments are mathematics and physics friendly to the majority of people in the physics community. Hence, it is fair and reasonable to raise the question of how one would check all these over sophisticated quantization conditions of the finite volume three-body problem presented in these works? In the present work, we aim to find a simple and exactly solvable three-body problem, so that the analytic results of quantization conditions in this simple case can be found. The result may serve as a calibration to more realistic treatments of the three-body problem in different approaches [69–76]. Moreover, an exactly solvable three-body problem in finite volume could be a very useful tool for understanding three-body dynamics in finite volume and it may also be very helpful for further development of the approximate method in more realistic three-body problems. In order to make the three-body scattering problem as simple as possible and exactly solvable, we will ignore the relativistic effect and also constrain ourself to one spatial dimension, so that analytic solutions can be found and an infinite sum in finite volume is easily carried out. We will further consider three nonrelativistic particles with equal masses, and the pairwise and short-range interactions among three particles. Under the above-mentioned assumptions, a simplified Faddeev's equation in free space (space with infinite volume) is established and solved analytically. Instead of attacking the finite volume problem in momentum representation, we employ the approach developed in [56,77], and work our

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way out with the wave function of three-body in the configuration representation. In our approach of solving the three-body problem in finite volume, there are three basic steps: first of all, the solutions of Faddeev's equation are used to construct the free space three-body wave function. Next, for the problem of three particles in a finite box, we show that the finite volume wave function of three-body interaction can be constructed through the free space three-body wave function. At last, the matching of the finite volume wave function and the free space wave function yields the Lusher's formula-like quantization conditions of the three-body interaction, which are sets of relations between the two-body phase shift and three-particle momenta in a finite box:

$$\cot\left(\frac{P}{3}+\frac{q_3}{2}\right)L+\cot\left(\delta(-q_{31})-\delta(-q_{23})\right)=0,$$
$$\cot\left(\frac{P}{3}+\frac{q_1}{2}\right)L+\cot\left(-\delta(-q_{31})-\delta(q_{12})\right)=0,$$
$$\cot\left(\frac{P}{3}+\frac{q_2}{2}\right)L+\cot\left(\delta(-q_{23})+\delta(q_{12})\right)=0,$$

where δ denotes the two-body scattering phase shift and L refers to the size of the box. The total momentum, P, relative momentum q_{ij} between ith and jth particles, and relative momenta q_k between kth particle and pair (ij) will be explained in Sec. II.

The advantage of using wave function in configuration representation is that first of all, the wave function approach is close to the way of solving traditional quantum mechanics problems in a periodic potential. A short review of our formalism for the two-body interaction in finite volume and its applications to exactly solvable quantum mechanical models are listed in Appendix D. Second, also the most important fact is that the asymptotic form of wave function displays physical on-shell transition amplitudes. As a well-known fact, the solutions of Faddeev's equation are not equivalent to physical transition amplitudes [27,78–80]. Three-body scattering amplitudes possess singularities of poles and δ-functions, the physical transition amplitudes are in fact associated to the residue functions of these singularities. These singularities are the consequence of existence of different distinct physical processes in the three-body system. For examples, unlike in the two-body system, the formation of a bound pair is not precluded by energy conservation, a pole is thus introduced by the presence of a two-body bound state. The singularity of δ-functions is associated with disconnected diagrams with an unscattered third particle. Because of these singularities in three-body amplitude, the three-body wave function in configuration representation may have several different pieces that decrease at a different rate and describe different physical processes. The physical transition amplitudes for different physical processes are thus also defined by asymptotic forms of the three-body wave function [79–83].

In this work, to describe the scattering of the three-body system, we adopt a one-dimensional model with the interaction of equal strength, pair-wise δ-function potential that was developed by McGuire in [84]. A brief review of McGuire's model is provided in Appendix C. McGuire's model is physically simple, but can still provide us a qualitative description of the three-body scattering process in finite volume. McGuire's model was originally solved by ray tracing and geometric optics consideration method [84], McGuire found that diffraction effects in this particular model are all canceled out, thus no new momenta are created over the scattering process, though momenta are allowed to be rearranged among three particles in the final state. In consequence, any dissociation or recombination of particles out of or into bound states is forbidden, bound states sectors are decoupled from the three free particles sector. The breakup process never happens when a particle is incident on a bound state [85–87]. Nevertheless, McGuire's model still encompasses rearrangement effects among three particles, it may even represent more realistic physical models of the short-range interaction. Although, the quantization conditions for the three-body problem in finite volume are obtained by considering a particular model, the final results are presented in terms of two-body phase shifts. The quantization conditions may be tested numerically in the future by one-dimensional lattice models, such as ones developed in [77,88].

The paper is organized as follows. In Sec. II we discuss the free space three-particle system. The finite volume three-particle system is presented in Sec. III. The summary and outlook are given in Sec. IV.

II. THREE-BODY SCATTERING FOR SHORT-RANGE INTERACTION IN FREE SPACE

Considering three spinless identical particles scattering, the short-range interactions among three-particle are pair-wise and equal strength for all pairs, $V(r)=V_0\delta(r)$. In general, the kernel for Faddeev's integral equation is off-shell two-body scattering amplitudes; the off-shell kernel usually complicates three-body integral equations even in one dimension and causes difficulties of solving Faddeev's equation. Fortunately, for the short-range δ-function potential, two-body scattering amplitude appears completely on energy shell, see Eq. (D24). This feature dramatically simplifies Faddeev's integral equation, so that finding an analytic solution is possible. For completeness, a brief review of formal scattering theory and the general framework of Faddeev's equation is presented in Appendix A.

The wave function of the scattering three-particle satisfies Schrödinger equation,

$$\left[-\frac{1}{2m}\sum_{i=1}^{3}\frac{d^2}{dx_i^2}+V_0\delta(r_{12})+V_0\delta(r_{23})+V_0\delta(r_{31})-E\right]\times\Psi(x_1,x_2,x_3;p_1,p_2,p_3)=0, \quad (1)$$

where the mass of particle is m, the total energy of the three-particle system is $E = \sum_{i=1}^{3} \frac{p_i^2}{2m}$, where p_i $(i = 1, 2, 3)$ stands for the particle's momenta in the initial state. The center of mass position is given by $R = \frac{x_1+x_2+x_3}{3}$. $r_{ij} = x_i - x_j$ refers to relative position between ith and jth particles, and $r_k = \frac{x_i+x_j}{2} - x_k$ denotes the relative position between the kth particle and pair (ij). The conjugate total and relative momenta are given by $P = p_1 + p_2 + p_3$, $q_{ij} = \frac{p_i-p_j}{2}$ and $q_k = \frac{p_i+p_j-2p_k}{3}$, respectively. A change of the pair of relative coordinates and corresponding conjugate momenta from $(ij)k$ configuration to other configurations, e.g. $(jk)i$ configuration in which relative coordinates and conjugate momenta are expressed in terms of (r_{jk}, r_i) and (q_{jk}, q_i), is accomplished by

$$r_{jk} = -\frac{1}{2} r_{ij} + r_k, \qquad r_i = -\frac{3}{4} r_{ij} - \frac{1}{2} r_k,$$
$$q_{jk} = -\frac{1}{2} q_{ij} + \frac{3}{4} q_k, \qquad q_i = -q_{ij} - \frac{1}{2} q_k, \tag{2}$$

where $(ij)k$ or $(jk)i$ always follows cyclic permutation of (1,2,3).

The total wave function of three particles can be expressed by the product of a plane wave, e^{iPR}, which describes center of mass motion, and the relative wave function, $\psi(r_{ij}, r_k; q_{ij}, q_k)$, which describes relative motions of three particles, $\Psi(x_1, x_2, x_3; p_1, p_2, p_3) = e^{iPR}\psi(r_{ij}, r_k; q_{ij}, q_k)$. For scattering with a free three-particle incoming wave, the wave function has the following form [26,27], $\Psi = \Psi_{(0)} + \sum_{\gamma=1}^{3} \Psi_{(\gamma)}$, where $\Psi_{(0)}$ refers to the incoming free wave, and $\Psi_{(\gamma)}$ satisfies the equation

$$\left[-\frac{1}{2m}\sum_{i=1}^{3}\frac{d^2}{dx_i^2} + V_0\delta(r_{\alpha\beta}) - E\right]\Psi_{(\gamma)} = -V_0\delta(r_{\alpha\beta})[\Psi_{(0)} + \Psi_{(\alpha)} + \Psi_{(\beta)}], \qquad \gamma \neq \alpha \neq \beta. \tag{3}$$

The integral representation of Eq. (3) for relative wave function, $\psi_{(\gamma)}(r_{\alpha\beta}, r_\gamma; q_{ij}, q_k) = e^{-iPR}\Psi_{(\gamma)}(x_1, x_2, x_3; p_1, p_2, p_3)$ is given by

$$\psi_{(\gamma)}(r_{\alpha\beta}, r_\gamma; q_{ij}, q_k) = \int_{-\infty}^{\infty} dr'_{\alpha\beta} dr'_\gamma G_{(\gamma)}(r_{\alpha\beta} - r'_{\alpha\beta}, r_\gamma - r'_\gamma; z_\sigma) \times mV_0\delta(r'_{\alpha\beta})[\psi_{(0)}(r'_{\alpha\beta}, r'_\gamma; q_{ij}, q_k) + \psi_{(\alpha)}(r'_{\beta\gamma}, r'_\alpha; q_{ij}, q_k) + \psi_{(\beta)}(r'_{\gamma\alpha}, r'_\beta; q_{ij}, q_k)], \tag{4}$$

where $z_\sigma = \sigma^2 + i\epsilon$ and $\sigma^2 = mE - \frac{P^2}{6} = q_{ij}^2 + \frac{3}{4}q_k^2$ $(k = 1, 2, 3)$, and the Green's function $G_{(\gamma)}$ satisfies the equation

$$\left[z_\sigma + \frac{d^2}{dr_{\alpha\beta}^2} + \frac{3}{4}\frac{d^2}{dr_\gamma^2} - mV_0\delta(r_{\alpha\beta})\right] G_{(\gamma)}(r_{\alpha\beta} - r'_{\alpha\beta}, r_\gamma; z_\sigma) = \delta(r_{\alpha\beta} - r'_{\alpha\beta})\delta(r_\gamma), \tag{5}$$

and the solution of Eq. (5) is given by

$$G_{(\gamma)}(r_{\alpha\beta} - r'_{\alpha\beta}, r_\gamma; z_\sigma) = \int_{-\infty}^{\infty} \frac{dq'_{\alpha\beta}}{2\pi}\frac{dq'_\gamma}{2\pi} \frac{\sum_{\mathcal{P}=\pm}\psi_{\mathcal{P}}(r_{\alpha\beta}; q'_{\alpha\beta})\psi^*_{\mathcal{P}}(r'_{\alpha\beta}; q'_{\alpha\beta})e^{iq'_\gamma r_\gamma}}{z_\sigma - q'^2_{\alpha\beta} - \frac{3}{4}q'^2_\gamma}. \tag{6}$$

The $\psi_\pm(r_{\alpha\beta}, q_{\alpha\beta})$ are parity two-body wave functions of pair $(\alpha\beta)$, and the solution of $\psi_\pm$ for the δ-function potential reads

$$\psi_{\mathcal{P}}(r; k) = \frac{e^{ikr} + \mathcal{P}e^{-ikr}}{2} + it_{\mathcal{P}}(\sqrt{z_k})Y_{\mathcal{P}}(k)e^{i\sqrt{z_k}|r|}, \tag{7}$$

where the on-shell two-body scattering amplitudes, $t_\pm$, are given in Eq. (D24): $t_+(k) = -\frac{mV_0}{2k+imV_0}$ and $t_-(k) = 0$. $t_\pm$ are normalized by relation $\frac{t_{\mathcal{P}}-t^*_{\mathcal{P}}}{2i} = t^*_{\mathcal{P}}t_{\mathcal{P}}$. Using the unitarity relation of two-body amplitude, it can be shown that

$$\int_{-\infty}^{\infty} dr'_\gamma G_{(\gamma)}(r_{\alpha\beta} - r'_{\alpha\beta}, r_\gamma - r'_\gamma; z_\sigma)V_0\delta(r'_{\alpha\beta})e^{iqr'_\gamma} = \frac{e^{i\sqrt{\sigma^2-\frac{3}{4}q^2}|r_{\alpha\beta}|}e^{iqr_\gamma}}{2i\sqrt{\sigma^2-\frac{3}{4}q^2}}\left[1 + it_+\left(\sqrt{\sigma^2 - \frac{3}{4}q^2}\right)\right]V_0\delta(r'_{\alpha\beta}). \tag{8}$$

Therefore, Eq. (4) can be written as

$$\psi_{(\gamma)}(r_{\alpha\beta}, r_\gamma; q_{ij}, q_k) = \int_{-\infty}^{\infty}\frac{dq}{2\pi}e^{i\sqrt{\sigma^2-\frac{3}{4}q^2}|r_{\alpha\beta}|}e^{iqr_\gamma} \times \frac{\left[1 + it_+\left(\sqrt{\sigma^2-\frac{3}{4}q^2}\right)\right]}{2i\sqrt{\sigma^2-\frac{3}{4}q^2}} \times \int_{-\infty}^{\infty} dr'_{\alpha\beta}dr'_\gamma e^{-iqr'_\gamma}mV_0\delta(r'_{\alpha\beta}) \times [\psi_{(0)}(r'_{\alpha\beta}, r'_\gamma; q_{ij}, q_k) + \psi_{(\alpha)}(r'_{\beta\gamma}, r'_\alpha; q_{ij}, q_k) + \psi_{(\beta)}(r'_{\gamma\alpha}, r'_\beta; q_{ij}, q_k)]. \tag{9}$$

Next, let us introduce amplitudes, $T_{(\gamma)}$, by

$$T_{(\gamma)}(k; q_{ij}, q_k) = -\int_{-\infty}^{\infty} dr_{\alpha\beta}dr_\gamma e^{-ikr_\gamma}mV_0\delta(r_{\alpha\beta}) \times \psi(r_{\alpha\beta}, r_\gamma; q_{ij}, q_k), \qquad \alpha \neq \beta \neq \gamma. \tag{10}$$

Using Eq. (9) and the property of two-body scattering amplitude, we find

$$T_{(\gamma)}(k;q_{ij},q_k) = -\left[1+it_+\left(\sqrt{\sigma^2-\frac{3}{4}k^2}\right)\right] \times \int_{-\infty}^{\infty} dr'_{\alpha\beta}dr'_\gamma e^{-ikr'_\gamma} mV_0\delta(r'_{\alpha\beta}) \times [\psi_{(0)}(r'_{\alpha\beta},r'_\gamma;q_{ij},q_k)+\psi_{(\alpha)}(r'_{\beta\gamma},r'_\alpha;q_{ij},q_k) + \psi_{(\beta)}(r'_{\gamma\alpha},r'_\beta;q_{ij},q_k)], \quad \alpha\neq\beta\neq\gamma, \tag{11}$$

therefore the wave function $\psi_{(\gamma)}$ is related to $T_{(\gamma)}$ amplitude by

$$\psi_{(\gamma)}(r_{\alpha\beta},r_\gamma;q_{ij},q_k) = i\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{e^{i\sqrt{\sigma^2-\frac{3}{4}q^2}|r_{\alpha\beta}|}e^{iqr_\gamma}}{2\sqrt{\sigma^2-\frac{3}{4}q^2}}T_{(\gamma)}(q;q_{ij},q_k). \tag{12}$$

Let us also define functions $v_{(\gamma)}$,

$$v_{(\gamma)}(k;q_{ij},q_k) = \int_{-\infty}^{\infty} dr_{\alpha\beta}dr_\gamma e^{-ikr_\gamma} \times mV_0\delta(r_{\alpha\beta})\psi_{(0)}(r_{\alpha\beta},r_\gamma;q_{ij},q_k). \tag{13}$$

Equations (11)–(13) all together thus yield coupled sets of integral equation of $T_{(\gamma)}$'s, which is exactly just Faddeev's equation for δ-function potential,

$$T_{(\gamma)}(k;q_{ij},q_k) = \left(2\sqrt{\sigma^2-\frac{3}{4}k^2}\right)it_+\left(\sqrt{\sigma^2-\frac{3}{4}k^2}\right)\left[\frac{v_{(\gamma)}(k;q_{ij},q_k)}{imV_0} + i\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{T_{(\alpha)}(q;q_{ij},q_k)+T_{(\beta)}(q;q_{ij},q_k)}{\sigma^2-\frac{3}{4}q^2-(k+\frac{q}{2})^2+i\epsilon}\right], \quad \alpha\neq\beta\neq\gamma. \tag{14}$$

At last, the total three-body scattering amplitude is defined by

$$T(k_{\alpha\beta},k_\gamma;q_{ij},q_k) = -\int_{-\infty}^{\infty} dr_{\alpha\beta}dr_\gamma e^{-ik_{\alpha\beta}r_{\alpha\beta}}e^{-ik_\gamma r_\gamma} \times mV_0[\delta(r_{\alpha\beta})+\delta(r_{\beta\gamma})+\delta(r_{\gamma\alpha})] \times \psi(r_{\alpha\beta},r_\gamma;q_{ij},q_k). \tag{15}$$

As suggested in [26,27], T is thus represented as the sum of three $T_{(\gamma)}$ amplitudes,

$$T(k_{\alpha\beta},k_\gamma;q_{ij},q_k) = \sum_{\delta=1}^{3} T_{(\delta)}(k_\delta;q_{ij},q_k), \tag{16}$$

where $k_\alpha = -k_{\alpha\beta}-\frac{k_\gamma}{2}$ and $k_\beta = k_{\alpha\beta}-\frac{k_\gamma}{2}$.

As already mentioned in the Introduction, unlike two-body scattering, the solution of Faddeev's equation defined in Eq. (15) is not equivalent to the physical transition amplitudes [27,78–80]. The physical transition amplitudes for different physical processes are associated to the residue functions of singularities of T-amplitude given in Eq. (15). In general, the T-amplitude has two distinct type singularities [27,79,80]. One type is called primary singularities, e.g. $(\sigma^2+\chi_{12}^2-\frac{3}{4}q_3^2)^{-1}$, where χ_{12} is bound state energy of pair (12). The pole of the $(\sigma^2+\chi_{12}^2-\frac{3}{4}q_3^2)^{-1}$ type presents in driving terms of Faddeev's equation and persists in all terms of an iterative series of amplitudes. It arises when relative momentum of the (12) pair hits the bound state pole position of two-body scattering amplitude, $t(q_{12})\sim(q_{12}^2+\chi_{12}^2)^{-1}$, and it describes the possibility of existence of the two-body bound state in both initial and final states. The other types of singularity, called secondary singularities [27,79,80], only present in driving terms and first a few iterations, and singularities are getting weaker and eventually disappear after a couple of iterations. A typical example is the δ-functions that are related to disconnected diagrams with the third particle remaining intact. The existence of singularities in T-amplitude is the consequence of the presence of multiple possible physical distinct processes in a three-body system. Hence, these singularities are directly associated with the different physically realizable asymptotic states of the system. The explicit decomposition of primary singularities of T-amplitude is given in [27,79,80] by

$$T(k_{12},k_3;q_{12},q_3) = \sum_{k=1}^{3}(2\pi)\delta(k_k-q_k)(2q_{ij})t(k_{ij};q_{ij}) + \sum_{\gamma,k=1}^{3}\left[\mathcal{F}_{(\gamma,k)}(k_{\alpha\beta},k_\gamma;q_{ij},q_k) + \frac{\phi_{(\gamma)}(k_{\alpha\beta})\mathcal{G}^*_{(\gamma,k)}(k_\gamma;q_{ij},q_k)}{\sigma^2+\chi_{\alpha\beta}^2-\frac{3}{4}k_\gamma^2} + \frac{\mathcal{G}_{(\gamma,k)}(k_{\alpha\beta},k_\gamma;q_k)\phi^*_{(k)}(q_{ij})}{\sigma^2+\chi_{ij}^2-\frac{3}{4}q_k^2} + \frac{\phi_{(\gamma)}(k_{\alpha\beta})\mathcal{K}_{(\gamma,k)}(k_\gamma;q_k)\phi^*_{(k)}(q_{ij})}{(\sigma^2+\chi_{\alpha\beta}^2-\frac{3}{4}k_\gamma^2)(\sigma^2+\chi_{ij}^2-\frac{3}{4}q_k^2)}\right], \tag{17}$$

where the t function in Eq. (17) stands for two-body off-shell scattering amplitude, the $\phi_{(\gamma)}$ function represents the vertex function of the two-body bound state wave function, $\phi_{(\gamma)}(k_{\alpha\beta}) = (k_{\alpha\beta}^2+\chi_{\alpha\beta}^2)\psi(k_{\alpha\beta})$. The residue functions that are associated to physical transition amplitudes, $\mathcal{F}_{(\gamma,k)}$, $\mathcal{G}_{(\gamma,k)}$ and $\mathcal{K}_{(\gamma,k)}$ in Eq. (17), do not have any primary singularities, though they may still have secondary singularities. It has been shown in [27,79,80] that the first term on the right-hand side of Eq. (17) is from a disconnected contribution and it describes the process that one of the incident free particles is unscattered. The $\mathcal{K}_{(\gamma,k)}$ function in the last term is the physical amplitude that describes the processes of either direct or rearrangement scattering on a bound state: $(ij)+k\to(\alpha\beta)+\gamma$. $\sum_{\gamma=1}^{3}[\mathcal{G}_{(\gamma,k)}+\frac{\phi_{(\gamma)}\mathcal{K}_{\gamma,k}}{\sigma^2+\chi^2_{\alpha\beta}-\frac{3}{4}k^2_\gamma}]$ is the physical transition amplitude for breakup or capture processes. The true $1+2+3\to1+2+3$ physical scattering amplitude is given by on-shell T-amplitude in the physical kinematic domain of three-particle momenta. The singularities in momentum space generate a more complicated asymptotic form of the three-body wave function than that of the two-body wave function in configuration space. The physical transition amplitudes can thus also be defined by asymptotic forms of the wave function in the configuration representation [80–83]. The asymptotic form of the three-body wave function depends on the type of initial state, and may behave quite differently and describe distinct physical processes at different domains in the (r_{ij},r_k) plane [79–83]. For example, scattering of the third particle on a bound state of (12) pair, the physical amplitudes of different processes are given by the asymptotic wave function in different domains: (i) direct channel scattering, $(12)+3\to(12)+3$, is described in the domain of finite r_{12} and large r_3, the scattering part of the asymptotic wave function is of the order of $e^{-\chi_{12}|r_{12}|}\mathcal{O}(|r_3|^{-1})$; (ii) rearrangement scattering processes, $(12)+3\to(23)+1$ or $(12)+3\to(31)+2$, are given in domains of finite r_{23} and large r_1 or finite r_{31} and large r_2 respectively, and the wave function behaves as $e^{-\chi_{23}|r_{23}|}\mathcal{O}(|r_1|^{-1})$ or $e^{-\chi_{31}|r_{31}|}\mathcal{O}(|r_2|^{-1})$ respectively; (iii) breakup process, $(12)+3\to1+2+3$, appears as both r_{12} and r_3 are large, but r_{12}/r_3 remains constant [79,80], the wave function for breakup is of the order of $\mathcal{O}((r^2_{12}+\frac{4}{3}r^2_3)^{-\frac{5}{4}})$. In the case of three free incident particles [81–83], the asymptotic form of the wave function consists of several different pieces that decrease at different rates and describes different distinct physical processes, and its falloff also depends on the direction in configuration space: (1) incident plane wave that does not decrease in any direction; (2) terms describe the scattering of a pair by itself without participation of the third particle, the wave function is of the order of $\mathcal{O}(|r_{ij}|^{-1})$ (kth particle as spectator). The disconnected terms and incident free wave must be subtracted out before other contributions become visible; (3) the terms that describe capture of pair (ij) as a bound state is of the order of $e^{-\chi_{ij}|r_{ij}|}\mathcal{O}(|r_k|^{-1})$. (4) The terms generated by on-shell double scattering is of the order of $\mathcal{O}((r^2_{12}+\frac{4}{3}r^2_3)^{-1})$. (5) The true three-body scattering terms are of the order of $\mathcal{O}((r^2_{12}+\frac{4}{3}r^2_3)^{-\frac{5}{4}})$. Nevertheless, it is clear that understanding of either singularities structure of T-amplitude in momentum space or asymptotic form of wave function in configuration space is a crucial step in order to extract physical transition amplitudes.

As mentioned early in the Introduction, McGuire's model permits no diffraction and no breakup or capture processes, only forward scattering (no new momenta are created after collision). Therefore, the physical processes in McGuire's model are split into two decoupled sectors: scattering on a bound state and three-to-three particles scattering. The analytic solution of Faddeev's equation for a particle scattering on a bound state has been discussed in [85–87]. In this work, we solve Faddeev's equation, Eq. (14), analytically for the $1+2+3\to1+2+3$ three particles scattering process. As will be shown in following sections, the T-amplitude for three-to-three scattering of identical bosons in McGuire's model has the form of

$$\begin{aligned}&T(k_{12},k_3;q_{12},q_3)\\&=\sum_{\gamma=1}^{3}2(2\pi)\delta(k_3-q_\gamma)\sum_{k=1}^{3}\left(2\sqrt{\sigma^2-\frac{3}{4}q_k^2}\right)t_+\left(\sqrt{\sigma^2-\frac{3}{4}q_k^2}\right)\\&\quad+\sum_{\gamma=1}^{3}\frac{R(k_\gamma)}{(k_\gamma-q_1)(k_\gamma-q_2)(k_\gamma-q_3)},\end{aligned}\tag{18}$$

where the first term again represents the disconnected contribution and the second term represents the sum of all the rescattering effects, and the $R(k_\gamma)$ function is a polynomial function of relative momenta of three-particle and free of poles, the explicit expression of the $R(k_\gamma)$ function will be made clear later on in Sec. II B. The pole structure in Eq. (18) yields the forward scattering of three particles in the end. Notice that the exact solution of Faddeev's equation in Eq. (18) has only singularities of poles and δ-function. This means that first of all, the branch cut contribution during the iteration of Faddeev's equation has to be all canceled out, it turns out to be true, see Sec. B 1. Second, higher order iterations of Faddeev's equation in one dimension do not completely eliminate the three-particle propagator singularities, this is a distinct feature from three-dimensional three-body physics. In both dimensions, off-shell double scattering displays the similar singularity structure of the three-particle propagator, e.g. $[q^2_{12}-(k_3+\frac{q_3}{2})^2+i\epsilon]^{-1}$, see Figs. 1(b) and 1(c). However, for triple scattering, see Fig. 1(d), the singularity structure starts diverging. A triple scattering in three dimensions has the typical singularities of type

$$\begin{aligned}&\int d^3q\frac{1}{\mathbf{k}^2_{12}-(\mathbf{q}+\frac{\mathbf{k}_3}{2})^2+i\epsilon}\frac{1}{\mathbf{q}^2_{12}-(\mathbf{q}+\frac{\mathbf{q}_3}{2})^2+i\epsilon}\\&=\frac{i\pi^2}{|\frac{\mathbf{k}_3}{2}-\frac{\mathbf{q}_3}{2}|}\ln\frac{\mathbf{q}^2_{12}+\mathbf{k}^2_{12}+|\frac{\mathbf{k}_3}{2}-\frac{\mathbf{q}_3}{2}|^2}{\mathbf{q}^2_{12}+\mathbf{k}^2_{12}-|\frac{\mathbf{k}_3}{2}-\frac{\mathbf{q}_3}{2}|^2}.\end{aligned}\tag{19}$$

In one dimension, triple scattering appears as a one-dimensional integral over the product of two three-particle propagators,

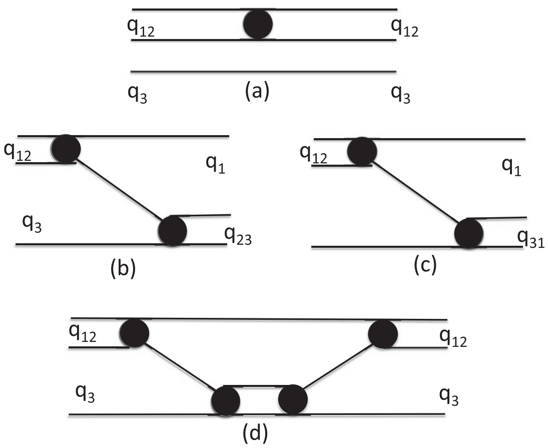

FIG. 1. (a) The disconnected diagram for the third particle as a spectator. (b) and (c) Double scattering contributions from pair (23) and (31) into (12) pair. (d) A triple scattering contribution.

$$\int_{-\infty}^{\infty} dq \frac{1}{k_{12}^2 - (q + \frac{k_3}{2})^2 + i\epsilon} \frac{1}{q_{12}^2 - (q + \frac{q_3}{2})^2 + i\epsilon}, \quad (20)$$

it is easy to see that after picking up the poles in the integrand, the result of the one-dimensional integral has only poles and branch cuts. As shown in [27], in three dimensions, the three-particle propagator singularities are smoothed out in higher iterations and are thus considered as secondary singularities. On the contrary, in one dimension, some poles survived higher iterations and all the branch cuts are canceled out after a sum over all the diagrams in McGuire's model. In the end, the exact solution of Faddeev's equation displays only singularities of poles and δ-function, as in Eq. (18).

On the other hand, it will also be shown in Sec. II B that the principal part of the pole term in Eq. (18) is proportional to a factor, $(\sigma^2 - k_{12}^2 - \frac{3}{4}k_3^2)$. As the consequence, the solutions of Faddeev's equation suffer no branch cut singularities, all the branch cut singularities in Faddeev's equation are canceled out. The three-body wave function consists of only six plane waves: $e^{iq_{ij}r_{12}}e^{iq_k r_3}$ $(k = 1, 2, 3)$, no diffraction effect is generated after scattering. When three-particle T-amplitude is put on the energy shell, $\sigma^2 = k_{12}^2 + \frac{3}{4}k_3^2$, the principal part of the pole term vanishes, thus, the on-shell physical three-body amplitude consists of only the terms that are proportional to the δ-function from both disconnected diagrams and on-shell three-body rescattering effect. Therefore, it allows us to define the on-shell scattering physical amplitude as a residue of T-amplitude at pole positions,

$$\left(-\sum_{\gamma=1}^{3} 2(2\pi)\delta(k - q_\gamma)\right)\left(2\sqrt{\sigma^2 - \frac{3}{4}k^2}\right)\mathcal{T}(k)$$
$$= T\left(\sqrt{\sigma^2 - \frac{3}{4}k^2}, k; q_{12}, q_3\right), \qquad k = q_{1,2,3}. \quad (21)$$

In general, there are six possible independent incoming plane waves in terms of permutation of incoming momenta, see [84]. In Appendix B, we show details of how Faddeev's equation is solved for an incoming plane wave $\psi_{(0)} = e^{iq_{12}r_{12}}e^{iq_3 r_3}$ as an example. In the end of Appendix B 1, the analytic solutions of Faddeev equation for scattering amplitudes $T_{(\gamma)}$'s and physical on-shell S-matrix are presented for all six possible incoming plane waves. The three-body wave function is constructed by using the solution of Faddeev's equation, $T_{(\gamma)}$'s, we also present the result of the constructed three-body wave function for incoming plane wave $\psi_{(0)} = e^{iq_{12}r_{12}}e^{iq_3 r_3}$ as an example in Appendix B 2. Although, there are six independent sets of solutions of Faddeev's equation corresponding to six independent incoming plane waves, for three spinless identical particles, only solutions for totally symmetric and totally antisymmetric wave functions have meaningful physical correspondences: scattering of three spinless bosons and three spinless fermions, respectively. Hence, in the following sections, the attention is focused on three spinless bosons and three spinless fermions scattering only.

A. Solutions of Faddeev's equation for three spinless fermions

For three spinless identical fermions, the wave function has to be totally antisymmetric under exchange of any two particles coordinates; the free incoming wave for totally antisymmetric three fermions is

$$\psi_{(0)}^{\text{anti}} = \sum_{k=1}^{3} (e^{iq_{ij}r_{12}} - e^{-iq_{ij}r_{12}})e^{iq_k r_3}. \quad (22)$$

Given the solutions of scattering amplitudes for each individual wave in Sec. B 1, it is easy to see that the solutions of Faddeev's equation for three spinless identical fermions all vanish, $T_{(\gamma)} = 0, \gamma = 1, 2, 3$. Therefore, the totally antisymmetric wave function for three identical fermions is given by the free incoming wave solution, $\psi_{\text{anti}}(r_{12}, r_3; q_{ij}, q_k) = \psi_{(0)}^{\text{anti}}$.

B. Solutions of Faddeev's equation for three spinless bosons

For three spinless identical bosons, the three-body wave function has to be totally symmetric under exchange of arbitrary two particles coordinates, the free incoming wave for totally symmetric three bosons is given by

$$\psi_{(0)}^{\text{sym}} = \sum_{k=1}^{3} (e^{iq_{ij}r_{12}} + e^{-iq_{ij}r_{12}})e^{iq_k r_3}, \quad (23)$$

therefore

$$v_{(1,2,3)}(k; q_{12}, q_3) = mV_0 \sum_{k=1}^{3} 2(2\pi)\delta(k - q_k). \quad (24)$$

Again using the solutions of scattering amplitudes for each individual wave in Sec. B 1, we found that the solution of Faddeev's equation for three identical bosons are

$$T_{(1,2,3)}(k;q_{ij},q_k)=2(2\pi i)\delta(k-q_1)\frac{imV_0}{1-\frac{imV_0}{2q_{23}}}+2(2\pi i)\delta(k-q_2)\frac{imV_0}{1-\frac{imV_0}{2q_{31}}}+2(2\pi i)\delta(k-q_3)\frac{imV_0(1+\frac{imV_0}{2q_{23}})}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})}-2\frac{\frac{6(imV_0)^2k}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})}}{(k-q_3-i\epsilon)(k-q_2-i\epsilon)(k-q_1+i\epsilon)}. \tag{25}$$

All three $T_{(\gamma)}$'s are identical due to Bose symmetry. By picking up the contribution of poles, $k=q_2+i\epsilon$, $q_3+i\epsilon$, and $q_1-i\epsilon$ in Eq. (25), we introduce three on-shell scattering amplitudes for later convenience of presentation,

$$i\mathcal{T}_3=\frac{(-\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}}\frac{imV_0}{2q_{31}})}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})},$$
$$i\mathcal{T}_1=\frac{(\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}}\frac{imV_0}{2q_{12}})}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})},$$
$$i\mathcal{T}_2=\frac{(\frac{imV_0}{2q_{31}})(1-\frac{imV_0}{2q_{23}}\frac{imV_0}{2q_{12}})}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})}, \tag{26}$$

and

$$i\mathbf{T}_1=i\mathcal{T}_1-it_+(-q_{23}),\quad i\mathbf{T}_2=i\mathcal{T}_2-it_+(-q_{31}),$$
$$i\mathbf{T}_3=i\mathcal{T}_3-it_+(q_{12})(1+2it_+(-q_{23}))=i\mathbf{T}_2-i\mathbf{T}_1, \tag{27}$$

where t_+ again is two-body scattering amplitude given in Eq. (D24). The on-shell scattering contribution of $T_{(\gamma)}$ is thus given by

$$T^{(\mathrm{phys})}_{(\gamma)}(k;q_{ij},q_k)=2(2\pi i)\delta(k-q_1)(2q_{23})i\mathcal{T}_1+2(2\pi i)\delta(k-q_2)(2q_{31})i\mathcal{T}_2-2(2\pi i)\delta(k-q_3)(2q_{12})i\mathcal{T}_3. \tag{28}$$

The on-shell amplitudes $\mathcal{T}_\gamma$ in Eq. (26) satisfy unitarity relations,

$$\mathrm{Im}\mathcal{T}_\gamma=\mathcal{T}^*_\gamma(\mathcal{T}_3+\mathcal{T}_1+\mathcal{T}_2),\qquad \gamma=1,2,3. \tag{29}$$

The three-body off-shell scattering amplitude thus reads

$$T(k_{12},k_3;q_{ij},q_k)=\sum_{\gamma=1}^{3}T^{(\mathrm{phys})}_{(\gamma)}(k_\gamma;q_{ij},q_k)-2\sum_{\gamma=1}^{3}\mathcal{P}\frac{\frac{6(imV_0)^2k_\gamma}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})}}{(k_\gamma-q_3)(k_\gamma-q_2)(k_\gamma-q_1)}, \tag{30}$$

where $\mathcal{P}$ stands for the principal part of poles, and $k_1=-k_{12}-\frac{1}{2}k_3$ and $k_2=k_{12}-\frac{1}{2}k_3$. It is easy to show that

$$\sum_{\gamma=1}^{3}\mathcal{P}\frac{k_\gamma}{(k_\gamma-q_3)(k_\gamma-q_2)(k_\gamma-q_1)}\propto\left(\sigma^2-k_{12}^2-\frac{3}{4}k_3^2\right), \tag{31}$$

therefore, the principal part on the right-hand side of Eq. (30) vanishes for on-shell scattering amplitude. The Bose symmetry warrants that all six on-shell S-matrices are identical and given by $S_{\mathrm{sym}}=1+2i\mathcal{T}$, where as defined in Eq. (21), $\mathcal{T}$ is physical scattering amplitude, and $\mathcal{T}=\sum_{k=1}^{3}\mathcal{T}_k$, thus, we find

$$S_{\mathrm{sym}}=\frac{(1-\frac{imV_0}{2q_{12}})(1+\frac{imV_0}{2q_{23}})(1+\frac{imV_0}{2q_{31}})}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})}. \tag{32}$$

The on-shell physical scattering amplitudes and S-matrix can be expressed in terms of a single two-body scattering phase shift, $t_+(q)=\frac{e^{2i\delta(q)}-1}{2i}$, thus, we obtain

$$i\mathcal{T}_3=\left(\frac{e^{2i\delta(q_{12})}-1}{2}\right)\left(\frac{e^{2i\delta(-q_{23})}+e^{2i\delta(-q_{31})}}{2}\right),$$
$$i\mathcal{T}_1=\left(\frac{e^{2i\delta(-q_{23})}-1}{2}\right)\left(\frac{1+e^{2i\delta(-q_{31})}e^{2i\delta(q_{12})}}{2}\right),$$
$$i\mathcal{T}_2=\left(\frac{e^{2i\delta(-q_{31})}-1}{2}\right)\left(\frac{1+e^{2i\delta(-q_{23})}e^{2i\delta(q_{12})}}{2}\right),$$
$$S_{\mathrm{sym}}=e^{2i(\delta(q_{12})+\delta(-q_{23})+\delta(-q_{31}))}. \tag{33}$$

It can be easily checked that the phase shift expressions of on-shell amplitudes $\mathcal{T}_\gamma$ in Eq. (33) are the consequence of unitarity relations in Eq. (29), therefore Eq. (33) may be more general for pairwise and short-range interactions of three identical particles scattering.

The totally symmetric wave function can be constructed by using Eq. (12) and solutions of Faddeev's equation given in Eq. (25). An example of construction of the wave function from solutions of Faddeev's equation is given in Appendix B 2, the construction is rather lengthy and tedious, so we do not present all the details in the text. The totally symmetric wave function is expressed in terms of a single independent coefficient,

$$\begin{aligned}
&\psi_{\mathrm{sym}}(r_{12}, r_3; q_{ij}, q_k) - \psi^{\mathrm{sym}}_{(0)} \\
&= (A_{\mathrm{sym}}(r_{12}, r_3) e^{iq_{12}r_{12}} + A_{\mathrm{sym}}(-r_{12}, r_3) e^{-iq_{12}r_{12}}) e^{iq_3 r_3} \\
&\quad + (A_{\mathrm{sym}}(r_{31}, r_2) e^{iq_{23}r_{12}} + A_{\mathrm{sym}}(-r_{23}, r_1) e^{-iq_{23}r_{12}}) e^{iq_1 r_3} \\
&\quad + (A_{\mathrm{sym}}(r_{23}, r_1) e^{iq_{31}r_{12}} + A_{\mathrm{sym}}(-r_{31}, r_2) e^{-iq_{31}r_{12}}) e^{iq_2 r_3},
\end{aligned} \tag{34}$$

where $r_{23} = -\frac{r_{12}}{2} + r_3$ and $r_{31} = -\frac{r_{12}}{2} - r_3$, and

$$\begin{aligned}
A_{\mathrm{sym}}(r_{12}, r_3) = 1 &+ \theta(r_{12}) 2it_+(q_{12})[1 + 2it_+(-q_{23})] \\
&+ \theta(-r_{23}) 2it_+(-q_{23}) + \theta(-r_{31}) 2it_+(-q_{31}) \\
&- \theta(r_{12})\theta(r_{23}) 4i\mathbf{T}_1 + \theta(r_{12})\theta(-r_{31}) 4i\mathbf{T}_2.
\end{aligned} \tag{35}$$

III. THREE-BODY SCATTERING IN FINITE VOLUME

For three particles interaction in a one-dimensional periodic box with the size of L, the wave function of the three-particle in finite volume, $\Psi^{(L)}(x_1, x_2, x_3; p_1, p_2, p_3)$, must satisfy the periodic boundary condition,

$$\begin{aligned}
&\Psi^{(L)}(x_1 + n_{x_1}L, x_2 + n_{x_2}L, x_3 + n_{x_3}L; p_1, p_2, p_3) \\
&= \Psi^{(L)}(x_1, x_2, x_3; p_1, p_2, p_3), \qquad n_{x_1,x_2,x_3} \in \mathbb{Z}.
\end{aligned} \tag{36}$$

The finite volume three-body wave function, $\Psi^{(L)}$, is constructed from the three-body free space wave function, Ψ, by

$$\begin{aligned}
&\Psi^{(L)}(x_1, x_2, x_3; p_1, p_2, p_3) \\
&= \frac{1}{V} \sum_{n_{x_1}, n_{x_2}, n_{x_3} \in \mathbb{Z}} \Psi(x_1 + n_{x_1}L, x_2 + n_{x_2}L, x_3 \\
&\quad + n_{x_3}L; p_1, p_2, p_3),
\end{aligned} \tag{37}$$

in this way, the periodic boundary condition in Eq. (36) is warranted. Factorizing the center of mass wave function and relative wave function, and also defining new variables, $n = n_{x_k}, n_{ij} = n_{x_i} - n_{x_j}$ and $n_k = (n_{x_i} + n_{x_j}) - 2n_{x_k}$, where $(n, n_{ij}, n_k) \in \mathbb{Z}$, thus, we find

$$\begin{aligned}
&\Psi^{(L)}(x_1, x_2, x_3; p_1, p_2, p_3) \\
&= \frac{1}{V}\left(\sum_{n\in\mathbb{Z}} e^{iPnL}\right) e^{iPR} \psi^{(L)}(r_{ij}, r_k; q_{ij}, q_k), \\
&\psi^{(L)}(r_{ij}, r_k; q_{ij}, q_k) \\
&= \sum_{n_{ij}, n_k \in \mathbb{Z}} e^{i\frac{P}{3}n_k L} \psi\left(r_{ij} + n_{ij}L, r_k + \frac{1}{2}n_k L; q_{ij}, q_k\right),
\end{aligned} \tag{38}$$

where $\psi^{(L)}$ represents the relative finite volume wave function, and the normalization factor of infinite summation V is given by $V = \sum_{n\in\mathbb{Z}} e^{iPnL} = \frac{2\pi}{L}\sum_{d\in\mathbb{Z}} \delta(P + \frac{2\pi}{L}d)$. The integer variables (n_{ij}, n_k) are related to relative coordinates (r_{jk}, r_k) in $(ij)k$ configuration. The relative variables in other configurations can be expressed in terms of variables (n_{ij}, n_k), e.g. integer variables (n_{jk}, n_i) in $(jk)i$ configuration are given by

$$n_{jk} = -\frac{1}{2}n_{ij} + \frac{1}{2}n_k, \qquad n_i = -\frac{3}{2}n_{ij} - \frac{1}{2}n_k, \qquad i \neq j \neq k. \tag{39}$$

After removal of center of mass motion, the periodic boundary condition for the relative finite volume wave function now reads

$$\begin{aligned}
&\psi^{(L)}\left(r_{ij} + n_{ij}L, r_k + \frac{1}{2}n_k L; q_{ij}, q_k\right) \\
&= e^{-i\frac{P}{3}n_k L} \psi^{(L)}(r_{ij}, r_k; q_{ij}, q_k), \qquad P = \frac{2\pi}{L}d, d \in \mathbb{Z}.
\end{aligned} \tag{40}$$

With the solution of wave functions, for instance, the totally symmetric wave function given in Eqs. (34) and (35), the finite volume three-body wave function is constructed by using Eq. (38). The infinite summation in one dimension can be performed by using the property of geometric series

$$\sum_{n=\alpha}^{\infty} x^n = \frac{x^\alpha}{1-x}, \qquad (n, \alpha) \in \mathbb{Z}. \tag{41}$$

Hence the analytic solutions in one dimension for the δ-function potential can be obtained. As have been mentioned in previous sections, not all six independent wave functions correspond to physical systems, except totally antisymmetric and symmetric wave functions given in Secs. II A and II B, which represent three identical fermions and bosons scattering respectively. The other four wave functions are not related to any physical processes, and indeed, we found no physical solutions in finite volume except three spinless bosons and fermion systems. In the case of three spinless identical fermions, because two-body interaction by the δ-function potential vanishes for identical spinless fermions, three identical fermions experience zero scattering effect, and behave as free particles. Therefore, the three-body wave function has a trivial solution in free space, as shown in Sec. II A. In finite volume, the periodic boundary condition leads to the quantization of the momenta of three fermions as a free particle in a finite box, $p_i = \frac{2\pi}{L}n_{x_i}, n_{x_i} \in \mathbb{Z}$. Nevertheless, in what follows, we will only work out all the details of the finite volume wave function for three the identical bosons system.

In the case of three spinless identical bosons, the expression of the three-body wave function is dramatically simplified by symmetry consideration, the three-body wave function in free space is expressed by a single independent coefficient only, see Eqs. (34) and (35). Therefore we only need to perform the infinite sum for a single plane wave, the rest of the components of the finite volume wave function are easily obtained by symmetry consideration. For instance, we can pick the plane wave $e^{iq_{12}r_{12}}e^{iq_3r_3}$ component, the corresponding coefficient of the plane wave in finite volume is then given by

$$A^{(L)}_{\rm sym}(r_{12},r_3)=\sum_{n_{12},n_3\in\mathbb{Z}}e^{iq_{12}n_{12}L}e^{i(\frac{P}{3}+\frac{q_3}{2})n_3L}\times A_{\rm sym}\left(r_{12}+n_{12}L,r_3+\frac{1}{2}n_3L\right). \quad (42)$$

For nontrivial solutions, only the last two terms in Eq. (35) survive in the finite box.

First of all, for the term proportional to $\theta(r_{12})\theta(r_{23})$ in Eq. (35), the infinite sum reads

$$\begin{aligned}&\sum_{n_{12},n_3\in\mathbb{Z}}^{n_{23}=\frac{n_3-n_{12}}{2}}e^{iq_{12}n_{12}L}e^{i(\frac{P}{3}+\frac{q_3}{2})n_3L}\theta(r_{12}+n_{12}L)\theta(r_{23}+n_{23}L)\\&=\sum_{n_{12}=\theta(-r_{12})}^{\infty}e^{-i(\frac{2}{3}P+q_1)n_{12}L}\sum_{n_{23}=\theta(-r_{23})}^{\infty}e^{i(\frac{2}{3}P+q_3)n_{23}L}\\&=\left[\theta(r_{12})+\frac{e^{-i(\frac{2}{3}P+q_1)L}}{1-e^{-i(\frac{2}{3}P+q_1)L}}\right]\left[\theta(r_{23})+\frac{e^{i(\frac{2}{3}P+q_3)L}}{1-e^{i(\frac{2}{3}P+q_3)L}}\right].\end{aligned} \quad (43)$$

Next, for the term proportional to $\theta(r_{12})\theta(-r_{31})$ in Eq. (35), we have

$$\begin{aligned}&\sum_{n_{12},n_3\in\mathbb{Z}}^{n_{31}=-\frac{n_3+n_{12}}{2}}e^{iq_{12}n_{12}L}e^{i(\frac{P}{3}+\frac{q_3}{2})n_3L}\theta(r_{12}+n_{12}L)\theta(-r_{31}-n_{31}L)\\&=\sum_{n_{12}=\theta(-r_{12})}^{\infty}e^{i(\frac{2}{3}P+q_2)n_{12}L}\sum_{n_{31}=-\infty}^{-\theta(r_{31})}e^{-i(\frac{2}{3}P+q_3)n_{31}L}\\&=\left[\theta(r_{12})+\frac{e^{i(\frac{2}{3}P+q_2)L}}{1-e^{i(\frac{2}{3}P+q_2)L}}\right]\left[\theta(-r_{31})+\frac{e^{i(\frac{2}{3}P+q_3)L}}{1-e^{i(\frac{2}{3}P+q_3)L}}\right].\end{aligned} \quad (44)$$

Putting everything together, we obtain the finite volume coefficient of plane wave $e^{iq_{12}r_{12}}e^{iq_3r_3}$,

$$\begin{aligned}A^{(L)}_{\rm sym}(r_{12},r_3)=&-\theta(r_{12})\theta(r_{23})4i\mathbf{T}_1+\theta(r_{12})\theta(-r_{31})4i\mathbf{T}_2\\&-\theta(r_{23})4i\mathbf{T}_1\frac{e^{-i(\frac{2}{3}P+q_1)L}}{1-e^{-i(\frac{2}{3}P+q_1)}}\\&+\theta(-r_{31})4i\mathbf{T}_2\frac{e^{i(\frac{2}{3}P+q_2)L}}{1-e^{i(\frac{2}{3}P+q_2)L}}\\&+\theta(r_{12})4i\mathbf{T}_3\frac{e^{i(\frac{2}{3}P+q_3)L}}{1-e^{i(\frac{2}{3}P+q_3)L}}\\&+4i\mathbf{T}_2\frac{e^{i(\frac{2}{3}P+q_2)L}}{1-e^{i(\frac{2}{3}P+q_2)L}}\frac{e^{i(\frac{2}{3}P+q_3)L}}{1-e^{i(\frac{2}{3}P+q_3)L}}\\&-4i\mathbf{T}_1\frac{e^{-i(\frac{2}{3}P+q_1)L}}{1-e^{-i(\frac{2}{3}P+q_1)L}}\frac{e^{i(\frac{2}{3}P+q_3)L}}{1-e^{i(\frac{2}{3}P+q_3)L}}.\end{aligned} \quad (45)$$

The coefficients for other waves in finite volume are obtained by symmetry consideration, e.g. for plane wave $e^{-iq_{12}r_{12}}e^{iq_3r_3}$, the coefficient is given by $A^{(L)}_{\rm sym}(-r_{12},r_3)$, etc. Therefore, the three-body wave function for spinless bosons system in finite volume yields

$$\begin{aligned}&\psi^{(L)}_{\rm sym}(r_{12},r_3;q_{ij},q_k)\\&=(A^{(L)}_{\rm sym}(r_{12},r_3)e^{iq_{12}r_{12}}+A^{(L)}_{\rm sym}(-r_{12},r_3)e^{-iq_{12}r_{12}})e^{iq_3r_3}\\&\quad+(A^{(L)}_{\rm sym}(r_{31},r_2)e^{iq_{23}r_{12}}+A^{(L)}_{\rm sym}(-r_{23},r_1)e^{-iq_{23}r_{12}})e^{iq_1r_3}\\&\quad+(A^{(L)}_{\rm sym}(r_{23},r_1)e^{iq_{31}r_{12}}+A^{(L)}_{\rm sym}(-r_{31},r_2)e^{-iq_{31}r_{12}})e^{iq_2r_3}.\end{aligned} \quad (46)$$

As demonstrated in the two-body scattering case in [56,77], the secular equations or quantization conditions for three-body interaction in the finite box are obtained by the matching condition, $\psi^{(L)}_{\rm sym}(r_{12},r_3;q_{ij},q_k)=\psi_{\rm sym}(r_{12},r_3;q_{ij},q_k)$. All six plane waves are independent in Eqs. (34) and (46), therefore, secular equations are equivalently obtained by matching coefficients of six independent plane waves. To obtain secular equations, we first consider the coefficient for a combination of $(e^{iq_{12}r_{12}}-e^{-iq_{12}r_{12}})e^{iq_3r_3}$, which is antisymmetric under exchange of $r_{12}\leftrightarrow -r_{12}$ and is obviously forbidden for the bosons system. The matching condition for this particular wave reads

$$\begin{aligned}&A_{\rm sym}(r_{12},r_3)-A_{\rm sym}(-r_{12},r_3)\\&\quad=A^{(L)}_{\rm sym}(r_{12},r_3)-A^{(L)}_{\rm sym}(-r_{12},r_3).\end{aligned} \quad (47)$$

Using Eqs. (35) and (45), the matching condition leads to

$$\begin{aligned}&4i\mathbf{T}_3[\theta(r_{12})-\theta(-r_{12})]\\&\quad\times\left[\frac{e^{i(\frac{2}{3}P+q_3)L}}{1-e^{i(\frac{2}{3}P+q_3)L}}-\frac{e^{2i(\delta(-q_{23})-\delta(-q_{31}))}}{1-e^{2i(\delta(-q_{23})-\delta(-q_{31}))}}\right]\\&\quad-4i\mathbf{T}_1[\theta(r_{23})-\theta(-r_{31})]\\&\quad\times\left[\frac{e^{-i(\frac{2}{3}P+q_1)L}}{1-e^{-i(\frac{2}{3}P+q_1)L}}-\frac{e^{-2i(\delta(-q_{31})+\delta(q_{12}))}}{1-e^{-2i(\delta(-q_{31})+\delta(q_{12}))}}\right]\\&\quad-4i\mathbf{T}_2[\theta(r_{23})-\theta(-r_{31})]\\&\quad\times\left[\frac{e^{i(\frac{2}{3}P+q_2)L}}{1-e^{i(\frac{2}{3}P+q_2)L}}-\frac{e^{-2i(\delta(-q_{23})+\delta(q_{12}))}}{1-e^{-2i(\delta(-q_{23})+\delta(q_{12}))}}\right]=0.\end{aligned} \quad (48)$$

The matching condition has to be satisfied in all regions in the (r_{12},r_3) plane, see Fig. 2, choosing region for an example (I): $r_{12}<0$, $r_{23}>0$ and $r_{31}<0$, thus, we obtain our first secular equation,

$$e^{i(\frac{2}{3}P+q_3)L}=e^{2i(\delta(-q_{23})-\delta(-q_{31}))}. \quad (49)$$

The other two secular equations are obtained similarly by considering the combination of $(e^{iq_{23}r_{12}}-e^{-iq_{23}r_{12}})e^{iq_1r_3}$ and $(e^{iq_{31}r_{12}}-e^{-iq_{31}r_{12}})e^{iq_2r_3}$ respectively,

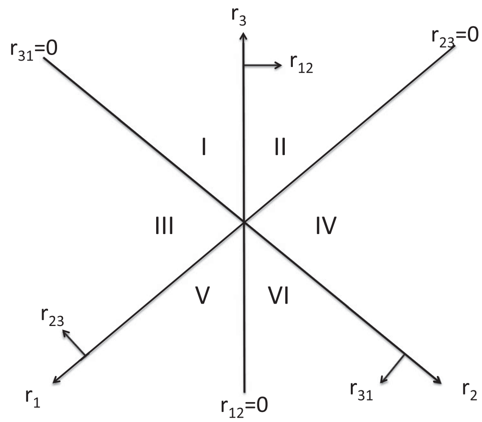

FIG. 2. Diagram show six segments from $(I)-(VI)$ in (r_{12}, r_3) plane, the δ-function potentials are nonzero only at lines, $r_{12}=0$, $r_{23}=0$ and $r_{31}=0$. The arrows show the positive direction of each variable.

$$e^{i(\frac{2}{3}P+q_1)L} = e^{2i(\delta(-q_{31})+\delta(q_{12}))}, \quad (50)$$

$$e^{i(\frac{2}{3}P+q_2)L} = e^{-2i(\delta(-q_{23})+\delta(q_{12}))}. \quad (51)$$

Although the above three secular equations for three-boson interaction are obtained by choosing a particular region, it is easy to check that Eqs. (49)–(51) are indeed the solutions of all six matching conditions in all regions on the (r_{12}, r_3) plane. As a matter of fact, the secular equations displayed in Eqs. (49)–(51) have been obtained long ago by Yang in [89] as a specific case of the N-particle system as $N=3$. In [89], based on Bethe's hypothesis [90–92], i.e. "no diffraction" hypothesis, Yang considered a more general situation of N identical particles problem in one dimension for δ-interaction. Nevertheless, all three secular equations for the three-boson system appear as Lüscher's formula-like quantization conditions,

$$\cot\left(\frac{P}{3}+\frac{q_3}{2}\right)L + \cot\left(\delta(-q_{31})-\delta(-q_{23})\right) = 0,$$
$$\cot\left(\frac{P}{3}+\frac{q_1}{2}\right)L + \cot\left(-\delta(-q_{31})-\delta(q_{12})\right) = 0,$$
$$\cot\left(\frac{P}{3}+\frac{q_2}{2}\right)L + \cot\left(\delta(-q_{23})+\delta(q_{12})\right) = 0. \quad (52)$$

As can be easily see from Eqs. (49)–(51), only two conditions in Eq. (52) are in fact independent that are both given by a single two-body phase shift and relative momenta of three particles. All the relative momenta of three particles are determined by two independent particle momenta as well. Therefore, the quantization conditions are finally given by two coupled equations that depend on two independent particle momenta and a two-body phase shift. Three-body energy spectrum in finite volume thus can be obtained by solving two independent particle momenta given that a two-body phase shift is known or can be modeled.

IV. DISCUSSION AND CONCLUSION

Using quantization conditions in Eqs. (49)–(51), we also obtain relations for relative momenta q_{ij}, for an example,

$$\cot\frac{q_{12}L}{2} + \cot\left(\delta(q_{12}) + \frac{\delta(-q_{23})+\delta(-q_{31})}{2}\right) = 0. \quad (53)$$

The $\delta(q_{12})$ comes from the disconnected scattering contribution in the (12) pair, see Fig. 1(a), $\frac{\delta(-q_{23})+\delta(-q_{31})}{2}$ is the net result of the sum over all rescattering contributions from other channels into the (12) pair. The physical picture is somehow quite similar to the three-body rescattering effect in three-body decay processes [32–42]. Based on the Khuri-Trieman equation approach, the decay process of a particle (0) into three final particles is described by a sum of all possible decay chains: $0 \to (12)3 + 1(23) + (31)2 \to 123$. For each individual decay chain, the amplitude is the product of two-body amplitude and a scalar function that describes the net effect of three-body rescattering corrections to the disconnected two-body contribution. The analogue to rescattering in three-body decay processes, $\frac{\delta(-q_{23})+\delta(-q_{31})}{2}$ may be interpreted as the three-body rescattering corrections to the disconnected two-body contribution, $\delta(q_{12})$.

Assuming that we can treat Faddeev's equation, Eq. (14), as a perturbation theory, and the leading order solution of Eq. (14) is a disconnected contribution,

$$T^{(0)}_{(\gamma)}(k; q_{ij}, q_k) = (2\pi)\delta(k-q_\gamma)\left(2\sqrt{\sigma^2-\frac{3}{4}q_\gamma^2}\right) t_+\left(\sqrt{\sigma^2-\frac{3}{4}q_\gamma^2}\right). \quad (54)$$

Therefore, the total scattering amplitude $T(k_{12}, k_3; q_{ij}, q_k) = \sum_{\gamma=1}^{3} T^{(0)}_{(\gamma)}(k_\gamma; q_{ij}, q_k)$ only has the contribution of three disconnected scattering amplitudes, $1+2\to 1+2$ with particle-3 as a spectator, $2+3\to 2+3$ with particle-1 as a spectator and $3+1\to 3+1$ as particle-2 as a spectator. Iterating Eq. (14) once, thus, the next-leading order contribution of $T_{(\gamma)}$ is given by

$$T^{(1)}_{(\gamma)}(k; q_{ij}, q_k) = \left(2\sqrt{\sigma^2-\frac{3}{4}k^2}\right) it_+\left(\sqrt{\sigma^2-\frac{3}{4}k^2}\right) \times \sum_{\alpha\neq\gamma} \frac{\left(2\sqrt{\sigma^2-\frac{3}{4}q_\alpha^2}\right) it_+\left(\sqrt{\sigma^2-\frac{3}{4}q_\alpha^2}\right)}{\sigma^2-\frac{3}{4}q_\alpha^2-(k+\frac{q_\alpha}{2})^2+i\epsilon}. \quad (55)$$

The diagrammatic representation of $T^{(1)}_{(3)}$ is shown in Figs. 1(b) and 1(c), which are double scattering contributions from pair (23) and (31) into the (12) pair. Now both leading and next-leading order contributions to the on-shell scattering amplitude $\mathcal{T}_3$ are

$$\mathcal{T}_3^{(0)} + \mathcal{T}_3^{(1)} = t_+(q_{12})[1 + it_+(-q_{23}) + it_+(-q_{31})]. \quad (56)$$

In Eq. (56), the perturbation result of scattering amplitude, $\mathcal{T}_3$, on right-hand side of the equation now indeed appears as the product of disconnected two-body scattering amplitude $t_+(q_{12})$ and the rescattering corrections to leading order contribution, $t_+(q_{12})$.

On the other hand, the asymptotic behavior of the two-body phase shift is given by $\delta(q) \to -\frac{mV_0}{2q}$ as $q \to \infty$, where $V_0 > 0$ for repulsive interaction and $V_0 < 0$ for attractive interaction. For large q_3 [the momentum of the third particle is well separated from relative momentum of pair (12)], thus, $q_{23} \to \frac{3}{4} q_3$ and $q_{31} \to -\frac{3}{4} q_3$, and

$$\frac{\delta(-q_{23})+\delta(-q_{31})}{2} \xrightarrow{q_3\to\infty} -\frac{mV_0}{3}\left(\frac{1}{q_3}-\frac{1}{q_3}\right) = 0,$$
$$\frac{\delta(-q_{23})-\delta(-q_{31})}{2} \xrightarrow{q_3\to\infty} -\frac{2mV_0}{3q_3} \sim \delta\left(\frac{3}{4}q_3\right),$$
$$it_+(-q_{23})+it_+(-q_{31}) \xrightarrow{q_3\to\infty} \frac{2imV_0}{3}\left(\frac{1}{q_3}-\frac{1}{q_3}\right) = 0. \quad (57)$$

Therefore, at large q_3, the rescattering between an energetic third particle and particles in pair (12) is less likely to happen. The quantization condition in Eq. (53) and the first condition in Eq. (52) are thus reduced to isobar model type conditions, $\cot\frac{q_{12}L}{2} + \cot\delta(q_{12}) = 0$ and $\cot(\frac{P}{3} + \frac{q_3}{2})L + \cot\delta(\frac{3}{4}q_3) = 0$, in which the rescattering effect from third particle is weak and neglected. The reduction of quantization conditions can be understood in the following arguments. Diagrammatically, rescattering amplitudes, such as Figs. 1(b) and 1(c), are proportional to propagators $\frac{1}{(k-q_3)(k-q_1)}$ and $\frac{1}{(k-q_3)(k-q_2)}$ respectively. When off-shell momentum q is taken close to q_3, the amplitude at the pole $k = q_3$ position leads to on-shell scattering amplitude $\mathcal{T}_3$ given in Eq. (33), meanwhile, the contributions from Figs. 1(b) and 1(c) are proportional to $\frac{1}{2q_{31}}$ and $\frac{1}{2q_{23}}$ respectively. Hence, for large q_3, the rescattering contribution from channel (23) and (31) into pair (12) are both highly suppressed by $\frac{1}{q_3}$, so that quantization conditions for both q_{12} in Eq. (53) and q_3 in the first condition in Eq. (52) are reduced to isobar model like quantization conditions, and the dominant contribution is from the disconnected diagram.

Although, McGuire's model displays no diffraction effect, our results given in Eq. (52) may still hold for a general short-range potential. This may be demonstrated by asymptotic behavior of the three-body wave function. The asymptotic form of the wave function in one dimension is quite different from that in three dimensions, e.g. the two-body scattering wave function in one dimension does not fall off in any direction, see Eq. (D5). For incoming three free particles, as in three dimensions, the one-dimensional three-body wave function also consists of several pieces that display the different asymptotic behavior and describe different physical processes: (1) the contribution from incoming free waves, disconnected diagrams and non-diffracted on-shell rescattering effects all have the form of nondiffraction waves, e.g. $e^{iq_{ij}r_{12}}e^{iq_k r_3}$; (2) the bound state capture process has the form of, e.g. $e^{-\chi_{12}|r_{12}|}e^{iq_3 r_3}$ with a bound state of (12) pair in the final state, which decays exponentially as $(r_{ij}, r_k) \to \infty$; (3) diffraction waves are of the order of

$$\int \frac{dk_{12}}{2\pi}\frac{dk_3}{2\pi}\frac{e^{ik_{12}r_{12}}e^{ik_3 r_3}}{\sigma^2 - k_{12}^2 - \frac{3}{4}k_3^2 + i\epsilon} \overset{(r_{12},r_3)\to\infty}{\propto} \left(r_{12}^2 + \frac{4}{3}r_3^2\right)^{-\frac{1}{4}} e^{i\sigma\sqrt{r_{12}^2+\frac{4}{3}r_3^2}}, \quad (58)$$

which describe the spherical wave of the three-body effect and are suppressed at a large distance [93]. Hence, at large separations of all three particles, the dominant contribution is from nondiffraction waves.

In summary, McGuire's model is adopted to describe three spinless identical particles scattering in one spatial dimension; we present the detailed solutions of Faddeev's equation for scattering of three free spinless particles. The three particles interaction in finite volume is derived in Sec. III. Our approach of solving three-body interaction in finite volume is a generalization of the approach developed in [56,77] by considering wave function in the configuration representation; the advantage is that the wave function contains only on-shell scattering amplitudes. The quantization conditions by matching wave function in free space and finite volume are given in terms of two-body scattering phase shifts in Eq. (52). The quantization conditions in McGuire's model is dramatically simplified due to Bethe's hypothesis, and the quantization conditions presented in Eq. (52) are Lüscher's formula-like and are consistent with results obtained in [89]. The results in Eq. (52) are presented in terms of two-body scattering phase shift. The quantization conditions may be tested in the near future by one-dimensional lattice models, such as ones studied in [77,88].

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APPENDIX A: FORMAL THEORY OF SCATTERING AND FADDEEV'S EQUATION

1. Formal theory of scattering

In the formal theory of scattering [94], assuming the Hamiltonian of the scattering system is given by the sum of a kinematic term and an interaction term, $\hat{H} = \hat{H}_{(0)} + \hat{V}$, the S-matrix is given in terms of the solution of the Schrödinger equation,

$$\langle f|\hat{S}|i\rangle = \langle f|\hat{U}(\infty,0)\hat{U}(0,-\infty)|i\rangle = \langle\Psi_f^{(-)}(0)|\Psi_i^{(+)}(0)\rangle, \tag{A1}$$

where the unitary operator $\hat{U}$ is given by $\hat{U}(t,t_0) = \mathcal{T}\{\exp[-i\int_{t_0}^{t} dt' e^{i\hat{H}_{(0)}t'}\hat{V}e^{-i\hat{H}_{(0)}t'}]\}$, it has the properties of $|\Psi(t)\rangle = \hat{U}(t,t_0)|\Psi(t_0)\rangle$. $|\Psi(t)\rangle$ is the solution of the time-dependent Schrödinger equation, which describes the wave vector of the scattering system. $|i\rangle$ and $|f\rangle$ are initial and final state vectors in the absence of interaction at distant past and future respectively. The incoming and outgoing wave vectors $|\Psi_i^{(+)}(0)\rangle = \hat{U}(0,-\infty)|i\rangle$ and $\langle\Psi_f^{(-)}(0)| = \langle f|\hat{U}(\infty,0)$ are also given by the Lippmann-Schwinger equation [94,95],

$$|\Psi_i^{(+)}(0)\rangle = \left[1+\frac{1}{E_i-\hat{H}_{(0)}-\hat{V}+i\epsilon}\hat{V}\right]|i\rangle = |i\rangle + \frac{1}{E_i-\hat{H}_{(0)}+i\epsilon}\hat{V}|\Psi_i^{(+)}(0)\rangle, \tag{A2}$$

$$\langle\Psi_f^{(-)}(0)| = \langle f|\left[1+\hat{V}\frac{1}{E_f-\hat{H}_{(0)}-\hat{V}+i\epsilon}\right] = \langle f| + \langle\Psi_f^{(-)}(0)|\hat{V}\frac{1}{E_f-\hat{H}_{(0)}+i\epsilon}, \tag{A3}$$

where E_i and E_f denote initial and final state energies respectively.

Using Eq. (A3), we first rewrite the S-matrix to

$$\langle f|\hat{S}|i\rangle = \langle f|\Psi_i^{(+)}(0)\rangle - \frac{1}{E_f-E_i+i\epsilon}\langle f|\hat{T}|i\rangle, \tag{A4}$$

where $\langle f|\hat{T}|i\rangle = -\langle f|\hat{V}|\Psi_i^{(+)}(0)\rangle$ is scattering T-matrix. With the help of Eq. (A2), we obtain the relations for the T-matrix,

$$\hat{T} = -\hat{V} + \hat{V}\frac{1}{E_i-\hat{H}+i\epsilon}\hat{V} = -\hat{V} + \hat{V}\frac{1}{E_i-\hat{H}_{(0)}+i\epsilon}\hat{T}. \tag{A5}$$

In terms of T-matrix, the incoming wave reads $|\Psi_i^{(+)}(0)\rangle = [1-\frac{1}{E_i-\hat{H}_{(0)}+i\epsilon}\hat{T}]|i\rangle$, therefore, the S- and T-matrix are related by

$$\langle f|\hat{S}|i\rangle = \langle f|i\rangle + 2\pi i\delta(E_i-E_f)\langle f|\hat{T}|i\rangle. \tag{A6}$$

2. Faddeev's equation

For three-particle scattering, the wave vector $|\Psi_E^{(+)}\rangle$ satisfies Schrödinger equation,

$$(E-\hat{H}_{(0)}-\hat{V})|\Psi_E^{(+)}\rangle = 0. \tag{A7}$$

Assuming pairwise interactions among each pair of particles, $\hat{V} = \sum_{\gamma=1}^{3}\hat{V}_{(\gamma)}$, where $\hat{V}_{(\gamma)}$ stands for the pairwise interaction between αth and βth particles. As shown in [26,27], the self-consistent equations for the three-body wave function depend on the free incoming waves, and are split into four classes according to four types of asymptotic free incoming waves: $|i\rangle$ and $|\Phi_{(\gamma)}\rangle$ ($\gamma = 1,2,3$), where $|i\rangle$ is solutions of $(E-\hat{H}_{(0)})|i\rangle = 0$ and represents incoming wave of three free particles, and $|\Phi_{(\gamma)}\rangle$ is the solution of $(E-\hat{H}_{(0)}-\hat{V}_{(\gamma)})|\Phi_{(\gamma)}\rangle = 0$ and represents the free γth particle plus a bound state in $(\alpha\beta)$ pair.

a. Scattering of three free particles

For three-body scattering with initial state of free incoming wave $|i\rangle$, the three-body scattering wave vector has the form of $|\Psi_E^{(+)}\rangle = |i\rangle + \sum_{\gamma=1}^{3}|\Psi_{(\gamma)}\rangle$ [26,27], where $|\Psi_{(\gamma)}\rangle$ satisfies the equation

$$|\Psi_{(\gamma)}\rangle = \hat{G}_{(\gamma)}\hat{V}_{(\gamma)}(|i\rangle + |\Psi_{(\alpha)}\rangle + |\Psi_{(\beta)}\rangle), \qquad \alpha\neq\beta\neq\gamma. \tag{A8}$$

The Green's function $\hat{G}_{(\gamma)} = (E-\hat{H}_{(0)}-\hat{V}_{(\gamma)}+i\epsilon)^{-1}$ is the solution of the equation $(E-\hat{H}_{(0)}-\hat{V}_{(\gamma)})\hat{G}_{(\gamma)} = 1$. Green's function $\hat{G}_{(\gamma)}$ is related to two-body scattering amplitude by $\hat{G}_{(\gamma)} = \hat{G}_{(0)}(1-\hat{t}_{(\gamma)}\hat{G}_{(0)})$, where $\hat{G}_{(0)} = (E-\hat{H}_{(0)}+i\epsilon)^{-1}$ and $\hat{t}_{(\gamma)} = -\hat{V}_{(\gamma)} + \hat{V}_{(\gamma)}\hat{G}_{(0)}\hat{t}_{(\gamma)}$ are free Green's function and two-body scattering T-matrix in the $(\alpha\beta)$ pair channel respectively. Therefore, we found relations $\hat{V}_{(\gamma)}\hat{G}_{(\gamma)} = -\hat{t}_{(\gamma)}\hat{G}_{(0)}$ and

$$\hat{V}_{(\gamma)}|\Psi_{(\gamma)}\rangle = -\hat{t}_{(\gamma)}\hat{G}_{(0)}\hat{V}_{(\gamma)}(|i\rangle + |\Psi_{(\alpha)}\rangle + |\Psi_{(\beta)}\rangle). \tag{A9}$$

The total three-body scattering amplitude is given by $\hat{T}|i\rangle = -\hat{V}|\Psi_E^{(+)}\rangle = \sum_{\gamma=1}^{3}\hat{T}_{(\gamma)}|i\rangle$ [26,27], where

$$\hat{T}_{(\gamma)}|i\rangle = -\hat{V}_{(\gamma)}|\Psi_E^{(+)}\rangle. \tag{A10}$$

Using Eq. (A9), we thus have

$$\hat{T}_{(\gamma)}|i\rangle = -(1-\hat{t}_{(\gamma)}\hat{G}_{(0)})\hat{V}_{(\gamma)}(|i\rangle + |\Psi_{(\alpha)}\rangle + |\Psi_{(\beta)}\rangle), \tag{A11}$$

$$|\Psi_{(\gamma)}\rangle = -\hat{G}_{(0)}\hat{T}_{(\gamma)}|i\rangle. \tag{A12}$$

Equations (A11) and (A12) together lead to the well-known Faddeev's equation for three particles scattering [26,27],

$$\hat{T}_{(\gamma)} = \hat{t}_{(\gamma)} - \hat{t}_{(\gamma)}\hat{G}_{(0)}(\hat{T}_{(\alpha)} + \hat{T}_{(\beta)}), \qquad \alpha\neq\beta\neq\gamma. \tag{A13}$$

b. Scattering by a bound state

For the case of the i-th particle incident on a bound state of other two particles pair, the initial state of the free incoming wave is given by $|\Phi_{(i)}\rangle$. The three-body wave vector has the form of $|\Psi_E^{(+)}\rangle = \sum_{\gamma=1}^{3} |\Psi_{(\gamma)}\rangle$ [26,27], where $|\Psi_{(\gamma)}\rangle$ satisfies the equation

$$|\Psi_{(\gamma)}\rangle = \delta_{\gamma,i}|\Phi_{(i)}\rangle + \hat{G}_{(\gamma)}\hat{V}_{(\gamma)}(|\Psi_{(\alpha)}\rangle + |\Psi_{(\beta)}\rangle), \quad \alpha \neq \beta \neq \gamma. \tag{A14}$$

The total scattering amplitude for a particle scattering with a bound state is given by $\hat{T}|\Phi_{(i)}\rangle = -\hat{V}|\Psi_E^{(+)}\rangle = \sum_{\gamma=1}^{3} \hat{T}_{(\gamma)}|\Phi_{(i)}\rangle$ [26,27], where

$$\hat{T}_{(\gamma)}|\Phi_{(i)}\rangle = -\hat{V}_{(\gamma)}|\Psi_E^{(+)}\rangle. \tag{A15}$$

Thus, we find

$$\hat{T}_{(\gamma)}|\Phi_{(i)}\rangle = -\delta_{\gamma,i}\hat{V}_{(\gamma)}|\Phi_{(i)}\rangle + \hat{t}_{(\gamma)}(|\Psi_{(\alpha)}\rangle + |\Psi_{(\beta)}\rangle), \tag{A16}$$

$$|\Psi_{(\gamma)}\rangle = -\hat{G}_{(0)}\hat{T}_{(\gamma)}|\Phi_{(i)}\rangle. \tag{A17}$$

The Faddeev's equation for the i-th particle incident on a bound state of other two particles pair yields

$$\hat{T}_{(\gamma)} = -\delta_{\gamma,i}\hat{V}_{(\gamma)} - \hat{t}_{(\gamma)}\hat{G}_{(0)}(\hat{T}_{(\alpha)} + \hat{T}_{(\beta)}), \qquad \alpha \neq \beta \neq \gamma. \tag{A18}$$

APPENDIX B: SOLUTIONS OF FADDEEV'S EQUATION FOR SHORT-RANGE INTERACTION IN FREE SPACE

In this section, we first show the details of the solution of Faddeev's equation, Eq. (14),

$$\begin{aligned} T_{(\gamma)}(k; q_{ij}, q_k) &= \left(2\sqrt{\sigma^2 - \frac{3}{4}k^2}\right) it_+\left(\sqrt{\sigma^2 - \frac{3}{4}k^2}\right)\left[\frac{v_{(\gamma)}(k; q_{ij}, q_k)}{imV_0}\right. \\ &\quad \left. + i\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{T_{(\alpha)}(q; q_{ij}, q_k) + T_{(\beta)}(q; q_{ij}, q_k)}{\sigma^2 - \frac{3}{4}q^2 - (k + \frac{q}{2})^2 + i\epsilon}\right], \\ &\quad \alpha \neq \beta \neq \gamma, \end{aligned}$$

where

$$v_{(\gamma)}(k; q_{ij}, q_k) = \int_{-\infty}^{\infty} dr_{\alpha\beta} dr_\gamma e^{-ikr_\gamma} \times mV_0\delta(r_{\alpha\beta})\psi_{(0)}(r_{\alpha\beta}, r_\gamma; q_{ij}, q_k).$$

Then, using the solutions obtained by solving Faddeev's equation, we demonstrate how the three-body scattering wave function is constructed.

1. Solution of T amplitudes

Let us first consider a free incoming wave,

$$\psi_{(0)} = e^{iq_{12}r_{12}}e^{iq_3r_3} = e^{iq_{23}r_{23}}e^{iq_1r_1} = e^{iq_{31}r_{31}}e^{iq_2r_2}, \tag{B1}$$

therefore

$$v_{(\gamma)}(k; q_{12}, q_3) = mV_0(2\pi)\delta(k - q_\gamma), \qquad \gamma = 1, 2, 3. \tag{B2}$$

First of all, let us introduce three new functions,

$$\begin{aligned} Z(k) &= \sum_{\gamma=1}^{3} T_{(\gamma)}(k; q_{ij}, q_k), \\ X(k) &= T_{(3)}(k; q_{ij}, q_k) - T_{(1)}(k; q_{ij}, q_k), \\ Y(k) &= T_{(3)}(k; q_{ij}, q_k) - T_{(2)}(k; q_{ij}, q_k), \end{aligned} \tag{B3}$$

thus Faddeev's equation, Eq. (14), can be reexpressed as three decoupled integral equations for (X, Y, Z) functions,

$$\begin{aligned} &\frac{1}{\left(2\sqrt{\sigma^2 - \frac{3}{4}k^2}\right) it_+\left(\sqrt{\sigma^2 - \frac{3}{4}k^2}\right)} Z(k) \\ &\quad = -i(2\pi)\sum_{\gamma=1}^{3}\delta(k - q_\gamma) \\ &\qquad + 2i\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{Z(q)}{\sigma^2 - \frac{3}{4}q^2 - (k + \frac{q}{2})^2 + i\epsilon}, \end{aligned} \tag{B4}$$

$$\begin{aligned} &\frac{1}{\left(2\sqrt{\sigma^2 - \frac{3}{4}k^2}\right) it_+\left(\sqrt{\sigma^2 - \frac{3}{4}k^2}\right)} X(k) \\ &\quad = -i(2\pi)\delta(k - q_3) + i(2\pi)\delta(k - q_1) \\ &\qquad - i\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{X(q)}{\sigma^2 - \frac{3}{4}q^2 - (k + \frac{q}{2})^2 + i\epsilon}, \end{aligned} \tag{B5}$$

and

$$\begin{aligned} &\frac{1}{\left(2\sqrt{\sigma^2 - \frac{3}{4}k^2}\right) it_+\left(\sqrt{\sigma^2 - \frac{3}{4}k^2}\right)} Y(k) \\ &\quad = -i(2\pi)\delta(k - q_3) + i(2\pi)\delta(k - q_2) \\ &\qquad - i\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{Y(q)}{\sigma^2 - \frac{3}{4}q^2 - (k + \frac{q}{2})^2 + i\epsilon}. \end{aligned} \tag{B6}$$

Next, let us solve Eq. (B4) first. According to [84], the three-body problem with equal-strength δ-function potentials is exactly solvable, diffraction effects are canceled out, the solution of wave function is expressed as the sum of six possible plane waves, see Eq. (C1). Therefore, the three-body scattering amplitudes can only be given by the sum of pole terms, see Eq. (C10). The strategy of solving Eqs. (B4)–(B6) is thus to make an ansatz of the solution as the sum of six possible pole terms, the pole positions are given in terms of the momenta of the incoming wave. Each pole term is then assigned with a constant coefficient. While the ansatz of the solution is plugged into integral equations, Eq. (B4)–(B6), by carefully defining the integration of the contour and also requiring that the branch cut contributions on both sides have to be canceled out as the consequence of Bethe's hypothesis, then the coefficients of pole terms can be fixed by matching both sides of the equations.

In what follows, we show how Eq. (B4) is satisfied by the ansatz,

$$Z(k) = (2\pi i)\sum_{\gamma=1}^{3}\kappa_\gamma \delta(k-q_\gamma) + \frac{\lambda k}{(k-q_3-i\epsilon)(k-q_2-i\epsilon)(k-q_1+i\epsilon)}. \quad \text{(B7)}$$

Instead of deforming the contour of integration in Eq. (B4), equivalently, we will adopt the $i\epsilon$ prescription in this work, and assign the small imaginary parts to relative momenta to avoid poles on the real axis. The left-hand side of Eq. (B4) is thus given by

$$LHS = \sum_{\gamma=1}^{3}\frac{\kappa_\gamma}{\left(2\sqrt{q^2_{\alpha\beta}}\right) it_+\left(\sqrt{q^2_{\alpha\beta}}\right)}(2\pi i)\delta(k-q_\gamma) - \frac{\lambda\left(\frac{1}{imV_0}+\frac{1}{2\sqrt{\sigma^2-\frac{3}{4}k^2}}\right)k}{(k-q_3-i\epsilon)(k-q_2-i\epsilon)(k-q_1+i\epsilon)}, \quad \text{(B8)}$$

where $\alpha \neq \beta \neq \gamma$. The integration on the right-hand side of Eq. (B4) is carried out by closing the contour in the upper half plane and picking up poles, $q = -\frac{k}{2} + \sqrt{\sigma^2 - \frac{3}{4}k^2} + i\epsilon$, $q_3 + i\epsilon$ and $q_2 + i\epsilon$, thus we find

$$RHS = -i\sum_{\gamma=1}^{3}(2\pi)\delta(k-q_\gamma) + 2\sum_{\gamma=1}^{3}\frac{\kappa_\gamma}{\left(k+\frac{q_\gamma}{2}+\sqrt{q^2_{\alpha\beta}}+i\epsilon\right)\left(k+\frac{q_\gamma}{2}-\sqrt{q^2_{\alpha\beta}}-i\epsilon\right)} + \frac{\lambda\left(1-\frac{k}{2\sqrt{\sigma^2-\frac{3}{4}k^2}}\right)}{(k-q_3-i\epsilon)(k-q_2-i\epsilon)(k-q_1+i\epsilon)} - \frac{\lambda\frac{q_3}{2q_{23}q_{31}}}{\left(k+\frac{q_3}{2}+\sqrt{q^2_{12}}+i\epsilon\right)\left(k+\frac{q_3}{2}-\sqrt{q^2_{12}}-i\epsilon\right)} - \frac{\lambda\frac{q_2}{2q_{12}q_{23}}}{\left(k+\frac{q_2}{2}+\sqrt{q^2_{31}}+i\epsilon\right)\left(k+\frac{q_2}{2}-\sqrt{q^2_{31}}-i\epsilon\right)}. \quad \text{(B9)}$$

We can clearly see that the branch cut contribution, the terms proportional to $\frac{1}{\sqrt{\sigma^2-\frac{3}{4}k^2}}$, on both sides of Eq. (B4) cancel out completely. Next, the square root terms, $\sqrt{q^2_{\alpha\beta}}$, are handled by assigning a small imaginary part to $q_{12} \to q_{12} + i0^+$, the imaginary part for $q_{23} \to q_{23} - i0^+$ and $q_{31} \to q_{31} - i0^+$ are determined completely by relations, $q_{23} = -\frac{1}{2}q_{12} + \frac{3}{4}q_3$ and $q_{31} = -\frac{1}{2}q_{12} - \frac{3}{4}q_3$ respectively. In addition, our convention for the complex square root is given by $\sqrt{q^2 \pm i0^+} = \pm\sqrt{q^2}$, therefore $\sqrt{(q\pm i0^+)^2} = \sqrt{q^2 \pm 2qi0^+} = \pm q$. Thus, with our assignment of the imaginary part to q_{12}, we obtain relations $\sqrt{(q_{12}+i0^+)^2} = q_{12}$, $\sqrt{(q_{23}-i0^+)^2} = -q_{23}$ and $\sqrt{(q_{31}-i0^+)^2} = -q_{31}$. Hence, the right-hand side of Eq. (B4) now can be reexpressed by

$$RHS = -\sum_{\gamma=1}^{3}(2\pi i)\delta(k-q_\gamma) - (2\pi i)\delta(k-q_3)\frac{\kappa_1}{q_{23}} + 2\frac{\sum_{\gamma=1}^{3}\kappa_\gamma(k-q_\gamma)}{(k-q_3-i\epsilon)(k-q_2-i\epsilon)(k-q_1+i\epsilon)} + \frac{\lambda\left(1-\frac{k}{2\sqrt{\sigma^2-\frac{3}{4}k^2}}-\frac{q_3(k-q_3)}{2q_{23}q_{31}}-\frac{q_2(k-q_2)}{2q_{12}q_{23}}\right)}{(k-q_3-i\epsilon)(k-q_2-i\epsilon)(k-q_1+i\epsilon)}. \quad \text{(B10)}$$

Comparing Eq. (B8) to Eq. (B10), the branch cut is canceled out, and the coefficients are given by

$$\kappa_1 = \frac{imV_0}{1-\frac{imV_0}{2q_{23}}}, \qquad \kappa_2 = \frac{imV_0}{1-\frac{imV_0}{2q_{31}}},$$
$$\kappa_3 = \frac{imV_0(1+\frac{imV_0}{2q_{23}})}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})},$$
$$\lambda = -\frac{6(imV_0)^2}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})}. \quad \text{(B11)}$$

The solutions of Eqs. (B5) and (B6) are found in a similar way,

$$X(k) = (2\pi i)\delta(k-q_3)\frac{imV_0}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})} - (2\pi i)\delta(k-q_1)\frac{imV_0}{1-\frac{imV_0}{2q_{23}}} + \frac{\frac{(2q_{31})(imV_0)^2}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})}}{(k-q_3-i\epsilon)(k-q_2-i\epsilon)(k-q_1+i\epsilon)}, \quad \text{(B12)}$$

and

$$Y(k) = (2\pi i)\delta(k - q_3)\frac{imV_0}{1 + \frac{imV_0}{2q_{12}}} - (2\pi i)\delta(k - q_2)\frac{imV_0}{1 - \frac{imV_0}{2q_{31}}} - \frac{\frac{(2q_{23})(imV_0)^2}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{31}})}}{(k - q_3 - i\epsilon)(k - q_2 - i\epsilon)(k - q_1 + i\epsilon)}. \quad \text{(B13)}$$

In the end, the solutions of $T_{(1,2,3)}$ for free incoming wave $\psi_{(0)} = e^{iq_{12}r_{12}}e^{iq_3r_3}$ are

$$T_{(3)}(k; q_{ij}, q_k) = (2\pi i)\delta(k - q_3)\frac{imV_0}{(1 + \frac{imV_0}{2q_{12}})(1 - \frac{imV_0}{2q_{23}})} - \frac{\frac{(imV_0)^2(2k+q_3)}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})}}{(k - q_3 - i\epsilon)(k - q_2 - i\epsilon)(k - q_1 + i\epsilon)}, \quad \text{(B14)}$$

$$T_{(1)}(k; q_{ij}, q_k) = (2\pi i)\delta(k - q_1)\frac{imV_0}{1 - \frac{imV_0}{2q_{23}}} - \frac{\frac{(imV_0)^2(2k+q_1-imV_0)}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})}}{(k - q_3 - i\epsilon)(k - q_2 - i\epsilon)(k - q_1 + i\epsilon)}, \quad \text{(B15)}$$

$$T_{(2)}(k; q_{ij}, q_k) = (2\pi i)\delta(k - q_2)\frac{imV_0}{1 - \frac{imV_0}{2q_{31}}} + (2\pi i)\delta(k - q_3)\frac{imV_0(\frac{imV_0}{2q_{23}})}{(1 + \frac{imV_0}{2q_{12}})(1 - \frac{imV_0}{2q_{23}})} - \frac{\frac{(imV_0)^2(2k+q_2+imV_0)}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})}}{(k - q_3 - i\epsilon)(k - q_2 - i\epsilon)(k - q_1 + i\epsilon)}. \quad \text{(B16)}$$

The total three-body scattering amplitude, $T(k_{\alpha\beta}, k_\gamma; q_{ij}, q_k)$, is determined by Eq. (16). As the consequence of Bethe's hypothesis, the physical scattering process for equal-strength δ-function potential and equal mass particles does not create any new momenta, see [84]. The final relative momenta in any pair configuration, for instance (k_{12}, k_3), can only be $(\pm q_{ij}, q_k)$ where $k = 1$, 2, 3 and $i \neq j \neq k$. Therefore, we may define the on-shell S-matrix by

$$(2\pi)\delta(k_{12} - q_{ij})(2\pi)\delta(k_3 - q_k)S(k_{12}, k_3) = (2\pi)\delta(k_{12} - q_{12})(2\pi)\delta(k_3 - q_3) + (2\pi \mathrm{i})\delta\left(\sigma^2 - k_{12}^2 - \frac{3}{4}k_3^3\right) \times T(k_{12}, k_3; q_{ij}, q_k). \quad \text{(B17)}$$

For free incoming wave $\psi_{(0)} = e^{iq_{12}r_{12}}e^{iq_3r_3}$, six possible on-shell S-matrix elements are

$$(S(q_{12}, q_3), S(-q_{12}, q_3), S(q_{23}, q_1), S(-q_{23}, q_1), S(q_{31}, q_2), S(-q_{31}, q_2)) = (s_1, s_2, s_3, s_4, s_5, s_6), \quad \text{(B18)}$$

where

$$s_1 = \frac{1}{(1 + \frac{imV_0}{2q_{12}})(1 - \frac{imV_0}{2q_{23}})(1 - \frac{imV_0}{2q_{31}})},$$
$$s_2 = \frac{(-\frac{imV_0}{2q_{12}})[1 + (\frac{imV_0}{2q_{23}})(\frac{imV_0}{2q_{31}})]}{(1 + \frac{imV_0}{2q_{12}})(1 - \frac{imV_0}{2q_{23}})(1 - \frac{imV_0}{2q_{31}})},$$
$$s_3 = \frac{(\frac{imV_0}{2q_{23}})(\frac{imV_0}{2q_{31}})}{(1 + \frac{imV_0}{2q_{12}})(1 - \frac{imV_0}{2q_{23}})(1 - \frac{imV_0}{2q_{31}})},$$
$$s_4 = \frac{(\frac{imV_0}{2q_{31}})}{(1 + \frac{imV_0}{2q_{12}})(1 - \frac{imV_0}{2q_{23}})(1 - \frac{imV_0}{2q_{31}})},$$
$$s_5 = s_3,$$
$$s_6 = \frac{(\frac{imV_0}{2q_{23}})}{(1 + \frac{imV_0}{2q_{12}})(1 - \frac{imV_0}{2q_{23}})(1 - \frac{imV_0}{2q_{31}})}. \quad \text{(B19)}$$

The solutions of T amplitudes of the Faddeev equation and S-matrix for the rest of the five independent free incoming waves can be obtained from solutions given in Eqs. (B14)–(B16) by relabeling subindices.

(1) For $\psi_{(0)} = e^{-iq_{12}r_{12}}e^{iq_3r_3}$, solutions of T amplitudes are given by $T_{(1)} \leftrightarrow T_{(2)}$, $T_{(3)}$ remains the same. The S-matrix elements are $(s_2, s_1, s_4, s_3, s_6, s_5)$.

(2) For $\psi_{(0)} = e^{iq_{23}r_{12}}e^{iq_1r_3}$, solutions of T amplitudes are given by $T_{(3)} \to T_{(2)}$, $T_{(1)} \to T_{(3)}$, and $T_{(2)} \to T_{(1)}$. The S-matrix elements are $(s_5, s_4, s_1, s_6, s_3, s_2)$.

(3) For $\psi_{(0)} = e^{-iq_{23}r_{12}}e^{iq_1r_3}$, solutions of T amplitudes are given by $T_{(1)} \leftrightarrow T_{(3)}$, $T_{(2)}$ remains the same. The S-matrix elements are $(s_4, s_5, s_6, s_1, s_2, s_3)$.

(4) For $\psi_{(0)} = e^{iq_{31}r_{12}}e^{iq_2r_3}$, solutions of T amplitudes are given by $T_{(3)} \to T_{(1)}$, $T_{(1)} \to T_{(2)}$, and $T_{(2)} \to T_{(3)}$. The S-matrix elements are $(s_3, s_6, s_5, s_2, s_1, s_4)$.

(5) For $\psi_{(0)} = e^{-iq_{31}r_{12}}e^{iq_2r_3}$, solutions of T amplitudes are given by $T_{(3)} \leftrightarrow T_{(2)}$, $T_{(1)}$ remains the same. The S-matrix elements are $(s_6, s_3, s_2, s_5, s_4, s_1)$.

In the end of this subsection, we also like to point out that the choice of imaginary part assignment for the complex square root is not unique, for instance, we could assign a small imaginary part to q_{23} instead of q_{12}. If so, the solutions obtained by assigning $i\epsilon$ to q_{23} are equivalent to relabel particle numbers by $1 \to 2$, $2 \to 3$ and $3 \to 1$ from the solutions obtained by assigning $i\epsilon$ to q_{12}.

2. Construction of wave function from solution of T's

With the solutions of scattering amplitudes in Eqs. (B14)–(B16), we are now at the position of constructing the wave function of three-body scattering. We show some details of wave function construction in this section for the incoming wave $\psi_{(0)} = e^{iq_{12}r_{12}}e^{iq_3 r_3}$ as an example. Using Eq. (12), we thus obtain

$$\psi(r_{12}, r_3; q_{ij}, q_k) = e^{iq_{12}r_{12}}e^{iq_3r_3} + \int_{-\infty}^{\infty} \frac{dk_{12}}{2\pi}\frac{dk_3}{2\pi}\frac{e^{ik_{12}r_{12}}e^{ik_3r_3}}{k_{12}^2 + \frac{3}{4}k_3^2 - \sigma^2 - i\epsilon} \times [T_{(3)}(k_3; q_{ij}, q_k) + T_{(1)}(k_1; q_{ij}, q_k) + T_{(2)}(k_1; q_{ij}, q_k)], \tag{B20}$$

where $k_1 = -k_{12} - \frac{k_3}{2}$ and $k_2 = k_{12} - \frac{k_3}{2}$. For each individual $\psi_{(1,2,3)}$, see in Eq. (12), the integration over $T_{(1,2,3)}$ amplitudes has both a branch cut contribution from the free three-body Green's function, see Eq. (12), and poles contribution from scattering amplitudes themselves. Only the branch cut contribution is responsible for the diffraction effect, in another word, only the branch cut integration creates new final momenta over scattering, pole terms do not create any new momenta. Branch cut integration is usually troublesome, fortunately, as we already know from [84], diffraction in the total wave function has to be canceled out. By some simple algebra in Eq. (B20), it is easy to see that $\sum_{\gamma=1}^{3} T_{(\gamma)}(k_\gamma) \propto (k_{12}^2 + \frac{3}{4}k_3^2 - \sigma^2)$, thus $\frac{\sum_{\gamma=1}^{3} T_{(\gamma)}(k_\gamma)}{k_{12}^2+\frac{3}{4}k_3^2-\sigma^2}$ has only pole terms. We first complete the integration of k_{12}, and pick up the poles in the upper half k_{12} plane for $r_{12} > 0$, and the poles in the lower half k_{12} plane for $r_{12} < 0$, so we get

$$\begin{aligned}\psi(r_{12}, r_3; q_{ij}, q_k) &= e^{iq_{12}r_{12}}e^{iq_3r_3} \\ &+ \frac{(\frac{imV_0}{2q_{23}})e^{-iq_{23}|r_{23}|}e^{iq_1r_1}}{1-\frac{imV_0}{2q_{23}}} + \frac{(\frac{imV_0}{2q_{31}})e^{-iq_{31}|r_{31}|}e^{iq_2r_2}}{1-\frac{imV_0}{2q_{31}}} - \frac{(\frac{imV_0}{2q_{12}})[(\frac{imV_0}{2q_{23}})e^{iq_{12}|r_{31}|}e^{iq_3r_2} + e^{iq_{12}|r_{12}|}e^{iq_3r_3}]}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})} \\ &+ \frac{(imV_0)}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})} i\int_{-\infty}^{\infty}\frac{dk_3}{2\pi} \\ &\times \Bigg[\frac{(-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})\theta(r_{12})e^{i(\frac{k_3}{2}+q_3)r_{12}}e^{ik_3r_3} + (\frac{imV_0}{2q_{31}})(1-\frac{imV_0}{2q_{23}})\theta(-r_{12})e^{-i(\frac{k_3}{2}+q_3)r_{12}}e^{ik_3r_3}}{(k_3-q_2-i\epsilon)(k_3-q_1+i\epsilon)} \\ &+ \frac{[(\frac{imV_0}{2q_{31}}) - (\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{31}})]\theta(r_{12})e^{-i(\frac{k_3}{2}+q_1)r_{12}}e^{ik_3r_3} + (-\frac{imV_0}{2q_{12}})(1+\frac{imV_0}{2q_{31}})\theta(-r_{12})e^{i(\frac{k_3}{2}+q_1)r_{12}}e^{ik_3r_3}}{(k_3-q_2-i\epsilon)(k_3-q_3-i\epsilon)} \\ &+ \frac{[(\frac{imV_0}{2q_{23}}) - (\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})]\theta(r_{12})e^{i(\frac{k_3}{2}+q_2)r_{12}}e^{ik_3r_3} + (-\frac{imV_0}{2q_{12}})(1+\frac{imV_0}{2q_{23}})\theta(-r_{12})e^{-i(\frac{k_3}{2}+q_2)r_{12}}e^{ik_3r_3}}{(k_3-q_1+i\epsilon)(k_3-q_3-i\epsilon)}\Bigg].\end{aligned} \tag{B21}$$

Next, we can perform k_3 integration and pick up all the poles in both the upper and the lower half k_3 plane in a similar manner as we did in k_{12} integration, thus, we finally get

$$\psi(r_{12}, r_3; q_{ij}, q_k) = (Ae^{iq_{12}r_{12}} + Be^{-iq_{12}r_{12}})e^{iq_3r_3} + (Ce^{iq_{23}r_{12}} + De^{-iq_{23}r_{12}})e^{iq_1r_3} + (Ee^{iq_{31}r_{12}} + Fe^{-iq_{31}r_{12}})e^{iq_2r_3}, \tag{B22}$$

where the coefficients are given by

$$\begin{aligned}A = 1 &+ \frac{\theta(-r_{23})(\frac{imV_0}{2q_{23}})}{1-\frac{imV_0}{2q_{23}}} + \frac{\theta(-r_{31})(\frac{imV_0}{2q_{31}})}{1-\frac{imV_0}{2q_{31}}} - \frac{\theta(r_{12})(\frac{imV_0}{2q_{12}})}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})} \\ &+ \theta(r_{12})\frac{-\theta(r_{23})(\frac{imV_0}{2q_{23}})[(\frac{imV_0}{2q_{31}}) - (\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{31}})] + \theta(-r_{31})(\frac{imV_0}{2q_{31}})[(\frac{imV_0}{2q_{23}}) - (\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})]}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})},\end{aligned} \tag{B23}$$

$$B = -\frac{\theta(-r_{12})(\frac{imV_0}{2q_{12}})}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})} + \theta(-r_{12})\frac{(\frac{imV_0}{2q_{12}})[\theta(-r_{31})(\frac{imV_0}{2q_{23}})(1+\frac{imV_0}{2q_{31}}) - \theta(r_{23})(\frac{imV_0}{2q_{31}})(1+\frac{imV_0}{2q_{23}})]}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})}, \tag{B24}$$

$$C = -\frac{\theta(r_{31})(\frac{imV_0}{2q_{12}})(\frac{imV_0}{2q_{23}})}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})} + \frac{(\frac{imV_0}{2q_{12}})[\theta(r_{12})\theta(r_{31})(\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}}) - \theta(-r_{12})\theta(-r_{23})(\frac{imV_0}{2q_{31}})(1+\frac{imV_0}{2q_{23}})]}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})}, \tag{B25}$$

$$D = \frac{\theta(r_{31})(\frac{imV_0}{2q_{31}})}{1-\frac{imV_0}{2q_{31}}} + \frac{(\frac{imV_0}{2q_{31}})\{-\theta(-r_{12})\theta(-r_{23})(\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}}) + \theta(r_{12})\theta(r_{31})[(\frac{imV_0}{2q_{23}}) - (\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})]\}}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})}, \tag{B26}$$

$$E = -\frac{\theta(-r_{12})\theta(r_{23})(\frac{imV_0}{2q_{12}})(\frac{imV_0}{2q_{31}})(1-\frac{imV_0}{2q_{23}}) + \theta(-r_{12})\theta(-r_{31})(\frac{imV_0}{2q_{23}})(\frac{imV_0}{2q_{12}})(1+\frac{imV_0}{2q_{31}})}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})}, \tag{B27}$$

and

$$F = \frac{\theta(r_{23})(\frac{imV_0}{2q_{23}})}{1-\frac{imV_0}{2q_{23}}} - \frac{\theta(-r_{31})(\frac{imV_0}{2q_{12}})(\frac{imV_0}{2q_{23}})}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})} + \frac{\theta(r_{12})(\frac{imV_0}{2q_{23}})\{\theta(-r_{31})(\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{31}}) + \theta(r_{23})[(\frac{imV_0}{2q_{31}}) - (\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{31}})]\}}{(1+\frac{imV_0}{2q_{12}})(1-\frac{imV_0}{2q_{23}})(1-\frac{imV_0}{2q_{31}})}. \tag{B28}$$

It can be shown that the coefficients given in Eq. (B28) are the solutions of McGuire's model, the coefficients obtained in six individual regions in the (r_{12}, r_3) plane, see Fig. 2, satisfy matrix transformation conditions in Eq. (C3).

The three-body wave functions for other free incoming waves are obtained in a similar way; because of length expression of these wave functions, we do not show them all in this work except the wave function for three fermions and three bosons system; the expression of three fermions and three bosons systems are listed in Secs. II A and II B respectively.

APPENDIX C: MCGUIRE'S MODEL

The one-dimensional three identical particles system interacting through the equal-strength δ-function potential has been solved by the ray-tracing method in [84]. After removal of the center-of-mass coordinate, the one-dimensional three-body problem resembles the motion of a single particle in a two-dimensional configuration space, e.g. (r_{12}, r_3) plane. The plane is divided symmetrically into six segments by interaction lines at $r_{ij}=0$ $(ij=12,23,31)$, see Fig. 2. According to ray-tracing arguments, the author in [84] shows that three-particle only exchange momenta during scattering, no new momenta are generated by collision, hence no diffraction. Therefore a general solution of the wave function is a linear combination of six possible plane waves,

$$\begin{aligned}\psi_\Lambda(r_{12}, r_3) = {} & (A_\Lambda e^{iq_{12}r_{12}} + B_\Lambda e^{-iq_{12}r_{12}})e^{iq_3 r_3} \\ & + (C_\Lambda e^{iq_{23}r_{12}} + D_\Lambda e^{-iq_{23}r_{12}})e^{iq_1 r_3} \\ & + (E_\Lambda e^{iq_{31}r_{12}} + F_\Lambda e^{-iq_{31}r_{12}})e^{iq_2 r_3},\end{aligned} \tag{C1}$$

where Λ stands for six segments from (I) up to (VI). The coefficients in six segments are related by boundary conditions of wave function, e.g. the boundary conditions at $r_{12}=0$ between segment (I) and (II) are given by

$$\begin{aligned}\psi_{II}(r_{12}, r_3)|_{r_{12}=0^+} &= \psi_I(r_{12}, r_3)|_{r_{12}=0^-}, \\ \left.\frac{\partial\psi_{II}(r_{12}, r_3)}{\partial r_{12}}\right|_{r_{12}=0^+} - \left.\frac{\partial\psi_I(r_{12}, r_3)}{\partial r_{12}}\right|_{r_{12}=0^-} & \\ &= mV_0\psi_{II}(r_{12}, r_3)|_{r_{12}=0^+},\end{aligned} \tag{C2}$$

the rest of the boundary conditions are given in a similar way. If we define the vector of coefficients by $\chi_\Lambda^T = (A_\Lambda, B_\Lambda, \ldots, F_\Lambda)$ in segment Λ, the two neighboring χ_Λ vectors are connected by matrix transformation,

$$\chi_\Lambda = \Gamma_{\Lambda,\Lambda'}\chi_{\Lambda'}, \tag{C3}$$

where $\Gamma_{\Lambda,\Lambda'}$ is determined by Eq. (C2).

For completeness, we give the expressions of six Γ matrices,

$$\Gamma_{II,I} = \begin{bmatrix} 1-\frac{imV_0}{2q_{12}} & -\frac{imV_0}{2q_{12}} & 0 & 0 & 0 & 0 \\ \frac{imV_0}{2q_{12}} & 1+\frac{imV_0}{2q_{12}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\frac{imV_0}{2q_{23}} & -\frac{imV_0}{2q_{23}} & 0 & 0 \\ 0 & 0 & \frac{imV_0}{2q_{23}} & 1+\frac{imV_0}{2q_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\frac{imV_0}{2q_{31}} & -\frac{imV_0}{2q_{31}} \\ 0 & 0 & 0 & 0 & \frac{imV_0}{2q_{31}} & 1+\frac{imV_0}{2q_{31}} \end{bmatrix}, \tag{C4}$$

$$\Gamma_{IV,II} = \begin{bmatrix} 1+\frac{imV_0}{2q_{23}} & 0 & 0 & 0 & 0 & \frac{imV_0}{2q_{23}} \\ 0 & 1-\frac{imV_0}{2q_{31}} & -\frac{imV_0}{2q_{31}} & 0 & 0 & 0 \\ 0 & \frac{imV_0}{2q_{31}} & 1+\frac{imV_0}{2q_{31}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\frac{imV_0}{2q_{12}} & -\frac{imV_0}{2q_{12}} & 0 \\ 0 & 0 & 0 & \frac{imV_0}{2q_{12}} & 1+\frac{imV_0}{2q_{12}} & 0 \\ -\frac{imV_0}{2q_{23}} & 0 & 0 & 0 & 0 & 1-\frac{imV_0}{2q_{23}} \end{bmatrix}, \tag{C5}$$

$$\Gamma_{VI,IV} = \begin{bmatrix} 1-\frac{imV_0}{2q_{31}} & 0 & 0 & -\frac{imV_0}{2q_{31}} & 0 & 0 \\ 0 & 1+\frac{imV_0}{2q_{23}} & 0 & 0 & \frac{imV_0}{2q_{23}} & 0 \\ 0 & 0 & 1-\frac{imV_0}{2q_{12}} & 0 & 0 & -\frac{imV_0}{2q_{12}} \\ \frac{imV_0}{2q_{31}} & 0 & 0 & 1+\frac{imV_0}{2q_{31}} & 0 & 0 \\ 0 & -\frac{imV_0}{2q_{23}} & 0 & 0 & 1-\frac{imV_0}{2q_{23}} & 0 \\ 0 & 0 & \frac{imV_0}{2q_{12}} & 0 & 0 & 1+\frac{imV_0}{2q_{12}} \end{bmatrix}, \tag{C6}$$

$$\Gamma_{V,VI} = \begin{bmatrix} 1+\frac{imV_0}{2q_{12}} & \frac{imV_0}{2q_{12}} & 0 & 0 & 0 & 0 \\ -\frac{imV_0}{2q_{12}} & 1-\frac{imV_0}{2q_{12}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\frac{imV_0}{2q_{23}} & \frac{imV_0}{2q_{23}} & 0 & 0 \\ 0 & 0 & -\frac{imV_0}{2q_{23}} & 1-\frac{imV_0}{2q_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\frac{imV_0}{2q_{31}} & \frac{imV_0}{2q_{31}} \\ 0 & 0 & 0 & 0 & -\frac{imV_0}{2q_{31}} & 1-\frac{imV_0}{2q_{31}} \end{bmatrix}, \tag{C7}$$

$$\Gamma_{III,V} = \begin{bmatrix} 1-\frac{imV_0}{2q_{23}} & 0 & 0 & 0 & 0 & -\frac{imV_0}{2q_{23}} \\ 0 & 1+\frac{imV_0}{2q_{31}} & \frac{imV_0}{2q_{31}} & 0 & 0 & 0 \\ 0 & -\frac{imV_0}{2q_{31}} & 1-\frac{imV_0}{2q_{31}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\frac{imV_0}{2q_{12}} & \frac{imV_0}{2q_{12}} & 0 \\ 0 & 0 & 0 & -\frac{imV_0}{2q_{12}} & 1-\frac{imV_0}{2q_{12}} & 0 \\ \frac{imV_0}{2q_{23}} & 0 & 0 & 0 & 0 & 1+\frac{imV_0}{2q_{23}} \end{bmatrix}, \tag{C8}$$

$$\Gamma_{I,III}=\begin{bmatrix} 1-\frac{imV_0}{2q_{31}} & 0 & 0 & -\frac{imV_0}{2q_{31}} & 0 & 0 \\ 0 & 1+\frac{imV_0}{2q_{23}} & 0 & 0 & \frac{imV_0}{2q_{23}} & 0 \\ 0 & 0 & 1-\frac{imV_0}{2q_{12}} & 0 & 0 & -\frac{imV_0}{2q_{12}} \\ \frac{imV_0}{2q_{31}} & 0 & 0 & 1+\frac{imV_0}{2q_{31}} & 0 & 0 \\ 0 & -\frac{imV_0}{2q_{23}} & 0 & 0 & 1-\frac{imV_0}{2q_{23}} & 0 \\ 0 & 0 & \frac{imV_0}{2q_{12}} & 0 & 0 & 1+\frac{imV_0}{2q_{12}} \end{bmatrix}. \tag{C9}$$

The scattering amplitudes $T_{(\gamma)}$'s can be constructed by using Eq. (10), therefore we obtain

$$\begin{aligned} T_{(3)}(q;q_{ij},q_k) &= imV_0\left[\frac{A_I+B_I}{q-q_3-i\epsilon}+\frac{C_I+D_I}{q-q_1-i\epsilon}+\frac{E_I+F_I}{q-q_2-i\epsilon}-\frac{A_{VI}+B_{VI}}{q-q_3+i\epsilon}-\frac{C_{VI}+D_{VI}}{q-q_1+i\epsilon}-\frac{E_{VI}+F_{VI}}{q-q_2+i\epsilon}\right], \\ T_{(1)}(q;q_{ij},q_k) &= imV_0\left[\frac{E_{III}+D_{III}}{q-q_3-i\epsilon}+\frac{A_{III}+F_{III}}{q-q_1-i\epsilon}+\frac{B_{III}+C_{III}}{q-q_2-i\epsilon}-\frac{E_{II}+D_{II}}{q-q_3+i\epsilon}-\frac{A_{II}+F_{II}}{q-q_1+i\epsilon}-\frac{B_{II}+C_{II}}{q-q_2+i\epsilon}\right], \\ T_{(2)}(q;q_{ij},q_k) &= imV_0\left[\frac{C_{VI}+F_{VI}}{q-q_3-i\epsilon}+\frac{B_{VI}+E_{VI}}{q-q_1-i\epsilon}+\frac{A_{VI}+D_{VI}}{q-q_2-i\epsilon}-\frac{C_I+F_I}{q-q_3+i\epsilon}-\frac{B_I+E_I}{q-q_1+i\epsilon}-\frac{A_I+D_I}{q-q_2+i\epsilon}\right]. \end{aligned} \tag{C10}$$

As we can see, the scattering amplitudes bear no branch cuts, but only pole terms as the consequence of Bethe's hypothesis.

APPENDIX D: TWO-BODY SCATTERING

For completeness, we also give the brief review of two-body interaction in finite volume in this section.

1. Two-body scattering in free space

We consider two spinless identical particles scattering, the positions and momenta of two particles are denoted by (x_1, x_2) and (p_1, p_2) respectively. The wave function of scattering two particles satisfies Schrödinger equation,

$$\left[-\frac{1}{2m}\frac{d^2}{dx_1^2}-\frac{1}{2m}\frac{d^2}{dx_2^2}+V(x_1-x_2)-E\right]\Psi(x_1,x_2)=0, \tag{D1}$$

where the mass of the particle is m, the total energy of the two-particle system is $E=\frac{p_1^2}{2m}+\frac{p_2^2}{2m}$. Let us denote the center of mass and relative positions by $R=\frac{x_1+x_2}{2}$ and $r=x_1-x_2$ respectively, and conjugate momenta by $P=p_1+p_2$ and $k=\frac{p_1-p_2}{2}$ respectively. Due to translational invariance of center of mass motion, the total wave function of two particles is described by the product of a plane wave, e^{iPR}, that describes center of mass motion and the wave function, $\psi(r;k)$, that only describes relative motion of two particles, $\Psi(x_1,x_2)=e^{iPR}\psi(r;k)$. It may be more convenient to use the Lippmann-Schwinger equation representation of solutions,

$$\psi(r;k)=e^{ikr}+\int_{-\infty}^{\infty}dr' G_{(0)}(r-r';z_k)mV(r')\psi(r';k), \tag{D2}$$

where $z_k=k^2+i\epsilon$ and $k^2=mE-\frac{P^2}{4}$, the free-particle Green's function is given by

$$G_{(0)}(r;z_k)=\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{e^{iqr}}{z_k-q^2}=-\frac{ie^{i\sqrt{k^2}|r|}}{2\sqrt{k^2}}. \tag{D3}$$

At large separation, $|r|\gg|r'|$, the Green's function can be approximated by

$$G_{(0)}(r-r';z_k)\overset{|r|\gg|r'|}{\simeq}-\frac{ie^{i\sqrt{k^2}|r|}}{2\sqrt{k^2}}e^{-i\sqrt{k^2}\frac{rr'}{|r|}}. \tag{D4}$$

Therefore, asymptotically,

$$\psi(r;k)\overset{\text{large}|r|}{=}e^{ikr}+it(k,k')e^{i\sqrt{k^2}|r|}, \tag{D5}$$

where $k'=\sqrt{k^2}\frac{r}{|r|}$ and the scattering amplitudes are given by

$$t(k,k')=-\frac{1}{2\sqrt{k^2}}\int_{-\infty}^{\infty}dr' e^{-ik'r'}mV(r')\psi(r';k). \tag{D6}$$

In this work, we only consider particles scattering in a symmetric potential, $V(r)=V(-r)$, therefore the Schrödinger equation exhibits a solution of even parity (two spinless bosons), $\psi_+(-r)=\psi_+(r)$, and a solution of odd parity (two spinless fermions), $\psi_-(-r) = -\psi_-(r)$, where $\psi_\pm = \frac{\psi_k \pm \psi_{-k}}{2}$. The parity amplitudes are given by $t(k,k') = t_+(\sqrt{k^2}) + \frac{kk'}{k^2} t_-(\sqrt{k^2})$, therefore

$$\psi_\pm(r;k) \overset{\text{large}|r|}{=} Y_\pm(k) \times \left[\frac{e^{i\sqrt{k^2}r} \pm e^{-i\sqrt{k^2}r}}{2} + it_\pm\left(\sqrt{k^2}\right) e^{i\sqrt{k^2}|r|} Y_\pm(r)\right], \quad (D7)$$

where $Y_+ = 1$ and $Y_-(k) = \frac{k}{\sqrt{k^2}}, Y_-(r) = \frac{r}{|r|}$. The general wave function thus is the linear superposition of both parity wave functions: $\psi = c_+\psi_+ + c_-\psi_-$.

2. Two-body scattering in finite volume

When the particles are placed in a one-dimensional periodic box with the size of L, the two-particle wave function in a finite box, $\Psi^{(L)}(x_1, x_2)$, has to satisfy the periodic boundary condition,

$$\Psi^{(L)}(x_1 + n_{x_1}L, x_2 + n_{x_2}L) = \Psi^{(L)}(x_1, x_2), \qquad n_{x_1,x_2} \in \mathbb{Z}. \quad (D8)$$

The finite volume wave function, $\Psi^{(L)}$, can be constructed from free space wave function Ψ by

$$\Psi^{(L)}(x_1, x_2) = \frac{1}{V} \sum_{n_{x_1}, n_{x_2} \in \mathbb{Z}} \Psi(x_1 + n_{x_1}L, x_2 + n_{x_2}L) = \left(\frac{1}{V}\sum_{n_{x_1}\in\mathbb{Z}} e^{iPn_{x_1}L}\right) e^{iPR}\psi^{(L)}(r;k),$$

$$\psi^{(L)}(r;k) = \sum_{n\in\mathbb{Z}} e^{-i\frac{P}{2}nL}\psi(r+nL;k), \quad (D9)$$

where $n = n_{x_1} - n_{x_2}$, and the volume of infinite summation, V, is given by $V = \sum_{n\in\mathbb{Z}} e^{iPnL} = \frac{2\pi}{L}\sum_{d\in\mathbb{Z}}\delta(P + \frac{2\pi}{L}d)$. The quantization of total momentum, $P = \frac{2\pi}{L}d$, is warranted by translational invariance of center of mass motion in a periodic box. By our construction, the general relative wave function in the finite box is given by $\psi^{(L)} = c_+\psi_+^{(L)} + c_-\psi_-^{(L)}$, the periodic boundary condition for $\psi^{(L)}$ reads

$$\psi^{(L)}(r+nL;k) = e^{i\frac{P}{2}nL}\psi^{(L)}(r;k). \quad (D10)$$

Applying Eq. (D7), the relative wave functions in the finite box, $\psi_\pm^{(L)}(r;k)$, are given by

$$\psi_\pm^{(L)}(r;k) \overset{\text{large}|r|}{=} it_\pm(\sqrt{k^2})Y_\pm(k)\sum_{n\in\mathbb{Z}} e^{-i\frac{P}{2}nL} \times [\pm\theta(-r-nL)e^{-i\sqrt{k^2}(r+nL)} + \theta(r+nL)e^{i\sqrt{k^2}(r+nL)}]. \quad (D11)$$

The summations can be carried out,

$$\sum_{n\in\mathbb{Z}} e^{-i\frac{P}{2}nL}\theta(-r-nL)e^{-i\sqrt{k^2}nL} = \theta(-r) + \frac{e^{i(\frac{P}{2}+\sqrt{k^2})L}}{1-e^{i(\frac{P}{2}+\sqrt{k^2})L}},$$

$$\sum_{n\in\mathbb{Z}} e^{-i\frac{P}{2}nL}\theta(r+nL)e^{i\sqrt{k^2}nL} = \theta(r) + \frac{e^{i(-\frac{P}{2}+\sqrt{k^2})L}}{1-e^{i(-\frac{P}{2}+\sqrt{k^2})L}}, \quad (D12)$$

therefore we find

$$\psi_\pm^{(L)}(r;k) \overset{\text{large}|r|}{=} it_\pm(\sqrt{k^2})Y_\pm(k) \times \left[e^{i\sqrt{k^2}|r|}Y_\pm(r) + \frac{e^{i(-\frac{P}{2}+\sqrt{k^2})L}}{1-e^{i(-\frac{P}{2}+\sqrt{k^2})L}}e^{i\sqrt{k^2}r} \pm \frac{e^{i(\frac{P}{2}+\sqrt{k^2})L}}{1-e^{i(\frac{P}{2}+\sqrt{k^2})L}}e^{-i\sqrt{k^2}r}\right]. \quad (D13)$$

The secular equation is obtained by matching $\psi^{(L)}(r)$ to $\psi(r)$ at an arbitrary r, larger than the range of the interaction. The matching procedure is equivalent to applying the periodic condition to both wave functions and the derivative of wave functions at nearest neighbor when solving periodic potential quantum mechanics problems. In addition, the matching condition $\psi^{(L)}(r) = \psi(r)$ also guarantees that $\psi^{(L)}(r)$ constructed by using Eq. (D9) is indeed the solution of the finite volume system for a short-range potential. Because wave functions are the linear superposition of two independent basis, $e^{\pm i\sqrt{k^2}r}$, by choosing $r > 0$ e.g., we obtain two matching equations,

$$\left[\frac{1}{2it_+} - \frac{e^{i(-\frac{P}{2}+\sqrt{k^2})L}}{1-e^{i(-\frac{P}{2}+\sqrt{k^2})L}}\right]c_+ + \left[\frac{1}{2it_-} - \frac{e^{i(-\frac{P}{2}+\sqrt{k^2})L}}{1-e^{i(-\frac{P}{2}+\sqrt{k^2})L}}\right]c_- = 0,$$

$$\left[\frac{1}{2it_+} - \frac{e^{i(\frac{P}{2}+\sqrt{k^2})L}}{1-e^{i(\frac{P}{2}+\sqrt{k^2})L}}\right]c_+ - \left[\frac{1}{2it_-} - \frac{e^{i(\frac{P}{2}+\sqrt{k^2})L}}{1-e^{i(\frac{P}{2}+\sqrt{k^2})L}}\right]c_- = 0. \quad (D14)$$

The above equations have nontrivial solutions when

$$1 - (2it_+ + 2it_-)\frac{1}{2}\left[\frac{e^{i(\frac{P}{2}+\sqrt{k^2})L}}{1-e^{i(\frac{P}{2}+\sqrt{k^2})L}} + \frac{e^{i(-\frac{P}{2}+\sqrt{k^2})L}}{1-e^{i(-\frac{P}{2}+\sqrt{k62})L}}\right] + 2it_+2it_-\frac{e^{i(\frac{P}{2}+\sqrt{k^2})L}}{1-e^{i(\frac{P}{2}+\sqrt{k^2})L}}\frac{e^{i(-\frac{P}{2}+\sqrt{k^2})L}}{1-e^{i(-\frac{P}{2}+\sqrt{k^2})L}} = 0. \quad (D15)$$

Due to $e^{iPL} = 1$, it is clear to see that the solutions of the secular equation, Eq. (D15), can be divided into classes of positive parity state solutions and negative parity state solutions, the definite parity state solutions are given by the equation

$$e^{-i(\frac{P}{2}+\sqrt{k^2})L} = 1 + 2it_{\mathcal{P}}(\sqrt{k^2}), \qquad \mathcal{P} = \pm. \tag{D16}$$

When scattering amplitudes are parametrized by phase shifts, $t_\pm = \frac{1}{\cot\delta_\pm - i}$, the secular equations, Eqs. (D15) and (D16), are reduced respectively to nonrelativistic versions of Lüscher's formula in one dimension [77],

$$\cos\frac{PL}{2} = \frac{\cos(\delta_+ + \delta_- + \sqrt{k^2}L)}{\cos(\delta_+ - \delta_-)}, \tag{D17}$$

$$\cot\delta_{\mathcal{P}} + \cot\frac{\frac{PL}{2} + \sqrt{k^2}L}{2} = 0. \tag{D18}$$

In the following subsections, we show the recovery of analytic solutions for two well-known one-dimensional models by applying the quantization condition obtained in Eq. (D16).

3. Solvable examples of two-body scattering in finite volume

a. Kronig Penney model

Let us consider the square well potential $V(r) = V_0$ for $|r| < \frac{b}{2}$, and $V(r) = 0$ otherwise. The symmetric wave functions in short range, $|r| < \frac{b}{2}$, are given by

$$\psi_\pm(r;k) = A_\pm \frac{e^{i\sqrt{\sigma_V^2}r} \pm e^{-i\sqrt{\sigma_V^2}r}}{2}, \qquad |r| < \frac{b}{2}, \tag{D19}$$

where $\sigma_V^2 = k^2 - mV_0$, continuity of wave functions at the boundary of the potential leads to relations,

$$A_\pm = \frac{2e^{-i\sqrt{k^2}\frac{b}{2}}}{\pm\left(1 - \frac{\sqrt{\sigma_V^2}}{\sqrt{k^2}}\right)e^{i\sqrt{\sigma_V^2}\frac{b}{2}} + \left(1 + \frac{\sqrt{\sigma_V^2}}{\sqrt{k^2}}\right)e^{-i\sqrt{\sigma_V^2}\frac{b}{2}}},$$
$$1 + 2it_\pm = e^{-i\sqrt{k^2}b}\frac{\cos\sqrt{\sigma_V^2}\frac{b}{2} + \left(\frac{\sqrt{\sigma_V^2}}{\sqrt{k^2}}\right)^{\pm1} i\sin\sqrt{\sigma_V^2}\frac{b}{2}}{\cos\sqrt{\sigma_V^2}\frac{b}{2} - \left(\frac{\sqrt{\sigma_V^2}}{\sqrt{k^2}}\right)^{\pm1} i\sin\sqrt{\sigma_V^2}\frac{b}{2}}. \tag{D20}$$

Easy to check, the scattering amplitudes $t_\pm$ are also the solutions of

$$t_\pm = -\frac{1}{2\sqrt{k^2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} dr' e^{-i\sqrt{k^2}r'} mV_0\psi_\pm(r';k). \tag{D21}$$

In the finite box with the periodic boundary condition, plugging the analytic expression of $t_\pm$ in Eq. (D20) into the secular equation Eq. (D15), we thus obtain the well-known energy quantization condition for the Kronig Penney model,

$$\cos\sqrt{k^2}a\cos\sqrt{\sigma_V^2}b - \frac{k^2+\sigma_V^2}{2\sqrt{k^2}\sqrt{\sigma_V^2}}\sin\sqrt{k^2}a\sin\sqrt{\sigma_V^2}b$$
$$= \cos\frac{PL}{2}, \qquad a = L - b. \tag{D22}$$

b. δ-function potential model

Now, let us consider a short-range interaction model with a delta potential, $V(r) = V_0\delta(r)$; the amplitudes for the δ-function potential thus are given by

$$t_\pm(\sqrt{k^2}) = -\frac{1}{2\sqrt{k^2}}mV_0\psi_\pm(0), \tag{D23}$$

where $\psi_+(0) = 1 + it_+$ and $\psi_-(0) = 0$. Therefore, we obtain

$$it_+(\sqrt{k^2}) = -\frac{\frac{imV_0}{2\sqrt{k^2}}}{1 + \frac{imV_0}{2\sqrt{k^2}}}, \qquad it_- = 0. \tag{D24}$$

Plugging the solution of it_+ into the secular equation, Eq. (D15), thus we obtain the well-known quantization condition for two-particle interaction in a finite box with a periodic boundary condition,

$$e^{-i(\frac{P}{2}+\sqrt{k^2})L} = \frac{1 - \frac{imV_0}{2\sqrt{k^2}}}{1 + \frac{imV_0}{2\sqrt{k^2}}}. \tag{D25}$$

The results of the delta potential can also be obtained from the Kronig Penny model by taking the limit of $b \to 0$, $V_0 \to \infty$ and $bV_0 = \text{const}$.

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