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Relativistic two-body calculation of $b\bar{b}$-mesons radiative decays

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This paper is a continuation of a previous work where we presented a unified two-fermion covariant scheme which produced very precise results for the masses of light and heavy mesons. We extend the analysis to some radiative decays of mesons Υ, χ_{b2}, h_b, χ_{b1}, χ_{b0}, and η_b, and we calculate their branching ratios and their widths. For most of them, we can make a comparison with experimental data, finding a good agreement. For the decays for which data are not available, we compare ours with other recent theoretical previsions.

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I. INTRODUCTION

Potential models have long been used to investigate the meson spectrum [1–5]. After the pioneering nonrelativistic work [1], it clearly emerged that the relativity effects are important for the description of mesons [6], both light and heavy. In order to reach a reasonable precision, the following papers have eventually included relativistic corrections, either by perturbation techniques or by covariant approaches [2,7]. The chromodynamic interactions of heavy quarks through order $(v/c)^2$ were introduced in Ref. [8] starting from a nonrelativistic treatment of QCD. An effective theory, called nonrelativistic QCD, was thus defined and used for lattice and continuum calculations [9,10]. This theory and the "potential nonrelativistic QCD" [11], a further effective theory derived from it, are among the most diffused methods for calculating meson spectra and decays [12]. The lattice techniques, on the other hand, have progressively improved up to the present day by including higher relativistic orders and QCD radiative effects. Increasingly accurate determinations of hyperfine splittings have then been calculated in this way [13–15].

Other types of approaches move directly from a covariant formulation. A short list of up-to-date available relativistic or semirelativistic treatments [16–19] was discussed in Ref. [20]. Starting from our previous results [21,22], we derived in Ref. [20] a covariant potential model for two relativistic quarks of arbitrary mass. The fermionic nature of the particles is explicitly considered, and each quark satisfies a Dirac equation; the two equations are then coupled by the interaction, described by the Cornell potential. As is well known, this potential is formed by a Coulomb-like part which appears in a vector coupling and by a linear term which must describe a scalar interaction in order to be confining [23]. Obviously, the two spin-orbit contributions for the two quarks are completely included and no problem concerning the reduced mass has to be posed. A first-order correction to the potential is added by means of the Breit term. In Ref. [20], we have shown that our wave equation is able to provide a unified framework to investigate all ranges of meson masses. For heavier mesons, the agreement with experimental data turns out to be really very precise up to the pair production threshold, not included in the Cornell potential. This has given suggestions for unknown spectroscopic classifications of some mesons and has allowed us to obtain a good accuracy when calculating the masses of light mesons, for which potential models usually fail.

All the methods so far described are currently applied to the radiative meson decays. The electromagnetic coupling is generally taken in the dipole approximation, electric or magnetic according to the considered transitions. Sometimes, the contributions due to the strong interactions are brought to bear to the calculation. In addition to the obvious comparison of the theoretical results with experimental data, in many papers previsions are also made about radiative transitions lacking direct data. In particular, this is the case of the transitions of the recently observed mesons $h_b(1p)$ and $h_b(2p)$ decaying into $\eta_b(1s)$ and substantiating the evidence of $\eta_b(2s)$ [24]. The results of some recent papers using a semirelativistic framework are found in Refs. [25,26], and those obtained by an effective potential are in Ref. [27]. QCD-based approaches are used in Refs. [28–31], and lattice calculations of three-point matrix elements for radiative bottomonium decays are presented in Ref. [32]. The width of the radiative decay of $\Upsilon(2s)$ into $\eta_b(1s)$ is given in Ref. [33]. The agreement between theoretical and experimental widths is generally not as good as it is for the spectra, even when taking into account the large errors that affect the experimental data [34].

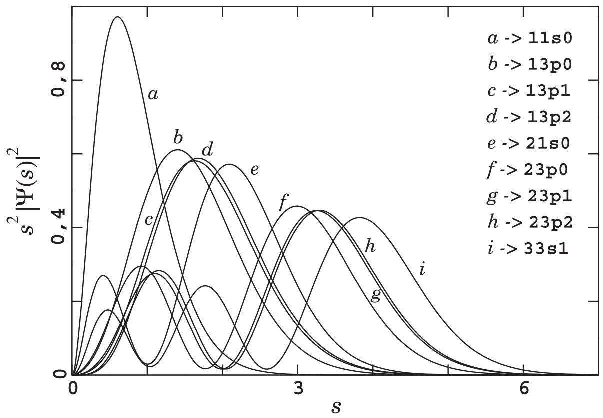

FIG. 1. The normalized densities of the states involved in the $\Upsilon(3)$ decay times the measure factor in the dimensionless coordinate s defined in Sec. A 4.

As we have stated in the abstract, this paper is a continuation of Ref. [20]. The purpose is to calculate the widths of the purely radiative decays of $b\bar{b}$, still assuming the Cornell potential as a constitutive interaction for the mesons. The electromagnetic coupling for the composite two-fermion system is then determined in analogy to the procedure established in Ref. [35]. In that paper, we calculated the hyperfine spacings for different hydrogenic atoms and the width of corresponding transitions: no additional corrections to the Coulomb interaction were included, apart from the first order of the Breit term, representing the spin-spin interaction responsible of the hyperfine splitting. The results we found are in extremely good agreement with the experimental data. It is therefore very tempting, if not compulsory, to have a look at the meson radiative decays by extending to mesons the treatment applied in Ref. [35] to atoms. We thus evaluate here the branching ratios and the widths of the measured radiative decays of $\Upsilon(3s)$, $\chi_{b2}(2p)$, $\chi_{b1}(2p)$, $\chi_{b0}(2p)$, and $\Upsilon(2s)$ (see Tables III and IV), and we make previsions for some decays of $h_b(1p)$, $h_b(2p)$, $\chi_{b1}(2p)$, $\Upsilon(2s)$, and $\Upsilon(1s)$ (see Table V) for which direct experimental data are not yet available. The results are rewarding, although we cannot expect to reach the same accuracy of the atomic transitions for several reasons. In the first place, the Cornell potential is itself an effective potential more suitable to the description of a stationary situation, such as the calculation of the spectrum, as opposed to the atomic interaction which comes from a fundamental theory. Second, for atoms the fine structure coupling constant $\alpha_{\rm em}$ is the same for the Coulomb potential, the Breit spin-spin interaction, and the decay process. In Fig. 1 we represent the normalized densities of the states related to some decays treated in the following. We are thus allowed to make a proper calculation of the first-order corrections to the wave functions due to the Breit term, as we did in Ref. [35]. These corrections turn out to be essential for getting very

TABLE I. The $b\bar{b}$ levels in MeV. First column: term symbol, $I^G(J^{PC})$ numbers, particle name. $\sigma = 1.111$ GeV/fm, $\alpha = 0.3272$, and $m_b = 4725.5$ MeV. Experimental data are taken from Ref. [34]. Our values can be compared with those obtained by different approaches, reported in Ref. [26].

State	Experiment	Numeric
$(1^1s_0)0^+(0^{-+})\eta_b$	9398.0 ± 3.2	19390.39
$(1^3s_1)0^-(1^{--})\Upsilon$	9460.30 ± 0.25	19466.10
$(1^3p_0)0^+(0^{++})\chi_{b0}$	9859.44 ± 0.73	19857.41
$(1^3p_1)0^+(1^{++})\chi_{b1}$	9892.78 ± 0.57	19886.70
$(1^1p_1)0^-(1^{+-})h_b$	9898.60 ± 1.4	19895.35
$(1^3p_2)0^+(2^{++})\chi_{b2}$	9912.21 ± 0.57	19908.14
$(2^1s_0)0^+(0^{-+})\eta_b$	9974.0 ± 4.4[a]	9971.14
$(2^3s_1)0^-(1^{--})\Upsilon$	10023.26 ± 0.0003	10009.04
$(2^3p_0)0^+(0^{++})\chi_{b0}$	10232.50 ± 0.0009	10232.36
$(2^3p_1)0^+(1^{++})\chi_{b1}$	10255.46 ± 0.0005	10256.58
$(2^1p_1)0^-(1^{+-})h_b$	10259.8 ± 1.6	110263.62
$(2^3p_2)0^+(2^{++})\chi_{b2}$	10268.65 ± 0.0007	10274.26
$(3^3s_1)0^-(1^{--})\Upsilon$	10355.20 ± 0.0005	10364.52

[a]See [36].

precise values of both the hyperfine levels of different hydrogenic atoms and the decay widths. Without them, the levels involved in hyperfine transitions would be degenerate, and first-order corrected wave functions are necessary to calculate the rate. This is not the situation for meson radiative decays, for which a rigorous perturbation expansion in the Breit term is not feasible. Indeed, in most quarkonium models the values assumed for the bottom mass m_b, the string tension σ, and $\alpha_{\rm QCD}$ are obtained from a fit of the experimental meson spectrum. In our case, the fit is calculated by including the first order of the Breit correction. Thus, a remnant of that correction is already present at the lowest order in the wave equation and in its solutions. In Table I, we report the masses of the mesons we will consider later on. In Table II, we show that the influence of the Breit term is actually very different on the different states. Because of the structure of the transition rate given in the following equation (2.11), if we use the physical (i.e. Breit-corrected) value for the transition frequency, it seems reasonable to take the corresponding spinors at the lowest perturbation order. However, for the states with $j = 0$, namely, η_b and χ_{b0}, the hyperfine shift is maximal and considerably larger than for the other states of their respective multiplets. These states are connected by a parity transformation, and in our model they are structurally distinguished from the other components of the respective multiplets, since they are determined by a second-order

TABLE II. The Breit corrections in MeV for the lowest states.

$\eta_b(1s)$	$\Upsilon_b(1s)$	$\chi_{b0}(1p)$	$\chi_{b1}(1p)$	$h_b(1p)$	$\chi_{b2}(1p)$
−92.13	−18.09	−44.3	−19.98	−15.95	−7.51

differential system instead of by a fourth-order one. Moreover, the inclusion of the first-order corrections in their wave functions makes a really great improvement on the results of the decay transition rate. Still in the context of radiative meson decays, an analogous situation was met in Ref. [5] for the relativistic corrections in $(v/c)^4$ "retained in the calculation of those rates where those terms make a substantial difference" (see [5], note 18). Thus, we shall assume unperturbed wave functions for all the $j \neq 0$ states and first-order corrected wave functions for all the $j = 0$ states. In Sec. III, the numerical way of calculating the corrections to the levels and to the states will be recalled.

We now give a sketchy summary of what follows. In Sec. II, we state the general formulas for calculating the radiative transitions. A plot of the radial probability density of the states is also presented so as to give an idea of the properties of the eigenfunctions. In Sec. III, we discuss some numerical aspects, describing how we calculate our spinors and giving some details on the numerical precision. In Sec. IV, we make some brief concluding remarks. As already stated, this paper is a continuation and a completion of our previous ones, and we have intended to give it a phenomenological exposition centered on the results. However, in order not to redirect too frequently the readers to our previous works [20–22,35] but also not to overwhelm the exposition with accessory technical details, we have added an Appendix. There we explain the notations, we recall the basic covariance properties of the model, and we give some references for treatments of related subjects of more mathematical flavor. Moreover, we contextualize the general relations of our previous papers to the present case, which is simpler because of the equal quark masses.

II. THE TRANSITION PROBABILITIES

Using the coordinates (A1), (A5), and (A6), the two-body relativistic wave equation for $b\bar{b}$, with eigenvalue λ, reads (see Secs. A 3 and A 4)

$$\left[(\gamma^0_{(1)}\gamma_{(1)a} - \gamma^0_{(2)}\gamma_{(2)a})q_a + \frac{1}{2}(\gamma^0_{(1)} + \gamma^0_{(2)})(2m_b + \sigma r) - \left(\lambda + \frac{b}{r}\right) + V_B(r)\right]\boldsymbol{\psi}(\boldsymbol{r}) = 0. \quad (2.1)$$

In (2.1), $\boldsymbol{\psi}(\boldsymbol{r})$ is a 16-component spherical spinor, explicitly given in Sec. A 3, with components ordered as specified in (A10) and (A11); $\gamma^0_{(i)}, \gamma_{(i)a} (i = 1, 2)$ are the γ matrices acting on the space of the quark and antiquark fermion; $b = (4/3)\alpha_{\mathrm{QCD}}$. The vector and scalar parts of the Cornell potential, respectively, give the $(\lambda + b/r)$ and $(2m_b + \sigma r)$ terms. Finally, the Breit potential $V_B(r)$ has the form

$$V_B(r) = \frac{b}{2r}\gamma^0_{(1)}\gamma_{(1)a}\gamma^0_{(2)}\gamma_{(2)b}\left(\delta_{ab} + \frac{r_a r_b}{r^2}\right). \quad (2.2)$$

The calculation of the meson mass spectrum reduces to the solution of two boundary value problems for the fourth-order systems [(A15)–(A17)], one for each parity.

We factorize the wave functions of initial and final bottomonium states, normalized in a box of volume V, into

$$\Psi_\ell(Z, \boldsymbol{r}) = V^{-1/2} e^{-iP_\ell \cdot Z} \boldsymbol{\psi}_\ell(\boldsymbol{r}), \qquad \ell = i, f, \quad (2.3)$$

where $\boldsymbol{\psi}_i(\boldsymbol{r})$ and $\boldsymbol{\psi}_f(\boldsymbol{r})$ are the spinors corresponding to initial and final energies, angular momenta, and parities. The Breit term (2.2) will always be considered a first-order perturbation term. As previously said, when necessary we will consider wave functions represented by the sum of a lowest-order contribution given by the exact eigenfunction of Eq. (2.1) with $V_B(r) = 0$ and a first-order correction generated by $V_B(r)$ itself. The general systems of equations for the case of mesons formed by quarks with possibly different masses is presented in Ref. [20]: we report in Sec. A 4 the systems for the $b\bar{b}$ case, not explicitly written in our previous papers. In Ref. [35], it was shown that the electromagnetic coupling for the fermion-antifermion bound system is introduced by means of the interaction Hamiltonian

$$H_{\mathrm{int}} = -e_b(\boldsymbol{\alpha}_{(1)} \cdot \boldsymbol{A}^{(1)} - \boldsymbol{\alpha}_{(2)} \cdot \boldsymbol{A}^{(2)}), \quad (2.4)$$

where $e_b = (1/3)e$ is the bottom charge, e being the electron charge. Here $\alpha_{(i)}$ is the vector of the α matrices for the ith fermion space, and $\boldsymbol{A}^{(i)}$ is the wave function of a photon with 4-momentum k and polarization $\boldsymbol{\epsilon}_\beta$ in the Coulomb gauge [37],

$$\boldsymbol{A}(k, \beta) = \frac{\sqrt{4\pi}}{\sqrt{2\omega V}}\boldsymbol{\epsilon}_\beta e^{-ik_\mu x^\mu}. \quad (2.5)$$

$\boldsymbol{A}^{(i)}$ is evaluated at the point $x = x_{(i)}$, where $\omega = k_0$ is the photon frequency. The invariant integration on the global coordinates, present in the calculation of the first perturbation order of the S-matrix element, yields

$$S_{fi} = -ie_b\frac{(2\pi)^4}{\sqrt{2\omega V}}\frac{\sqrt{4\pi}}{\sqrt{V^2}}\delta^4(P_f + k - P_i)(\boldsymbol{\epsilon}^*_\beta \cdot \boldsymbol{M}_{fi}). \quad (2.6)$$

The δ^4 function represents the conservation of the global 4-momentum of the three particles

$$P^\mu_i = P^\mu_f + k^\mu \quad (2.7)$$

and contains the recoil of the meson due to the radiation emission. Since the experimental value of the physical width of the decay is given in the rest frame of the initial meson, we choose $\boldsymbol{P}_i = 0$, and we obtain

$$\boldsymbol{M}_{fi} = \int d^3\boldsymbol{r}\boldsymbol{\psi}_f^*(\boldsymbol{r})(\tilde{\boldsymbol{\alpha}}_{(1)}e^{-\frac{i}{2}\boldsymbol{k}\cdot\boldsymbol{r}} - \tilde{\boldsymbol{\alpha}}_{(2)}e^{\frac{i}{2}\boldsymbol{k}\cdot\boldsymbol{r}})\boldsymbol{\psi}_i(\boldsymbol{r}). \quad (2.8)$$

Here $\tilde{\boldsymbol{\alpha}}_{(j)}$ are the transformed matrices in the basis of the spinors with components ordered as in (A10) and (A11).

Using (A12), the integrals over the angular variables are calculated analytically, without any approximation on the exponential, in terms of elementary functions that correspond to the lowest-order Bessel functions appearing in the usual series expansions. The radial integrals are calculated numerically for all the allowed transitions as described in more detail in the next section.

Summing over the polarizations and the possible final states and averaging over the initial states, the differential transition rate therefore reads

$$dw = \frac{e_b^2}{2\pi\omega}\delta^4(P_f + k - P_i)\sum\frac{|\boldsymbol{\epsilon}_\beta^*\cdot\boldsymbol{M}_{fi}|^2}{2j_i+1}d^3\boldsymbol{k}d^3\boldsymbol{P}_f. \quad (2.9)$$

The dipole approximation for $\boldsymbol{M}_{fi}$ is obtained by letting the exponentials equal to unity in (2.8). For the transitions for which the dipole approximation gives a nonvanishing result, the sum over the polarizations is more easily calculated by using the Wigner-Eckart theorem.

As $\int d^3\boldsymbol{P}/2P^0 = \int d^4P\theta(P^0)\delta(P^2-\lambda^2)$, integrating over the final global momentum we get

$$\frac{dw}{d\omega d\Omega_n} = \frac{e_b^2\omega}{2\pi\lambda_i}(\lambda_i-\omega)\delta\left(\omega - \frac{\lambda_i^2-\lambda_f^2}{2\lambda_i}\right)\sum\frac{|\boldsymbol{\epsilon}_\beta^*\cdot\boldsymbol{M}_{fi}|^2}{2j_i+1}, \quad (2.10)$$

where $d\Omega_n$ is the unit solid angle in the direction $\boldsymbol{n} = \boldsymbol{k}/|\boldsymbol{k}|$. Reinserting the $\hbar$ and c factors, the final integration over the solid angle gives the total transition rate

$$w = \frac{4}{3}\frac{e_b^2}{\hbar c}\omega_{fi}\Lambda_{fi}^2\sum\frac{|\boldsymbol{\epsilon}_\beta^*\cdot\boldsymbol{M}_{fi}|^2}{2j_i+1}, \quad (2.11)$$

while

$$\omega_{fi} = \frac{c}{\hbar}\frac{\lambda_i+\lambda_f}{2\lambda_i}(\lambda_i-\lambda_f) \quad (2.12)$$

is the frequency of the emitted photon that completely includes the recoil. Finally,

$$\Lambda_{fi}^2 = \frac{\lambda_i^2+\lambda_f^2}{2\lambda_i^2} \quad (2.13)$$

is the relativistic correction factor coming from kinematics. This is not very far from unity but for the transitions between states with a large mass difference.

III. DISCUSSION OF THE RESULTS

The levels and the eigenstates of $b\bar{b}$ are obtained by a numerical solution of the singular boundary value problem posed by the Hamiltonian (2.1). The unknown functions are the eight radial function coefficients $\{a_i(r), b_i(r), c_i(r), d_i(r)\}_{i=0,1}$. The symmetries of the problem imply four algebraic relations, so that the previous radial functions can be expressed in terms of only four unknown functions $\{u_i(r)\}_{i=1,4}$ giving rise to the 4×4 system described in Sec. A 4. For the $j=0$ states, the 4×4 system actually reduces to a 2×2 one. The corrections due to the Breit term (2.2) are calculated by solving the spectral problem for $\varepsilon V_B(r)$ and taking the first-order expansion in ε for both levels and spinors normalized to unity. Because of the value of the bottom mass, the parameters entering the system are such that the solutions can be expressed directly in terms of Padé approximants, so that we get a complete control of the numerical error. The Padé we have calculated are of the order of [260, 260], and, since the arithmetical precision has also been taken with a sufficiently large number of digits, our numerical errors can safely be assumed to be vanishing. We have used our eigenfunctions to give some estimates of the average meson radii and quark velocity. For the radii we have seen a monotonic increase with mass, from 0.156 fm of the $\eta_b(1s)$ to 0.609 fm of $\Upsilon(3p)$. The estimate of the quark velocity has been approximated by $\beta \simeq (\langle\boldsymbol{q}^2\rangle/(m^2c^2+\langle\boldsymbol{q}^2\rangle))^{1/2}$, where $q = (p_1-p_2)/2$ is the conjugate relative momentum of the two fermions (A5) and $\langle\boldsymbol{q}^2\rangle$ has been taken as the average of $-\hbar^2\boldsymbol{\nabla}_r^2$ over the corresponding state. Averaging then over all the states, we have found a $\beta \simeq 0.29$ in good agreement with the estimate of Ref. [31]. Such a value, in our opinion, strengthens the idea that a relativistic treatment

TABLE III. Comparison of our calculated branching ratios of the radiative decays with experimental data. The experimental errors have been linearly combined.

Branching ratios	Theory	Experiment
$\Upsilon(3s)\to\gamma\chi_{b1}(2p)/\Upsilon(3s)\to\gamma\chi_{b2}(2p)$	0.812	0.96 ± 0.21
$\Upsilon(3s)\to\gamma\chi_{b0}(2p)/\Upsilon(3s)\to\gamma\chi_{b2}(2p)$	0.433	0.45 ± 0.10
$\Upsilon(3s)\to\gamma\eta_b(2s)/\Upsilon(3s)\to\gamma\chi_{b2}(2p)$	0.002	<0.005
$\Upsilon(3s)\to\gamma\chi_{b2}(1p)/\Upsilon(3s)\to\gamma\chi_{b2}(2p)$	0.042	0.075 ± 0.019
$\Upsilon(3s)\to\gamma\chi_{b1}(1p)/\Upsilon(3s)\to\gamma\chi_{b2}(2p)$	0.010	0.007 ± 0.005
$\Upsilon(3s)\to\gamma\chi_{b0}(1p)/\Upsilon(3s)\to\gamma\chi_{b2}(2p)$	0.010	0.021 ± 0.006
$\Upsilon(3s)\to\gamma\eta_b(1s)/\Upsilon(3s)\to\gamma\chi_{b2}(2p)$	0.003	0.004 ± 0.001
$\Upsilon(2s)\to\gamma\chi_{b1}(1p)/\Upsilon(2s)\to\gamma\chi_{b2}(1p)$	0.812	$0.96\pm.10$
$\Upsilon(2s)\to\gamma\chi_{b0}(1p)/\Upsilon(2s)\to\gamma\chi_{b2}(1p)$	0.410	0.53 ± 0.08
$\Upsilon(2s)\to\gamma\eta_b(1s)/\Upsilon(2s)\to\gamma\chi_{b2}(1p)$	0.006	0.006 ± 0.002
$\chi_{b2}(2p)\to\gamma\Upsilon(1s)/\chi_{b2}(2p)\to\gamma\Upsilon(2s)$	0.55	0.66 ± 0.23
$\chi_{b1}(2p)\to\gamma\Upsilon(1s)/\chi_{b1}(2p)\to\gamma\Upsilon(2s)$	0.46	0.46 ± 0.08
$\chi_{b0}(2p)\to\gamma\Upsilon(1s)/\chi_{b0}(2p)\to\gamma\Upsilon(2s)$	0.13	(0.20 ± 0.20)[a]

[a]This case is not very meaningful due to the large error by which it is affected.

TABLE IV. Values in keV of the widths of radiative decays of the mesons $\Upsilon(3s)$, $\chi_{b2}(2p)$, $\chi_{b1}(2p)$, $\chi_{b0}(2p)$, and $\Upsilon(2s)$. In the second column, we give our theoretical data obtained from Eqs. (6)–(11). In the third column, we report the results for the calculations in the electric dipole approximation (D.A.). The final column shows the experimental data.

Decay	Theory	D.A.	Experiment
$\Upsilon(3s) \to \gamma\chi_{b2}(2p)$	3.51	2.99	2.70 ± 0.57
$\Upsilon(3s) \to \gamma\chi_{b1}(2p)$	2.85	2.16	2.58 ± 0.48
$\Upsilon(3s) \to \gamma\chi_{b0}(2p)$	1.52	1.26	1.21 ± 0.23
$\Upsilon(3s) \to \gamma\eta_b(2s)$	0.006		<0.013
$\Upsilon(3s) \to \gamma\chi_{b2}(1p)$	0.149	0.204	0.204 ± 0.045
$\Upsilon(3s) \to \gamma\chi_{b1}(1p)$	0.036	0.067	0.019 ± 0.012
$\Upsilon(3s) \to \gamma\chi_{b0}(1p)$	0.032	0.060	0.056 ± 0.013
$\Upsilon(3s) \to \gamma\eta_b(1s)$	0.009		0.011 ± 0.003
$\Upsilon(2s) \to \gamma\chi_{b2}(1p)$	2.13	1.85	2.30 ± 0.20
$\Upsilon(2s) \to \gamma\chi_{b1}(1p)$	1.73	1.35	2.22 ± 0.21
$\Upsilon(2s) \to \gamma\chi_{b0}(1p)$	0.87	0.71	1.22 ± 0.15
$\Upsilon(2s) \to \gamma\eta_b(1s)$	0.013		0.013 ± 0.04
$\chi_{b2}(2p) \to \gamma\Upsilon(2s)$	18.77	23.34	15.10 ± 5.60
$\chi_{b2}(2p) \to \gamma\Upsilon(1s)$	10.27	12.01	9.80 ± 2.30
$\chi_{b1}(2p) \to \gamma\Upsilon(2s)$	16.80	12.75	14.40 ± 5.00
$\chi_{b1}(2p) \to \gamma\Upsilon(1s)$	7.68	5.08	8.96 ± 2.24
$\chi_{b0}(2p) \to \gamma\Upsilon(2s)$	11.77	4.03	…
$\chi_{b0}(2p) \to \gamma\Upsilon(1s)$	1.49	1.38	…
$\chi_{b2}(1p) \to \gamma\Upsilon(1s)$	3.73		…
$\chi_{b1}(1p) \to \gamma\Upsilon(1s)$	29.48		…
$\chi_{b0}(1p) \to \gamma\Upsilon(1s)$	19.65		…

is appropriate. Finally, we have calculated the average values of the orbital angular momentum from the radial part of the Laplace operator, and we have found values no larger than 0.115 for s states and values between 0.976 and 1.045 for p states.

In Table III, we present our results for "relative" branching ratios of bottomonium radiative decays, namely, for branching ratios not using the total width of the decaying particle, and we make a comparison with experimental data whose errors have been linearly combined. It appears that the agreement between the theoretical and experimental data is very good for most of the decays and that the worst results are different for a factor not greater than 1.5.

In the second column of Table IV, we report the values in keV we have calculated for the $b\bar{b}$-meson radiative decays. We get from Ref. [34] the total widths $\Gamma_{\Upsilon(3s)} = 20.32 \pm 1.85$ keV and $\Gamma_{\Upsilon(2s)} = 31.98 \pm 2.63$ keV. The total widths of $\Gamma_{\chi_{b2}(2p)} = 138. \pm 19.$ keV and $\Gamma_{\chi_{b1}(2p)} = 96. \pm 16.$ keV are taken from Ref. [38]. Again we have assumed a linear composition of the errors of the experimental data. The agreement is again very good even for the decays involving the η_b and the χ_{b0} states. The widths of the allowed electric dipole transition are reported in the third column of the table. As expected, they show that the dipole approximation gives also results in a very good agreement with the data.

TABLE V. Comparison of the previsions for the theoretical widths of some radiative decays of χ_{b2}, h_b, χ_{b1}, χ_{b0}, and Υ. Units are eV.

Decay	Ours	Ref. [27]	Ref. [22]
$h_b(2p) \to \gamma\eta_b(2s)$	20 681	17 600	16 600
$h_b(2p) \to \gamma\eta_b(1s)$	16 884	14 900	17 500
$\Upsilon(2s) \to \gamma\eta_b(2s)$	0.369	0.58	0.59
$\eta_b(2s) \to \gamma\Upsilon(1s)$	65.41	45	64
$\chi_{b2}(1p) \to \gamma h_b(1p)$	0.015	0.089	
$\chi_{b2}(1p) \to \gamma\Upsilon(1s)$	33 731	39 150	31 800
$h_b(1p) \to \gamma\chi_{b1}(1p)$	0.050	0.012	0.0094
$h_b(1p) \to \gamma\chi_{b0}(1p)$	0.124	0.86	0.90
$h_b(1p) \to \gamma\eta_b(1s)$	39 318	43 660	35 800
$\Upsilon(1s) \to \gamma\eta_b(1s)$	3.101[a]	9.34	10

[a]This value is in agreement with the value (3.6 ± 2.9) eV of Ref. [29].

The decay widths obtained by several different approaches can be found in Ref. [26].

In Table V, we give the results of our previsions in comparison with those of Refs. [26,31].

IV. CONCLUSIONS

We close the paper with some short observations on the positive aspects of the potential model we have constructed and on its main limitations. We have constructed a covariant model for two fermions with the correct nonrelativistic limit and the single-body limit when the mass of one fermion tends to infinity. It includes in a nonperturbative scheme all the relativistic effects. It uses coupled Dirac equations to describe the component quarks and to establish the correct vector or scalar couplings. The model is conceptually simple: indeed, although the expressions (A12)–(A20) may look awkward, they are directly obtained by extending to a two-fermion system the standard procedure of the Dirac equation in a central potential. It contains the Breit correction responsible for the hyperfine splittings: in fact, this is the only first-order correction needed to achieve a very good agreement of the computed mass spectrum with the experimental data. It is efficient, without any change in its structure, for dealing with both heavy and light mesons. It involves a minimal number of fitted parameters. On the other hand, it is not suited for an analytic development. However, it is rather manageable for obtaining numerical results in a combined environment of computational methods and computer algebra. We believe, therefore, that the framework in which it operates and the type of effects it is able to take into account are very transparent. The quality of the results depends then essentially upon the capability of the potential to give an accurate description of the physical interaction. Although very simple, the Cornell potential has proved to be very effective in this respect, both for the determination of the meson spectra and for the calculation of the radiative decay widths. However, in the latter application, additional caution is in order when dealing with the Breit correction to the wave functions. Indeed, since the connection with QCD is only through the potential, we meet two general types of difficulties. In the first place, since we give a nonperturbative treatment of some classes of effects and the potential contains the fitted parameters, it is not straightforward to single out the dominant effects in the description. Second, we obviously do not yet have a direct way to determine an systematic expansion for adding further corrections. We finally could say that the framework of the two coupled Dirac equation plus Breit term is a starting point quite well suited to study meson physics. A closer connection with QCD is certainly necessary to improve the agreement with experimental data and to enlarge the types of phenomena to be treated.

APPENDIX

The kinematics and the classical dynamics of two relativistic interacting bodies constituted an active field of research during the 1970s and 1980s. One of the main approaches was the use of the theory of constrained systems [39] where the phase space was cut by imposing some invariant relations (the "first- and second-class constraints"). The motion was described in a covariant way, although not all physical problems were completely settled. This framework keeps being used up to present day to produce quantum potential models for atoms and quarkonium systems (see, e.g., [39], where the choice of the potential is different from ours). Our approach to the two-body relativistic problems was also developed long ago [40] using the ideas of the reduction of Hamiltonian systems with symmetry and, more generally, of the induced representations of groups on homogeneous spaces and of geometrical quantization [41]. The kinematics and the dynamics were thus obtained from an invariant canonical reduction of the direct product of the phase space of a system of two particles with masses $m_{(1)}, m_{(2)}$. The reduced phase space, therefore, turns out to be a manifold supporting a canonical action of the Poincaré group, and no choice of reference frame is necessary. The quantum mechanical picture has been developed accordingly in more recent papers [20–22], which we refer to for proofs and details. We briefly describe the main ideas in coordinates, and we specify some properties for the case of $b\bar{b}$ mesons.

1. The classical setting

We first sketch the classical setting. We denote by $(x^\mu_{(1)}, p^\mu_{(1)}, x^\mu_{(2)}, p^\mu_{(2)})$ the coordinates of the two-particle phase space. Letting

$$P^\mu = p^\mu_{(1)} + p^\mu_{(2)} \tag{A1}$$

and $a = 1, 2, 3$, we define the "vierbein"

$$\varepsilon^\mu_0(P) = \frac{P^\mu}{\sqrt{P^2}}, \qquad \varepsilon^\mu_a(P) = \eta^\mu_a - \frac{P_a(P^\mu + \eta^\mu_0\sqrt{P^2})}{\sqrt{P^2}(P_0 + \sqrt{P^2})}. \tag{A2}$$

They satisfy the identities (μ, $\alpha = 0, 3$)

$$\eta_{\mu\nu}\varepsilon^\mu_\alpha(P)\varepsilon^\nu_\beta(P) = \eta_{\alpha\beta}, \qquad \eta^{\alpha\beta}\varepsilon^\mu_\alpha(P)\varepsilon^\nu_\beta(P) = \eta^{\mu\nu}, \tag{A3}$$

$\eta_{\mu\nu}$ being the Lorentz metric tensor. We then make a canonical transformation ($a = 1, 2, 3$)

$$\{(x^\mu_{(1)}, p^\mu_{(1)}), (x^\mu_{(2)}, p^\mu_{(2)})\} \mapsto \{(Z^\mu, P^\mu), (\check{r}, \check{p}), (r_a, q_a)\}, \tag{A4}$$

where, letting $X^\mu = (x^\mu_{(1)} + x^\mu_{(2)})/2$,

$$Z^\mu = X^\mu + \frac{\varepsilon_{abc}P_a\eta^\mu_b L_c}{\sqrt{P^2}(P_0 + \sqrt{P^2})} + \frac{\varepsilon^\mu_a}{\sqrt{P^2}}(q_a\check{r} - r_a\check{q}) + \frac{P^\mu}{P^2}\check{q}\,\check{r},$$
$$\check{r} = \varepsilon^\mu_0(P)(x_{(1)\mu} - x_{(2)\mu}, \quad \check{q} = \varepsilon^\mu_0(P)(p_{(1)\mu} - p_{(2)\mu})/2,$$
$$r_a = \varepsilon^\mu_a(P)(x_{(1)\mu} - x_{(2)\mu}), \quad q_a = \varepsilon^\mu_a(P)(p_{(1)\mu} - p_{(2)\mu})/2. \tag{A5}$$

$(Z_k)_{k=1,2,3}$ is a Newton-Wigner position vector for a particle with angular momentum $L_a = \varepsilon_{abc}r_b q_c$; $\boldsymbol{r} = (r_a)$ and $\boldsymbol{q} = (q_a)$ are Wigner vectors of spin one and

$$\check{r}, \check{q}, r = \sqrt{\boldsymbol{r}^2}, \qquad q = \sqrt{\boldsymbol{q}^2}, \boldsymbol{q}\cdot\boldsymbol{r}, \qquad P_\mu Z^\mu = P_\mu X^\mu + \check{q}\,\check{r} \tag{A6}$$

are Lorentz invariant. The reduction of the phase space is then generated by the invariant $\check{q} = (m_1^2 - m_2^2)/(2\sqrt{P^2})$ ($\check{q} = 0$ for $b\bar{b}$). Correspondingly, the invariant relative time coordinate $\check{r}$ becomes cyclic. Observe, for instance, that

$$p_{(1)}{}^\mu x_{(1)\mu} + p_{(2)}{}^\mu x_{(2)\mu} = P^\mu Z_\mu - \boldsymbol{q}\cdot\boldsymbol{r}. \tag{A7}$$

Thus, $\check{r}$ disappears from the treatment and can be assigned any arbitrary value. We can choose a vanishing relative time $\check{r}$. For a free system, the mass-shell conditions $p_1^2 = m_1^2$, $p_2^2 = m_2^2$ immediately yield $P^2 = (\sqrt{q_a q_a + m_1^2} + \sqrt{q_a q_a + m_2^2})^2$. In the presence of an interaction, the obvious changes must be applied to this invariant.

2. The coupled Dirac equations

The quantum description of two relativistic fermions is obtained by means of two Dirac operators coupled by the interaction. The wave functions are therefore 16-dimensional spinors. When the fermions are free, we combine the two Dirac operators written in terms of (A5). The matrices $\check{\gamma}_{(i)} = \varepsilon^\mu_0(P)\gamma^\mu_{(i)}$ and $\gamma_{(i)a} = \varepsilon^\mu_a(P)\gamma^\mu_{(i)}$

are unitary transformed and can be given the usual representation. Finally, the wave equation is expressed by an invariant operator:

$$(\breve{\gamma}_{(1)}\gamma_{(1)_a} - \breve{\gamma}_{(2)}\gamma_{(2)_a})q_a + \breve{\gamma}_{(1)}m_{(1)} + \breve{\gamma}_{(2)}m_{(2)} = \sqrt{P^2} \equiv \lambda. \tag{A8}$$

The eigenvalues of (A8) are the invariant masses given by the four possible combinations

$$\lambda = \pm\sqrt{q_a q_a + m^2_{(1)}} \pm \sqrt{q_a q_a + m^2_{(2)}}, \tag{A9}$$

each one with multiplicity four. The relative momentum is again fixed to $\breve{q} = (m^2_{(1)} - m^2_{(2)})/(2\lambda)$, and the relative time is cyclic. We finally mention that (a) the Schrödinger equation is obtained in the nonrelativistic limit and (b) the Dirac equation for a single particle in an external potential is reproduced when the mass of the other tends to infinity. This second limit is generally not obtained in approaches *à la* Bethe-Salpeter.

3. The 16-component spinors

In analogy to the usual treatment of the Dirac equation in a central field [37], we define a 16-component spherical spinor by diagonalizing angular momentum and parity. We then reorder its components by collecting them in four groups labeled by the free eigenvalues (A9) with the choice of signs $(++), (--), (-+), (+-)$, respectively:

$$\Psi_\pm = {}^t(\Psi^{(++)}_\pm, \Psi^{(--)}_\pm, \Psi^{(-+)}_\pm, \Psi^{(+-)}_\pm). \tag{A10}$$

Here the subscript $(\pm)$ refers to the parity. In each group, the components are in the singlet-triplet order:

$$\Psi^{(**)}_\pm = {}^t(\Psi^{(**)}_{\pm,0}, \Psi^{(**)}_{\pm,1_+}, \Psi^{(**)}_{\pm,1_0}, \Psi^{(**)}_{\pm,1_-}), \tag{A11}$$

where $(**)$ indicates any combination of $+$ and $-$, the subscript 0 refers to the singlet component, while 1_+, 1_0, and 1_- denote the triplet components. The operator are reordered accordingly. The parity operator is given by the inner parity—which is the reordered $\breve{\gamma} \otimes \breve{\gamma}$—combined with the change $\boldsymbol{r} \to -\boldsymbol{r}$ times a global arbitrary phase. In our previous papers dealing with atomic states, we had called "even" or "odd" the states with eigenvalues of P equal to $(-)^j$ or $(-)^{j+1}$, respectively, choosing the arbitrary phase equal to unity. With such a choice, the ground state of an atomic system has the parity eigenvalue equal to unity. We maintain this terminology, but we observe that we have to choose the global phase equal to -1 in order to agree with the usual meson classification scheme. The explicit form of reordered spinor we have called even for general masses is given in Refs. [21,35]. For equal masses, as in the present case where $m_{(1)} = m_{(2)} = m_b$, some simplifications are in order and the spinor reads

$$\begin{aligned}
\psi^{(++)}_{+0} &= Y^j_m(\theta,\phi)a_0(r),\\
\psi^{(++)}_{+1_+} &= -\frac{\sqrt{j-m+1}\sqrt{j+m}}{\sqrt{2j}\sqrt{j+1}}Y^j_{m-1}(\theta,\phi)b_0(r),\\
\psi^{(++)}_{+1_0} &= \frac{m}{\sqrt{j}\sqrt{1+j}}Y^j_m(\theta,\phi)b_0(r),\\
\psi^{(++)}_{+1_-} &= \frac{\sqrt{j-m}\sqrt{j+m+1}}{\sqrt{2j}\sqrt{j+1}}Y^j_{m+1}(\theta,\phi)b_0(r),\\
\psi^{(--)}_{+0} &= Y^j_m(\theta,\phi)a_1(r),\\
\psi^{(--)}_{+1_+} &= -\frac{\sqrt{j-m+1}\sqrt{j+m}}{\sqrt{2j}\sqrt{j+1}}Y^j_{m-1}(\theta,\phi)b_1(r),\\
\psi^{(--)}_{+1_0} &= \frac{m}{\sqrt{j}\sqrt{1+j}}Y^j_m(\theta,\phi)b_1(r),\\
\psi^{(--)}_{+1_-} &= \frac{\sqrt{j-m}\sqrt{j+m+1}}{\sqrt{2j}\sqrt{j+1}}Y^j_{m+1}(\theta,\phi)b_1(r),\\
\psi^{(-+)}_{+0} &= 0,\\
\psi^{(-+)}_{+1_+} &= \frac{\sqrt{j+m-1}\sqrt{j+m}}{\sqrt{2j}\sqrt{2j-1}}Y^{j-1}_{m-1}(\theta,\phi)c_0(r)\\
&\quad+\frac{\sqrt{j-m+1}\sqrt{j-m+2}}{\sqrt{2j+2}\sqrt{2j+3}}Y^{j+1}_{m-1}(\theta,\phi)d_0(r),\\
\psi^{(-+)}_{+1_0} &= \frac{\sqrt{j-m}\sqrt{j+m}}{\sqrt{j}\sqrt{2j-1}}Y^{j-1}_m(\theta,\phi)c_0(r)\\
&\quad-\frac{\sqrt{j-m+1}\sqrt{j+m+1}}{\sqrt{1+j}\sqrt{2j+3}}Y^{j+1}_m(\theta,\phi)d_0(r),\\
\psi^{(-+)}_{+1_-} &= \frac{\sqrt{j-m-1}\sqrt{j-m}}{\sqrt{2j}\sqrt{2j-1}}Y^{j-1}_{m+1}(\theta,\phi)c_0(r)\\
&\quad+\frac{\sqrt{j+m+1}\sqrt{j+m+2}}{\sqrt{2j+2}\sqrt{2j+3}}Y^{j+1}_{m+1}(\theta,\phi)d_0(r),\\
\psi^{(+-)}_{+0} &= 0,\\
\psi^{(+-)}_{+1_+} &= \frac{\sqrt{j+m-1}\sqrt{j+m}}{\sqrt{2j}\sqrt{2j-1}}Y^{j-1}_{m-1}(\theta,\phi)c_1(r)\\
&\quad+\frac{\sqrt{j-m+1}\sqrt{j-m+2}}{\sqrt{2j+2}\sqrt{2j+3}}Y^{j+1}_{m-1}(\theta,\phi)d_1(r),\\
\psi^{(+-)}_{+1_0} &= \frac{\sqrt{j-m}\sqrt{j+m}}{\sqrt{j}\sqrt{2j-1}}Y^{j-1}_m(\theta,\phi)c_1(r)\\
&\quad-\frac{\sqrt{j-m+1}\sqrt{j+m+1}}{\sqrt{j+1}\sqrt{2j+3}}Y^{j+1}_m(\theta,\phi)d_1(r),\\
\psi^{(+-)}_{+1_-} &= \frac{\sqrt{j-m-1}\sqrt{j-m}}{\sqrt{2j}\sqrt{2j-1}}Y^{j-1}_{m+1}(\theta,\phi)c_1(r)\\
&\quad+\frac{\sqrt{j+m+1}\sqrt{j+m+2}}{\sqrt{2j+2}\sqrt{2j+3}}Y^{j+1}_{m+1}(\theta,\phi)d_1(r).
\end{aligned} \tag{A12}$$

We get a state with opposite parity by applying to (A12) the block matrix with the 8×8 zero matrices on the diagonal and the 8×8 identity on the antidiagonal.

4. The 4 × 4 spectral problem

When considering two quarks interacting by the Cornell potential depending upon r, the invariant equation (A8) is replaced by (2.1). Obviously, as in the solution of the Dirac equation for the hydrogen spectrum, when determining meson masses or atomic levels, the mass contribution to the eigenvalue is given the maximal positive value [37].

We introduce the dimensionless variables Ω, w, and s by

$$\sigma = m_b^2 \Omega^{\frac{3}{2}}, \qquad \lambda = m_b(2+\Omega w), \qquad r = m_b^{-1}\Omega^{-\frac{1}{2}} s, \tag{A13}$$

and the dimensionless functions

$$h(s) = (2+\Omega w)/\sqrt{\Omega} + b/s, \quad k(s) = (2+\Omega s)/(2\sqrt{\Omega}). \tag{A14}$$

The coefficient functions $\{a_i(s), b_i(s), c_i(s), d_i(s)\}_{i=0,1}$ for the eigenstates are obtained by solving a boundary value problem. Because of the symmetries of the Hamiltonian, this is actually equivalent to solving a reduced 4×4 system for each different parity. The system is [20]

$$\begin{pmatrix} u_1'(s) \\ u_2'(s) \\ u_3'(s) \\ u_4'(s) \end{pmatrix} + \begin{pmatrix} 0 & A_0(s) & -B_0(s) & 0 \\ A_\varepsilon(s) & 1/s & 0 & B_\varepsilon(s) \\ C_\varepsilon(s) & 0 & 2/s & A_\varepsilon(s) \\ 0 & D_\varepsilon(s) & A_0(s) & 1/s \end{pmatrix} \times \begin{pmatrix} u_1(s) \\ u_2(s) \\ u_3(s) \\ u_4(s) \end{pmatrix} = 0. \tag{A15}$$

In (A15), $A_0 = A_\varepsilon|_{\varepsilon=0}$, $B_0 = B_\varepsilon|_{\varepsilon=0}$, and the four unknown functions $\{u_i(s)\}_{i=1,4}$ determine the above eight $\{a_i(s), b_i(s), c_i(s), d_i(s)\}_{i=0,1}$ as reported below. The even parity coefficients for $b\bar{b}$ are

$$\begin{aligned} A_\varepsilon(s) &= 0, \\ B_\varepsilon(s) &= \frac{1}{2s}(h(s)s + 2\varepsilon b), \\ C_\varepsilon(s) &= \frac{1}{2s}(h(s)s + 2\varepsilon b) - \frac{2J^2}{s(h(s)s - 2\varepsilon b)} - \frac{2k^2(s)s}{sh(s) - 4\varepsilon b}, \\ D_\varepsilon(s) &= -\frac{1}{2s}(h(s)s + 2\varepsilon b) + \frac{2J^2}{h(s)s^2} + \frac{2k^2(s)s}{sh(s) - 2\varepsilon b}. \end{aligned} \tag{A16}$$

The odd parity coefficients for $b\bar{b}$ read [42]

$$\begin{aligned} A_\varepsilon(s) &= -\frac{2\sqrt{J^2}k(s)}{h(s)s - 2\varepsilon b}, \\ B_\varepsilon(s) &= \frac{1}{2s}(h(s)s + 2\varepsilon b) - \frac{2k^2(s)s}{h(s)s - 2\varepsilon b}, \\ C_\varepsilon(s) &= \frac{1}{2s}(h(s)s + 4\varepsilon b) - \frac{2J^2}{s(h(s)s - 2\varepsilon b)}, \\ D_\varepsilon(s) &= -\frac{h(s)}{2} + \frac{2J^2}{h(s)s^2} - \frac{\varepsilon b}{s}. \end{aligned} \tag{A17}$$

In (A16) and (A17), $J^2 = j(j+1)$.

5. The coefficients of the even states

For integer n, we let

$$\Delta_n(s) = (2+\Omega w)s + \sqrt{\Omega} b(1 - n\varepsilon). \tag{A18}$$

The relations among the four u_i and the eight a_i, b_i, c_i, d_i variables for the even states are then

$$\begin{aligned} a_0(s) &= \frac{1}{2}\left(1 + \frac{(\Omega s + 2)s}{\Delta_4(s)}\right) u_1(s), \\ a_1(s) &= \frac{1}{2}\left(1 - \frac{(\Omega s + 2)s}{\Delta_4(s)}\right) u_1(s), \\ b_0(s) &= \frac{1}{2}\left(1 + \frac{(\Omega s + 2)s}{\Delta_2(s)}\right) u_2(s), \\ b_1(s) &= \frac{1}{2}\left(-1 + \frac{(\Omega s + 2)s}{\Delta_2(s)}\right) u_2(s), \\ c_0(s) &= -\frac{\sqrt{\Omega}(j+1)\sqrt{j}u_1(s)}{\sqrt{2j+1}\Delta_2(s)} + \frac{\sqrt{j+1}\sqrt{\Omega}ju_2(s)}{\sqrt{2j+1}\Delta_0(s)} \\ &\quad - \frac{1}{2}\frac{\sqrt{j}u_3(s)}{\sqrt{2j+1}} - \frac{1}{2}\frac{\sqrt{j+1}u_4(s)}{\sqrt{2j+1}}, \\ c_1(s) &= -\frac{\sqrt{\Omega}(j+1)\sqrt{j}u_1(s)}{\sqrt{2j+1}\Delta_2(s)} - \frac{\sqrt{j+1}\sqrt{\Omega}ju_2(s)}{\sqrt{2j+1}\Delta_0(s)} \\ &\quad - \frac{1}{2}\frac{\sqrt{j}u_3(s)}{\sqrt{2j+1}} + \frac{1}{2}\frac{\sqrt{j+1}u_4(s)}{\sqrt{2j+1}}, \\ d_0(s) &= -\frac{\sqrt{j+1}\sqrt{\Omega}ju_1(s)}{\sqrt{2j+1}\Delta_2(s)} - \frac{\sqrt{j}(j+1)\sqrt{\Omega}u_2(s)}{\sqrt{2j+1}\Delta_0(s)} \\ &\quad + \frac{1}{2}\frac{\sqrt{j+1}u_3(s)}{\sqrt{2j+1}} - \frac{1}{2}\frac{\sqrt{j}u_4(s)}{\sqrt{2j+1}}, \\ d_1(s) &= -\frac{\sqrt{j+1}\sqrt{\Omega}ju_1(s)}{\sqrt{2j+1}\Delta_2(s)} + \frac{\sqrt{j}(j+1)\sqrt{\Omega}u_2(s)}{\sqrt{2j+1}\Delta_0(s)} \\ &\quad + \frac{1}{2}\frac{\sqrt{j+1}u_3(s)}{\sqrt{2j+1}} + \frac{1}{2}\frac{\sqrt{j}u_4(s)}{\sqrt{2j+1}}. \end{aligned} \tag{A19}$$

6. The coefficients of the odd states

The relations for the odd states are

$$
\begin{aligned}
a_0(s) &= a_1(s) = \frac{1}{2}u_1(s),\\
b_0(s) &= -b_1(s) = \frac{1}{2}u_2(s),\\
c_0(s) &= -\frac{\sqrt{\Omega}\sqrt{j+1}\sqrt{j}}{\sqrt{2j+1}}\left(\frac{\sqrt{j+1}u_1(s)}{\Delta_2(s)} - \frac{\sqrt{j}u_2(s)}{\Delta_0(s)}\right)\\
&\quad -\frac{1}{2}\frac{1}{\sqrt{2j+1}}\left[\sqrt{j}\left(1+\frac{(\Omega s+2)s}{\Delta_0(s)}\right)u_3(s) + \left(1+\frac{(\Omega s+2)s}{\Delta_2(s)}\right)u_4(s)\sqrt{j+1}\right],\\
c_1(s) &= -\frac{\sqrt{\Omega}\sqrt{j+1}\sqrt{j}}{\sqrt{2j+1}}\left(\frac{\sqrt{j+1}u_1(s)}{\Delta_2(s)} + \frac{\sqrt{j}u_2(s)}{\Delta_0(s)}\right)\\
&\quad -\frac{1}{2}\frac{1}{\sqrt{2j+1}}\left[\sqrt{j}\left(1-\frac{(\Omega s+2)s}{\Delta_0(s)}\right)u_3(s) - \sqrt{j+1}\left(1-\frac{(\Omega s+2)s}{\Delta_2(s)}\right)u_4(s)\right],\\
d_0(s) &= -\frac{\sqrt{\Omega}\sqrt{j+1}\sqrt{j}}{\sqrt{2j+1}}\left(\frac{\sqrt{j}u_1(s)}{\Delta_2(s)} + \frac{\sqrt{j+1}u_2(s)}{\Delta_0(s)}\right)\\
&\quad +\frac{1}{2}\frac{1}{\sqrt{2j+1}}\left[\sqrt{j+1}\left(1+\frac{(\Omega s+2)s}{\Delta_0(s)}\right)u_3(s) - \sqrt{j}\left(1+\frac{(\Omega s+2)s}{\Delta_2(s)}\right)u_4(s)\right],\\
d_1(s) &= -\frac{\sqrt{\Omega}\sqrt{j+1}\sqrt{j}}{\sqrt{2j+1}}\left(\frac{\sqrt{j}u_1(s)}{\Delta_2(s)} - \frac{\sqrt{j+1}u_2(s)}{\Delta_0(s)}\right)\\
&\quad +\frac{1}{2}\frac{1}{\sqrt{2j+1}}\left[\sqrt{j+1}\left(1-\frac{(\Omega s+2)s}{\Delta_0(s)}\right)u_3(s) + \sqrt{j}\left(1-\frac{(\Omega s+2)s}{\Delta_2(s)}\right)u_4(s)\right].
\end{aligned}
\tag{A20}
$$

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