

# New approach to the resummation of logarithms in Higgs-boson decays to a vector quarkonium plus a photon

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We present a calculation of the rates for Higgs-boson decays to a vector heavy-quarkonium state plus a photon, where the heavy-quarkonium states are the  $J/\psi$  and the  $\Upsilon(nS)$  states, with  $n = 1, 2$ , or  $3$ . The calculation is carried out in the light-cone formalism, combined with nonrelativistic QCD factorization, and is accurate at leading order in  $m_Q^2/m_H^2$ , where  $m_Q$  is the heavy-quark mass and  $m_H$  is the Higgs-boson mass. The calculation contains corrections through next-to-leading order in the strong-coupling constant  $\alpha_s$  and the square of the heavy-quark velocity  $v$ , and includes a resummation of logarithms of  $m_H^2/m_Q^2$  at next-to-leading logarithmic accuracy. We have developed a new method, which makes use of Abel summation, accelerated through the use of Padé approximants, to deal with divergences in the resummed expressions for the quarkonium light-cone distribution amplitudes. This approach allows us to make definitive calculations of the resummation effects. Contributions from the order- $\alpha_s$  and order- $v^2$  corrections to the light-cone distribution amplitudes that we obtain with this new method differ substantially from the corresponding contributions that one obtains from a model light-cone distribution amplitude [M. König and M. Neubert, *J. High Energy Phys.* **08** (2015) 012]. Our results for the real parts of the direct-process amplitudes are considerably smaller than those from one earlier calculation [G. T. Bodwin, H. S. Chung, J.-H. Ee, J. Lee, and F. Petriello, *Phys. Rev. D* **90**, 113010 (2014)], reducing the sensitivity to the Higgs-boson-heavy-quark couplings, and are somewhat smaller than those from another earlier calculation [M. König and M. Neubert, *J. High Energy Phys.* **08** (2015) 012]. However, our results for the standard-model Higgs-boson branching fractions are in good agreement with those in M. König and M. Neubert, *J. High Energy Phys.* **08** (2015) 012.

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## I. INTRODUCTION

Several years ago, it was pointed out that Higgs-boson ( $H$ ) decays into a vector charmonium state ( $V$ ) plus a photon ( $\gamma$ ) proceed through two processes [1]. One process is the “direct process,” in which the Higgs boson decays into a heavy quark-antiquark ( $Q\bar{Q}$ ) pair, followed by the radiation of a real photon by the  $Q$  or  $\bar{Q}$  and the subsequent evolution of the  $Q\bar{Q}$  pair into the quarkonium. The other process is the “indirect process,” in which the Higgs boson decays via a  $W$ -boson loop or a quark loop into a  $\gamma$  and a virtual photon ( $\gamma^*$ ), followed by the decay of the  $\gamma^*$  into a  $Q\bar{Q}$  pair, which evolves into the quarkonium.

The direct amplitude is proportional to the  $HQ\bar{Q}$  coupling. However, its standard-model (SM) value is generally too small to lead to a rate that is measurable at the LHC. In the case in which the quarkonium is a  $J/\psi$ , the SM indirect amplitude is much larger than the SM direct amplitude and leads to a rate that is potentially measurable

in a high-luminosity LHC [1]. Furthermore, the contribution from interference between the direct and indirect amplitudes, which is destructive, may also be within the realm of measurement at a high-luminosity LHC [1] and could lead to a determination of the  $Hc\bar{c}$  coupling. In the cases in which the quarkonium is an  $\Upsilon(nS)$  state, the SM rates are too small to be measured even at a high-luminosity LHC [1]. However, owing to the destructive interference between the direct and indirect amplitudes, the rates are very sensitive to deviations of the direct amplitudes from the SM values [1]. Because the direct and indirect amplitudes for the decays  $H \rightarrow V + \gamma$  are comparable in size, these decays can give information about the phases of the  $HQ\bar{Q}$  couplings. They are the only processes that have been identified so far that can yield that phase information.

The indirect amplitude can be obtained, up to corrections of relative order  $m_Q^2/m_H^2$ , from the amplitude for  $H \rightarrow \gamma\gamma$  [1], which is known in the SM with a precision of a few percent [2,3]. Here,  $m_Q$  is the heavy-quark mass and  $m_H$  is the Higgs-boson mass.

In Ref. [1], the direct amplitude was computed through next-to-leading order (NLO) in the strong coupling  $\alpha_s$  by making use of the result of Shifman and Vysotsky [4]. That result was derived by making use of light-cone methods

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[5,6] that are valid up to corrections of order  $m_Q^2/m_H^2$ . In addition, in Ref. [1], logarithms of  $m_H^2/m_Q^2$  were resummed at leading logarithmic (LL) accuracy to all orders in  $\alpha_s$  by making use of the LL resummed expression for the direct amplitude in Ref. [4].

The largest single uncertainty in the calculation of Ref. [1] was due to uncalculated relativistic corrections to the direct amplitude of relative order  $v^2$ , where  $v$  is the velocity of the  $Q$  or  $\bar{Q}$  in the quarkonium rest frame. Those order- $v^2$  corrections were computed in Ref. [7] in the nonrelativistic QCD (NRQCD) formalism [8] and, also, in the light-cone formalism [5,6], so as to make contact with the light-cone calculation of Ref. [4].

Logarithms of  $m_H^2/m_Q^2$  can be resummed by evolving the  $HQ\bar{Q}$  coupling, which is proportional to  $m_Q(\mu)$ , the quarkonium decay constant, and the light-cone distribution amplitude (LCDA) from the renormalization scale  $\mu = m_Q$  to the renormalization scale  $\mu = m_H$ . The standard method for carrying out the evolution of the LCDA is to expand the LCDA in a series of eigenfunctions of the lowest-order evolution kernel. The eigenfunctions are proportional to Gegenbauer polynomials [9]. In Ref. [7], it was noticed that the eigenfunction series is not convergent in the case of the order- $v^2$  corrections to the direct amplitude. Consequently, for the order- $v^2$  correction, logarithms of  $m_H^2/m_Q^2$  were summed only through relative order  $\alpha_s^2$  in Ref. [7].

Resummation of logarithms of  $m_H^2/m_Q^2$  at next-to-leading-logarithmic (NLL) accuracy requires a calculation in the light-cone formalism of the order- $\alpha_s$  corrections to both the hard-scattering kernel for the direct process and the LCDA. That calculation was accomplished in Ref. [10] at leading order (LO) in  $v$ . (The calculation of the order- $\alpha_s$  correction to the hard-scattering kernel in Ref. [10] was confirmed in Ref. [11].) The calculation of the LCDA was carried out in the NRQCD framework, and the result was expressed in terms of the NRQCD nonperturbative long-distance matrix elements (LDMEs) [12].

The actual resummation of logarithms of  $m_H^2/m_Q^2$  at NLL accuracy was carried out in Ref. [11], in which it was found that the NLL corrections have a substantial impact on the numerical results for the rates. In that work, the calculational strategy involved introducing a model LCDA whose nonzero second moment would take into account the known order- $v^2$  and order- $\alpha_s$  corrections to the LCDA at a scale of 1 GeV. This approach avoids the problem of the lack of convergence of the eigenfunction expansion in a calculation of the order- $v^2$  corrections to the LCDA. However, as we will see, the model wave function does not give a very accurate accounting of the order- $v^2$  and order- $\alpha_s$  corrections to the LCDA, even after evolution to the scale  $m_H$ .

In this paper, we present a new method for calculating the evolution of the order- $v^2$  corrections to the LCDA. The method introduces a regulator that defines the generalized

functions (distributions) that appear in the initial LCDAs as sequences of ordinary functions. The regulator method is equivalent to Abel summation of the eigenfunction expansion. In order to accelerate the convergence of the Abel summation, we introduce Padé approximants to obtain an approximate analytic continuation in the regulator variable that converges rapidly as the regulator is removed. We refer to this method that makes use of a combination of Abel summation and Padé approximants as the ‘‘Abel-Padé method.’’ The Abel-Padé method gives very accurate results in cases for which analytic results are known for the LCDAs, even in situations in which the eigenfunction expansion diverges. The Abel-Padé method solves the general problem of carrying out the scale evolution in a nonrelativistic expansion of the LCDA for heavy-quarkonium systems, and it should be applicable in other situations in which series of orthogonal polynomials fail to converge.

The results that we obtain with the Abel-Padé method agree reasonably well with the perturbative estimates of Ref. [7]. However, the Abel-Padé method gives results that differ significantly from those that are obtained by making use of the model of Ref. [11]. We use the Abel-Padé method to obtain a complete calculation of the rates for  $H \rightarrow V + \gamma$  through orders  $\alpha_s$  and  $v^2$  and to all orders in  $\alpha_s$  through order  $v^2$  at NLL accuracy.

The remainder of this paper is organized as follows. In Sec. II, we discuss the light-cone amplitude for the direct process through orders  $\alpha_s$  and  $v^2$ . In Sec. III, we describe the resummation of logarithms of  $m_H^2/m_Q^2$  and give resummed expressions for the contributions to the direct amplitude in terms of sums over eigenfunctions of the LO evolution kernel. Section IV contains a discussion of the problem of the nonconvergence of the eigenfunction series and a presentation of a solution of the problem, which leads to the Abel-Padé method for summing the series. In Sec. V, we compare results from the Abel-Padé method with those that follow from the model LCDA that was proposed in Ref. [11]. In Sec. VI we give the expressions that we use to compute the direct amplitudes and the indirect amplitudes and discuss the numerical inputs that we use and the sources of uncertainties. We also present a novel method to compute uncertainties in the decay rates that allows us to deal with the highly nonlinear dependences of the decay rates on the input parameters. We give our numerical results in Sec. VII, and we summarize and discuss our results in Sec. VIII.

## II. LIGHT-CONE AMPLITUDE FOR THE DIRECT PROCESS

In the light-cone approach, the direct amplitude for  $H \rightarrow V + \gamma$  is given, up to corrections of relative order  $m_Q^2/m_H^2$ , by<sup>1</sup>

<sup>1</sup>See, for example, Ref. [1].

$$\begin{aligned}
i\mathcal{M}_{\text{dir}}^{\text{LC}}[H \rightarrow V + \gamma] &= \frac{i}{2} e e_Q \kappa_Q \bar{m}_Q(\mu) (\sqrt{2} G_F)^{1/2} f_V^\perp(\mu) \\
&\times \left( -\epsilon_V^* \cdot \epsilon_\gamma^* + \frac{\epsilon_V^* \cdot p_\gamma p \cdot \epsilon_\gamma^*}{p_\gamma \cdot p} \right) \\
&\times \int_0^1 dx T_H(x, \mu) \phi_V^\perp(x, \mu), \quad (1)
\end{aligned}$$

where  $e$  is the electric charge,  $e_Q$  is the fractional charge of the heavy quark  $Q$ ,  $\kappa_Q$  is an adjustable parameter in the  $HQ\bar{Q}$  coupling whose SM value is 1,  $\bar{m}_Q$  is the mass of  $Q$  in the modified minimal subtraction ( $\overline{\text{MS}}$ ) scheme,  $G_F$  is the Fermi constant,  $f_V^\perp$  is the decay constant of the vector quarkonium  $V$ ,  $\epsilon_V$  and  $p$  are the quarkonium polarization and momentum, respectively,  $\epsilon_\gamma$  and  $p_\gamma$  are the photon polarization and momentum, respectively,  $\mu$  is the renormalization scale, and  $x$  is the  $Q\bar{Q}$  momentum fraction of  $V$ , which runs from 0 to 1.  $\phi_V^\perp(x, \mu)$  is the vector-quarkonium LCDA, which is defined by

$$\begin{aligned}
\frac{1}{2} \langle V | \bar{Q}(z) [\gamma^\mu, \gamma^\nu] [z, 0] Q(0) | 0 \rangle &= f_V^\perp(\mu) (\epsilon_V^{*\mu} p_V^\nu - \epsilon_V^{*\nu} p_V^\mu) \\
&\times \int_0^1 dx e^{i p^- z x} \phi_V^\perp(x, \mu) \quad (2)
\end{aligned}$$

and has the normalization  $\int_0^1 dx \phi_V^\perp(x, \mu) = 1$ . The coordinate  $z$  lies along the plus light-cone direction, and the gauge link

$$[z, 0] = P \exp \left[ i g_s \int_0^z dx A_a^+(x) T^a \right] \quad (3)$$

makes the nonlocal operator gauge invariant. In Eq. (3),  $g_s = \sqrt{4\pi\alpha_s}$ ,  $A_a^+$  is the gluon field with the color index  $a = 1, 2, \dots, N_c^2 - 1$ ,  $T^a$  is the generator of the fundamental representation of  $\text{SU}(N_c)$  color, and the symbol  $P$  denotes path ordering. The nonrelativistic expansion of  $\phi_V^\perp(x, \mu)$ , through linear orders in  $\alpha_s$  and  $v^2$ , is

$$\begin{aligned}
\phi_V^\perp(x, \mu) &= \phi_V^{\perp(0)}(x, \mu) + \langle v^2 \rangle_V \phi_V^{\perp(v^2)}(x, \mu) \\
&+ \frac{\alpha_s(\mu)}{4\pi} \phi_V^{\perp(1)}(x, \mu) + \mathcal{O}(\alpha_s^2, \alpha_s v^2, v^4), \quad (4)
\end{aligned}$$

where the LO contribution is given by

$$\phi_V^{\perp(0)}(x, \mu) = \delta\left(x - \frac{1}{2}\right) \quad (5)$$

and  $\delta$  is the Dirac delta function.  $\langle v^2 \rangle_V$  is proportional to the ratio of the NRQCD LDME of order  $v^2$  to the LDME of order  $v^0$ :

$$\langle v^2 \rangle_V = \frac{1}{m_Q^2} \frac{\langle V(\epsilon_V) | \psi^\dagger(-\frac{i}{2}\vec{\nabla})^2 \sigma \cdot \epsilon_V \chi | 0 \rangle}{\langle V(\epsilon_V) | \psi^\dagger \sigma \cdot \epsilon_V \chi | 0 \rangle}. \quad (6)$$

Here,  $\psi$  is the two-component (Pauli) spinor field that annihilates a heavy quark,  $\chi^\dagger$  is the two-component spinor field that annihilates a heavy antiquark,  $\sigma_i$  is a Pauli matrix,  $|V(\epsilon_V)\rangle$  denotes the vector quarkonium state in the quarkonium rest frame with spatial polarization  $\epsilon_V$ , and  $m_Q$  denotes the quark pole mass. The coefficient of the order- $v^2$  contribution,  $\phi_V^{\perp(v^2)}$ , was computed in Ref. [7] and is given by

$$\phi_V^{\perp(v^2)}(x, \mu) = \frac{1}{24} \delta^{(2)}\left(x - \frac{1}{2}\right), \quad (7)$$

where  $\delta^{(n)}$  is the  $n$ th derivative of the Dirac delta function. The coefficient of the order- $\alpha_s$  contribution,  $\phi_V^{\perp(1)}(x, \mu)$ , was computed in Ref. [10] and is given by<sup>2</sup>

$$\begin{aligned}
\phi_V^{\perp(1)}(x, \mu) &= C_F \theta(1 - 2x) \\
&\times \left\{ \left[ \frac{8x}{1 - 2x} \left( \log \frac{\mu^2}{m_Q^2 (1 - 2x)^2} - 1 \right) \right]_+ \right. \\
&\left. + \left[ \frac{16x(1 - x)}{(1 - 2x)^2} \right]_{++} \right\} + (x \leftrightarrow 1 - x), \quad (8)
\end{aligned}$$

where  $C_F = (N_c^2 - 1)/(2N_c)$ ,  $N_c = 3$  is the number of colors, and the plus and plus-plus distributions are defined by

$$\int_0^1 dx f(x) [g(x)]_+ = \int_0^1 dx \left[ f(x) - f\left(\frac{1}{2}\right) \right] g(x), \quad (9a)$$

$$\begin{aligned}
&\int_0^1 dx f(x) [g(x)]_{++} \\
&= \int_0^1 dx \left[ f(x) - f\left(\frac{1}{2}\right) - f'\left(\frac{1}{2}\right) \left(x - \frac{1}{2}\right) \right] g(x). \quad (9b)
\end{aligned}$$

Although  $\phi_V^{\perp(0)}(x, \mu)$  and  $\phi_V^{\perp(v^2)}(x, \mu)$  are independent of  $\mu$ , we keep  $\mu$  explicit in their arguments as a reminder that a single scale  $\mu$  applies to all of the terms in  $\phi_V^\perp(x, \mu)$  [Eq. (4)].

The quarkonium decay constant  $f_V^\perp(\mu)$  is given by

$$\begin{aligned}
f_V^\perp(\mu) &= \frac{\sqrt{2N_c} \sqrt{2m_V}}{2m_Q} \Psi_V(0) \left[ 1 - \frac{5}{6} \langle v^2 \rangle_V - \frac{C_F \alpha_s(\mu)}{4\pi} \right. \\
&\times \left. \left( \log \frac{\mu^2}{m_Q^2} + 8 \right) + \mathcal{O}(\alpha_s^2, \alpha_s v^2, v^4) \right], \quad (10)
\end{aligned}$$

<sup>2</sup>Equation (3.17) of Ref. [10] applies to the case in which  $\Delta$  in Eq. (3.16) of Ref. [10] is set equal to zero. We thank the authors of Ref. [10] for confirming that this is the case.

where the order- $v^2$  term was computed in Ref. [7] and the order- $\alpha_s$  term was computed in Ref. [10]. Here,  $\Psi_V(0)$  is the quarkonium wave function at the origin, which is given in terms of an NRQCD LDME by [8]

$$\Psi_V(0) = \frac{1}{\sqrt{2N_c}} \langle V(\epsilon_V) | \psi^\dagger \boldsymbol{\sigma} \cdot \epsilon_V \chi | 0 \rangle. \quad (11)$$

The hard-scattering kernel  $T_H$  for the process  $H \rightarrow V + \gamma$  is given by

$$T_H(x, \mu) = T_H^{(0)}(x, \mu) + \frac{\alpha_s(\mu)}{4\pi} T_H^{(1)}(x, \mu) + \mathcal{O}(\alpha_s^2), \quad (12a)$$

where

$$T_H^{(0)}(x, \mu) = \frac{1}{x(1-x)}, \quad (12b)$$

$$T_H^{(1)}(x, \mu) = C_F \frac{1}{x(1-x)} \left[ 2 \left( \log \frac{m_H^2}{\mu^2} - i\pi \right) \log x(1-x) + \log^2 x + \log^2(1-x) - 3 \right]. \quad (12c)$$

The order- $\alpha_s$  term in  $T_H$  was computed in Ref. [10] by taking the quark mass to be the pole mass and in Ref. [11] by taking the quark mass to be the  $\overline{\text{MS}}$  mass.<sup>3</sup> The expression in Eq. (12c) is for the case in which the quark mass is taken to be the  $\overline{\text{MS}}$  mass.

### III. RESUMMATION OF LOGARITHMS IN THE DIRECT AMPLITUDE

Our strategy for resumming logarithms of  $m_H^2/m_Q^2$  is the following. In Eq. (1) we take the scale  $\mu$  to be  $m_H$ . Then  $T_H$  [Eq. (12)] contains no large logarithms. Note that, if one takes the quark mass in the computation of  $T_H$  to be the pole mass, then the order- $\alpha_s$  correction to  $T_H(x, \mu)$  contains a term that is proportional to  $\log(m_H^2/m_Q^2)$ , as can be seen from the corrected version of Eq. (4.23) of Ref. [10]. Such large NLLs would slow, or even spoil, the convergence of the perturbation expansion. We initially evaluate  $\phi_V^\perp(x, \mu)$  and  $f_V^\perp(\mu)$  at a scale  $\mu_0$  of order  $m_Q$ , so that the perturbative expressions in Eqs. (5), (7), and (8) do not contain any logarithms of  $m_H^2/m_Q^2$ . Then, we evolve  $\phi_V^\perp(x, \mu)$  and  $f_V^\perp(\mu)$  to the scale  $\mu = m_H$ , along with  $\bar{m}_Q(\mu)$ . Expressions for the evolution of  $\bar{m}_Q(\mu)$  and  $f_V^\perp(\mu)$  are given in Appendix A. We now address the evolution of  $\phi_V^\perp(x, \mu)$ .

<sup>3</sup>Equation (4.23) of Ref. [10] contains a typo:  $3 \ln[\mu^2/(-m_h^2)]$  should be replaced with  $3 \ln(\mu^2/m_Q^2)$ . This typo was noted in Ref. [11]. We thank the authors of Ref. [10] for confirming the existence of this typo.

### A. Evolution of the LCDA

The LCDA  $\phi_V^\perp(x, \mu)$  satisfies the evolution equation [5]

$$\mu^2 \frac{\partial}{\partial \mu^2} \phi_V^\perp(x, \mu) = C_F \frac{\alpha_s(\mu)}{2\pi} \int_0^1 dy V_T(x, y) \phi_V^\perp(y, \mu), \quad (13)$$

where the LO evolution kernel  $V_T(x, y)$  is given by [5]

$$V_T(x, y) = V_0(x, y) - \frac{1-x}{1-y} \theta(x-y) - \frac{x}{y} \theta(y-x), \quad (14a)$$

$$V_0(x, y) = V_{\text{BL}}(x, y) - \delta(x-y) \int_0^1 dz V_{\text{BL}}(z, x), \quad (14b)$$

$$V_{\text{BL}}(x, y) = \frac{1-x}{1-y} \left( 1 + \frac{1}{x-y} \right) \theta(x-y) + \frac{x}{y} \left( 1 + \frac{1}{y-x} \right) \theta(y-x). \quad (14c)$$

As is well known, the eigenfunctions of LO evolution kernel for  $\phi_V^\perp(x, \mu)$  are given by [9]

$$G_n(x) = w(x) C_n^{(3/2)}(2x-1), \quad (15)$$

where  $w(x) = x(1-x)$  is the weighting function and the  $C_n^{(3/2)}$  are Gegenbauer polynomials. The corresponding eigenvalues (anomalous dimensions) are

$$\gamma_n^{\perp(0)} = 8C_F(H_{n+1} - 1), \quad (16)$$

where the  $H_n$  are harmonic numbers. The orthogonality relation of the Gegenbauer polynomials is given by

$$\begin{aligned} N_n \int_0^1 dx w(x) C_n^{(3/2)}(2x-1) C_m^{(3/2)}(2x-1) \\ = N_n \int_0^1 dx G_n(x) C_m^{(3/2)}(2x-1) \\ = \delta_{nm}, \end{aligned} \quad (17)$$

where the normalization factor  $N_n$  is given by

$$N_n = \frac{4(2n+3)}{(n+1)(n+2)}. \quad (18)$$

In order to work out the evolution of the LCDAs, it is convenient to write them in terms of the eigenfunctions. Using Eq. (17), we have

$$\phi_V^\perp(x, \mu) = \sum_{n=0}^{\infty} \phi_n^\perp(\mu) G_n(x), \quad (19a)$$



where the moments  $\phi_n^\perp(\mu)$  are given by

$$\phi_n^\perp(\mu) = N_n \int_0^1 dx C_n^{(3/2)}(2x-1) \phi_V^\perp(x, \mu). \quad (19b)$$

In a similar fashion, we can write  $T_H$  in terms of Gegenbauer polynomials:

$$T_H(x, \mu) = \sum_{n=0}^{\infty} N_n T_n(\mu) C_n^{(3/2)}(2x-1), \quad (20a)$$

where

$$T_n(\mu) = \int_0^1 dx T_H(x, \mu) G_n(x). \quad (20b)$$

Then, using Eq. (17), we can write the light-cone amplitude, at least formally, as a sum over moments of  $T_H$  and  $\phi_V^\perp$ :

$$\int_0^1 dx T_H(x, \mu) \phi_V^\perp(x, \mu) = \sum_{n=0}^{\infty} T_n(\mu) \phi_n^\perp(\mu). \quad (21)$$

The moments  $\phi_n^\perp(\mu)$  can be written in terms of the moments  $\phi_n^\perp(\mu_0)$  as

$$\phi_n^\perp(\mu) = \sum_{k=0}^n U_{nk}(\mu, \mu_0) \phi_k^\perp(\mu_0), \quad (22)$$

where we are using the notation of Ref. [11]. The expressions for  $U_{nk}(\mu, \mu_0)$  at LL and NLL accuracies are given in Appendix B. Note that the off-diagonal elements of  $U_{nk}(\mu, \mu_0)$  are nonvanishing only for even  $n-k$  [13,14].

We decompose the light-cone amplitude according to the powers of  $\alpha_s$  and  $v^2$ :

$$\begin{aligned} \int_0^1 dx T_H(x, \mu) \phi_V^\perp(x, \mu) &= \mathcal{M}^{(0,0)}(\mu) + \frac{\alpha_s(\mu)}{4\pi} \mathcal{M}^{(1,0)}(\mu) \\ &+ \frac{\alpha_s(m_Q)}{4\pi} \mathcal{M}^{(0,1)}(\mu) \\ &+ \langle v^2 \rangle_V \mathcal{M}^{(0,v^2)}(\mu) \\ &+ \mathcal{O}(\alpha_s^2, \alpha_s v^2, v^4), \end{aligned} \quad (23a)$$

where

$$\begin{aligned} \mathcal{M}^{(i,j)}(\mu) &= \int_0^1 dx T_H^{(i)}(x, \mu) \phi_V^{\perp(j)}(x, \mu) \\ &= \sum_{n=0}^{\infty} T_n^{(i)}(\mu) \phi_n^{\perp(j)}(\mu). \end{aligned} \quad (23b)$$

$T_n^{(0)}(\mu)$  and  $T_n^{(1)}(\mu)$  vanish for  $n$  odd and are given for  $n$  even by

$$T_n^{(0)}(\mu) = 1, \quad (24a)$$

$$\begin{aligned} T_n^{(1)}(\mu)/C_F &= -4(H_{n+1} - 1) \left( \log \frac{m_H^2}{\mu^2} - i\pi \right) \\ &+ 4H_{n+1}^2 - 3 + 4\pi i, \end{aligned} \quad (24b)$$

where the expression for  $T_n^{(1)}(\mu)$  was first given in Ref. [11]. The  $\phi_n^{\perp(i)}(\mu)$  also vanish for  $n$  odd.

For  $\mathcal{M}^{(0,0)}(\mu)$ , we use the NLL expression for  $U_{nk}(\mu, \mu_0)$  to compute  $\phi_n^{\perp(0)}(\mu)$ , while, for the other  $\mathcal{M}^{(i,j)}(\mu)$ , we use the LL expression for  $U_{nk}(\mu, \mu_0)$ .

As was noted in the appendix of Ref. [7], the eigenfunction series for  $\mathcal{M}^{(0,v^2)}(\mu)$  is not convergent. Some of the eigenfunction series for the other  $\mathcal{M}^{(i,j)}(\mu)$  converge rather slowly. We address these issues of nonconvergence and slow convergence in Sec. IV.

#### IV. NONCONVERGENCE OF THE EIGENFUNCTION SERIES AND SUMMATION BY THE ABEL-PADÉ METHOD

##### A. The problem of nonconvergence

From the theory of orthogonal polynomials on a finite interval, we know that a series of Gegenbauer polynomials  $C_n^{(3/2)}(2x-1)$  can represent sufficiently smooth functions over the interval  $0 < x < 1$ . That is,  $C_n^{(3/2)}(2x-1)$  are a complete set of functions and satisfy the completeness relation

$$\sum_{n=0}^{\infty} N_n w(x) C_n^{(3/2)}(2x-1) C_n^{(3/2)}(2y-1) = \delta(x-y). \quad (25)$$

It follows that the sum over  $n$  on the right side of Eqs. (21) or (23b) is well defined and is equal to the left side of Eqs. (21) or (23b) when  $T_H(x, \mu)$  and  $\phi_V^\perp(x, \mu)$  are sufficiently smooth functions of  $x$  [15]. A difficulty can arise because the nonrelativistic expansion of  $\phi_V^\perp(x, \mu)$  contains generalized functions (distributions) in  $x$  about the point  $x = 1/2$ . For example, the factor  $\delta^{(2)}(x - \frac{1}{2})$  in  $\phi_V^{\perp(v^2)}$  [Eq. (7)] causes the sum over  $n$  in the expression for  $\mathcal{M}^{(0,v^2)}(\mu)$  to diverge, as was shown in the appendix of Ref. [7]. Nevertheless,  $\mathcal{M}^{(0,v^n)}(\mu)$  remains well defined as  $\mu$  evolves.

In order to demonstrate this, we define the quantity

$$\mathcal{M}^{(i,j)}(\mu_f, \mu) = \int_0^1 dx T_H^{(i)}(x, \mu_f) \phi_V^{\perp(j)}(x, \mu), \quad (26)$$

which gives the projection of  $\phi_V^{\perp(j)}(x, \mu)$  onto the hard-scattering amplitude evaluated at the final scale in the evolution  $\mu_f$ . Note that  $\mathcal{M}^{(i,j)}(\mu_f, \mu_f) = \mathcal{M}^{(i,j)}(\mu_f)$ . Now,

$\mathcal{M}^{(0,v^n)}(\mu_f, \mu)$  satisfies the same evolution equation as does  $\phi_V^\perp(x, \mu)$ , namely,

$$\begin{aligned} \mu^2 \frac{\partial}{\partial \mu^2} \mathcal{M}^{(0,v^n)}(\mu_f, \mu) \\ = C_F \frac{\alpha_s(\mu)}{2\pi} \int_0^1 dx \int_0^1 dy T_H^{(0)}(x, \mu_f) V_T(x, y) \phi_V^{\perp(v^n)}(y, \mu). \end{aligned} \quad (27)$$

First, we note that  $\mathcal{M}^{(0,v^n)}(\mu_f, \mu_0)$  is well defined. This follows from the definition of  $\mathcal{M}^{(0,v^n)}(\mu_f, \mu_0)$  in Eq. (26), the fact that  $\phi_V^{\perp(v^n)}(x, \mu_0)$  is proportional to  $\delta^{(n)}(x - \frac{1}{2})$ , and the fact that  $T_H^{(0)}(x, \mu_f)$  is infinitely differentiable at  $x = 1/2$ . [We remind the reader that  $T_H^{(0)}(x, \mu)$  is actually independent of  $\mu$ .] Furthermore, it is easy to see that  $\int_0^1 dx T_H^{(0)}(x, \mu) V_T(x, y)$  is infinitely differentiable with respect to  $y$  at  $y = 1/2$ . It then follows from the evolution equation (27) that  $\mu^2(\partial/\partial\mu^2)\mathcal{M}^{(0,v^n)}(\mu_f, \mu)$  is well defined for all  $\mu$  between  $\mu_0$  and  $\mu_f$ . Therefore,  $\mathcal{M}^{(0,v^n)}(\mu_f, \mu_f) = \mathcal{M}^{(0,v^n)}(\mu_f)$  is well defined.

## B. Solution of the problem and the Abel-Padé method

In order to address the difficulty of nonconvergent eigenfunction series, we first define a smearing function  $S(x, y, z)$  by modifying the completeness relation (25). We introduce a factor  $z^n$  into each term in the sum over  $n$ :

$$S(x, y, z) = \sum_{n=0}^{\infty} z^n N_n w(x) C_n^{(3/2)}(2x-1) C_n^{(3/2)}(2y-1), \quad (28)$$

where  $z$  is a complex parameter. For  $|z| < 1$ , the sum over  $n$  in Eq. (28) is absolutely convergent, and  $S(x, y, z)$  is an ordinary function of  $x$  and  $y$ . As  $z$  approaches 1,  $S(x, y, z)$  becomes more and more sharply peaked around  $x = y$  and, in the limit  $z \rightarrow 1$ , is a representation of  $\delta(x - y)$ . We use the smearing function to define a smeared distribution amplitude:

$$\begin{aligned} \phi_S(x, z, \mu) &= \int_0^1 dy S(x, y, z) \phi_V^\perp(y, \mu) \\ &= \sum_{n=0}^{\infty} \phi_n^\perp(\mu) \sum_{m=0}^{\infty} z^m w(x) C_m^{(3/2)}(2x-1) N_m \\ &\quad \times \int_0^1 dy w(y) C_m^{(3/2)}(2y-1) C_n^{(3/2)}(2y-1) \\ &= \sum_{n=0}^{\infty} \phi_n^\perp(\mu) \sum_{m=0}^{\infty} z^m w(x) C_m^{(3/2)}(2x-1) \delta_{nm} \\ &= \sum_{n=0}^{\infty} \phi_n^\perp(\mu) z^n G_n(x), \end{aligned} \quad (29)$$

where we have used the orthogonality relation (17). For  $|z| < 1$ ,  $\phi_S(x, z, \mu)$  is an ordinary function of  $x$ . Because  $S(x, y, z)$  is a representation of  $\delta(x - y)$  in the limit  $z \rightarrow 1$ ,  $\phi_S(x, z, \mu)$  is a representation of  $\phi_V^\perp(x, \mu)$  in the limit  $z \rightarrow 1$ . That is, Eq. (29) can be used to define generalized functions in  $\phi_V^\perp(x, \mu)$  as a limit of a sequence of ordinary functions. It then follows, from the theory of orthogonal functions, that, for any  $z < 1$ ,<sup>4</sup>

$$\int_0^1 dx T_H(x, \mu) \phi_S(x, z, \mu) = \sum_{n=0}^{\infty} T_n(\mu) z^n \phi_n^\perp(\mu). \quad (30)$$

Then, we obtain the light-cone amplitude  $\mathcal{M}$  that corresponds to the distribution  $\phi_V^\perp(x, \mu)$  by taking the limit of the sequence of ordinary functions that we use to define  $\phi_V^\perp(x, \mu)$ :

$$\begin{aligned} \mathcal{M} &= \int_0^1 dx T_H(x, \mu) \phi_V^\perp(x, \mu) \\ &= \lim_{z \rightarrow 1} \int_0^1 dx T_H(x, \mu) \phi_S(x, z, \mu) \\ &= \lim_{z \rightarrow 1} \sum_{n=0}^{\infty} T_n(\mu) z^n \phi_n^\perp(\mu). \end{aligned} \quad (31)$$

We note that Eq. (31) amounts to Abel summation of the eigenfunction series. A mathematical proof of Eq. (31) is beyond the scope of this paper. However, we will describe several numerical tests that strongly support the validity of the Abel summation in Eq. (31).

In principle, one can use Eq. (31) to compute the light-cone amplitude, making use of Eq. (22) to take into account the scale evolution of the LCDA. In order to do this, one would need carry out the sum in Eq. (31) before taking limit  $z \rightarrow 1$ . In practice, in carrying out a numerical evaluation, one must include enough terms in the sum to guarantee that the remainder is small for a given value of  $|1 - z|$ . For the functions  $T_H(x, \mu)$  and  $\phi_V^\perp(x, \mu)$  that we consider, this typically requires that one include thousands of terms in order to achieve percent-level precision.<sup>5</sup>

A much more efficient procedure is to use Padé approximants to approximate the sum in Eq. (31). As we have mentioned, we refer to this method that makes use of a combination of Abel summation and Padé approximants as the Abel-Padé method. The sum in Eq. (31) defines a function of  $z$  that is analytic for  $|z| < 1$ . The Padé

<sup>4</sup>It can be seen from the analysis of the appendix of Ref. [7] that, for  $\phi_V^\perp(x, \mu) \rightarrow \phi_V^{\perp(0)}(x, \mu) \equiv \delta^{(0)}(x - \frac{1}{2})$  and  $T_H(x, \mu) \rightarrow T_H^{(0)}(x, \mu)$ , the sum on the right side of Eq. (30) is absolutely convergent for arbitrary  $\mu$  when  $z < 1$ .

<sup>5</sup>We have verified numerically, for the cases  $\mathcal{M}^{(0,0)}$  and  $\mathcal{M}^{(0,v^2)}$ , with  $\mu = m_Q, m_H/2, m_H, 2m_H, 1 \text{ TeV}$ , and  $2 \text{ TeV}$ , that the Abel summation does converge, although very slowly, to the result that is given by the Abel-Padé procedure.

approximant gives an approximate analytic continuation of that function to larger values of  $|z|$ . In particular, the Padé approximant can give precise values of Eq. (31) for  $z = 1$ , even when poles in the disc  $|z| < 1$  render the radius of convergence of the series to be less than 1. Consequently, a Padé-approximant expression that is based on a given partial sum can give much better precision as  $z \rightarrow 1$  than does the original partial sum. For the functions  $T_H(x, \mu)$  and  $\phi_V^\perp(x, \mu)$  that we consider, one can typically achieve much better than percent-level precision by keeping 20 terms in the partial sum and generating a  $10 \times 10$  Padé approximant.

In Appendix C 3, we have tested the Abel-Padé method for the cases  $\phi_V^\perp(x, \mu) \rightarrow \phi_V^\perp(x, \mu_0) \rightarrow \delta^{(k)}(x - \frac{1}{2})$ , with  $k = 0, 2, \dots, 10$ , and  $T_H(x, \mu) \rightarrow T_H^{(0)}(x, \mu_0)$ , i.e., with no evolution. Analytic results are easily obtained in these cases, and the Abel-Padé expression converges quickly to them, even though the eigenfunction series are not convergent for  $k > 0$ . As can be seen from the appendix of Ref. [7], evolution of  $\phi_V^{\perp(0)}(x, \mu_0)$  to a higher scale generally improves that convergence of the eigenfunction series. (This general property is confirmed numerically in Appendix C 3.) It seems, therefore, that the zero-evolution tests of the Abel-Padé method that we have made are particularly demanding. We have also tested the Abel-Padé method by expanding the LL evolved expression for  $c_2(\mu) = f_V^\perp(\mu) \mathcal{M}^{(0, v^2)}(\mu)$  as a series in  $\alpha_s$ , using the Abel-Padé method to compute the first three terms in the series from their eigenfunction expansions (taking  $\mu_0 = m_c, m_b$  and  $\mu = m_H$ ), and comparing the results with the analytic expressions for the first three terms in the series in Eq. (39b) of Ref. [7]. Again, the Abel-Padé expressions converge rapidly to the analytic results, even though the eigenfunction series themselves are not convergent.

We conclude that the Abel-Padé method is reliable, and we use it in this paper to sum all of the eigenvalue series for the LCDAs.

## V. COMPARISON WITH A MODEL LCDA

In Ref. [11], it was proposed to incorporate the effects of the order- $v^2$  and order- $\alpha_s$  corrections to the LCDA by making use of a model LCDA:

$$\phi_V^{\perp M}(x, \mu_0) = N_\sigma \frac{4x(1-x)}{\sqrt{2\pi}\sigma_V(\mu_0)} \exp\left[-\frac{(x-\frac{1}{2})^2}{2\sigma_V^2(\mu_0)}\right]. \quad (32)$$

Here,  $N_\sigma$  is chosen so that

$$\int_0^1 dx \phi_V^{\perp M}(x, \mu_0) = 1. \quad (33)$$

It is stated in Ref. [11] that the width parameter  $\sigma_V(\mu_0)$  is chosen so that  $\phi_V^{\perp M}(x, \mu_0)$  yields the second moment of  $\phi_V^\perp(x, \mu)$  through linear order in  $v^2$  and  $\alpha_s$ :

$$4\sigma_V^2(\mu_0) = \int_0^1 dx (2x-1)^2 \phi_V^{\perp M}(x, \mu_0) \\ \equiv \frac{\langle v^2 \rangle_V}{3} + \frac{C_F \alpha_s(\mu_0)}{4\pi} \left( \frac{28}{9} - \frac{2}{3} \ln \frac{m_Q^2}{\mu_0^2} \right). \quad (34)$$

The initial scale is chosen to be  $\mu_0 = 1$  GeV.

The model LCDA circumvents the problem of the non-convergence of the eigenfunction series for  $\mathcal{M}^{(0, v^2)}(\mu)$ : Because  $\phi_V^{\perp M}(x, \mu_0)$  is an ordinary function of  $x$ , the eigenfunction series converges. However, a number of assumptions go into the construction of the model LCDA. We now discuss the validity of those assumptions.

First, we note that the first equality in Eq. (34) holds only in the zero-width ( $\sigma_V \rightarrow 0$ ) limit. In Ref. [11], numerical values of  $\sigma_V(1 \text{ GeV})$  were computed by equating  $4\sigma_V^2$  to the expression on the right side of the second equality in Eq. (34). This procedure leads to values for the second  $x$  moments of  $\phi_V^{\perp M}(x, 1 \text{ GeV})$  that differ substantially from the true values of second  $x$  moments of  $\phi_V^\perp(x, 1 \text{ GeV})$  through linear order in  $v^2$  and  $\alpha_s$ . For example, in the case of the  $J/\psi$ , with  $m_c = 1.4$  GeV and  $\langle v^2 \rangle_{J/\psi} = 0.225$ , the second  $x$  moment of  $\phi_V^{\perp M}(x, 1 \text{ GeV})$  is 0.120256, while the second  $x$  moment of  $\phi_V^\perp(x, 1 \text{ GeV})$  through linear order in  $v^2$  and  $\alpha_s$  is 0.207729. In fact, in this case, there is no choice of  $\sigma_V(1 \text{ GeV})$  that yields the correct second  $x$  moment through linear order in  $v^2$  and  $\alpha_s$ .

Second, we note that only the second  $x$  moment of the order- $\alpha_s$  correction to the LCDA enters into the model LCDA. That is, there is an implicit assumption that the order- $\alpha_s$  correction can be adequately characterized by its second  $x$  moment alone. However, the order- $\alpha_s$  correction to the LCDA has substantial  $x$  moments beyond the second moment, and, so, this assumption seems to be questionable. In contrast, only the second  $x$  moment of the order- $v^2$  correction to the LCDA is nonvanishing.

Third, the functional form of the LCDA has implications for the higher  $x$  moments of the LCDA. These higher  $x$  moments are related to corrections to the LCDA of higher order in  $v^2$  (see Refs. [16–18] and Appendix C) and to higher  $x$  moments of the corrections to the LCDA of order  $\alpha_s$  and higher. It is not clear that the functional form of the LCDA accounts adequately for these corrections. In Appendix C 2, we examine  $x$  moments of the model LCDA in order  $\alpha_s^0$ , using the relationships between the  $x$  moments of the LCDA and the NRQCD LDMEs that are given in Refs. [16–18]. We find that  $x$  moments of the model LCDA are much larger than expectations from the NRQCD velocity-scaling rules, suggesting that the model LCDA leads to spuriously large corrections of higher order in  $v^2$ .<sup>6</sup>

<sup>6</sup>Strictly speaking, the velocity-scaling rules state that a LDME  $\langle v^n \rangle_V$ , which is defined by the obvious generalization of Eq. (6), vanishes as  $v^n$  in the limit  $v \rightarrow 0$ . However, in phenomenology, the velocity-scaling rules are usually taken to mean that  $\langle v^n \rangle_V$  is equal to  $v^n$  times a coefficient of order 1. This point of view is supported by the generalized Gremm-Kapustin relation [19].

The ultimate test of the model LCDA is whether it leads to an accurate numerical result for the light-cone amplitude. We will carry out such a test by comparing the results for the light-cone amplitude that are obtained from the model LCDA with the results for the light-cone amplitude that are obtained from our calculation through orders  $\alpha_s$  and  $v^2$ . In doing so, we are implicitly assuming that the expansions in the small parameters  $\alpha_s$  and  $v^2$  are valid and that corrections beyond those in orders  $\alpha_s$  and  $v^2$  are small in comparison with the corrections of orders  $\alpha_s$  and  $v^2$ . One could question whether the evolution from the scale  $\mu_0$  to the scale  $m_H$  could invalidate the  $\alpha_s$  and  $v^2$  expansions. Regarding the  $\alpha_s$  expansion, evolution from the scale 1.0 GeV to the scale  $m_H$  changes the order- $\alpha_s$  correction from 16% of the order- $\alpha_s^0$  contribution to 9% of the order- $\alpha_s^0$  contribution, suggesting that evolution does not spoil the  $\alpha_s$  expansion. Further tests of the  $\alpha_s$  expansion would require the computation of corrections of still higher orders in  $\alpha_s$ . We can investigate the convergence of the  $v^2$  expansion (nonrelativistic expansion) and the effects of evolution on it more completely, and we do so in Appendix C. There, we test the numerical convergence of the nonrelativistic expansion in order  $\alpha_s^0$  for the example of the model LCDA. We find that the nonrelativistic expansion converges rapidly to the exact result for the model LCDA at the scale  $\mu = \mu_0$  and that it converges even more rapidly at the scale  $\mu = m_H$ . The expansion through order  $v^2$  gives a good approximation to the exact result. We conclude that the model LCDA, if it is valid, should not produce corrections beyond the leading order in  $\alpha_s$  and  $v^2$  that deviate significantly from the sum of the corrections of order  $\alpha_s$  and order  $v^2$  that we compute in this paper.

We can assess whether the contributions of higher order that arise from the model LCDA  $\phi_V^{\perp M}(x, \mu)$  agree with the contributions of order  $v^2$  and order  $\alpha_s$  that we compute by examining the quantity

$$\Delta(\mu) = \frac{\alpha_s(\mu_0)}{4\pi} \mathcal{M}^{(0,1)}(\mu) + \langle v^2 \rangle_V \mathcal{M}^{(0,v^2)}(\mu), \quad (35)$$

where, in order to compare with  $\phi_V^{\perp M}(x, \mu)$ , we take  $\mu_0 = 1$  GeV in  $\alpha_s(\mu_0)$  and, implicitly, in  $\mathcal{M}^{(0,1)}(\mu)$  and  $\mathcal{M}^{(0,v^2)}(\mu)$ . The equivalent expression for the model LCDA  $\phi_V^{\perp M}(x, \mu)$ , is given, up to corrections of higher orders in  $\alpha_s$  and  $v^2$ , by

$$\Delta^M(\mu) = \int_0^1 dx T_H^{(0)}(x, \mu) [\phi_V^{\perp M}(x, \mu) - \phi_V^{\perp(0)}(x, \mu)]. \quad (36)$$

In Table I we compare the values of  $\Delta(\mu_0)$  and  $\Delta^M(\mu_0)$  for the  $J/\psi$  and  $\Upsilon(nS)$  states, using the values of the input parameters that are given in Ref. [11]. In the case of  $\Delta^M(\mu_0)$ , we also show the values that result from varying  $\sigma_V(\mu_0)$  by  $\pm 25\%$ , as was suggested in Ref. [11].

TABLE I. Numerical values of  $\Delta(\mu_0)$  and  $\Delta^M(\mu_0)$  for  $V = J/\psi$  and  $\Upsilon(nS)$  at  $\mu_0 = 1$  GeV. In the last two columns, we have evaluated  $\Delta^M(\mu_0)$  by replacing  $\sigma_V(\mu_0)$  by 0.75 and 1.25 times its nominal value, respectively.

$V$	$\Delta(\mu_0)$	$\Delta^M(\mu_0)$	$\Delta^M(\mu_0) _{\sigma_V \rightarrow 0.75\sigma_V}$	$\Delta^M(\mu_0) _{\sigma_V \rightarrow 1.25\sigma_V}$
$J/\psi$	0.971375	0.843339	0.510365	1.12087
$\Upsilon(1S)$	0.0770658	0.211269	0.116175	0.338490
$\Upsilon(2S)$	0.209066	0.359150	0.195740	0.563622
$\Upsilon(3S)$	0.295732	0.458135	0.250834	0.697510

As can be seen from Table I, the central value of  $\Delta^M(\mu_0)$  deviates from the value of  $\Delta(\mu_0)$  by  $-13\%$  for the  $J/\psi$ ,  $+174\%$  for the  $\Upsilon(1S)$ ,  $+72\%$  for the  $\Upsilon(2S)$ , and  $+55\%$  for the  $\Upsilon(3S)$ . We also see that the result is very sensitive to the choice of  $\sigma_V(\mu_0)$ : The values of  $\Delta^M(\mu_0)$  vary by factors of 2 or more as  $\sigma_V(\mu_0)$  is varied by  $\pm 25\%$ . [In contrast,  $\Delta(\mu_0)$  would vary by less than  $\pm 25\%$  if the input parameter  $\langle v^2 \rangle_V$  were varied by  $\pm 25\%$ .] Therefore, we regard the approximate agreement of the central value of  $\Delta^M(\mu_0)$  with the value of  $\Delta(\mu_0)$  for the case of the  $J/\psi$  as accidental.

In Table II we compare the values of  $\Delta(m_H)$  and  $\Delta^M(m_H)$  for the  $J/\psi$  and  $\Upsilon(nS)$  states, using the values of the input parameters at 1 GeV that are given in Ref. [11]. Again, in the case of  $\Delta^M(m_H)$ , we also show the values that result from varying  $\sigma_V(\mu_0)$  by  $\pm 25\%$ . We make use of the Abel-Padé method in carrying out the evolution of  $\mu$  from  $\mu_0 = 1$  GeV to  $m_H = 125.09$  GeV, taking 100 terms in the eigenfunction expansion and using a  $50 \times 50$  Padé approximant.

In Ref. [11], it was suggested that the evolution of the model LCDA to the scale  $\mu = m_H$  would reduce the dependence on the specifics of the model. As can be seen from Table II, the central value of  $\Delta^M(m_H)$  deviates from value of  $\Delta(m_H)$  by  $-24\%$  for the  $J/\psi$ ,  $+46\%$  for the  $\Upsilon(1S)$ ,  $+23\%$  for the  $\Upsilon(2S)$ , and  $+16\%$  for the  $\Upsilon(3S)$ . Comparison with Table I shows that, in the case of the  $J/\psi$ , the deviation of  $\Delta^M(m_H)$  from  $\Delta(m_H)$  actually increases as  $\mu$  is evolved from  $\mu_0 = 1$  GeV to  $m_H$ . While the deviations in the case of the  $\Upsilon(nS)$  states decrease as  $\mu$  is evolved from 1 GeV to  $m_H$ , they are still rather large, especially in the case of the  $\Upsilon(1S)$ . Furthermore, the results are very sensitive to the choice of  $\sigma_V(1 \text{ GeV})$ : The values of

TABLE II. Numerical values of  $\Delta(\mu)$  and  $\Delta^M(\mu)$  for  $V = J/\psi$  and  $\Upsilon(nS)$  at  $\mu = m_H$ . In the last two columns, we have evaluated  $\Delta^M(\mu)$  by replacing  $\sigma_V(\mu_0)$  by 0.75 and 1.25 times its nominal value, respectively.

$V$	$\Delta(\mu)$	$\Delta^M(\mu)$	$\Delta^M(\mu) _{\sigma_V \rightarrow 0.75\sigma_V}$	$\Delta^M(\mu) _{\sigma_V \rightarrow 1.25\sigma_V}$
$J/\psi$	0.684103	0.522962	0.337973	0.666378
$\Upsilon(1S)$	0.103008	0.150110	0.084148	0.233466
$\Upsilon(2S)$	0.200579	0.246479	0.139542	0.368862
$\Upsilon(3S)$	0.264641	0.307054	0.176647	0.444124



$\Delta^M(m_H)$  vary by factors of 2 or more as  $\sigma_V(1 \text{ GeV})$  is varied by  $\pm 25\%$ .

We would expect the uncalculated corrections of higher orders in  $\alpha_s$  and  $v^2$  to be of size  $\alpha_s$  or  $v^2$  relative to the corrections that we have calculated. We see that the model LCDA of Ref. [11] produces results that deviate from ours by amounts that are much larger than the expected sizes of these uncalculated corrections. Therefore, we conclude that the model LCDA of Ref. [11] does not lead to reliable results for contributions to the light-cone amplitude of the order- $\alpha_s$  and order- $v^2$  corrections to the LCDA. However, because the value of  $\Delta(m_H)$  is small in comparison with the

leading contribution to the leading light-cone amplitude  $\mathcal{M}^{(0,0)} = 4$ , the deviations of  $\Delta^M(m_H)$  from  $\Delta(m_H)$  affect the light-cone amplitude only at the level of about 4% for the  $J/\psi$  and at the level of about 1% for the  $\Upsilon(nS)$  states.

## VI. COMPUTATION OF THE DECAY RATES

### A. Direct amplitude

Our formula for the light-cone direct amplitude through order  $\alpha_s$ , with NLL resummation of logarithms of  $m_H^2/m_Q^2$ , is

$$i\mathcal{M}_{\text{dir}}^{\text{LC}}[H \rightarrow V + \gamma] = \frac{i}{2} e e_Q \kappa_Q \bar{m}_Q(\mu) (\sqrt{2} G_F)^{1/2} \left( -\epsilon_V^* \cdot \epsilon_\gamma^* + \frac{\epsilon_V^* \cdot p_\gamma p \cdot \epsilon_\gamma^*}{p_\gamma \cdot p} \right) \frac{f_V^\perp(m_H)}{f_V^\perp(\mu_0)} \frac{\sqrt{2N_c} \sqrt{2m_V}}{2m_Q} \Psi_V(0) \\ \times \left\{ \left[ 1 - \frac{5}{6} \langle v^2 \rangle_V + \frac{C_F \alpha_s(\mu_0)}{4\pi} \left( \log \frac{m_Q^2}{\mu_0^2} - 8 \right) \right] \mathcal{M}^{(0,0)}(\mu) \right. \\ \left. + \frac{\alpha_s(\mu)}{4\pi} \mathcal{M}^{(1,0)}(\mu) + \frac{\alpha_s(\mu_0)}{4\pi} \mathcal{M}^{(0,1)}(\mu) + \langle v^2 \rangle_V \mathcal{M}^{(0,v^2)}(\mu) \right\}, \quad (37)$$

where, in computing  $i\mathcal{M}_{\text{dir}}^{\text{LC}}[H \rightarrow V + \gamma]$ , we take  $e = \sqrt{4\pi\alpha(0)}$ .

We note that the formula (37) does not contain any cross terms of order  $\alpha_s^2$ ,  $\alpha_s v^2$ , or  $v^4$ . In contrast, the expressions in Ref. [11] do contain such cross terms because the expansions of  $T_H$  and the ratio  $f_V^\perp/f_V$  in powers of  $\alpha_s$  and  $\langle v^2 \rangle_V$  appear as factors in the expression that was used in Ref. [11] for the direct amplitude. On the other hand, our computation contains cross terms that arise from the ratio  $f_V^\perp/f_V$  that are not contained in the expression for  $f_V^\perp/f_V$  in Ref. [11]. That is because we use the values of the LDMEs that were extracted in Refs. [20,21] by making use of a formula for the quarkonium leptonic width that contains the expansion of the factor  $f_V$  in powers of  $\alpha_s$  and  $\langle v^2 \rangle_V$ . All of the cross terms that we have mentioned appear at orders that are beyond the claimed precision of our calculation or the calculation of Ref. [11]. In our calculation, they are taken into account in our estimates of uncertainties from uncalculated higher-order corrections.

In the evolution of the expression in Eq. (37), we choose the initial scale to be  $\mu_0 = m_Q$  and the final scale to be  $\mu = m_H$ . This choice incorporates the logarithms of  $m_H^2/m_Q^2$  into the evolved expressions. We will discuss the effect of using the choice of scale  $\mu_0 = 2m_Q$  in Sec. VII.

We note that, in Ref. [11], the initial scales were taken to be 1 GeV for the LCDAs and 2 GeV for the ratio of decay constants  $f_V^\perp/f_V$ . This latter choice is somewhat inconsistent with the use of values of  $\langle v^2 \rangle_V$  from Refs. [20,21], as they were extracted by making use of the expansion of  $f_V$  in powers of  $\alpha_s$  and  $\langle v^2 \rangle_V$ , with  $\alpha_s(\mu)$  evaluated at the scale  $m_V$ .

### B. Indirect amplitude

In computing the indirect amplitude, we follow Refs. [1,7], taking

$$i\mathcal{M}_{\text{ind}} = i\mathcal{A}_{\text{ind}} \left( -\epsilon_V^* \cdot \epsilon_\gamma^* + \frac{\epsilon_V^* \cdot p_\gamma p_V \cdot \epsilon_\gamma^*}{p_\gamma \cdot p_V} \right), \quad (38a)$$

where

$$\mathcal{A}_{\text{ind}} = \frac{g_{V\gamma} \sqrt{4\pi\alpha(m_V)} m_H}{m_V^2} \left[ 16\pi \frac{\alpha(m_V)}{\alpha(0)} \Gamma(H \rightarrow \gamma\gamma) \right]^{\frac{1}{2}}, \quad (38b)$$

and  $g_{V\gamma}$  is expressed in terms of the width of  $V$  into leptons [1]:

$$g_{V\gamma} = -\frac{e_Q}{|e_Q|} \left[ \frac{3m_V^3 \Gamma(V \rightarrow \ell^+ \ell^-)}{4\pi\alpha^2(m_V)} \right]^{\frac{1}{2}}. \quad (38c)$$

We obtain  $\Gamma(H \rightarrow \gamma\gamma)$  from the values of the Higgs-boson total width and branching fraction to  $\gamma\gamma$  in Refs. [2,3]. In the expression (38b) for  $\mathcal{A}_{\text{ind}}$ , we neglect a small phase that is about 0.005. As in Ref. [1], we have chosen the scales of the electromagnetic coupling as follows: we use  $\alpha(m_V)$  to compute  $g_{V\gamma}$  from the  $V$  leptonic width, we use  $e = \sqrt{4\pi\alpha(m_V)}$  for the couplings of the virtual photon, and we use  $e = \sqrt{4\pi\alpha(0)}$  for the coupling of the real photon. We have also compensated for the fact that  $\Gamma(H \rightarrow \gamma\gamma)$  was computed in Refs. [2,3] using  $e = \sqrt{4\pi\alpha(0)}$ .

In contrast with the calculations in Refs. [1,11], our calculation of  $\mathcal{A}_{\text{ind}}$  does not include contributions that are

TABLE III. Values of  $|\Psi_V(0)|^2$  in units of  $\text{GeV}^3$  and  $\langle v^2 \rangle_V$  for  $V = J/\psi$  and  $\Upsilon(nS)$ . These values have been taken from Refs. [20,21], except for the uncertainties in  $\langle v^2 \rangle_{\Upsilon(1S)}$  and  $\langle v^2 \rangle_{\Upsilon(2S)}$ , which are described in the text.

$V$	$ \Psi_V(0) ^2(\text{GeV}^3)$	$\langle v^2 \rangle_V$
$J/\psi$	$0.0729 \pm 0.0109$	$0.201 \pm 0.064$
$\Upsilon(1S)$	$0.512 \pm 0.035$	$-0.00920 \pm 0.0105$
$\Upsilon(2S)$	$0.271 \pm 0.019$	$0.0905 \pm 0.0109$
$\Upsilon(3S)$	$0.213 \pm 0.015$	$0.157 \pm 0.017$

suppressed as  $m_V^2$  divided by combinations of  $m_H^2$ ,  $m_t^2$ ,  $m_Z^2$ , or  $m_W^2$ , where  $m_t$ ,  $m_W$ , and  $m_Z$  are the masses of the top quark,  $W^\pm$  boson, and  $Z^0$  boson, respectively. Such contributions can arise not only from explicit  $m_V$  terms in the amplitude for  $H \rightarrow \gamma\gamma^*$ , but also from electroweak corrections to the amplitude for  $H \rightarrow V + \gamma$ . In the latter, it is not possible to distinguish between direct and indirect processes in a gauge-invariant way.

### C. Numerical inputs

We take the pole masses to be the one-loop values  $m_c = 1.483 \text{ GeV}$  and  $m_b = 4.580 \text{ GeV}$ , we take the  $\overline{\text{MS}}$  masses to be  $\bar{m}_c = 1.275 \text{ GeV}$  and  $\bar{m}_b = 4.18 \text{ GeV}$ , and we take  $m_H = 125.09 \pm 0.21(\text{stat}) \pm 0.11(\text{syst}) \text{ GeV}$ , which implies, from the tables in Refs. [2,3], that  $\Gamma(H \rightarrow \gamma\gamma) = (9.308 \pm 0.120) \times 10^{-6} \text{ GeV}$ . Here, we have included a 1% uncertainty from uncalculated higher-order terms in the theoretical expression, an uncertainty of 0.022% from the uncertainty in  $m_t$ , an uncertainty of 0.024% from the uncertainty in  $m_W$ , and an uncertainty of 0.82% from the uncertainty in  $m_H$ . Our values for  $|\Psi_V(0)|^2$  and  $\langle v^2 \rangle_V$  are shown in Table III. Following Ref. [1], we use the values from Refs. [20,21], except that we have increased the uncertainties in  $\langle v^2 \rangle_{\Upsilon(1S)}$  and  $\langle v^2 \rangle_{\Upsilon(2S)}$  from those in Ref. [21]. The uncertainty from uncalculated corrections of order  $v^4$  was estimated in Ref. [21] by multiplying the central value of  $\langle v^2 \rangle_{\Upsilon(nS)}$  by  $v^2$ , where  $v^2 = 0.1$  was used for the  $\Upsilon(nS)$  states. Because the central value of  $\langle v^2 \rangle_{\Upsilon(1S)}$  is anomalously small (much less than  $v^2$ ), owing to an accidental cancellation in the  $\overline{\text{MS}}$  subtraction scheme, the estimate of the uncalculated order- $v^4$  corrections in Ref. [21] considerably understates the uncertainty from this source. The uncertainty for  $\langle v^2 \rangle_{\Upsilon(2S)}$  was also slightly underestimated. Instead of using the estimates in Ref. [21], we take the uncertainties in  $\langle v^2 \rangle_{\Upsilon(1S)}$  and  $\langle v^2 \rangle_{\Upsilon(2S)}$  from uncalculated order- $v^4$  corrections to be  $v^4 = 0.01$ .

### D. Sources of uncertainties

In calculating the decay rates, we take into account uncertainties in both the direct and indirect amplitudes, as is

described below. In computing branching fractions, we also take into account the uncertainty in the total decay width of the Higgs boson [2,3].

### 1. Direct amplitude

In the direct amplitude, we include the uncertainties that arise from the uncertainties in  $\Psi_V(0)$  and the uncertainties in  $\langle v^2 \rangle_V$ . We also include the uncertainties that arise from uncalculated corrections of order  $\alpha_s^2$ , order  $\alpha_s v^2$ , and order  $v^4$ . We estimate the uncertainties from these uncalculated corrections, relative to the lowest nontrivial order in the direct amplitude, to be  $\{[C_F C_A \alpha_s^2(m_Q)/\pi^2]^2 + [C_F \alpha_s(m_Q) v^2/\pi]^2 + [v^4]^2\}^{1/2}$  for the real part of the direct amplitude and  $\{[C_A \alpha_s(m_Q)/\pi]^2 + [v^2]^2\}^{1/2}$  for the imaginary part of the direct amplitude. (Note that the real part of the direct amplitude starts in absolute order  $\alpha_s^0$  and the imaginary part of the direct amplitude starts in absolute order  $\alpha_s$ .) We take  $v^2 = 0.3$  for the  $J/\psi$  and  $v^2 = 0.1$  for the  $\Upsilon(nS)$  states.

In Ref. [11], it was suggested that the uncertainties in  $\Psi_V(0)$  and  $\langle v^2 \rangle_V$  were underestimated in Refs. [20,21]. We now address these issues.

One difficulty that was raised in Ref. [11] is that one-loop pole masses were used in Refs. [20,21] in the one-loop expression for  $\Gamma(V \rightarrow \ell^+ \ell^-)$ , which was used to compute  $\Psi_V(0)$ . The objection is that the pole mass is ill defined outside of perturbation theory and is subject to renormalon ambiguities. However, in Refs. [20,21], the pole mass was used in conjunction with one-loop corrections to  $\Gamma(V \rightarrow \ell^+ \ell^-)$  that are calculated using the pole mass. This is equivalent, up to corrections of higher order in  $\alpha_s$ , to the use of the  $\overline{\text{MS}}$  mass in conjunction with one-loop corrections to  $\Gamma(V \rightarrow \ell^+ \ell^-)$  that are calculated using the  $\overline{\text{MS}}$  mass. At one-loop order, the numerical difference between the two procedures is small.

Another difficulty that was raised in Ref. [11] is that the perturbation series for  $\Gamma(V \rightarrow \ell^+ \ell^-)$  has very large corrections at two-loop and three-loop orders [22–25]. The perturbation series was truncated at one-loop order in Refs. [20,21]. While an understanding of the large two-loop and three-loop corrections to  $\Gamma(V \rightarrow \ell^+ \ell^-)$  is still lacking, it should be noted that the analyses in Refs. [20,21] of the wave functions at the origin for the vector states  $V$  and the pseudoscalar states  $P$ , which make use of the one-loop expressions for  $\Gamma(V \rightarrow \ell^+ \ell^-)$  and  $\Gamma(P \rightarrow \gamma\gamma)$ , result in the same values for the corresponding  $V$  and  $P$  wave functions at the origin, up to differences whose numerical sizes are of order  $v^2$ , in agreement with NRQCD velocity scaling. This agreement was obtained in spite of the fact that both  $\Gamma(V \rightarrow \ell^+ \ell^-)$  and  $\Gamma(P \rightarrow \gamma\gamma)$  receive different large corrections in two-loop order [24], and it suggests that one-loop truncation is a reasonable procedure at the current level of precision.

In Ref. [11], the ratio  $f_V^\perp(\mu)/f_V$  appears, where the direct amplitude is proportional to  $f_V^\perp(\mu)$  and  $\Gamma(V \rightarrow \ell^+ \ell^-)$  is

proportional to  $f_V^2$ . The expression for this ratio through order  $\alpha_s$  (one-loop order) and through order  $v^2$  was used in Ref. [11], rather than the separate expressions for the numerator and the denominator. At the one-loop order, for which the perturbation series for the numerator and the denominator are separately well behaved, the use of the ratio confers no particular advantage. At the two-loop order, at which the perturbation series for  $\Gamma(V \rightarrow \ell^+ \ell^-) \propto f_V^2$  is badly behaved, the ratio could conceivably be better behaved than either the numerator or the denominator. However, this conjecture has not yet been validated, as the two-loop corrections to  $f_V^\pm(\mu)$  have yet to be calculated.

Finally, we mention that, even if we assume that the uncertainty in the perturbative expression for  $\Gamma(V \rightarrow \ell^+ \ell^-)$  is as large as 100% of the contribution of the one-loop term, the resulting uncertainty in  $\langle v^2 \rangle_V$  is comparable to that from other sources of uncertainty. If we repeat the analyses of Refs. [20,21], but allow the perturbative expression for  $\Gamma(V \rightarrow \ell^+ \ell^-)$  to vary by 100% of the contribution of the one-loop term, then the values for  $\langle v^2 \rangle_V$  deviate from the central value by a maximum of 88%, 143%, 62%, and 135% of the error bars in Table III for the  $J/\psi$ ,  $\Upsilon(1S)$ ,  $\Upsilon(2S)$ , and  $\Upsilon(3S)$ , respectively. Hence, the uncertainties in  $\langle v^2 \rangle_V$  that are given in Table III seem to be ample to take into account the uncertainties in the perturbative expression for  $\Gamma(V \rightarrow \ell^+ \ell^-)$ .

## 2. Indirect amplitude

In estimating the uncertainties in the indirect amplitude, we follow the method that is given in footnote 2 of Ref. [1]. As we have already mentioned, we include in  $\Gamma(H \rightarrow \gamma\gamma)$  the uncertainties that arise from uncalculated higher-order terms in the theoretical expression, the uncertainty in  $m_t$ , the uncertainty in  $m_W$ , and the uncertainty in  $m_H$ . We assume that the uncertainties in the leptonic decay widths are 2.5% for the  $J/\psi$ , 1.3% for the  $\Upsilon(1S)$ , and 1.8% for the  $\Upsilon(2S)$  and  $\Upsilon(3S)$  states. We take the relative uncertainty in the indirect amplitude from uncalculated mass corrections to be  $m_V^2/m_H^2$ .

## E. Method for computing uncertainties in the decay rates

Owing to cancellations between the direct and indirect amplitudes, small variations in those amplitudes can result in very nonlinear changes in  $\Gamma(H \rightarrow V + \gamma)$ . Hence, one cannot reliably estimate the total uncertainty in  $\Gamma(H \rightarrow V + \gamma)$  simply by adding the uncertainties from the individual sources in quadrature. Instead, we use the following method to estimate the total uncertainty in  $\Gamma(H \rightarrow V + \gamma)$ . We write  $\Gamma(H \rightarrow V + \gamma)$  as a function of the various uncertain input parameters and the normalizations of the direct and indirect amplitudes. Then, we find the global maximum and global minimum of  $\Gamma(H \rightarrow V + \gamma)$

in a region about the central values of the input parameters and normalizations that is constrained as

$$\sum_i \left| \frac{c_i - c_{i0}}{\Delta c_i} \right|^2 \leq 1, \quad (39)$$

where the  $c_i$  are the input parameters and normalizations, the  $c_{i0}$  are the central values of the  $c_i$ , and the  $\Delta c_i$  are the uncertainties in the  $c_i$ . We take the upper (lower) error bar on  $\Gamma(H \rightarrow V + \gamma)$  to be the global maximum (minimum) of  $\Gamma(H \rightarrow V + \gamma)$  minus the central value of  $\Gamma(H \rightarrow V + \gamma)$ .

## VII. RESULTS

Our results for the direct and indirect amplitudes are given in Table IV, where the evolution of the direct amplitudes has been computed by the Abel-Padé method, and we have retained 100 terms in the eigenvalues series and used  $50 \times 50$  Padé approximants.

We note that, had we made the choice of initial scale  $\mu_0 = 2m_Q$ , that would have shifted our results for the real parts of the direct amplitudes by +13%, +4%, +4%, and +4% for the  $J/\psi$ ,  $\Upsilon(1S)$ ,  $\Upsilon(2S)$ , and  $\Upsilon(3S)$ , respectively. These shifts are within our estimated uncertainties for the real parts of the direct amplitudes, which are 15%, 4%, 4%, and 4% for the  $J/\psi$ ,  $\Upsilon(1S)$ ,  $\Upsilon(2S)$ , and  $\Upsilon(3S)$ , respectively. The choice of initial scale  $\mu_0 = 2m_Q$  would have shifted our results for the imaginary parts of the direct amplitudes by +0.1% and -1.6% for the  $J/\psi$  and  $\Upsilon(nS)$  states, respectively. These shifts are well within our estimated uncertainties for the imaginary parts of the direct amplitudes.

The results in Ref. [7] for the real parts of the direct amplitudes are considerably larger than our results, by 66%, 20%, 22%, and 23% for the  $J/\psi$ ,  $\Upsilon(1S)$ ,  $\Upsilon(2S)$ , and  $\Upsilon(3S)$ , respectively. These differences are due, primarily, to the use of LL evolution, rather than NLL evolution, for  $\bar{m}(\mu)$  and  $f_V^\pm(\mu)$  in Ref. [7]. The differences are larger than the values that one obtains simply by considering the generic size of a next-to-leading logarithm, namely,  $[\alpha_s(m_Q)/\pi]^2 \log(m_H^2/m_Q^2)$ . In the case of  $\phi_V^\pm(x, \mu)$ , the use of NLL evolution, rather than LL evolution, changes the direct amplitude by about 0.12% for the  $J/\psi$  and about 0.16%–0.17% for the  $\Upsilon(nS)$  states. These changes are negligible in comparison with the uncertainties in the direct

TABLE IV. Values of the parameters  $\alpha_V$  and  $\beta_V$  in  $\Gamma(H \rightarrow V + \gamma) = |\alpha_V - \beta_V \kappa_Q|^2 \times 10^{-10}$  GeV for  $V = J/\psi$  and  $\Upsilon(nS)$ .

$V$	$\alpha_V$	$\beta_V$
$J/\psi$	$11.71 \pm 0.16$	$(0.627_{-0.094}^{+0.092}) + (0.118_{-0.054}^{+0.054})i$
$\Upsilon(1S)$	$3.283 \pm 0.035$	$(2.908_{-0.124}^{+0.122}) + (0.391_{-0.092}^{+0.092})i$
$\Upsilon(2S)$	$2.155 \pm 0.028$	$(2.036_{-0.089}^{+0.087}) + (0.293_{-0.069}^{+0.069})i$
$\Upsilon(3S)$	$1.803 \pm 0.023$	$(1.749_{-0.078}^{+0.077}) + (0.264_{-0.062}^{+0.062})i$



amplitudes. The use of the Abel-Padé method to sum the logarithms of  $c_2(\mu) = f_V^\perp(\mu)\mathcal{M}^{(0,v^2)}(\mu)$  to all orders in  $\alpha_s$ , rather than through order  $\alpha_s^2$ , as in Ref. [7], amounts to about a 10% change in the case of the  $J/\psi$  and to about a 4% change in the case of the  $\Upsilon(nS)$  states. Since the corrections to the direct amplitude that arise from  $c_2(\mu)$  are about 4% in the case of the  $J/\psi$  and about 3% in the case of the  $\Upsilon(nS)$  states, the changes to the direct amplitude that result from the use of the Abel-Padé method are negligible in comparison to the uncertainties.

The results in Ref. [11] for the ratio of the real part of the direct amplitude to the indirect amplitude are slightly larger than our results for that ratio, by 17%, 7%, 7%, and 8.5% for the  $J/\psi$ ,  $\Upsilon(1S)$ ,  $\Upsilon(2S)$ , and  $\Upsilon(3S)$ , respectively. These differences are somewhat larger than our relative uncertainties in the real parts of the direct amplitudes, and they are also larger than the uncertainties that are given in Ref. [11] for the ratio of the real part of the direct amplitude to the indirect amplitude.

The results in Ref. [11] for the ratio of the imaginary part of the direct amplitude to the indirect amplitude differ from our results for that ratio by  $-12\%$ ,  $9\%$ ,  $4\%$ , and  $1\%$  for the  $J/\psi$ ,  $\Upsilon(1S)$ ,  $\Upsilon(2S)$ , and  $\Upsilon(3S)$ , respectively. These differences are well within our relative uncertainties for the imaginary parts of the direct amplitudes.

As we have already mentioned, there are several possible sources of these differences between our results for the direct amplitudes and those of Ref. [11]. (1) Our initial scales for the evolution of  $f_V^\perp(\mu)$  and the LCDAs are different from those in Ref. [11]. (2) Our formula for the direct amplitude (37) treats cross terms of order  $\alpha_s^2$ ,  $\alpha_s v^2$ , and  $v^4$  differently than does the corresponding formula in Ref. [11]. (3) Our treatment of the order  $\alpha_s$  and order  $v^2$  corrections to the LCDA is different from the model-LCDA treatment of Ref. [11].

Our results for the SM decay rates and branching fractions ( $\kappa_Q = 1$ ) are given in Table V. In computing the uncertainties in the branching fractions, we have included the effect of the uncertainty in the Higgs-boson total width.

Our results for the SM decay rates agree with those in Ref. [7], within the uncertainties that are given in Ref. [7], except in the case of the  $\Upsilon(1S)$ . In this case, the real parts of the SM direct and indirect amplitudes nearly cancel, and so,

TABLE V. SM values of  $\Gamma(H \rightarrow V + \gamma)$  in units of GeV and  $\text{Br}(H \rightarrow V + \gamma)$  for  $V = J/\psi$  and  $\Upsilon(nS)$ .

$V$	$\Gamma(H \rightarrow V + \gamma)$ (GeV)	$\text{Br}(H \rightarrow V + \gamma)$
$J/\psi$	$1.228^{+0.042}_{-0.042} \times 10^{-8}$	$3.01^{+0.16}_{-0.15} \times 10^{-6}$
$\Upsilon(1S)$	$2.94^{+1.25}_{-1.02} \times 10^{-11}$	$7.19^{+3.07}_{-2.52} \times 10^{-9}$
$\Upsilon(2S)$	$1.00^{+0.48}_{-0.39} \times 10^{-11}$	$2.45^{+1.18}_{-0.96} \times 10^{-9}$
$\Upsilon(3S)$	$7.27^{+3.67}_{-2.93} \times 10^{-12}$	$1.78^{+0.90}_{-0.72} \times 10^{-9}$

as was pointed out in Ref. [11], the inclusion of the imaginary part of the direct amplitude results in a significant increase in the rate.

Our results for the SM branching fractions agree with those in Ref. [11], within our uncertainties. Note that our estimated uncertainties in the branching fractions are comparable to those of Ref. [11], except in the case of the  $\Upsilon(1S)$ , for which our uncertainty is considerably larger. Since, in the  $\Upsilon(1S)$  case, our uncertainty in the ratio of the direct amplitude to the indirect amplitude is essentially the same as Ref. [11], we suspect that the difference between the uncertainty estimates arises because of the highly nonlinear dependences of the decay rate on the input parameters. (See Sec. VI E.)

## VIII. SUMMARY AND DISCUSSION

In this paper, we have presented new calculations of Higgs-boson decay rates to vector heavy-quarkonium states plus a photon, where we have considered the vector quarkonium states  $J/\psi$  and  $\Upsilon(nS)$ , with  $n = 1, 2$ , or  $3$ . As was pointed out in Ref. [1], these decay rates, when compared with data from a high-luminosity LHC run, can provide information about the  $Hc\bar{c}$  and  $Hb\bar{b}$  couplings. Our calculation is carried out in the light-cone formalism in which the nonperturbative parts of the quarkonium LCDAs are expressed in terms of NRQCD long-distance matrix elements [10]. Our calculations of the direct decay amplitudes take into account corrections through order  $\alpha_s$  and order  $v^2$  and include resummations of logarithms of  $m_H^2/m_Q^2$  to all orders in  $\alpha_s$  through order  $v^2$  at NLL accuracy.

In order to resum logarithms that are associated with the quarkonium LCDAs, we have devised a new method, called the Abel-Padé method, which makes use of Abel summation, accelerated through the use of Padé approximants. The new method allows us to compute formally divergent sums over the eigenfunctions of the LO evolution kernels. These divergences arise because the LCDAs at initial scale of the evolution are generalized functions (distributions) of the light-cone fractions, rather than ordinary functions. The Abel-Padé method defines these distributions as sequences of ordinary functions and, hence, gives finite and unambiguous results for the formally divergent sums. We have tested this method numerically against known analytic results for the LCDAs, and we find that it converges quickly and reliably to the values from analytic calculations. It solves the general problem of carrying out the scale evolution in a nonrelativistic expansion of the LCDA for heavy-quarkonium systems, and it should be applicable in other situations in which series of orthogonal polynomials fail to converge when they are used to represent generalized functions. Using the Abel-Padé method, we were able to make definitive calculations of the LCDA-evolution effects in Higgs-boson decays to a quarkonium plus a photon.



We have compared the Abel-Padé method with the approach of Ref. [11], in which a model LCDA is used to take into account relativistic and QCD corrections to the LCDA. In contrast with the model approach, the Abel-Padé method makes use only of the calculated nonrelativistic corrections [7] and QCD corrections [10], and does not introduce any new model assumptions. We find that the model of Ref. [11] gives results that disagree substantially with those from the Abel-Padé method and that the model results are very sensitive to the choices of model parameters. It turns out that the relativistic and QCD corrections to the LCDA have only small effects on the direct decay amplitude, and so the large differences between the model and Abel-Padé calculations of the relativistic and QCD corrections to the LCDA have only small effects on the decay rates.

Our results for the ratios of the direct decay amplitudes to the indirect decay amplitudes are in reasonable agreement with those in Ref. [11]. Since the indirect decay amplitude can be determined quite precisely, this implies that our direct decay amplitudes are in reasonable agreement with those in Ref. [11]. Our results for the real parts of the direct decay amplitudes are considerably smaller than those in Ref. [7], owing to the use in Ref. [7] of LL resummation, rather than NLL resummation, of the logarithms of  $m_H^2/m_Q^2$ . Our result implies that the sensitivities of the decay rates to the  $HQ\bar{Q}$  couplings are considerably smaller than the sensitivities that were suggested in Ref. [7], especially in the case of the  $J/\psi$ .

Our results for the SM decay rates are in good agreement with those of Ref. [7], except in the case of the  $\Upsilon(1S)$ . As was pointed out in Ref. [11], it is important to include the imaginary part of the direct amplitude in the case of the decay to  $\Upsilon(1S)$  because there is an almost exact cancellation between the real parts of the direct and indirect amplitudes. The inclusion of the imaginary part of the direct amplitude in our calculation increases the decay rate in the  $\Upsilon(1S)$  case substantially in comparison to the rate that is given in Ref. [7].

The branching fractions that we find are in good agreement with those in Ref. [11]. Our uncertainty estimate in the case of the  $\Upsilon(1S)$  differs from that in Ref. [11], possibly owing to the highly nonlinear dependence of the rate on the input parameters. In Sec. VI E, we have presented a novel method for estimating the uncertainties in the presence of such nonlinearities.

In the calculations that we have described, there is one important theoretical issue that remains unresolved. The direct amplitude is proportional to the quarkonium wave function at the origin. The wave function at the origin is usually determined by comparing the theoretical expression for the quarkonium decay rate to leptons with the measured rate. In Refs. [7,11], and in the present work, the one-loop expression for the decay rate was used. Two- and three-loop expressions exist [23–25], but the higher-loop corrections

apparently destroy the convergence of the perturbation series. As we have mentioned, the one-loop analyses in Refs. [20,21] result in values for the corresponding vector and pseudoscalar wave functions at the origin that agree, up to differences whose numerical sizes are of relative order  $v^2$ . This agreement, which is predicted by the NRQCD velocity-scaling rules, is obtained in spite of the fact that the two-loop corrections to the vector decays to leptons and the pseudoscalar decays to two photons are large and different in relative size. The agreement suggests that the one-loop truncations of the perturbation series may lead to reasonable results for the wave functions at the origin at a level of precision of order  $v^2$ .

In Ref. [11], the ratio of decay constants  $f_V^\perp/f_V$  appears. The direct  $H \rightarrow V + \gamma$  amplitude is proportional to  $f_V^\perp$ , and the leptonic width of the vector quarkonium is proportional to  $f_V^2$ . This ratio is evaluated through order  $\alpha_s$  (one-loop order) and order  $v^2$ . Hence, the calculation in Ref. [11] also truncates the perturbation series for the leptonic width at one-loop level. It is conceivable that the ratio  $f_V^\perp/f_V$  is better behaved than either the numerator or the denominator. A calculation of two-loop QCD corrections to  $f_V^\perp$  would help to test this conjecture.

Higgs-boson decays to a vector quarkonium plus a photon provide important opportunities to measure the  $HQ\bar{Q}$  couplings at the LHC and are the only known processes that can provide phase information about those couplings. In order to take advantage of these opportunities to determine the  $HQ\bar{Q}$  couplings, it is essential to have the theoretical calculations of the decay rates under good control. In this paper, we have addressed the issue of the divergences that appear when one uses conventional eigenfunction-expansion methods to resum the logarithms of  $m_H^2/m_Q^2$  that appear in the nonrelativistic expansions of the quarkonium light-cone distribution amplitudes. With the resolution of this issue, we believe that, aside from the matter of the determination of quarkonium wave functions at the origin that we have mentioned above, calculations of the rates for Higgs-boson decays to vector quarkonia plus a photon are now on a sound theoretical footing.

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### APPENDIX A: EVOLUTION OF THE RUNNING MASS AND DECAY CONSTANT

Here, we collect formulas at NLL accuracy for the evolution of the running  $\overline{\text{MS}}$  mass  $\bar{m}(\mu)$  [26] and the decay constant  $f_V^\perp(\mu)$  [27]:

$$\frac{\bar{m}(\mu)}{\bar{m}(\mu_0)} = \left[ \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right]^{-\gamma_0^m/(2\beta_0)} \times \left[ 1 - \frac{\gamma_1^m \beta_0 - \beta_1 \gamma_0^m}{2\beta_0^2} \frac{\alpha_s(\mu) - \alpha_s(\mu_0)}{4\pi} + \dots \right], \quad (\text{A1a})$$

$$\frac{f_V^\perp(\mu)}{f_V^\perp(\mu_0)} = \left[ \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right]^{+\gamma_0^T/(2\beta_0)} \times \left[ 1 + \frac{\gamma_1^T \beta_0 - \beta_1 \gamma_0^T}{2\beta_0^2} \frac{\alpha_s(\mu) - \alpha_s(\mu_0)}{4\pi} + \dots \right], \quad (\text{A1b})$$

where

$$\gamma_0^m = -6C_F, \quad \gamma_1^m = -3C_F^2 - \frac{97}{3}C_F C_A + \frac{20}{3}C_F T_F n_f, \quad (\text{A2a})$$

$$\gamma_0^T = 2C_F, \quad \gamma_1^T = -19C_F^2 + \frac{257}{9}C_F C_A - \frac{52}{9}C_F T_F n_f. \quad (\text{A2b})$$

Here,  $\beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f$  is the one-loop coefficient of the QCD beta function,  $\beta_1 = \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_F n_f - 4C_F T_F n_f$  is the two-loop coefficient of the QCD beta function,  $C_F = (N_c^2 - 1)/(2N_c)$ ,  $C_A = 3$ ,  $N_c = 3$  is the number of colors,  $T_F = 1/2$ , and  $n_f$  is the number of active quark flavors.

### APPENDIX B: EVOLUTION MATRIX

At NLL accuracy, the evolution matrix  $U_{nk}(\mu, \mu_0)$  is given by [13]

$$U_{nk}(\mu, \mu_0) = \begin{cases} E_n^{\text{NLO}}(\mu, \mu_0), & \text{if } k = n, \\ \frac{\alpha_s(\mu)}{4\pi} E_n^{\text{LO}}(\mu, \mu_0) d_{nk}(\mu, \mu_0), & \text{if } k < n, \end{cases} \quad (\text{B1})$$

where

$$E_n^{\text{LO}}(\mu, \mu_0) = \left[ \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right]^{\frac{\gamma_n^{\perp(0)}}{2\beta_0}}, \quad (\text{B2a})$$

$$E_n^{\text{NLO}}(\mu, \mu_0) = E_n^{\text{LO}}(\mu, \mu_0) \times \left[ 1 + \frac{\alpha_s(\mu) - \alpha_s(\mu_0)}{4\pi} \frac{\gamma_n^{\perp(1)} \beta_0 - \gamma_n^{\perp(0)} \beta_1}{2\beta_0^2} \right], \quad (\text{B2b})$$

$$d_{nk}(\mu, \mu_0) = \frac{M_{nk}}{\gamma_n^{\perp(0)} - \gamma_k^{\perp(0)} - 2\beta_0} \left\{ 1 - \left[ \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right]^{\frac{\gamma_n^{\perp(0)} - \gamma_k^{\perp(0)} - 2\beta_0}{2\beta_0}} \right\}, \quad (\text{B2c})$$

$$M_{nk} = \frac{(k+1)(k+2)(2n+3)}{(n+1)(n+2)} (\gamma_n^{\perp(0)} - \gamma_k^{\perp(0)}) \times \left[ \frac{8C_F A_{nk} - \gamma_k^{\perp(0)} - 2\beta_0}{(n-k)(n+k+3)} + 4C_F \frac{A_{nk} - \psi(n+2) + \psi(1)}{(k+1)(k+2)} \right], \quad (\text{B2d})$$

$$A_{nk} = \psi\left(\frac{n+k+4}{2}\right) - \psi\left(\frac{n-k}{2}\right) + 2\psi(n-k) - \psi(n+2) - \psi(1). \quad (\text{B2e})$$

Here  $\psi(n)$  is the digamma function. The LO and NLO anomalous dimensions,  $\gamma_n^{\perp(0)}$  and  $\gamma_n^{\perp(1)}$ , respectively, are given by

$$\gamma_n^{\perp(0)} = \gamma_n^{(0)} - \gamma_0^T, \quad (\text{B3a})$$

$$\gamma_n^{\perp(1)} = \gamma_n^{(1)} - \gamma_1^T, \quad (\text{B3b})$$

where, from Refs. [4,28], we have

$$\gamma_n^{(0)} = 8C_F(H_{n+1} - 3/4), \quad (\text{B4})$$

and, from Refs. [29,30], we have

$$\begin{aligned}
\gamma_n^{(1)} \equiv & 4C_F^2 \left[ H_{n+1}^{(2)} - 2H_{n+1} - \frac{1}{4} \right] + C_F C_A \left[ -16H_{n+1}H_{n+1}^{(2)} - \frac{58}{3}H_{n+1}^{(2)} + \frac{572}{9}H_{n+1} - \frac{20}{3} \right] \\
& - 8 \left( C_F^2 - \frac{1}{2}C_F C_A \right) \left[ 4H_{n+1} \left( S_{(n+1)/2}'^{(2)} - H_{n+1}^{(2)} - \frac{1}{4} \right) - 8\tilde{S}_{n+1} + S_{(n+1)/2}'^{(3)} - \frac{5}{2}H_{n+1}^{(2)} \right] \\
& + \frac{1 + (-1)^n}{(n+1)(n+2)} + \frac{1}{4} + \frac{32}{9}C_F \frac{n_f}{2} \left[ 3H_{n+1}^{(2)} - 5H_{n+1} + \frac{3}{8} \right], \tag{B5a}
\end{aligned}$$

where

$$H_n^{(k)} \equiv \sum_{j=1}^n \frac{1}{j^k}, \quad \text{with } H_n^{(1)} \equiv H_n, \tag{B5b}$$

$$S_{n/2}'^{(k)} \equiv \begin{cases} H_{n/2}^{(k)}, & \text{if } n \text{ is even,} \\ H_{(n-1)/2}^{(k)}, & \text{if } n \text{ is odd,} \end{cases} \tag{B5c}$$

$$\tilde{S}_n \equiv \sum_{j=1}^n \frac{(-1)^j}{j^2} H_j. \tag{B5d}$$

Here, the  $H_n^{(k)}$  are the generalized harmonic numbers. Note that the off-diagonal matrix elements, which are proportional to  $d_{nk}(\mu, \mu_0)$ , are nonvanishing only for even  $n - k$  [13,14]. One can obtain  $U_{nk}(\mu, \mu_0)$  at LL accuracy by replacing  $E_n^{\text{NLO}}(\mu, \mu_0)$  in Eq. (B1) with  $E_n^{\text{LO}}(\mu, \mu_0)$  and setting the off-diagonal terms to zero.

## APPENDIX C: NONRELATIVISTIC EXPANSION

In this appendix we discuss the nonrelativistic expansion of the light-cone amplitude in order  $\alpha_s^0$  and investigate the convergence of that expansion numerically.

### 1. Formulation of the expansion

In Ref. [7], a formal expansion of the LCDA was given. Making the change of light-cone variables  $x \rightarrow 2x - 1$ , we write that expansion as

$$\phi_V^\perp(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \langle x^k \rangle}{2^k k!} \delta^{(k)} \left( x - \frac{1}{2} \right), \tag{C1}$$

where the normalization condition is

$$\int_0^1 dx \phi_V^\perp(x) = 1. \tag{C2}$$

Here,  $\langle x^k \rangle$  is defined by

$$\langle x^k \rangle = 2^k \int_0^1 dx \left( x - \frac{1}{2} \right)^k \phi_V^\perp(x). \tag{C3}$$

As we will see in Appendix C 2, the  $k$ th  $x$  moment in Eq. (C3) is proportional, in order  $\alpha_s^0$ , to the NRQCD LDME

$\langle v^k \rangle$ . Hence, the expansion in Eq. (C1) is the nonrelativistic expansion of the LCDA in order  $\alpha_s^0$ . In the following discussions, we will assume that  $\phi_V^\perp(x)$  is even under the replacement  $x \leftrightarrow 1 - x$  (charge-conjugation parity), in which case, only the moments  $\langle x^k \rangle$  with  $k$  even are nonvanishing.

The meaning of this formal expansion is that, if one integrates  $\phi_V^\perp(x)$  against a test function  $f(x)$ , then that integral is replaced by the sum of the integrals of  $\phi_V^\perp(x)$  against each term in the Taylor expansion of  $f(x)$ ,

$$\begin{aligned}
\int_0^1 dx f(x) \phi_V^\perp(x) &= \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \frac{d^k}{dx^k} f(x) \right] \Big|_{x=1/2} \\
&\times \int_0^1 dx (x - 1/2)^k \phi_V^\perp(x) \\
&= \sum_{k=0}^{\infty} f^{(k)} \langle x^k \rangle, \tag{C4a}
\end{aligned}$$

where

$$f^{(k)} = \frac{1}{2^k k!} \left[ \frac{d^k}{dx^k} f(x) \right] \Big|_{x=1/2}. \tag{C4b}$$

In our case, we wish to compute the light-cone amplitude

$$\mathcal{M}^{(0)}(\mu) = \int_0^1 dx T_H(x, \mu) \phi_V^\perp(x, \mu), \tag{C5}$$

where the superscript (0) denotes order  $\alpha_s^0$ .  $\mathcal{M}^{(0)}(\mu)$  has the nonrelativistic expansion

$$\mathcal{M}^{(0)}(\mu) = \sum_{k=0}^{\infty} \mathcal{M}^{(0, v^{2k})}(\mu), \tag{C6}$$

where

$$\mathcal{M}^{(0, v^{2k})}(\mu) = f^{(2k)} \langle x^{2k} \rangle, \tag{C7}$$

and we make the identification

$$f(x) = \sum_{n=0}^{\infty} \sum_{m=0}^n T_n(\mu) U_{nm}(\mu, \mu_0) N_m C_m^{(3/2)}(2x - 1). \tag{C8}$$

We compute the derivatives of this quantity by making use of the Abel summation in Eq. (31). That is, we compute

$$f^{(2k)} = \lim_{z \rightarrow 1} \sum_{n=0}^{\infty} \sum_{m=0}^n z^n T_n(\mu) U_{nm}(\mu, \mu_0) N_m \times \frac{1}{2^{2k} (2k)!} \frac{d^{2k}}{dx^{2k}} C_m^{(3/2)}(2x-1) \Big|_{x=1/2} \quad (\text{C9})$$

and we accelerate the convergence of the sum of  $m$  by making use of Padé approximants, as we have described earlier.

Making use of the identities

$$\frac{d}{dx} C_n^{\lambda/2}(x) = \lambda C_{n-1}^{(\lambda+2)/2}(x) \quad (\text{C10a})$$

and

$$C_{2n}^{\lambda/2}(0) = \frac{(-1)^n (\lambda + 2n - 2)!!}{(2n)!! (\lambda - 2)!!}, \quad (\text{C10b})$$

we obtain a convenient expression for the even derivatives of the even-order Gegenbauer polynomials:

$$\frac{d^{2k}}{dx^{2k}} C_{2n}^{(3/2)}(2x-1) \Big|_{x=1/2} = (-1)^{n-k} 2^{2k} \frac{(2n+2k+1)!!}{(2n-2k)!!}. \quad (\text{C11})$$

## 2. Sizes of the nonrelativistic moments

In order  $\alpha_s^0$ , the  $x$  moments of the LCDA [Eq. (C3)] have the following relationships to the NRQCD LDMEs [16–18]:

$$\langle x^{2k} \rangle = \frac{\langle v^{2k} \rangle}{2k+1}. \quad (\text{C12})$$

As we have mentioned in footnote 6, the NRQCD velocity-scaling rules, in their strictest sense, state that  $\langle v^n \rangle_V$  vanishes as  $v^n$  in the limit  $v \rightarrow 0$ . However, in phenomenology, the velocity-scaling rules are usually taken to mean that

$$\langle v^{2k} \rangle \sim \langle v^2 \rangle^k, \quad (\text{C13})$$

where  $\sim$  means equal up to a coefficient of order 1. These approximate sizes of the LDMEs are consistent with the generalized Gremm-Kapustin relation [19].

Now let us consider the  $x$  moments of the model LCDA in Eq. (32), which we denote by  $\langle x^k \rangle_M$ . We compute  $\sigma_{J/\psi}(\mu_0)$  using Eq. (34), but we drop the order- $\alpha_s$  term so as to obtain the behavior at order  $\alpha_s^0$ . Then, using  $\langle v^2 \rangle_{J/\psi} = 0.201$  we obtain  $\sigma_{J/\psi} = 0.129422$ . The first several  $x$  moments are then

$$\begin{aligned} \langle x^0 \rangle_M &= 1, \\ \langle x^2 \rangle_M &= 0.0573955, \\ \langle x^4 \rangle_M &= 0.00962303, \\ \langle x^6 \rangle_M &= 0.00259973, \\ \langle x^8 \rangle_M &= 0.000943655, \\ \langle x^{10} \rangle_M &= 0.000419855. \end{aligned} \quad (\text{C14})$$

On the other hand, from the relationship between the  $x$  moments and the LDMEs at order  $\alpha_s^0$  [Eq. (C12)] and the NRQCD velocity-scaling rules [Eq. (C13)], we expect that

$$\begin{aligned} \langle x^0 \rangle &= 1, \\ \langle x^2 \rangle &= 0.0573955, \\ \langle x^4 \rangle &\sim 0.00592964, \\ \langle x^6 \rangle &\sim 0.000729288, \\ \langle x^8 \rangle &\sim 0.0000976684, \\ \langle x^{10} \rangle &\sim 0.0000137595. \end{aligned} \quad (\text{C15})$$

The expression for  $\langle x^{2k} \rangle_M$  in the limit  $\sigma_V \rightarrow 0$  is given by

$$\langle x^{2k} \rangle_M = (2\sigma_V)^{2k} (2k-1)!! [1 + O(\sigma_V^2)]. \quad (\text{C16})$$

Hence, the model LCDA satisfies the NRQCD velocity-scaling rules in the strict sense that the  $2k$ th moment vanishes as the  $k$ th power of a quantity that could be interpreted as the square of the velocity. However, we see from Eq. (C14) that the first several  $x$  moments of the model LCDA badly violate the broader expectation that the LDMEs satisfy the relationship in Eq. (C15).

The crucial issue for the convergence of the velocity expansion is the behavior of the  $2k$ th  $x$  moment of the LCDA in the limit  $k \rightarrow \infty$  for fixed  $\sigma_V$ . We can derive an asymptotic expansion for the  $x$  moments of the model LCDA by integrating the definition in Eq. (C3) twice by parts. The result for even moments is

$$\begin{aligned} \langle x^{2k} \rangle_M &= \frac{[\frac{\partial}{\partial x} \phi_V^{LM}(x)]|_{x=0} - [\frac{\partial}{\partial x} \phi_V^{LM}(x)]|_{x=1}}{4(2k+1)(2k+2)} + O[1/(2k)^3] \\ &= N_\sigma \frac{\sqrt{2/\pi} e^{-1/(8\sigma_V^2)}}{\sigma_V (2k+1)(2k+2)} + O[1/(2k)^3]. \end{aligned} \quad (\text{C17})$$

Hence, we see that the  $2k$ th moment falls as  $1/k^2$  in the limit  $k \rightarrow \infty$ , while we expect, from Eqs. (C12) and (C15), that the  $2k$ th moment should fall faster than  $v^{2k}$ . Nevertheless, Eq. (C17) shows that the nonrelativistic expansion converges for the model LCDA, in the absence of evolution, provided that



$$T_H^{(2k)} \equiv \frac{1}{2^{2k}(2k)!} \left[ \frac{d^{2k}}{dx^{2k}} T_H(x) \right] \Big|_{x=1/2} \quad (\text{C18})$$

grows more slowly than a power of  $k$ .<sup>7</sup>

We record here the values for  $\langle x^{2k} \rangle$  that we obtain by retaining both the order- $\alpha_s$  term and the order- $v^2$  term in Eq. (34), which corresponds to taking  $\sigma_{J/\psi} = 0.228$ ,

$$\begin{aligned} \langle x^0 \rangle_M &= 1, \\ \langle x^2 \rangle_M &= 0.120256, \\ \langle x^4 \rangle_M &= 0.0373473, \\ \langle x^6 \rangle_M &= 0.0166916, \\ \langle x^8 \rangle_M &= 0.00909954, \\ \langle x^{10} \rangle_M &= 0.00561735. \end{aligned} \quad (\text{C19})$$

These  $x$  moments, of course, lead to a slower convergence of the nonrelativistic expansion than those for the case  $\sigma_{J/\psi} = 0.129422$ .

### 3. Numerical tests of the convergence of the nonrelativistic expansion

Now let us test numerically the convergence of the nonrelativistic expansion of the light-cone amplitude in order  $\alpha_s^0$ , which is given in Eq. (C6). We do this by comparing the numerical results from the nonrelativistic expansion of the light-cone amplitude with the numerical results that are obtained by computing the light-cone amplitude directly from a model LCDA. For this purpose, we make use of the model LCDA in Eq. (32). As we have pointed out, the  $x$  moments of this model LCDA decrease much more slowly with increasing moment number than would be expected from the NRQCD velocity-scaling rules. Therefore, we expect the nonrelativistic expansion to converge more slowly for this model LCDA than for a more realistic LCDA. However, as we will see, even for this model LCDA, the convergence of the nonrelativistic expansion is quite rapid.

#### a. Without evolution

We first take the case of no evolution, i.e.,  $\mu = \mu_0$ . We consider  $T_H(\mu)$  at leading order in  $\alpha_s$ . Then,  $f(x) = T_H^{(0)}$ , and we can compute  $f^{(2k)}$  analytically from Eq. (C4b), with the result

$$\mathcal{M}^{(0,v^{2k})}(\mu_0) = 4\langle x^{2k} \rangle \quad (\text{C20})$$

for all  $k$ . We note that we can also compute the  $f^{(2k)}$  in Eq. (C20) by making use of the Abel summation in Eq. (C9). If we accelerate the convergence of the sum

over  $m$  by employing a  $50 \times 50$  Padé approximant, then, through  $\mathcal{M}^{(0,v^{10})}$ , the agreement with the coefficient 4 in Eq. (C20) holds to greater than 5 places after the decimal. This agreement provides strong confirmation of the validity of the Abel summation in Eq. (C9), as supplemented by the use of Padé approximants.

Using the  $x$  moments of the model LCDA that correspond to  $\sigma_{J/\psi} = 0.129422$  [Eq. (C14)], we find that

$$\sum_{k=0}^5 \mathcal{M}^{(0,v^{2k})}(\mu_0)|_M = 4.28393. \quad (\text{C21})$$

On the other hand, if we evaluate  $\mathcal{M}(\mu_0)$  directly in Gegenbauer-moment space, taking the first 20 Gegenbauer moments, we obtain

$$\mathcal{M}(\mu_0)|_M \approx 4.28670. \quad (\text{C22})$$

This value agrees very well with the one that is obtained from the first 5 terms in the nonrelativistic expansion. [It also agrees very well with the value that is obtained by direct computation of the amplitude in  $x$  space as, in Eq. (C5).] The order- $v^2$  term in the expansion accounts for 80% of the higher-order corrections. As we have noted, the  $x$  moments of the model LCDA severely violate the velocity-scaling relation in Eq. (C13), and, so, we would expect that, in the case of a more realistic LCDA, the order- $v^2$  term in the expansion would account more fully for the higher-order corrections. If we use the values of the  $x$  moments in Eq. (C15), which are based on the NRQCD velocity-scaling rules, then we find that the order- $v^2$  term in the expansion accounts for 89% of the higher-order corrections.

We can evaluate these same quantities for the  $x$  moments in Eq. (C19), which correspond to the choice  $\sigma_{J/\psi} = 0.228$ . We remind the reader that this value of  $\sigma_{J/\psi}$  corresponds to the inclusion of the order- $\alpha_s$  corrections, as well as the order- $v^2$  corrections, in the model LCDA. Hence, for this value of  $\sigma_{J/\psi}$ , the relationship between the  $x$  moments of the model LCDA and the NRQCD LDMEs in Eq. (C12) does not hold, and the  $x$ -moment expansion is not, strictly speaking, a nonrelativistic expansion. Nevertheless, it is interesting to examine the convergence of the  $x$ -moment expansion in this case. The result for the  $x$ -moment expansion is

$$\sum_{k=0}^5 \mathcal{M}^{(0,v^{2k})}(\mu_0)|_M = 4.75605, \quad (\text{C23})$$

and the result for the direct evaluation, using the first 20 Gegenbauer moments, is

$$\mathcal{M}(\mu_0)|_M \approx 4.84334. \quad (\text{C24})$$

<sup>7</sup>We note that the limits  $k \rightarrow \infty$  and  $\sigma_V \rightarrow 0$  cannot be interchanged, as can be seen explicitly from Eqs. (C16) and (C17).

Again, there is good agreement between the results from  $x$ -moment expansion and the direct evaluation, although, as expected, the  $x$ -moment expansion converges more slowly with this choice of  $\sigma_{J/\psi}$ .

### b. With evolution

In this section, we compute the same quantities as in the preceding section, but taking  $\mu = m_H$  and  $\mu_0 = 1$  GeV. We use LL evolution. In order to compute the coefficients of the  $\langle x^{2k} \rangle$  in the presence of evolution, we use the Abel summation in Eq. (C9), accelerating the convergence to the limit by employing a  $50 \times 50$  Padé approximant. The result is

$$\begin{aligned}\mathcal{M}^{(0,0)}(\mu) &= 4.91403 \langle x^0 \rangle, \\ \mathcal{M}^{(0,v^2)}(\mu) &= 2.95670 \langle x^2 \rangle, \\ \mathcal{M}^{(0,v^4)}(\mu) &= 2.31150 \langle x^4 \rangle, \\ \mathcal{M}^{(0,v^6)}(\mu) &= 1.96596 \langle x^6 \rangle, \\ \mathcal{M}^{(0,v^8)}(\mu) &= 1.74271 \langle x^8 \rangle, \\ \mathcal{M}^{(0,v^{10})}(\mu) &= 1.58320 \langle x^{10} \rangle.\end{aligned}\quad (\text{C25})$$

We note that the evolution results in a decreasing sequence of coefficients, and, so we expect the nonrelativistic expansion to converge more rapidly than in the absence of evolution. With choice  $\sigma_{J/\psi} = 0.129422$ , the nonrelativistic expansion gives

$$\sum_{k=0}^5 \mathcal{M}^{(0,v^{2k})}(\mu)|_M = 5.11340, \quad (\text{C26})$$

and the direct evaluation, using the first 20 Gegenbauer moments, gives

$$\mathcal{M}(\mu)|_M = 5.11425. \quad (\text{C27})$$

There is good agreement between the nonrelativistic expansion and the direct evaluation. As expected, the nonrelativistic expansion converges more rapidly than in the case of no evolution. In this case, the order- $v^2$  term in the expansion accounts for 85% of the higher-order corrections. We would expect that, in the case of a more realistic LCDA, the order- $v^2$  term in the expansion would account more fully for the higher-order corrections. If we use the values of the  $x$  moments in Eq. (C15), which are based on the NRQCD velocity-scaling rules, then we find that order- $v^2$  term in the expansion accounts for 92% of the higher-order corrections.

Finally, we carry out the same computation with the choice  $\sigma_{J/\psi} = 0.228$ . Again, we remind the reader that this value of  $\sigma_{J/\psi}$  corresponds to the inclusion the order- $\alpha_s$  corrections, as well as the order- $v^2$  corrections, in the model LCDA, and, so, for this value of  $\sigma_{J/\psi}$ , the expansion the  $x$ -moment expansion of the LCDA is not, strictly speaking, a nonrelativistic expansion. The result for the  $x$ -moment expansion is

$$\sum_{k=0}^5 \mathcal{M}^{(0,v^{2k})}(\mu)|_M = 5.41349, \quad (\text{C28})$$

and the result from the direct evaluation is

$$\mathcal{M}(\mu)|_M \approx 5.43700. \quad (\text{C29})$$

Again, the  $x$ -moment expansion converges rapidly to the result from the direct evaluation, although, as expected, not as rapidly as with the choice  $\sigma_{J/\psi} = 0.129422$ .

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