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The large-N Yang-Mills S matrix is ultraviolet finite, but the large-N QCD S matrix is only renormalizable

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Yang-Mills (YM) theory and QCD are known to be renormalizable, but not ultraviolet (UV) finite, order by order, in perturbation theory. It is a fundamental question whether YM theory or QCD is UV finite, or only renormalizable, order by order, in the large-N 't Hooft or Veneziano expansions. We demonstrate that the renormalization group (RG) and asymptotic freedom imply that in 't Hooft large-N expansion the S matrix in YM theory is UV finite, while in both 't Hooft and Veneziano large-N expansions, the S matrix in confining massless QCD is renormalizable but not UV finite. By the same argument, the large-N $\mathcal{N}=1$ supersymmetry (SUSY) YM S matrix is UV finite as well. Besides, we demonstrate that, in both 't Hooft and Veneziano large-N expansions, the correlators of local gauge-invariant operators, as opposed to the S matrix, are renormalizable but, in general, not UV finite, either in YM theory and $\mathcal{N}=1$ SUSY YM theory or *a fortiori* in massless QCD. Moreover, we compute explicitly the counterterms that arise from renormalizing the 't Hooft and Veneziano expansions by deriving in confining massless QCD-like theories a low-energy theorem of the Novikov-Shifman-Vainshtein-Zakharov type that relates the log derivative with respect to the gauge coupling of a k-point correlator, or the log derivative with respect to the RG-invariant scale, to a $(k+1)$-point correlator with the insertion of $\mathrm{Tr}F^2$ at zero momentum. Finally, we argue that similar results hold in the large-N limit of a vast class of confining massive QCD-like theories, provided a renormalization scheme exists—as, for example, $\overline{\mathrm{MS}}$—in which the beta function is not dependent on the masses. Specifically, in both 't Hooft and Veneziano large-N expansions, the S matrix in confining massive QCD and massive $\mathcal{N}=1$ SUSY QCD is renormalizable but not UV finite.

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I. INTRODUCTION

SU(N) Yang-Mills (YM) theory and SU(N) QCD with N_f quark flavors are known to be renormalizable but not UV finite in perturbation theory. It is a fundamental question that has not been considered previously whether their large-N 't Hooft or Veneziano expansions (Sec. II) enjoy better UV properties nonperturbatively, perhaps limited only to the large-N S matrix, once the lowest $\frac{1}{N}$ order has been made finite by renormalization, as defined in Secs. III and IV. Answering this question sets the strongest constraints on the solution, which is yet to come, of large-N YM theory and QCD. The first main result of this paper is that the renormalization group (RG) and asymptotic freedom (AF) imply that, in the 't Hooft expansion, the large-N YM S matrix is UV finite, while in both 't Hooft and Veneziano expansions, the large-N S matrix in confining massless QCD[1] is renormalizable but not UV finite (Sec. III): in the 't Hooft expansion due to log divergences of meson loops (Sec. II) starting from the order of $\frac{N_f}{N}$, and in the Veneziano expansion due to log-log divergences of "overlapping" meson-glueball loops (Sec. II) starting from the order of $\frac{N_f}{N^3}$. By the same argument, the large-N $\mathcal{N}=1$ supersymmetry (SUSY) YM S matrix is UV finite as well. Correlators (Sec. IV), as opposed to the S matrix, turn out to be renormalizable but log-log divergent in general, in addition to the possible divergences of the S matrix in the aforementioned large-N expansions, but at the lowest order, even in large-N YM and $\mathcal{N}=1$ SUSY YM theory. The second main result is a low-energy theorem (Sec. V) of the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) type in confining massless QCD-like theories[2] that allows us to compute explicitly the lowest-order large-N counterterms implied by the RG and AF, as opposed to perturbation theory. Finally, we argue that similar results hold (Sec. VI) for the large-N S matrix in a vast class of confining QCD-like theories with massive matter fields, provided that a renormalization scheme exists in which the beta function is not dependent on the masses. $\overline{\mathrm{MS}}$ is an example of such a scheme. In addition, the asymptotic results in Sec. IV also extend to the correlators of the massive theory, provided that the massless limit of the massive theory exists smoothly.

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[1]By massless QCD, we mean QCD with massless quarks.

[2]By QCD-like theory, we mean a confining asymptotically free gauge theory admitting the large-N 't Hooft or Veneziano limits. We call such a theory massive if its matter fields are massive, and massless if a choice of parameters exists for which the theory is massless at all orders in perturbation theory.

II. LARGE-N'T HOOFT AND VENEZIANO EXPANSIONS

We recall briefly the 't Hooft [1] and Veneziano [2] expansions in large-N YM theory and QCD with N_f quark flavors. Nonperturbatively, the 't Hooft large-N limit is defined by computing the QCD functional integral in a neighborhood of $N = \infty$, with the 't Hooft gauge coupling $g^2 = g^2_{\mathrm{YM}}N$ and N_f fixed. The corresponding perturbative expansion, once expressed in terms of g^2, can be reorganized in such a way that each power of $\frac{1}{N}$ contains the contribution of an infinite series in g^2 [1,2]. The lowest-order contribution in powers of $\frac{1}{N}$ to connected correlators of local single-trace gauge-invariant operators $\mathcal{G}_i(x_i)$ and of quark bilinears $\mathcal{M}_i(x_i)$, both normalized in such a way that the two-point correlators are on the order of 1, turns out to be on the order of

$$\langle \mathcal{G}_1(x_1)\mathcal{G}_2(x_2)\cdots\mathcal{G}_n(x_n)\rangle_{\mathrm{conn}} \sim N^{2-n}; \qquad \langle \mathcal{M}_1(x_1)\mathcal{M}_2(x_2)\cdots\mathcal{M}_k(x_k)\rangle_{\mathrm{conn}} \sim N^{1-\frac{k}{2}}$$
$$\langle \mathcal{G}_1(x_1)\mathcal{G}_2(x_2)\cdots\mathcal{G}_n(x_n)\mathcal{M}_1(x_1)\mathcal{M}_2(x_2)\cdots\mathcal{M}_k(x_k)\rangle_{\mathrm{conn}} \sim N^{1-n-\frac{k}{2}}. \tag{1}$$

This is the 't Hooft planar theory, which perturbatively sums Feynman graphs triangulating, respectively, a sphere with n punctures, a disk with k punctures on the boundary, and a disk with k punctures on the boundary and n punctures in the interior. The punctured disk arises in the 't Hooft large-N expansion from Feynman diagrams whose boundary is exactly one quark loop. Higher-order contributions correspond to summing the Feynman graphs triangulating orientable Riemann surfaces with a smaller fixed Euler characteristic. They correct additively 't Hooft planar theory with a weight N^χ, where $\chi = 2 - 2g - h - n - \frac{k}{2}$ is the Euler characteristic of an orientable Riemann surface of genus g (i.e., a sphere with g handles), with h holes (or boundaries), n punctures in the interior, and k punctures on the boundary of some hole that the Feynman graphs triangulate. Nonperturbatively, a handle is interpreted as a glueball loop, and a hole as a meson loop [1,2]. On the contrary, nonperturbatively, the Veneziano large-N limit is defined by computing the QCD functional integral in a neighborhood of $N = \infty$, with g^2 and $\frac{N_f}{N}$ fixed. Since factors of the ratio $\frac{N_f}{N}$, which is kept fixed and considered on the order of 1, may arise perturbatively only from quark loops, the Veneziano large-N expansion contains perturbatively already at the lowest order Feynman graphs that triangulate a punctured sphere or a punctured disk with any number of holes, i.e., it contains the sum of all the Riemann surfaces that are geometrically planar. This is the Veneziano planar theory. Higher orders contain higher-genus Riemann surfaces.

III. LARGE-N YM AND MASSLESS QCD S MATRIX

We assume that YM theory and QCD have been regularized in a way that we leave undefined, introducing a common cutoff scale Λ, both perturbatively, in the large-N expansion, and nonperturbatively. The details of the regularization do not matter for our arguments. Perturbatively, pure YM theory and massless QCD need only gauge-coupling renormalization in the classical action in order to get a finite large-Λ limit since, in massless QCD, there is no quark-mass renormalization because chiral symmetry is exact in perturbation theory. In addition, local gauge-invariant operators also need, in general, multiplicative renormalizations, associated with the anomalous dimensions of the operators, in order to make their correlators finite. However, multiplicative renormalizations must cancel in the S matrix because of the Lehmann-Symanzik-Zimmermann reduction formulas since the S matrix cannot depend on the choice of interpolating fields for a given asymptotic state in the external lines [3]. Therefore, only gauge-coupling renormalization is necessary in the YM and the massless QCD S matrix, but nonperturbatively according to the RG,[3] because, otherwise, every physical mass scale in the S matrix is set to zero. Indeed, nonperturbatively gauge-coupling renormalization is equivalent to make finite and (asymptotically) constant the RG-invariant scale: $\Lambda_{\mathrm{RG}} = \mathrm{const}\Lambda \exp(-\frac{1}{2\beta_0 g^2})(\beta_0 g^2)^{-\frac{\beta_1}{2\beta_0^2}}(1+\cdots)$, uniformly in a neighborhood of $\Lambda = \infty$ and $g = 0$, where the dots represent an asymptotic series in g^2 of renormalization-scheme dependent terms that obviously vanish as $g \to 0$. The overall constant is scheme dependent as well. Moreover, RG requires that every physical mass scale of the theory is proportional to Λ_{RG}. Therefore, the UV finiteness of the large-N S matrix is equivalent to the existence of a renormalization scheme for g in which the large-N expansion of Λ_{RG} is finite because, if the S matrix is divergent, its divergences ought to be reabsorbed into a divergent redefinition of Λ_{RG}, which is the only parameter of the S matrix both in large-N YM theory and in confining massless QCD. This is decided as follows. Firstly, we

[3] We assume that the aforementioned theories actually exist mathematically and are renormalizable, that the $\frac{1}{N}$ expansion is at least asymptotic, and that the standard RG is actually asymptotic in the UV to the exact result because of AF. Though these statements are universally believed, no rigorous mathematical construction of YM theory or of QCD—or of their large-N limits—presently exists, let alone a mathematically rigorous proof of these statements.

consider the 't Hooft expansion in large-N YM theory. In this case, $\beta_0 = \beta_0^P = \frac{1}{(4\pi)^2}\frac{11}{3}$, $\beta_1 = \beta_1^P = \frac{1}{(4\pi)^4}\frac{34}{3}$, where the superscript P stands for 't Hooft planar theory. Now, both in the 't Hoof planar theory and to all the $\frac{1}{N}$ orders, the first-two coefficients of the beta function β_0, β_1 get contributions only from 't Hooft planar diagrams. This implies that, in large-N YM theory, the $\frac{1}{N}$ expansion of Λ_{YM} is, in fact, finite [4], as the nonplanar $\frac{1}{N}$ corrections occurring in the dots or in the const contribute only, at most, a finite change of renormalization scheme to the 't Hooft planar RG-invariant scale, $\Lambda^P_{\mathrm{YM}} = \mathrm{const}\,\Lambda\,\exp(-\frac{1}{2\beta_0^P g^2})(\beta_0^P g^2)^{-\frac{\beta_1^P}{2\beta_0^{P2}}}(1+\cdots)$. Thus, all glueball loops in the S matrix of large-N YM theory are UV finite in the 't Hooft expansion around the planar theory once the planar theory has been made finite by the gauge-coupling renormalization implicit in the finiteness of Λ^P_{YM} [4]. A similar argument implies that, in the 't Hooft expansion, the large-N $\mathcal{N}=1$ SUSY YM S matrix is UV finite as well. The 't Hooft expansion of large-N massless QCD is deeply different. In this case, $\beta_0 = \beta_0^P + \beta_0^{NP} = \frac{1}{(4\pi)^2}\frac{11}{3} - \frac{1}{(4\pi)^2}\frac{2}{3}\frac{N_f}{N}$ and $\beta_1 = \beta_1^P + \beta_1^{NP} = \frac{1}{(4\pi)^4}\frac{34}{3} - \frac{1}{(4\pi)^4}(\frac{13}{3} - \frac{1}{N^2})\frac{N_f}{N}$, where the superscript NP stands for non–'t Hooft planar theory. Since quark-loop contributions are on the order of $\frac{1}{N}$, the first coefficient of the beta function, β_0^P, gets an additive non–'t Hooft planar $\frac{1}{N}$ correction, $\beta_0^{NP} = -\frac{1}{(4\pi)^2}\frac{2}{3}\frac{N_f}{N}$. As a consequence, it is impossible to find a renormalization scheme for g that makes Λ_{QCD} finite in the 't Hooft planar theory and at the next order of the $\frac{1}{N}$ expansion at the same time, as the following computation shows [4]:

$$\Lambda_{\mathrm{QCD}} \sim \Lambda \exp\left(-\frac{1}{2\beta_0^P(1+\frac{\beta_0^{NP}}{\beta_0^P})g^2}\right) \sim \Lambda \exp\left(-\frac{1}{2\beta_0^P g^2}\right)\left(1+\frac{\frac{\beta_0^{NP}}{\beta_0^P}}{2\beta_0^P g^2}\right) \sim \Lambda^P_{\mathrm{QCD}}\left(1+\frac{\beta_0^{NP}}{\beta_0^P}\log\left(\frac{\Lambda}{\Lambda^P_{\mathrm{QCD}}}\right)\right). \tag{2}$$

The symbol $\sim$ means asymptotic equality in a sense specified by the context, up to perhaps a nonzero constant overall factor. The equalities in Eq. (2) hold asymptotically, uniformly for any large finite Λ and small g even before planar renormalization, without needing to actually take the limits $\Lambda \to \infty$, $g \to 0$, as they are obtained expressing g identically in terms of Λ^P_{QCD}, which is the new free parameter of the planar theory, in the last asymptotic equality. The log divergence in Eq. (2) occurs precisely because of the AF of the planar theory. In Sec. V we will compute explicitly the large-N counterterm due to the renormalization of Λ^P_{QCD}, which turns out to agree exactly, within leading-log accuracy, with the perturbative counterterm due to quark loops. Indeed, were Λ^P_{QCD} to get only a finite renormalization, the complete large-N QCD and the 't Hooft planar theory would have the same β_0, which is false. Hence, as Λ_{QCD} is the only physical mass scale, glueball and meson masses receive $\frac{1}{N}$ log-divergent self-energy corrections proportional to the one of Λ_{QCD}, which can arise only from a log divergence of the meson loops. This is a physical fact that characterizes the meson interactions in the UV, reflecting the corresponding perturbative quark interactions in the UV. Therefore, the 't Hooft expansion of the QCD S matrix, though renormalizable, starting from the order of $\frac{N_f}{N}$, is log divergent due to log divergences of the meson loops. The chances of finiteness would seem more promising in the Veneziano expansion. In this case, $\beta_0 = \beta_0^{VP} = \frac{1}{(4\pi)^2}\frac{11}{3} - \frac{1}{(4\pi)^2}\frac{2}{3}\frac{N_f}{N}$ and $\beta_1 = \beta_1^{VP} + \beta_1^{NVP}$, with $\beta_1^{VP} = \frac{1}{(4\pi)^4}(\frac{34}{3} - \frac{13}{3}\frac{N_f}{N})$ and $\beta_1^{NVP} = \frac{1}{(4\pi)^4}\frac{N_f}{N^3}$, where the superscripts, VP and NVP, stand for Veneziano planar theory and non-Veneziano planar theory. Since the Veneziano planar theory already contains all quark loops, the first coefficient of the Veneziano planar beta function and of the complete beta function coincide. Thus, there is no log divergence in the expansion of Λ_{QCD}. Yet, in the Veneziano expansion as well, it is impossible to find a renormalization scheme for g in which both $\Lambda^{VP}_{\mathrm{QCD}}$ and its $\frac{1}{N}$ corrections are finite at the same time because of log-log divergences starting from the order of $\frac{N_f}{N^3}$ due to overlapping glueball-meson loops, as the following computation shows:

$$\begin{aligned}\Lambda_{\mathrm{QCD}} &\sim \Lambda \exp\left(-\frac{1}{2\beta_0 g^2}\right)(g^2)^{-\frac{\beta_1^{VP}}{2\beta_0^2}}(g^2)^{-\frac{\beta_1^{NVP}}{2\beta_0^2}} \\ &\sim \Lambda^{VP}_{\mathrm{QCD}}\left(1+\frac{\beta_1^{NVP}}{2\beta_0^2}\log\log\left(\frac{\Lambda}{\Lambda^{VP}_{\mathrm{QCD}}}\right)\right). \end{aligned} \tag{3}$$

Thus, the large-N Veneziano expansion of the S matrix in confining[4] massless QCD is not UV finite as well. In any case, both the 't Hooft and Veneziano expansions of the S matrix are renormalizable, as the aforementioned

[4]In fact, Eq. (3) may be valid only for $\frac{N_f}{N}$ and g in a certain neighborhood of 0. Indeed, it is believed that there is a critical value of $\frac{N_f}{N}$ and of g at which massless QCD becomes exactly conformal because of an infrared zero of the beta function, where Λ_{QCD} may vanish due to the infrared zero. Similar considerations may apply to other massless QCD-like theories.

divergences are reabsorbed order by order in the $\frac{1}{N}$ expansions by a redefinition of $\Lambda_{\rm QCD}$.

IV. LARGE-N YM AND MASSLESS QCD CORRELATORS

The asymptotic structure of the two-point correlators of (Hermitian) operators of spin s and mass dimension D in the $\frac{1}{N}$ expansions is obtained by dividing the renormalized correlators by the multiplicative renormalization factor in the complete massless QCD-like theory [5,6] $(\frac{g^2(\Lambda)}{g^2(\mu)})^{\frac{\gamma_0}{\beta_0}}$ to get the bare correlators, and by multiplying by the same factor in the planar theory because of planar renormalization (the superscript $\mathcal{P}$ stands for P or VP):

$$\langle\mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle_{\rm conn}\sim\frac{\mathcal{P}^{(s)}(\frac{x_\alpha}{x})}{x^{2D}}\left(\frac{g^2(\Lambda)}{g^2(\mu)}\right)^{\frac{\gamma_0^{\mathcal{P}}}{\beta_0^{\mathcal{P}}}}\left(\frac{g^2(x)}{g^2(\Lambda)}\right)^{\frac{\gamma_0}{\beta_0}}=\frac{\mathcal{P}^{(s)}(\frac{x_\alpha}{x})}{x^{2D}}\left(\frac{g^2(\Lambda)}{g^2(\mu)}\right)^{\frac{\gamma_0^{\mathcal{P}}}{\beta_0^{\mathcal{P}}}}\left(\frac{g^2(x)}{g^2(\Lambda)}\right)^{\frac{\gamma_0^{\mathcal{P}}}{\beta_0^{\mathcal{P}}}}\left(\frac{g^2(x)}{g^2(\Lambda)}\right)^{\frac{\gamma_0}{\beta_0}-\frac{\gamma_0^{\mathcal{P}}}{\beta_0^{\mathcal{P}}}}$$
$$\sim\langle\mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle^{\mathcal{P}}\left(1+\left(\frac{\gamma_0}{\beta_0}-\frac{\gamma_0^{\mathcal{P}}}{\beta_0^{\mathcal{P}}}\right)\log\frac{g^2(x)}{g^2(\Lambda)}\right)$$
$$\sim\langle\mathcal{O}^{(s)}(x)\mathcal{O}^{(s)}(0)\rangle^{\mathcal{P}}\left(1+\left(\frac{\gamma_0}{\beta_0}-\frac{\gamma_0^{\mathcal{P}}}{\beta_0^{\mathcal{P}}}\right)\log\frac{\log(\frac{\Lambda^2}{\Lambda^2_{\rm QCD}})}{\log(\frac{1}{x^2\Lambda^2_{\rm QCD}})}\right), \tag{4}$$

where $\mathcal{P}^{(s)}(\frac{x_\alpha}{x})$ is the spin projector in the coordinate representation in the conformal limit. Thus, the expansion of the correlators around the planar theory has, in general, log-log divergences due to the $\frac{1}{N}$ corrections to the anomalous dimensions. Remarkably, the correlator of ${\rm Tr}F^2$ [5,6],

$$\langle{\rm Tr}F^2(x){\rm Tr}F^2(0)\rangle_{\rm conn}\sim\frac{1}{x^8}\left(\frac{g^4(x)}{g^4(\mu)}\right)\sim\frac{1}{x^8}\left(\frac{1}{\beta_0\log(\frac{1}{x^2\Lambda^2_{\rm QCD}})}\left(1-\frac{\beta_1}{\beta_0^2}\frac{\log\log(\frac{1}{x^2\Lambda^2_{\rm QCD}})}{\log(\frac{1}{x^2\Lambda^2_{\rm QCD}})}\right)\right)^2, \tag{5}$$

has no such log-log corrections since $\gamma_0=2\beta_0$ for ${\rm Tr}F^2$, both in the complete theory and in the planar theory, and thus the change of the anomalous dimension is always compensated for by the change of the beta function. Hence, the only renormalization in Eq. (5) is due to the $\frac{1}{N}$ expansion of $\Lambda_{\rm QCD}$ described in Sec. III.

V. LARGE-N MASSLESS QCD COUNTERTERMS FROM A LOW-ENERGY THEOREM, AS OPPOSED TO PERTURBATION THEORY

A new version of a NSVZ low-energy theorem [7] is obtained as follows. For a set of operators $\mathcal{O}_i$, deriving

$$\langle\mathcal{O}_1\cdots\mathcal{O}_i\rangle=Z^{-1}\int\mathcal{O}_1\cdots\mathcal{O}_i e^{-\frac{N}{2g^2}\int{\rm Tr}F^2(x)d^4x+\cdots} \tag{6}$$

with respect to $-\frac{1}{g^2}$, we get a NSVZ low-energy theorem:

$$\frac{\partial\langle\mathcal{O}_1\cdots\mathcal{O}_i\rangle}{\partial\log g}=\frac{N}{g^2}\int\langle\mathcal{O}_1\cdots\mathcal{O}_i{\rm Tr}F^2(x)\rangle-\langle\mathcal{O}_1\cdots\mathcal{O}_i\rangle\langle{\rm Tr}F^2(x)\rangle d^4x. \tag{7}$$

Since, nonperturbatively, in massless QCD-like theories the only parameter is $\Lambda_{\rm QCD}$, we can trade g for $\Lambda_{\rm QCD}$ on the lhs. Hence, $\frac{\partial\langle\mathcal{O}_1\cdots\mathcal{O}_i\rangle}{\partial(-\frac{1}{g^2})}=\frac{\partial\langle\mathcal{O}_1\cdots\mathcal{O}_i\rangle}{\partial\Lambda_{\rm QCD}}\frac{\partial\Lambda_{\rm QCD}}{\partial(-\frac{1}{g^2})}$, with $(\frac{\partial}{\partial\log\Lambda}+\beta(g)\frac{\partial}{\partial g})\Lambda_{\rm QCD}=0$ and $\beta(g)=-\beta_0g^3-\beta_1g^5+\cdots$:

$$\frac{\partial\langle\mathcal{O}_1\cdots\mathcal{O}_i\rangle}{\partial\log\Lambda_{\rm QCD}}=-\frac{N\beta(g)}{g^3}\int\langle\mathcal{O}_1\cdots\mathcal{O}_i{\rm Tr}F^2(x)\rangle-\langle\mathcal{O}_1\cdots\mathcal{O}_i\rangle\langle{\rm Tr}F^2(x)\rangle d^4x. \tag{8}$$

Now we specialize to multiplicatively renormalized operators in the planar theory, $\mathcal{O}_i={\rm Tr}F^2$, in such a way that the only source of divergences is the renormalization of $\Lambda_{\rm QCD}$ [see below Eq. (5)], as $\frac{N\beta(g)}{g^3}{\rm Tr}F^2(x)$ is, in fact, RG invariant. Since the divergent part of the correlator at the lowest $\frac{1}{N}$ order is $[\langle{\rm Tr}F^2\cdots{\rm Tr}F^2\rangle^{\mathcal{NP}}]_{\rm div}=\frac{\partial\langle{\rm Tr}F^2\cdots{\rm Tr}F^2\rangle^{\mathcal{P}}}{\partial\Lambda_{\rm QCD}}\Lambda^{\mathcal{NP}}_{\rm QCD}$, where $\mathcal{NP}$ stands for

NP or NVP, with $\Lambda_{\rm QCD}^{NP}=\frac{\beta_0^{NP}}{\beta_0^P}\Lambda_{\rm QCD}^P\log(\frac{\Lambda}{\Lambda_{\rm QCD}^P})+\cdots$ and $\Lambda_{\rm QCD}^{NVP}=\frac{\beta_1^{NVP}}{2\beta_0^2}\Lambda_{\rm QCD}^{VP}\log\log(\frac{\Lambda}{\Lambda_{\rm QCD}^{VP}})+\cdots$, the new low-energy theorem follows from Eq. (8):

$$[\langle {\rm Tr}F^2\cdots{\rm Tr}F^2\rangle^{\mathcal{NP}}]_{\rm div}=\frac{N\beta^{\mathcal{P}}(g)\Lambda_{\rm QCD}^{\mathcal{NP}}}{g^3\Lambda_{\rm QCD}^{\mathcal{P}}}\int\langle {\rm Tr}F^2\cdots{\rm Tr}F^2\rangle^{\mathcal{P}}\langle {\rm Tr}F^2(x)\rangle^{\mathcal{P}}-\langle {\rm Tr}F^2\cdots{\rm Tr}F^2{\rm Tr}F^2(x)\rangle^{\mathcal{P}}d^4x. \tag{9}$$

Thus, the divergence in the correlator arises, up to finite corrections, from the divergent counterterm in the action: $-\frac{\beta_0^{\mathcal{P}}N\Lambda_{\rm QCD}^{\mathcal{NP}}}{\Lambda_{\rm QCD}^{\mathcal{P}}}\int{\rm Tr}F^2(x)$. In the 't Hooft expansion, including in Eq. (2) the contributions from β_1, we get $-\frac{\beta_0^{\mathcal{P}}N\Lambda_{\rm QCD}^{\mathcal{NP}}}{\Lambda_{\rm QCD}^{\mathcal{P}}}=-N[\beta_0^{NP}\log(\frac{\Lambda}{\Lambda_{\rm QCD}^P})+\frac{1}{2\beta_0^P}(\beta_1^{NP}-\beta_0^{NP}\frac{\beta_1^P}{\beta_0^P})\log\log(\frac{\Lambda}{\Lambda_{\rm QCD}^P})]=\frac{1}{(4\pi)^2}\frac{2}{3}N_f\log(\frac{\Lambda}{\Lambda_{\rm QCD}^P})+\cdots$, which coincides exactly, within the leading-log accuracy, with the perturbative counterterm arising from the quark loops.

VI. S MATRIX IN LARGE-N MASSIVE QCD-LIKE THEORIES

We may wonder whether the results for massless theories described in Secs. III, IV, and V also apply to confining massive QCD-like theories, particularly to the large-N limit of massive QCD. Introducing mass scales may involve extra renormalizations associated with the mass parameters. However, supposing that the further parameters have been already renormalized, we may ask whether the large-N expansion of the massive theory may get milder UV divergences than the massless one. The simple answer is negative, provided a renormalization scheme exists in which the beta function is not dependent on the masses, as is appropriate for the UV-complete massive theory, as opposed to the "low-energy" effective theory at scales much smaller than the masses. In such a scheme, the renormalization of $\Lambda_{\rm QCD}$ goes through exactly as in the massless theory, as described in Secs. III, IV, and V. An example is the $\overline{\rm MS}$ scheme in massive QCD-like theories. Specifically, the large-N massive QCD S matrix is renormalizable but not UV finite, as it is not its massless limit. Moreover, both the 't Hooft and Veneziano expansions of the $\mathcal{N}=1$ SUSY massive QCD S matrix in the confining/Higgs phase [8] are renormalizable but not UV finite because the first-two coefficients of the beta function, $\beta_0=\frac{1}{(4\pi)^2}3-\frac{1}{(4\pi)^2}\frac{N_f}{N}$ and $\beta_1=\frac{1}{(4\pi)^4}6-\frac{1}{(4\pi)^4}(4-\frac{2}{N^2})\frac{N_f}{N}$, imply that $\beta_0^{NP}=-\frac{1}{(4\pi)^2}\frac{N_f}{N}$ and $\beta_1^{NVP}=\frac{2}{(4\pi)^4}\frac{N_f}{N^3}$. Another question regards what happens when regularizing and renormalizing a QCD-like theory by embedding into an UV finite theory. For example, this is feasible concretely for $\mathcal{N}=1$ SUSY QCD with $1\leq N_f\leq N$ and for $\mathcal{N}=1$ SUSY YM theory by embedding into a suitable finite $\mathcal{N}=2$ SUSY theory [9] containing massive multiplets on the order of M that act as regulators, and that may eventually be decoupled in the limit $M\to\infty$, in order to recover the original theory [9]. In this respect, the Veneziano limit of massive $\mathcal{N}=1$ SUSY QCD with $1\leq N_f\leq N$ is particularly interesting since, in this case, both the Veneziano planar theory and the next orders in the large-N expansion of the regularizing $\mathcal{N}=2$ theory are UV finite. In this case, the UV finiteness depends only on the vanishing of β_0 [9] and on the soft breaking of the $\mathcal{N}=2$ SUSY by the massive multiplets [9]. However, being asymptotically conformal in the deep UV, the regularizing $\mathcal{N}=2$ theory is not $\mathcal{N}=1$ SUSY QCD, which instead is asymptotically free, which means that the conformal behavior is corrected, in general, in the correlators by fractional powers of logs, according to Eq. (4). Thus, despite the finiteness of the $\mathcal{N}=2$ theory, what we want really to discover is the gauge-coupling renormalization of its $\mathcal{N}=1$ "low-energy limit" in the Veneziano expansion as the mass M of the regulator multiplets goes to infinity. This is again the original question that we have already answered above, as the only difference is that the effective cutoff of the regularized $\mathcal{N}=1$ theory is now on the order of M instead of Λ. Hence, though the regularized massive $\mathcal{N}=1$ SUSY QCD S matrix is finite for a finite M, it is UV divergent in the Veneziano expansion as $M\to\infty$. Finally, we should add that the asymptotic estimates for the correlators in the massless theory in Sec. IV apply without modification to massive QCD-like theories—provided that the massless limit exists smoothly—since, in this case, the leading UV asymptotics of the correlators is not dependent on the masses. Yet, some modification may possibly arise in massive $\mathcal{N}=1$ SUSY QCD with $1\leq N_f<N$ because the massless limit in the correlators may not necessarily be smooth, as the massless limit for certain SUSY meson one-point correlators is divergent [8].

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[1] G. 't Hooft, A planar diagram theory for strong interactions, Nucl. Phys. **B72**, 461 (1974).
[2] G. Veneziano, Some aspects of a unified approach to gauge, dual and Gribov theories, Nucl. Phys. **B117**, 519 (1976).
[3] M. Bochicchio, An asymptotic solution of large-N QCD, EPJ Web Conf. **80**, 00010 (2014).
[4] M. Bochicchio, Asymptotic freedom versus open/closed duality in large-N QCD, arXiv:1606.04546.
[5] M. Bochicchio, Glueball and meson propagators of any spin in large-N QCD, Nucl. Phys. **B875**, 621 (2013).
[6] M. Bochicchio and S. P. Muscinelli, Ultraviolet asymptotics of glueball propagators, J. High Energy Phys. 08 (2013) 064.
[7] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Are all hadrons alike?, Nucl. Phys. **B191**, 301 (1981).
[8] K. Intriligator and N. Seiberg, Lectures on supersymmetric gauge theories and electric-magnetic duality, Nucl. Phys. B, Proc. Suppl. **45**, 1 (1996).
[9] N. Arkani-Hamed and H. Murayama, Holomorphy, rescaling anomalies and exact beta functions in supersymmetric gauge theories, J. High Energy Phys. 06 (2000) 030.