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Reply to "Comment on 'Fermion production in a magnetic field in a de Sitter universe'"

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In this paper, we study the problem of gauge invariance of the first order transition amplitudes in de Sitter QED in the Coulomb gauge. We consider the gauge transformations which preserve the Coulomb gauge, that contain the gradient of the gauge function. The final results prove that the first order transition amplitudes do not change at a gradient transformation of the vector potential because the only allowed transformation is $\Lambda = 0$. Our results suggest that the remarks made in the comment by Nicolaevici and Farkas [this issue, Phys. Rev. D **95**, 048501 (2017)] are not directly applicable to the results in our paper since their proposed gauge transformations do not preserve the Coulomb gauge.

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I. INTRODUCTION

The problem of gauge invariance of the transition amplitudes in Minkowski QED was the subject of an intense debate some decades ago [1–13]. While some of the authors argue that the surface terms that appear in the variation of the amplitude in the first order, at a gauge transformation of the potential A^μ, are vanishing due to the fact that the fields vanish if we impose boundary conditions [5,8–11,14], others prove that the problem of gauge invariance of the amplitudes must be discussed in relation with the renormalization of the Minkowski QED [1–3], since the renormalization constants also depend on the chosen gauge. In quantum field theory on curved spacetime [15], the problem of gauge invariance was not studied in detail, and the existing results do not allow us to reach definitive conclusions regarding the gauge dependence of the amplitudes. The perturbative QED on the de Sitter spacetime was constructed in [16] where it is shown that a mandatory condition for quantifying the whole theory is to choose and fix the Coulomb gauge. Moreover, this seems to be a specific feature of the de Sitter geometry as long as the massless limit of the free Proca field on this background [17] gives just the free Maxwell field in the Coulomb gauge.

Another result obtained recently [18] shows that there are indications that the transition amplitudes in the first order of perturbation theory are gauge dependent in de Sitter spacetime. The comment [19] to our paper [20] is a continuation of the result obtained in [18], and the cause of the gauge dependence of the amplitudes was indicated to be contained in the temporal part of the Dirac current which loses its oscillatory behavior in the infinite future. In this reply, we want to reconsider the problem of gauge invariance of transition amplitudes in de Sitter QED. This is done by discussing the situation from our paper [20], where we use Coulomb gauge and by making some observations about the results obtained in the comment [19].

The paper is organized as follows: In Sec. II, we discuss the problem of the transition amplitudes when the Coulomb gauge is used. In Sec. III, the problem of gauge invariance is discussed in relation with the vector potential decomposition in perpendicular and parallel components. In Sec. IV, we discuss the problem of gauge invariance in relation with the renormalization of the theory, and our conclusions are presented in Sec. V.

II. AMPLITUDES IN THE COULOMB GAUGE

The line element [21] which describes the de Sitter universe is

$$ds^2 = dt^2 - e^{2Ht}d\vec{x}^2 = \frac{1}{(H\eta)^2}(d\eta^2 - d\vec{x}^2), \qquad (1)$$

where H is the expansion factor, and $H > 0$, while $\eta = -\frac{1}{H}e^{-Ht}$, is the conformal time.

It is known that Minkowski QED is a gauge invariant theory in the sense that any transformations of the type

$$A_\mu \to A_\mu + \partial_\mu \Lambda,$$
$$\psi \to e^{ie\Lambda}\psi \qquad (2)$$

leave the field equations invariant, where Λ is the gauge function which is a scalar function dependent both on time and spatial coordinates [9,11,12,14,22]. The second transformation from (2) refers to the matter fields, and this second transformation must be accompanied by the transformation of the potential in the case when the electromagnetic field is coupled with a matter field for leaving the field equations unchanged. We will assume the usual boundary conditions in space and time, such that the gauge function vanishes at infinity [10,14]:

$$\Lambda \to 0, \qquad (3)$$

for $t \to \infty, x \to \infty$.

Before proceeding in our analysis, let us make a few observations related to the Coulomb gauge. It is true that in our paper about the fermion production in the field of the magnetic dipole [20], we omit to mention that we use the Coulomb gauge. The reasons for using the Coulomb gauge will be detailed in what follows, and an extended discussion can also be found in our previous papers [16,23,24]. The significance of the Coulomb gauge in de Sitter geometry is very important if we take into account the conformal invariance of the Maxwell equations. Then it seems that the Coulomb gauge is the only gauge which opens the way to conformally relate the whole theory of the Maxwell field written in the chart with conformal time $\{\eta, \vec{x}\}$, to the usual electrodynamics from Minkowski spacetime. The Lorentz condition is conformally invariant only in the Coulomb gauge in de Sitter geometry, and this allows us to obtain the solutions of the free Maxwell equations in the helicity-momentum basis as in flat space theory [24]. The canonical quantization of the free Maxwell field can be performed in the Coulomb gauge where the Lorentz condition becomes conformally invariant [24], as we point out above. This means that it is useful to maintain this gauge for constructing the theory of interacting fields [16]. However there is a lot of work to be done if we want to speak about measurable quantities in this geometry, but an important result is that we recover in the limit of the zero expansion factor the results from flat space QED. This means that in the limit of the zero expansion factor, the transition amplitudes and probabilities computed in de Sitter QED in the Coulomb gauge reduce to those from Minkowski theory. To the best of our knowledge, it is not known how the theory of the free Maxwell field looks in other gauges in de Sitter geometry, and, therefore, further studies must be done to fully understand the theory of the Maxwell field in this geometry. For the above-mentioned reasons, we restrict ourselves to use the Coulomb gauge for studying the perturbative QED in de Sitter geometry.

All the details related to the construction of perturbative QED in de Sitter geometry using the Coulomb gauge can be found in [16], and we remind here only the main steps. The construction of the de Sitter QED in the Coulomb gauge starts with the Lagrangian theory that gives the field equations and the principal conserved observables of the interacting fields [16]. Then the equal time commutators and anticommutators are postulated, and the equation of the time-dependent evolution operator is derived, obtaining the perturbation series of the scattering operator in terms of free fields. This is generated by the interaction Hamiltonian which does not depend on the Coulomb potential. So the Coulomb gauge allows a natural quantization separating of the Coulomb potential [16]. Finally, the asymptotic fields are defined, and we obtain the *in-out* amplitudes by using the reduction formalism and the scattering operator [16].

Thus, one first chooses a gauge for solving the free Maxwell equations, then the quantization can be done. The theory of field interactions is also constructed by choosing a gauge since we have to find the solutions for the interacting field equations, which in de Sitter geometry are strongly dependent on the gauge [16]. Then the perturbation theory can be constructed, and finally we can obtain the expressions for the transition amplitudes in any order. It is then clear that once a gauge is fixed and the quantization procedure is done, one could not do a gauge transformation in the transition amplitudes for passing to another gauge. In these circumstances, any gauge transformations related to the electromagnetic potentials that are allowed are just those that preserve the chosen gauge in which the theory of interacting fields was constructed and the quantization was done. The authors of the comment [19] to our paper seem to miss this important observation, since the only allowed gauge transformations are those that preserve the Coulomb gauge, and their discussion should consider these transformations only when commenting on our paper [20]. Instead, at the beginning of the comment, the authors leave the impression that they work in the Coulomb gauge taking into account that they consider $A_i \neq 0, A_0 = 0$, and then they choose to make the discussion in another gauge such that the amplitude is defined in general with A_μ. This comment should construct the theory of the free electromagnetic field in another gauge, in de Sitter geometry, then make the quantization, and finally construct the QED in this new gauge. Therefore, a comment to our papers [16,20,23] should prove the following: First, take another gauge and solve the Maxwell equations and then construct the perturbative QED in this new gauge. Then in this new gauge, take a potential which gives the same dipolar magnetic field (as the one used in our paper [20]), compute the first order amplitude corresponding to the fermion pair production, and finally compare the results from this new gauge with our results obtained in the Coulomb gauge. To conclude, the comment [19] did not prove that the amplitudes from our paper [20] are gauge dependent. In what follows, we will clarify what it means to do a gauge transformation in our case [20].

The amplitude of pair production in an external magnetic field from our paper [20] will be further considered. In [20], the vector potential that describes the magnetic field produced by a dipole was taken as

$$\vec{A} = \frac{\vec{\mathcal{M}} \times \vec{x}}{|\vec{x}|^3} e^{-Ht}, \qquad A^0 = 0, \tag{4}$$

where $\vec{\mathcal{M}}$ is the magnetic dipole moment. We also observe that $\nabla\vec{A} = 0, A_0 = 0$. The first order transition amplitude corresponding to pair creation in external field, assuming the minimal coupling, is [20]

$$\mathcal{A}_{e^-e^+} = -ie \int d^4x[-g(x)]^{1/2} \bar{U}_{\vec{p},\lambda}(x)\vec{\gamma} \cdot \vec{A}(x) V_{\vec{p}',\lambda'}(x), \tag{5}$$

where $U_{\vec{p},\lambda}(x), V_{\vec{p}',\lambda'}(x)$ are the solutions of the Dirac equation in momentum basis in de Sitter geometry [25].

Our analysis in the de Sitter case is done by preserving the Coulomb gauge, and the same is true for our paper [20]. Let us denote by ∇_α the covariant derivative, with $\alpha = 0$, 1, 2, 3. The condition $\partial_i A^i = 0$ from Minkowski space is replaced by the vanishing of the covariant derivative in the de Sitter case $\nabla_i(\sqrt{-g}A^i) = 0$ [16,24], but it is sufficient to apply the covariant derivative only on A^i since $\sqrt{-g}$ depends only on time, and we obtain

$$\nabla_i A^i = 0 = \partial_i A^i + \Gamma^i_{i\alpha}A^\alpha = \partial_i A^i + \Gamma^i_{ij}A^j + \Gamma^i_{i0}A^0. \qquad (6)$$

Considering the de Sitter line element (1), with $g_{00} = 1$, $g_{ij} = -\delta_{ij}e^{2Ht}$, we obtain that $\Gamma^i_{ij} = 0, \Gamma^i_{i0} = H$, and the above equation becomes

$$\nabla_i A^i = \partial_i A^i + HA^0 = 0. \qquad (7)$$

Since $A^0 = 0$, this implies $\partial_i A^i = 0$.

It is known that further gauge transformations that preserve the Coulomb gauge condition can be made, and these gauge transformations which contain the components of A^μ can be written as

$$A^i \to A^i + \partial^i\Lambda,$$
$$A^0 \to A^0 + \partial^0\Lambda. \qquad (8)$$

Since $A^0 = 0$, we observe that the condition $\partial^t\Lambda = 0$ is mandatory (this conclusion can be reached following similar arguments like in the Minkowski theory; see [7]).

Let us apply the covariant derivative to the potential transformation given in Eq. (8)

$$\nabla_i A^i \to \nabla_i A^i + \nabla_i(\partial^i\Lambda) \qquad (9)$$

and compute $\nabla_i(\partial^i\Lambda)$ to obtain

$$\nabla_i(\partial^i\Lambda) = \partial_i\partial^i\Lambda + \Gamma^i_{ij}\partial^j\Lambda + \Gamma^i_{i0}\partial^0\Lambda = \partial_i\partial^i\Lambda + H\partial^t\Lambda. \qquad (10)$$

Since we impose the condition $\nabla_i A^i = 0$, we observe that we must take $\nabla_i(\partial^i\Lambda) = 0$ for preserving the Coulomb gauge and finally obtain the equation for Λ in de Sitter geometry:

$$\partial_i^2\Lambda - He^{2Ht}\partial_t\Lambda = 0. \qquad (11)$$

A similar situation is encountered in Minkowski QED, where for preserving the Coulomb gauge the condition $\partial_i^2\Lambda = 0$ is imposed [10,12,14,26]. In Eq. (11), we observe that Λ is a time-independent function, i.e., $\partial_t\Lambda = 0$, as shown above, and the first term of the equation reproduces the situation from the flat space case, giving for the gauge function an equation of the Laplace type [10,12,14,26]:

$$\triangle\Lambda = 0. \qquad (12)$$

From the analysis above, we observe that in the Coulomb gauge the gauge function Λ depends only on spatial coordinates. But Eq. (12) has a nice property: its solutions are unique [14]. In other words, if we can find a solution to the Laplace equation which satisfies the boundary conditions, then it is clear that this is the only solution. The physical criterion that the gauge function must accomplish is for it to vanish when the spatial distances and the time become infinite, where the Dirac fields and the potential used in our calculations also vanish [9,10,12,14]. In these circumstances, the unique solution that accomplishes these criteria is [12,14,22]

$$\Lambda = 0, \qquad (13)$$

and no gauge arbitrary remains in this case. So our transition amplitudes of fermion production in the magnetic field on de Sitter spacetime obtained in [20] do not change if we add to the potential the gradient of the gauge function, as long as we impose to remain in the Coulomb gauge. To the best of our knowledge, how physics looks in de Sitter geometry if we choose other gauges is not studied in the literature. Instead, it seems that for the moment, the Coulomb gauge is the only gauge in which one could construct the theory of free electromagnetic field, imposing then the canonical quantization and further making a coherent perturbative QED [16].

An interesting observation about our vector potential is that we know its divergence and curl, and, in addition, we know that it vanishes when the spatial distances become infinite. Then, a vector field which vanishes at infinity is completely specified once its divergence and its curl are given ($\nabla\vec{A} = 0, \nabla\times\vec{A} = \vec{B}$) [14]. There are no additional terms due to the transformation (8) in our amplitude of fermion production in magnetic field, and this is the result of the use of the Coulomb gauge and of the use of boundary conditions in space and time. So we prove that any gradient transformation leaves the transition amplitude invariant in the Coulomb gauge. Or, in other words, the only gauge transformation allowed which preserves the Coulomb gauge is $\Lambda = 0$, as in the Minkowski QED [12].

III. GAUGE INVARIANCE AND VECTOR POTENTIAL DECOMPOSITION

Another way to tell the above story is as follows. Consider that the vector potential is decomposed in longitudinal and transversal parts [27],

$$\vec{A} = \vec{A}_\perp + \vec{A}_\parallel, \qquad (14)$$

such that [27] $\nabla\vec{A}_\perp = 0, \nabla\times\vec{A}_\parallel = 0$, which is known since the magnetic field is purely transversal $\vec{B}_\parallel = 0$. Then, the gauge transformation

$$\vec{\mathcal{A}} \to \vec{\mathcal{A}} + \nabla\Lambda \tag{15}$$

will give the transformation rules for the longitudinal and transversal components [27]:

$$\vec{\mathcal{A}}_{\perp} \to \vec{\mathcal{A}}_{\perp},$$
$$\vec{\mathcal{A}}_{\parallel} \to \vec{\mathcal{A}}_{\parallel} + \nabla\Lambda. \tag{16}$$

We observe from the above equation that the transversal component of the potential vector is gauge invariant [27], and only the longitudinal component transforms. In our calculations, we use only the transversal component $\vec{A} = \vec{\mathcal{A}}_{\perp}$ as given in Eq. (4). Moreover in the Coulomb gauge $\vec{A}_{\parallel} = 0$ [27], and only the transversal components are not vanishing.

Finally, we conclude that working in the Coulomb gauge, only with the transversal components of the potential vector can one obtain results which are gauge independent in de Sitter QED. One needs to fix a gauge for studying the theory of free electromagnetic field and perturbative QED and then compute the amplitudes.

IV. GAUGE INVARIANCE AND RENORMALIZATION

Since we know from flat space QED that for computing observable quantities one needs to do the renormalization of the theory, which depends on the gauge [1–4,7], the same observation could be also valid in de Sitter QED. Then the discussion of gauge invariance in de Sitter QED is not at all an easy task since a complete proof of the gauge independence/dependence of the amplitudes/probabilities could depend on the renormalization of the theory, which is an issue not clear at this moment in this geometry. It is known from flat space QED that the unrenormalized S-matrix elements obtained from Feynman diagrams always appear in physical-scattering amplitudes multiplied by the renormalization constants Z_2 and Z_3. As we know, Z_3 is gauge invariant, but Z_2 is a gauge-dependent quantity [1–4]. This means that the unrenormalized transition amplitudes could be gauge dependent in order to secure the gauge invariance of the product between the renormalization constants and the unrenormalized matrix elements [1–4]. So a proof of the gauge invariance could be carried out on the renormalized amplitudes/probabilities in de Sitter QED. For that, we must study the Maxwell and Dirac propagators including their radiative corrections. The above program must be completed, and only then a definite conclusion could be addressed properly about the gauge dependence of the amplitudes in de Sitter QED.

V. CONCLUSIONS

The final conclusion is that the comment [19] needs to be considered with care, and the authors do not present valid arguments regarding the gauge dependence of the amplitude from our paper [20]. In our paper, we work in the Coulomb gauge in which the only allowed gauge transformations are $\Lambda = 0$. In quantum field theory, the standard procedure is to establish the gauge first and only afterwards perform the quantization. Once a gauge is fixed and the quantization procedure is done, one could not do a gauge transformation in the transition amplitudes which alter the chosen gauge.

Another important observation is that the analysis of the amplitudes variation in other gauges and with given external electromagnetic fields must be done in de Sitter geometry in order to understand the problem of gauge invariance, but there are no concrete results at the present time in the literature.

Since the authors of the comment do not say or prove how the analytical results or physical interpretation of our results will be modified by their result, we disagree with the implication of the comment [19] that the analysis in our paper [20] and our previous papers [16,23] are physically incorrect or there are ambiguities about the quantities that were computed.

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