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Two-loop gravity amplitudes from four dimensional unitarity

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We compute the polylogarithmic parts of the two-loop four- and five-graviton amplitudes where the external helicities are positive and express these in a simple compact analytic form. We use these to extract the $\ln(\mu^2)$ terms from both the four- and five-point amplitudes and show that these match the same R^3 counterterm.

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I. INTRODUCTION

Computing the scattering amplitudes of a quantum theory using its singular and analytic structure has a long history with many notable successes [1]. Recently the five-point all-plus helicity amplitude has been computed in QCD: first the integrands were determined using the method of maximal cuts [2] and D-dimensional unitarity, and then the integrals were evaluated yielding a compact analytic form [3]. In D-dimensional unitarity the cuts are computed in $D = 4 - 2\epsilon$ dimensions where, typically, the components of the cuts are considerably more complicated than in four dimensions. In [4] it was demonstrated that four dimensional unitarity techniques [5,6] blended with a knowledge of the singular structure of the amplitude could reproduce this form in a straightforward way. The four dimensional approach was also used to calculate the six-point amplitude [7,8] which was subsequently verified [9].

Here we apply these techniques to gravity amplitudes with a particular emphasis on their ultraviolet (UV) behavior. Understanding the UV structure of quantum gravity necessitates studying the two-loop amplitude. 't Hooft and Veltman [10] demonstrated the one-loop finiteness of on-shell amplitudes in quantum gravity by showing the available counterterms in four dimensions made no contribution to perturbative amplitudes. This cancellation does not persist to two loop, and Goroff and Sagnotti [11,12] in a landmark calculation were able to compute a UV infinity of the form

$$\sim \frac{209}{80\epsilon} \times A^R, \tag{1.1}$$

where A^R is the functional form generated by an R^3 counterterm. A feature of the computation was the appearance of subdivergences which were canceled diagram by diagram. In [13] this computation was revisited using evanescent operators which arise at one-loop to remove the subdivergences in the four-point two-loop amplitude. For gravity coupled to various matter multiplets they found UV terms

$$\left(\frac{a_1}{\epsilon} + a_2 \ln(\mu^2)\right) \times A^R \tag{1.2}$$

and noted that the coefficient of $\ln(\mu^2)$, a_2, was more robust and simpler than a_1. Specifically, when coupled to n_3 (nonpropagating) three form fields a_1 obtained a contribution proportional to n_3 but a_2 did not. Additionally when coupled to scalars and vectors, a_2 was simply proportional to the difference between the number of bosonic and fermionic degrees of freedom. As was argued in Ref. [13], the coefficient a_2 has physical content since after renormalization the amplitude depends upon a_2 but not a_1.

In this article we explore and compute, up to rational terms, the four- and five-point two-loop amplitudes in quantum gravity where all the external gravitons have positive helicity. We use the techniques which have proven successful for the gluon amplitude: the amplitude is organized using a knowledge of its singularity structure, and then four dimensional unitarity is used to determine the logarithmic and dilogarithmic parts. Since $\ln(\mu^2)$ only appears in the combination $\ln(K^2/\mu^2)$ we can extract the coefficient of $\ln(\mu^2)$ using unitarity. We obtain the same coefficient of $\ln(\mu^2)$ for the four-point amplitude as in Ref. [13]. We also obtain the coefficient of $\ln(\mu^2)$ for the five-point amplitude and show that this matches the same R^3 counterterm.

II. STRUCTURE OF THE AMPLITUDE

As a convention we remove the coupling constant factors from the full n-point L-loop amplitude, $\mathcal{M}_n^{(L)}$, using

$$\mathcal{M}_n^{(L)}(1,\ldots,n) = \frac{i(\kappa/2)^{n-2+2L}(r_\Gamma)^L}{(4\pi)^{L(2-\epsilon)}} M_n^{(L)}(1,\ldots,n), \tag{2.1}$$

where $r_\Gamma = \Gamma^2(1-\epsilon)\Gamma(1+\epsilon)/\Gamma(1-2\epsilon)$.

We then organize the amplitude according to its singularity structure. The amplitude has both infrared (IR) and UV singularities in the dimensional regulation parameter ϵ. The all-plus amplitudes, which are finite at one-loop, can be divided as

$$M_n^{(2)} = M_n^{(1)} I_n^{(1)} + G_n^{(2)} + F_n^{(2)} + R_n^{(2)} + \mathcal{O}(\epsilon), \tag{2.2}$$

where the first term contains the IR singularities of the amplitude. The function $I_n^{(1)}$ is [14–17]

$$I_n^{(1)} = \left[-\sum_{i<j}^{n} \frac{1}{\epsilon^2} s_{ij} \left(\frac{\mu^2}{-s_{ij}} \right)^{\epsilon} \right], \tag{2.3}$$

and $M_n^{(1)}$ is the all-ϵ form of the one-loop amplitude [with $s_{ij} = (k_i + k_j)^2$]. The leading singularity of $I_n^{(1)}$ is only ϵ^{-1} since

$$\frac{1}{\epsilon^2} \left(\sum_{i<j}^{n} s_{ij} \right) = \frac{1}{\epsilon^2} \times 0 \tag{2.4}$$

by momentum conservation, and the leading singularity is then

$$\frac{1}{\epsilon} \times \left(\sum_{i<j}^{n} s_{ij} \ln(-s_{ij}/\mu^2) \right) \times M_n^{(1)}. \tag{2.5}$$

The amplitude also has finite logarithmic terms. In our four dimensional formulation these arise in two-particle cuts and have the form of one-loop bubble integral functions,

$$G_n^{(2)} = \sum_{i<j}^{n} c_{ij} \frac{1}{\epsilon} \left(\frac{\mu^2}{-s_{ij}} \right)^{\epsilon}. \tag{2.6}$$

where c_{ij} are rational functions of λ_k and $\bar{\lambda}_k$.[1]

The all-plus two-loop amplitude in QCD does not contain this term [19]. This gives rise to ϵ^{-1} and $\ln(\mu^2)$ terms in the combination

$$\left(\sum_{i<j} c_{ij} \right) \times \left(\frac{1}{\epsilon} + \ln(\mu^2) \right). \tag{2.7}$$

There may be other sources of ϵ^{-1} terms not directly determined by unitarity. The function $F_n^{(2)}$ contains the remaining polylogarithms of the amplitude and $R_n^{(2)}$ contains the remaining rational terms. In dimensional regularization the internal momenta lie in $D = 4 - 2\epsilon$, and it is really D-dimensional unitarity which should be used to reconstruct the amplitude. Consequently, four dimensional unitarity is not sensitive to the rational terms; however, it does give considerable simplification. The rational terms may be computed by complementary methods such as recursion. For the case of QCD, the rational terms for $n = 5$, 6 were computed by recursion starting from the four-point amplitude. We will not compute $R_5^{(2)}$ here: it not being necessary for our analysis and $R_4^{(2)}$ not being available.

The all-plus two-loop amplitude is a particularly simple amplitude. The all-plus helicity tree amplitude vanishes,

$$M_n^{(0)}(1^+, 2^+, \ldots, n^+) = 0. \tag{2.8}$$

This can be seen as a consequence of supersymmetric Ward identities [20]. These imply that this amplitude vanishes to all orders in perturbation theory in supersymmetric theories. Since the n-graviton amplitudes for pure gravity coincide with those for supersymmetric theories at tree level, then the gravity tree amplitude also vanishes.

The one-loop four-point amplitude for pure gravity is [21][2]

$$M_4^{(1)}(1^+, 2^+, 3^+, 4^+) = -\left(\frac{st}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \right)^2 \frac{(s^2 + t^2 + u^2)}{120} + O(\epsilon), \tag{2.9}$$

and the n-point amplitude can be expressed as [22]

$$M_n^{(1)}(1^+, 2^+, 3^+, \ldots, n^+) = \frac{(-1)^n}{960} \sum_{a,b,M,N} h(a, M, b) h(b, N, a) \mathrm{tr}^3(aMbN) + O(\epsilon), \tag{2.10}$$

where $h(a, M, b)$ are the "half-soft" functions of Ref. [22]. The summation is over pairs of legs (a, b) and partitions (M, N) of the remaining legs where both the sets M and N have at least one element. The half-soft functions we need for the five-point amplitude are

$$h(a, \{c\}, b) = \frac{1}{\langle ac \rangle^2 \langle cb \rangle^2},$$
$$h(a, \{c, d\}, b) = \frac{1}{\langle ac \rangle \langle ad \rangle} \frac{[cd]}{\langle cd \rangle} \frac{1}{\langle cb \rangle \langle db \rangle}. \tag{2.11}$$

When coupled minimally to additional bosons and fermions these amplitudes are multiplied by a factor $(N_B - N_F)/2$ where $N_{B/F}$ is the total number of bosonic/fermionic degrees of freedom. A key feature of the one-loop amplitudes is that they are, to order ϵ^0, rational

[1] As usual, a null momentum is represented as a pair of two component spinors $p^\mu = \sigma^\mu_{\alpha\dot{\alpha}} \lambda^\alpha \bar{\lambda}^{\dot{\alpha}}$. For real momenta $\lambda = \pm \bar{\lambda}^*$ but for complex momenta λ and $\bar{\lambda}$ are independent [18]. We are using a spinor helicity formalism with the usual spinor products $\langle ab \rangle = \epsilon_{\alpha\beta} \lambda_a^\alpha \lambda_b^\beta$ and $[ab] = -\epsilon_{\dot{\alpha}\dot{\beta}} \bar{\lambda}_a^{\dot{\alpha}} \bar{\lambda}_b^{\dot{\beta}}$.

[2] We use for four-point kinematics $s \equiv s_{12}$, $t \equiv s_{14}$, and $u \equiv s_{13}$.

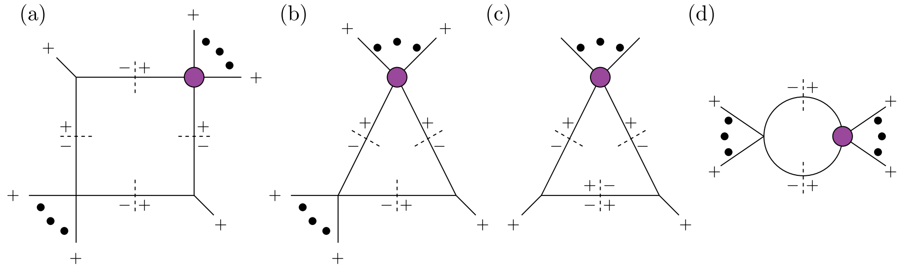

FIG. 1. Four dimensional cuts of the two-loop all-plus amplitude involving an all-plus one-loop vertex (indicated by the solid circles). Contributions can arise from (a) boxes, (b) two-mass triangles, (c) one-mass triangles, and (d) bubbles.

functions and as such have no cuts in four dimensions. Thus if computing amplitudes using cuts in four dimensions, they are indivisible and can be treated as a vertex. The only four dimensional cuts of the n-point all-plus amplitude are shown in Fig. 1. For this helicity amplitude the only tree amplitudes necessary to compute the cuts are the three-point amplitudes and the maximally-helicity-violating (MHV) tree amplitudes [23].

The four dimensional calculation gives the coefficient of $I_n^{(1)}$ to be $M_n^{(1)}$ to leading order in ϵ. As in the QCD case, we promote this to the all-ϵ form. The all-plus one-loop amplitudes in Eq. (2.2) for four and five points are known to all orders in ϵ [22,24],

$$
\begin{aligned}
&M_4^{(1)}(1^+,2^+,3^+,4^+)\\
&\quad=2\frac{[12]^2[34]^2}{\langle 12\rangle^2\langle 34\rangle^2}(I_4^{1234}[\mu_0^8]+I_4^{1243}[\mu_0^8]+I_4^{1423}[\mu_0^8]),\\
&M_5^{(1)}(1^+,2^+,3^+,4^+,5^+)\\
&\quad=\beta_{123(45)}I_4^{123(45)}[\mu_0^8]-2\frac{[12][23][34][45][51]}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}I_5^{12345}[\mu_0^{10}]\\
&\qquad+\text{Perms},
\end{aligned}
\tag{2.12}
$$

where

$$
\beta_{123(45)}=-[12]^2[23]^2h(1,\{4,5\},3) \tag{2.13}
$$

and $I_n[\mu_0^{2m}]$ are the n-point integral functions with μ_0^{2m} inserted where μ_0 are the -2ϵ coordinates in dimensional regularization, $\int d^Dx=\int d^4xd^{-2\epsilon}\mu_0$. The superscripts denote the ordering and clustering of the external legs. These are related to scalar integrals in higher dimensions [24,25],

$$
I_m[\mu_0^{2r}]=-\epsilon(1-\epsilon)\cdots(r-1-\epsilon)(4\pi)^rI_m^{D=4+2r-2\epsilon}. \tag{2.14}
$$

III. THE FOUR-POINT ALL-PLUS HELICITY AMPLITUDE

The four-point all-plus helicity amplitude has some significant simplifications. Specifically, the quadruple cuts vanish[3] and there are only one-mass triangle and bubble contributions. In fact, this amplitude is sufficiently simple that using the one-loop amplitude as a vertex both the triangle and bubble functions can be obtained simply from the two-particle cuts [21] with the result

$$
\begin{aligned}
&M_4^{(2)}(1^+,2^+,3^+,4^+)\\
&=M_4^{(1)}(1^+,2^+,3^+,4^+)\times\bigg(\frac{2(-s)^{1-\epsilon}+2(-t)^{1-\epsilon}+2(-u)^{1-\epsilon}}{\epsilon^2}\\
&\quad-\frac{2s(3u^2+3t^2-2s^2)}{(s^2+t^2+u^2)}\frac{(-s/\mu^2)^{-\epsilon}}{\epsilon}\\
&\quad-\frac{2t(3s^2+3u^2-2t^2)}{(s^2+t^2+u^2)}\frac{(-t/\mu^2)^{-\epsilon}}{\epsilon}\\
&\quad-\frac{2u(3t^2+3s^2-2u^2)}{(s^2+t^2+u^2)}\frac{(-u/\mu^2)^{-\epsilon}}{\epsilon}+\text{rational terms}\bigg).
\end{aligned}
\tag{3.1}
$$

This expression contains ϵ^{-1} and the $\ln(\mu^2)$ terms in the combination,

$$
\begin{aligned}
&M_4^{(1)}(1^+,2^+,3^+,4^+)\times\frac{30stu}{(s^2+t^2+u^2)}\times\left(\frac{1}{\epsilon}+\ln(\mu^2)\right)\\
&\quad=-\frac{1}{4}\left(\frac{st}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right)^2\times stu\times\left(\frac{1}{\epsilon}+\ln(\mu^2)\right).
\end{aligned}
\tag{3.2}
$$

If gravity is coupled to matter with N_B-2 additional bosonic degrees of freedom and N_F fermionic degrees of freedom, the one-loop all-plus amplitude is multiplied by a factor of $(N_B-N_F)/2$ and the calculation follows through as above. The all-plus amplitude in this theory is thus the expression above with the replacement

$$
-\frac{1}{4}\rightarrow-\frac{N_B-N_F}{8}, \tag{3.3}
$$

[3]In computing the quadruple cuts for a four-point amplitude the only nonvanishing product of cut amplitudes has alternating MHV and $\overline{\text{MHV}}$ three-point vertices at the corners. This precludes any box functions for the four-point all-plus amplitude.

which matches the result of [13]. Consequently, we note that four dimensional unitarity gives the correct amplitude up to rational terms although, as was known in Ref. [21], the coefficient of ϵ^{-1} does not match the field theory calculation.

IV. THE FIVE-POINT ALL-PLUS AMPLITUDE

This amplitude contains functions, particularly dilogarithms, that are not present in the four-point amplitude. These are contained in the box contributions shown in Fig. 2. The box contribution is readily evaluated using a quadruple cut [26].

With the labeling of Fig. 2 the cut momenta are

$$\ell_1 = \frac{\langle cd\rangle}{\langle ec\rangle}\bar{\lambda}_d\lambda_e, \qquad \ell_2 = \frac{\langle c|P_{de}|}{\langle ec\rangle}\lambda_e,$$
$$\ell_3 = \frac{\langle e|P_{cd}|}{\langle ec\rangle}\lambda_c, \qquad \ell_4 = \frac{\langle ed\rangle}{\langle ec\rangle}\bar{\lambda}_d\lambda_c, \tag{4.1}$$

giving the coefficient of the box function,

$$\begin{aligned}\mathcal{C}_{\{a,b\},c,d,e} &= \frac{1}{2}M_4^{(1)}(a^+,b^+,\ell_3^+,-\ell_2^+)\times M_3^{(0)}(-l_3^-,c^+,l_4^+)\\ &\quad\times M_3^{(0)}(-l_4^-,d^+,l_1^-)\times M_3^{(0)}(-l_1^+,e^+,l_2^-)\\ &= \frac{1}{240}\left(\frac{[ab]^6[cd]^2[ed]^2}{\langle ec\rangle^4}\right)\\ &\quad\times(\langle ab\rangle^2\langle ec\rangle^2+\langle ae\rangle^2\langle bc\rangle^2+\langle ac\rangle^2\langle be\rangle^2).\end{aligned} \tag{4.2}$$

This is the coefficient of the integral function $I_4^{1\mathrm{m}}(s_{cd},s_{de},s_{ab})$ where [27]

$$\begin{aligned}&I_4^{1\,\mathrm{m}}(S,T,M^2)\\ &\quad= -\frac{2}{ST}\left[-\frac{(\mu^{2\epsilon})2}{\epsilon^2}[(-S)^{-\epsilon}+(-T)^{-\epsilon}-(-M^2)^{-\epsilon}]\right.\\ &\qquad\left.+\mathrm{Li}_2\left(1-\frac{M^2}{S}\right)+\mathrm{Li}_2\left(1-\frac{M^2}{T}\right)+\frac{1}{2}\ln^2\left(\frac{S}{T}\right)+\frac{\pi^2}{6}\right]\\ &\qquad+\mathcal{O}(\epsilon),\end{aligned} \tag{4.3}$$

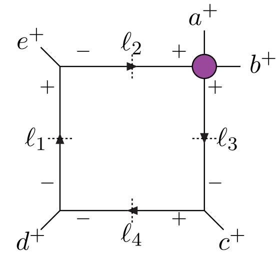

FIG. 2. The labeling and internal helicities of the quadruple cut.

and overall factors have been removed according to the normalization of Eq. (2.1).

This integral function splits into singular terms plus a remainder $I_4^{1\,\mathrm{m}} = I_4^{1\,\mathrm{m:IR}} + I_4^{1\,\mathrm{m:F}}$ where

$$\begin{aligned}&I_4^{1\,\mathrm{m:IR}}(S,T,M^2)\\ &\quad\equiv -\frac{2}{ST}\left[-\frac{(\mu^{2\epsilon})2}{\epsilon^2}[(-S)^{-\epsilon}+(-T)^{-\epsilon}-(-M^2)^{-\epsilon}]\right].\end{aligned} \tag{4.4}$$

The one-mass integral function is

$$I_3^{1\,\mathrm{m}}(K^2) = \frac{(\mu^{2\epsilon})2}{\epsilon^2}(-K^2)^{-1-\epsilon}, \tag{4.5}$$

and the two-mass triangle function is

$$I_3^{2\,\mathrm{m}}(K_1^2,K_2^2) = \frac{(\mu^{2\epsilon})2}{\epsilon^2}\frac{(-K_1^2)^{-\epsilon}-(-K_2^2)^{-\epsilon}}{(-K_1^2)-(-K_2^2)}. \tag{4.6}$$

The boxes, one-mass triangles, and two-mass triangles all have IR infinite terms of the form

$$\frac{(\mu^{2\epsilon})2}{\epsilon^2}(-K^2)^{-\epsilon}. \tag{4.7}$$

The coefficients of the triangle contributions can be evaluated using triple cuts [28–31] and a canonical basis [32]. Summation over the box and triangle contributions gives an overall coefficient of $M_5^{(1),\epsilon^0}(a^+,b^+,c^+,d^+,e^+)$, i.e.,

$$\left(\sum\mathcal{C}_{\{a,b\},c,d,e}I_4^{1\,\mathrm{m}}+\sum\mathcal{C}_{\{a,b,c\},d,e}I_3^{1\,\mathrm{m}}+\sum\mathcal{C}_{\{a,b\},c,\{d,e\}}I_3^{2\,\mathrm{m}}\right)_{\mathrm{IR}} = M_5^{(1),\epsilon^0}(a^+,b^+,c^+,d^+,e^+)\times\sum_{i<j}^{n}-\frac{1}{\epsilon^2}s_{ij}\left(\frac{\mu^2}{-s_{ij}}\right)^{\epsilon}, \tag{4.8}$$

where $M_5^{(1),\epsilon^0}(a^+,b^+,c^+,d^+,e^+)$ is the order ϵ^0 truncation of the one-loop amplitude. A key step is to promote the coefficient of these terms to the all-ϵ form of the one-loop amplitude, which then gives the correct singular structure of the amplitude. We have confirmed relationship (4.8) for the n-point amplitude by computing the triple and quadruple cuts at specific kinematic points up to $n = 10$.

Consequently, we can obtain the compact explicit analytic form for the dilogarithmic remainder part of the amplitude

$$F_5^{(2)} = \frac{1}{240}\sum_{P_{30}}\left(\frac{[ab]^6[cd]^2[ed]^2}{\langle ec\rangle^4}\right) \times (\langle ab\rangle^2\langle ec\rangle^2 + \langle ae\rangle^2\langle bc\rangle^2 + \langle ac\rangle^2\langle be\rangle^2) \times\left(-\frac{2}{s_{cd}s_{de}}\right)\left[\mathrm{Li}_2\left(1-\frac{s_{ab}}{s_{cd}}\right)+\mathrm{Li}_2\left(1-\frac{s_{ab}}{s_{de}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{cd}}{s_{de}}\right)+\frac{\pi^2}{6}\right], \quad (4.9)$$

where the permutation sum is over the 30 independent permutations of the legs $(\{a,b\},c,d,e)$ after factoring out for the symmetries $(\{a,b\},c,d,e) \equiv (\{b,a\},c,d,e) \equiv (\{a,b\},e,d,c)$.

Note that the coefficients in the $F_5^{(2)}$ term contain $\langle ec\rangle^{-4}$ singularities. On this singularity the integral function vanishes and $F_5^{(2)}$ has $\langle ec\rangle^{-3}$ singularities. These are spurious and not present in the full amplitude. They cancel against the $G_5^{(2)}$ terms as we will discuss at the end of the next section.

V. COEFFICIENT OF $\ln(s/\mu^2)$

We determine the presence of the $\ln(s_{ab}/\mu^2)$ functions using two-particle cuts. The coefficient has two contributions as shown in Fig. 3.

We determine these using canonical forms. The canonical basis approach [32] is a systematic method to determine the coefficients of triangle and bubble integral functions in a one-loop amplitude from the three- and two-particle cuts. A two-particle cut is of the form

$$C_2 \equiv i\int d^4\ell_1\delta(\ell_1^2)\delta(\ell_2^2)A_1^{\mathrm{tree}}(-\ell_1,a,\ldots,b,\ell_2) \times A_2^{\mathrm{tree}}(-\ell_2,\ldots,\ell_1). \quad (5.1)$$

The product of tree amplitudes appearing in the two-particle cut can be decomposed in terms of canonical forms $\mathcal{H}_i$,

$$A_1^{\mathrm{tree}}(-\ell_1,\ldots,\ell_2)\times A_2^{\mathrm{tree}}(-\ell_2,\ldots,\ell_1) = \sum e_i\mathcal{H}_i(\rho_k,\ell_j), \quad (5.2)$$

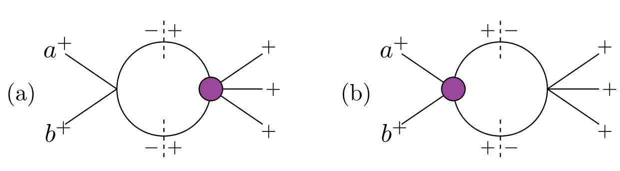

FIG. 3. Contributions to the two-loop amplitudes involving an all-plus loop (indicated by the solid circle). Contributions arise from (a) five-point and (b) four-point all-plus one-loop insertions.

where the e_i are coefficients independent of ℓ_j. We then use substitution rules to replace the $\mathcal{H}_i(\rho_k,\ell_j)$ by rational functions $H_i[\rho_k,P]$ to obtain the coefficient of the bubble integral function as

$$\sum_i e_iH_i[\rho_k,P]. \quad (5.3)$$

Here we use this technique treating the one-loop all-plus vertex as a tree amplitude.

The canonical forms we need and their substitutions are

$$\mathcal{H}_{1,1}^0(A,C;a,c,\ell_1,\ell_2) \equiv \frac{\langle a\ell_1\rangle}{\langle A\ell_1\rangle}\frac{\langle c\ell_2\rangle}{\langle C\ell_2\rangle} \to H_{1,1}^0[A,C;a,c;P] = \frac{\langle Cc\rangle[C|P|a\rangle}{\langle CA\rangle[C|P|C\rangle}+\frac{\langle Ac\rangle[A|P|a\rangle}{\langle AC\rangle[A|P|A\rangle},$$
$$\mathcal{H}_{2x}^0(A;B_1;C_1;\ell_1,\ell_2) \equiv \frac{\langle B_1\ell_1\rangle\langle C_1\ell_2\rangle}{\langle A\ell_1\rangle\langle A\ell_2\rangle} \to H_{2x}^0[A;B_1;C_1;P] = \frac{[A|P|B_1\rangle[A|P|C_1\rangle}{[A|P|A\rangle^2}, \quad (5.4)$$

where P is the cut-momentum $P = k_a + \ldots k_b$. We take the product of amplitudes in the cut and split them into a sum of terms of type in Eq. (5.4) using partial fractioning,

$$\frac{\prod_{j=1}^{n-1}\langle\ell X_j\rangle}{\prod_{i=1}^{n}\langle\ell Y_i\rangle} = \sum_{i=1}^{n}\left(\frac{\prod_{j=1}^{n-1}\langle Y_iX_j\rangle}{\prod_{l\neq i}\langle Y_lY_i\rangle}\right)\times\frac{1}{\langle\ell Y_i\rangle} = \sum_{i=1}^{n}\alpha_i\frac{1}{\langle\ell Y_i\rangle}. \quad (5.5)$$

For the five-point amplitude the cuts have $P = k_a + k_b$. Working specifically with $s_{ab} = s_{45}$, the two-particle cut has two contributions,

$$A: M_5^{(0)}(1^+,2^+,3^+,l_2^-,l_1^-)\times M_4^{(1)}(4^+,5^+,l_1^+,l_2^+),$$
$$B: M_5^{(1)}(1^+,2^+,3^+,l_2^+,l_1^+)\times M_4^{(0)}(4^+,5^+,l_1^-,l_2^-). \quad (5.6)$$

From the term A we obtain a contribution to the coefficient of

$$C_{45}^A = \sum_{r=1,2}c_r\sum_{i=1}^{3}\sum_{j=1,j\neq i}^{2}\alpha_i^r\beta_j^rH_{1,1}^0[A_i^r,a_j^r;B_3^r,b_3^r;P] + \sum_{r=1,2}c_r\sum_{i=1}^{2}\alpha_i^r\beta_i^rH_{2x}^0[A_i^r;B_3^r,b_3^r;P] \quad (5.7)$$

with

$$\alpha_i^r = \frac{\prod_{l=1}^{2}\langle B_l^rA_i^r\rangle}{\prod_{l\neq i}\langle A_l^rA_i^r\rangle}, \qquad \beta_i^r = \frac{\langle b_1^ra_i^r\rangle}{\prod_{l\neq i}\langle a_l^ra_i^r\rangle}, \quad (5.8)$$

where

$$c_1 = -\frac{[45]^6[23]}{120\langle 23\rangle\langle 13\rangle}, \qquad c_2 = \frac{[45]^6[12]}{120\langle 12\rangle\langle 13\rangle}, \tag{5.9}$$

and

$$\begin{aligned}
&\{|A_i^1\rangle\} = \{|1\rangle,|2\rangle,|3\rangle\}, \quad \{|A_i^2\rangle\} = \{|2\rangle,|3\rangle,|1\rangle\},\\
&\{|a_i^1\rangle\} = \{|1\rangle,|2\rangle\}, \quad \{a_i^2\} = \{|2\rangle,|3\rangle\},\\
&\{|B_i^1\rangle\} = \{|5\rangle,|5\rangle,-K_{45}|1]\}, \quad \{|B_i^2\rangle\} = \{|5\rangle,|5\rangle,-K_{45}|3]\},\\
&\{|b_i^1\rangle\} = \{|b_i^2\} = \{|4\rangle,|4\rangle\}.
\end{aligned} \tag{5.10}$$

Contributions from the second configuration are more complicated, but using relationships between the different terms to simplify the final expression we obtain

$$C_{45}^B = T_{1,2,3,4,5}^B + T_{2,1,3,4,5}^B + T_{3,2,1,4,5}^B \tag{5.11}$$

with

$$\begin{aligned}
T_{1,2,3,4,5}^B &= \frac{[23][45]}{120\langle 23\rangle\langle 45\rangle}\left(\sum_{i=2}^{5}\sum_{j=2,j\neq i}^{5}\alpha_i\beta_j H_{1,1}^0[i,j;A_4,B_4,\{4,5\}]\right.\\
&\quad\left.+\sum_{i=2}^{5}\alpha_i\beta_i H_{2x}^0[i;A_4,B_4,\{4,5\}]\right),
\end{aligned} \tag{5.12}$$

where

$$\alpha_i = \frac{\prod_{k=1}^3\langle iA_k\rangle}{\prod_{l\in\{2,3,4,5\}-\{i\}}\langle il\rangle}, \qquad \beta_j = \frac{\prod_{k=1}^3\langle jB_k\rangle}{\prod_{l\in\{2,3,4,5\}-\{j\}}\langle jl\rangle}, \tag{5.13}$$

and

$$\{|A_k\rangle\} = \{|B_k\rangle\} = \{|1\rangle, K|1], K|1], K|1]\}. \tag{5.14}$$

Thus the amplitude contains

$$(C_{45}^A + C_{45}^B)\left(\frac{1}{\epsilon} - \ln(s_{45}/\mu^2)\right). \tag{5.15}$$

The full amplitude thus has ϵ^{-1} and $\ln(\mu^2)$ terms of

$$\sum_{i<j}(C_{ij}^A + C_{ij}^B)\ln(s_{ij}) + \sum_{i<j}(C_{ij}^A + C_{ij}^B)\left(\frac{1}{\epsilon} + \ln(\mu^2)\right). \tag{5.16}$$

We have determined the above expression using four dimensional unitarity which isolates the coefficient of $\ln(s_{ij})$. The value of $\ln(\mu^2)$ then follows. The coefficient of ϵ^{-1} in this is tied to the $\ln(\mu^2)$ but is, presumably, not the value which would be obtained from a field theory calculation.

The bubble coefficients contain spurious $\langle ec\rangle^{-3}$ singularities which must cancel [33] against the singularities in the $F_5^{(2)}$ term. Specifically the singularities are of the form

$$\frac{1}{\langle ec\rangle^3} \times \{\ln(s_{ab}), \ln(s_{dc}), \ln(s_{ed}) + \text{permutations}\}, \tag{5.17}$$

where the permutations are of a, b, d. The $F_5^{(2)}$ term contains dilogarithms, but near the point $\langle ec\rangle = 0$ these simplify and

$$\begin{aligned}
F_5^{(2)} = \frac{1}{\langle ec\rangle^4}(0 + s_{ec}\{&\ln(s_{ab}), \ln(s_{dc}), \ln(s_{ed})\\
&+ \text{permutations}\} + \mathcal{O}(s_{ec}^2)).
\end{aligned} \tag{5.18}$$

We have explicitly checked that within $F_5^{(2)} + G_5^{(2)}$ both the $\langle ec\rangle^{-3}$ *and* $\langle ec\rangle^{-2}$ singularities cancel, leaving the full amplitude free of spurious singularities: this is a strong consistency check. We have also checked the collinear limit of the five-point amplitude.

VI. COUNTERTERM LAGRANGIAN

In this section we enumerate the possible independent counterterms for pure gravity in four dimensions. In general, graviton scattering amplitudes, in D dimensions at L loops, require the introduction of counterterms of the form

$$\nabla^n R^m, \tag{6.1}$$

where $n + 2m = (D-2)L + 2$ and we have suppressed the indices on R. R may stand for the Riemann tensor, R_{abcd}; the Ricci tensor, $R_{ab} \equiv g^{cd}R_{acbd}$; or the curvature scalar, $R \equiv g^{ab}R_{ab}$. Although there are a large number of tensor structures which may appear, fortunately, the symmetries of the Riemann tensor reduce these considerably. First, there are the basic symmetries of R_{abcd},

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}, \tag{6.2}$$

and the cyclic symmetry,

$$R_{abcd} + R_{acdb} + R_{adbc} = 0. \tag{6.3}$$

Second, we have the Bianchi identity for $\nabla_e R_{abcd}$,

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0. \tag{6.4}$$

There are also "derivative identities" which involve two covariant derivatives,

$$\begin{aligned}\nabla_e\nabla_f R_{abcd} - \nabla_f\nabla_e R_{abcd} &= R^g{}_{aef}R_{gbcd} + R^g{}_{bef}R_{agcd} \\ &\quad + R^g{}_{cef}R_{abgd} + R^g{}_{def}R_{abcg}, \\ \nabla^2 R_{abcd} &= 2R^f{}_{ace}R^e{}_{dbf} - 2R^f{}_{bce}R^e{}_{daf} \\ &\quad - R^e{}_{dab}R_{ce} + R^e{}_{cab}R_{de} \\ &\quad + \nabla_c\nabla_a R_{bd} - \nabla_c\nabla_b R_{ad} \\ &\quad - \nabla_d\nabla_a R_{bc} + \nabla_d\nabla_b R_{ac}. \end{aligned} \tag{6.5}$$

These symmetries will be used to determine the minimal set of inequivalent counterterms.

From power counting the possible two-loop counterterms in $D = 4$ are of the form R^3 or $\nabla^2 R^2$. The independent terms involving R_{abcd}, R_{ab}, and R are [34,35]

$$\begin{aligned} &T_1 = \nabla_a R\nabla^a R, \quad T_2 = \nabla_a R_{bc}\nabla^a R^{bc}, \\ &T_3 = \nabla_e R_{abcd}\nabla^e R^{abcd}, \quad T_4 = \nabla_c R_{ab}\nabla^b R^{ac}, \\ &T_5 = R^3, \quad T_6 = RR_{ab}R^{ab}, \\ &T_7 = RR_{abcd}R^{abcd}, \quad T_8 = R_{abcd}R^{abce}R^d_e, \\ &T_9 = R_{abcd}R^{ac}R^{bd}, \quad T_{10} = R_a{}^b R_b{}^c R_c{}^a, \\ &T_{11} = R^{ab}{}_{cd}R^{cd}{}_{ef}R^{ef}{}_{ab}, \quad T_{12} = R_{abcd}R^a{}_e{}^c{}_f R^{bedf}. \end{aligned} \tag{6.6}$$

For the case of pure gravity, the counterterm structure can be represented as a single counterterm with a numerical coefficient. We review the argument leading to the conclusion that a single counterterm is sufficient. (When matter is coupled to gravity, this is no longer the case.)

For pure gravity the equation of motion is

$$R_{ab} = 0. \tag{6.7}$$

Hence terms involving the Ricci tensor or curvature scalar will not contribute to the S-matrix, and such terms can be discarded when calculating the counterterms. (If calculating an off-shell object, such counterterms can, and do, appear.) Ignoring such terms leaves us with three tensors—T_3, T_{11}, and T_{12}. The term T_3,

$$T_3 = \nabla_e R_{abcd}\nabla^e R^{abcd} \equiv -R_{abcd}\nabla^2 R^{abcd}, \tag{6.8}$$

can be rearranged using the identity in Eq. (6.5) into terms involving the Ricci tensor plus cubic terms in the Riemann tensor. Thus for pure gravity this term is equivalent to a combination of T_{11} and T_{12} and can be eliminated from the list of inequivalent counterterms.

Finally, in six dimensions the scalar topological density can be written

$$\delta^{abcdef}_{[mnpqrs]}R^{mn}{}_{ab}R^{pq}{}_{cd}R^{rs}{}_{ef}, \tag{6.9}$$

which implies that the combination

$$\sum_{i=5}^{12} a_i T_i \equiv 0 \tag{6.10}$$

is topological for some coefficients a_i in dimensions $D \leq 6$. Hence for pure gravity amplitudes we can replace T_{12} with T_{11} (or vice versa). Thus we are led to the fact that the counterterm can be taken as a single tensor with a coefficient. Thus the counterterm can be chosen to be

$$\frac{C_{R^3}}{60} \times \left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^4} \int d^4x\sqrt{-g}R^{ab}{}_{cd}R^{cd}_{ef}R^{ef}{}_{ab} \tag{6.11}$$

with the free coefficient C_{R^3}.

Computing with this Lagrangian, the parts of the four-point amplitudes proportional to C_{R^3} are [36]

$$\begin{aligned} &\mathcal{M}_4^{R^3}(1^+,2^+,3^+,4^+) \\ &\quad = C_{R^3} \times \left(\frac{\kappa}{2}\right)^6 \times \frac{1}{(4\pi)^4}\left(\frac{st}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right)^2 stu, \\ &\mathcal{M}_4^{R^3}(1^-,2^+,3^+,4^+) \\ &\quad = C_{R^3} \times \left(\frac{\kappa}{2}\right)^6 \times \frac{1}{(4\pi)^4}\left(\frac{[24]^2 st^2}{[12]\langle 23\rangle\langle 34\rangle[41]}\right)^2 \frac{t}{10su}, \\ &\mathcal{M}_4^{R^3}(1^-,2^-,3^+,4^+) = C_{R^3} \times 0. \end{aligned} \tag{6.12}$$

Comparing this to the coefficient of $\ln(\mu^2)$ in Eq. (3.2) we find

$$C_{R^3} = -\frac{1}{4}. \tag{6.13}$$

We also require the five-point amplitude computed using the above Lagrangian. Perturbative gravity calculations based upon Feynman diagrams are notoriously difficult; however, we can compute the higher point functions using recursion.

The original Britto, Cachazo, Feng and Witten (BCFW) shift [37],

$$\lambda_i \to \lambda_i + z\lambda_j, \qquad \bar\lambda_j \to \bar\lambda_j - z\bar\lambda_i, \tag{6.14}$$

does not lead to an expression with the correct symmetry; however, the shift [38,39]

$$\begin{aligned} \lambda_a \to \lambda_{\hat a} &= \lambda_a + z[bc]\lambda_\eta, \\ \lambda_b \to \lambda_{\hat b} &= \lambda_b + z[ca]\lambda_\eta, \\ \lambda_c \to \lambda_{\hat c} &= \lambda_c + z[ab]\lambda_\eta, \end{aligned} \tag{6.15}$$

where λ_η is an arbitrary spinor does. Using this shift we can obtain an expression for the five-point amplitude. The amplitude has two factorizations,

$$M_3^{\rm tree}(\hat{a}^+, d^+, K^-) \times \frac{1}{K^2} \times M_4^{R^3}(\hat{b}^+, \hat{c}^+, d^+, -K^+),$$
$$M_3^{\rm tree}(\hat{b}^+, \hat{c}^+, K^-) \times \frac{1}{K^2} \times M_4^{R^3}(\hat{a}^+, d^+, e^+, -K^+) \quad (6.16)$$

with six terms of the first type and three of the second. With $(a, b, c, d, e) = (1, 2, 3, 4, 5)$ we find that the first term gives

$$T^A_{1,2,3,4,5} = N_4 \frac{[14]}{\langle 14\rangle} \frac{[53][52]}{\langle 1\eta\rangle^2 \langle 4\eta\rangle} \frac{[23]^2}{\langle 45\rangle} \times [5|K_{14}|\eta\rangle [2|K_{14}|\eta\rangle [3|K_{14}|\eta\rangle, \quad (6.17)$$

which is symmetric under $2 \leftrightarrow 3$. The second is

$$T^B_{1,2,3,4,5} = -N_4 \frac{[14][15][23][1|K_{23}|\eta\rangle^2 [5|K_{23}|\eta\rangle [4|K_{23}|\eta\rangle}{\langle 23\rangle \langle 2\eta\rangle^2 \langle 3\eta\rangle^2} \frac{[45]}{\langle 45\rangle}, \quad (6.18)$$

which is symmetric under $2 \leftrightarrow 3$ and $4 \leftrightarrow 5$. The normalization is

$$N_4 = \left(\frac{\kappa}{2}\right)^6 \times \frac{1}{(4\pi)^4} \times C_{R^3}. \quad (6.19)$$

The resultant contribution to the amplitude is

$$\mathcal{M}_5^{R^3}(1^+, 2^+, 3^+, 4^+, 5^+) = N_4 \left(\sum_{P_6} T^A_{1,2,3,4,5} + \sum_{P_3} T^B_{1,2,3,4,5} \right), \quad (6.20)$$

where the summation is over the six independent T^A and the three independent T^B.

The expression for $\mathcal{M}_5^{R^3}(1^+, 2^+, 3^+, 4^+, 5^+)$ is

(i) Fully crossing symmetric between external legs;
(ii) Independent of the spinor η.

These are strong indicators that we have computed the correct expression. We have checked this construction for the n-point all-plus amplitude up to $n = 10$.

Additionally

(i) As $z \to \infty$ for the BCFW shift the amplitude does not vanish but behaves as z^2. This is the reason why the shift (6.14) does not generate this amplitude.
(ii) The expression has soft limits with

$$\mathcal{M}_5^{R^3} = \left(\frac{1}{t^3} S^{(0)} + \frac{1}{t^2} S^{(1)} + \frac{1}{t} S^{(2)} \right) \mathcal{M}_4^{R^3} + O(t^0), \quad (6.21)$$

where $S^{(i)}$ are the leading, subleading, and sub-subleading soft operators [40]. As a two-loop amplitude, there is a possibility that the sub-subleading would not satisfy this so we regard this as a feature of the constructed amplitudes rather than as a necessary constraint.

Comparing expression (6.20) with (5.16) we find

$$\left(\sum_{P_6} T^A_{1,2,3,4,5} + \sum_{P_3} T^B_{1,2,3,4,5} \right) = -4 \times \sum_{i<j} (C^A_{ij} + C^B_{ij}), \quad (6.22)$$

and we therefore obtain $C_{R^3} = -1/4$. The counterterm is thus consistent with that required for the four-point amplitude.

VII. CONCLUSION

Computing quantum gravity amplitudes is notoriously difficult. Only a small number of on-shell scattering amplitudes have been computed analytically. For pure gravity only the four- and five-point one-loop amplitudes have been presented for all helicity configurations with all-n expressions for the all-plus and single-minus amplitudes. Progress beyond one-loop has been confined to theories which are supersymmetric where the enhanced symmetries significantly simplify the amplitudes.

In this article we have shown how four dimensional cutting techniques allow us to compute large and interesting parts of two-loop pure gravity amplitudes and have obtained the (poly)logarithmic parts of the all-plus helicity amplitude for four and five points in compact analytic forms. We also obtain the associated $\ln(\mu^2)$ terms which as argued in Ref. [13] determine the nonrenormalizability of the amplitudes. We have matched these to the same R^3 counterterm for both the four- and five-point amplitudes. Given that the $\ln(\mu^2)$ terms are key to renormalizability, this technique provides a straightforward method to study the UV behavior of gravity theories.

Our approach has been entirely based upon physical on-shell amplitudes and is very different from a field theory approach where the one-loop renormalization utilizes "evanescent" operators [13]. We do not obtain the ϵ^{-1} term found there, but we do reproduce the four-point *renormalized* amplitude and present a five-point amplitude correctly renormalized.

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