

Uncolored random tensors, melon diagrams, and the Sachdev-Ye-Kitaev models

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(Received 19 December 2016; published 13 February 2017)

Certain models with rank-3 tensor degrees of freedom have been shown by Gurau and collaborators to possess a novel large N limit, where $g^2 N^3$ is held fixed. In this limit the perturbative expansion in the quartic coupling constant, g , is dominated by a special class of “melon” diagrams. We study “uncolored” models of this type, which contain a single copy of real rank-3 tensor. Its three indices are distinguishable; therefore, the models possess $O(N)^3$ symmetry with the tensor field transforming in the tri-fundamental representation. Such uncolored models also possess the large N limit dominated by the melon diagrams. The quantum mechanics of a real anticommuting tensor therefore has a similar large N limit to the model recently introduced by Witten as an implementation of the Sachdev-Ye-Kitaev (SYK) model which does not require disorder. Gauging the $O(N)^3$ symmetry in our quantum mechanical model removes the nonsinglet states; therefore, one can search for its well-defined gravity dual. We point out, however, that the model possesses a vast number of gauge-invariant operators involving higher powers of the tensor field, suggesting that the complete gravity dual will be intricate. We also discuss the quantum mechanics of a complex 3-index anticommuting tensor, which has $U(N)^2 \times O(N)$ symmetry and argue that it is equivalent in the large N limit to a version of SYK model with complex fermions. Finally, we discuss similar models of a commuting tensor in dimension d . While the quartic interaction is not positive definite, we construct the large N Schwinger-Dyson equation for the two-point function and show that its solution is consistent with conformal invariance. We carry out a perturbative check of this result using the $4 - \epsilon$ expansion.

DOI: [10.1103/PhysRevD.95.046004](https://doi.org/10.1103/PhysRevD.95.046004)

I. INTRODUCTION

An important tool in theoretical physics is the study of certain models in the limit where they have a large number of degrees of freedom. Several different broad classes of such “large N limits” have been explored. Perhaps the most tractable large N limit applies to theories where the degrees of freedom transform as N -component vectors under a symmetry group. A well-known example is the $O(N)$ symmetric theory of N scalar fields ϕ^a in d dimensions with interaction $g(\phi^a \phi^a)^2$ (for reviews, see [1,2]). It is exactly solvable in the large N limit where gN is held fixed, since summation over the necessary class of bubble diagrams is not hard to evaluate. Another famous class of examples are models of interacting $N \times N$ matrix fields, so that the number of degrees of freedom scales as N^2 ; here, one can introduce single-trace interactions like $\text{tr} \phi^4$. A significant simplification occurs in the ’t Hooft large N limit where gN is held fixed: the perturbative expansion is dominated by the planar diagrams [3]. While such planar matrix theories are exactly solvable in some special low-dimensional cases [4], the problem does not appear to be solvable in general.

In view of these classic results, it is natural to study theories with rank- m tensor degrees of freedom $\phi^{a_1 \dots a_m}$, where each index takes N values so that the net number of degrees of freedom scales as N^m [5–7]. Since the

complexity of taking the large N limit increases from $m = 1$ to $m = 2$, one might expect that the tensor models with $m > 2$ are much more difficult than the matrix models. However, Gurau and collaborators [8–13] have discovered that, by adjusting the interactions appropriately, it is possible to find models with $m > 2$ where a large N limit is solvable. The perturbative expansion is then dominated by a special class of “melon diagrams” (for some examples with $m = 3$, see Fig. 1).

Gurau’s original example [8] was a so-called colored tensor model where complex fermionic tensors $\psi_A^{a_1 \dots a_m}$ carry an additional label A which takes $m + 1$ possible values $0, 1, \dots, m$. In the smallest nontrivial case $m = 3$ this model has the interaction

$$g \psi_0^{abc} \psi_1^{ade} \psi_2^{fbe} \psi_3^{fdc} + \text{c.c.} \quad (1.1)$$

The label A may be thought of as corresponding to the four different vertices of a tetrahedron. Each pair of fields has one pair of indices in common, just as every pair of vertices of a tetrahedron is connected by one edge. The interaction (1.1) has $U(N)^6$ symmetry, where each $U(N)$ corresponds to one of the edges of the tetrahedron. Including the quadratic piece $\psi_A^{abc} \bar{\psi}_A^{abc}$ and integrating over the fermionic tensors with interaction (1.1) generates a summation over a particular class of three-dimensional intrinsic geometries

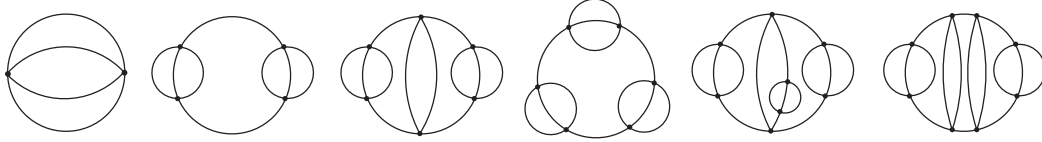


FIG. 1. Some melonic contributions to the free energy.

made out of tetrahedra. Apart from this interpretation, this model is of much interest because it exhibits a novel type of large N limit, where the coupling constant is scaled so that $g^2 N^3$ is held constant, and the theory has N^3 degrees of freedom.¹ Thus, it is interesting to try generalizing Gurau's construction² from the $d = 0$ tensor integral case to d -dimensional quantum theories. An important step in this direction was recently made by Witten [19], who studied a quantum mechanical model of colored anticommuting tensors and observed that its large N limit is similar to that in the Sachdev-Ye-Kitaev (SYK) model [20–23].

The quantum mechanical model introduced by Witten uses, in the $m = 3$ case, real fermionic tensors ψ_A^{abc} and possesses $O(N)^6$ symmetry. The action of this model is

$$S_{\text{Gurau-Witten}} = \int dt \left(\frac{i}{2} \psi_A^{abc} \partial_t \psi_A^{abc} + g \psi_0^{abc} \psi_1^{ade} \psi_2^{fbc} \psi_3^{fdc} \right). \quad (1.2)$$

It was shown [19,24] that, in the large N limit of this model, only the “melonic” Feynman graphs survive, just as in the SYK model. Very importantly, gauging the $O(N)^6$ symmetry gets rid of the nonsinglet states. This removes a crucial conceptual obstacle in the search for the gravity dual of this model, in the spirit of the AdS/CFT correspondence for gauge theories [25–27].

In work subsequent to [8] it was shown that the “coloring” is not necessary for obtaining a large N limit where the melon graphs dominate, and theories of just one complex bosonic tensor were shown to have this property [13,28–30]. More recently, a model of a single real rank-3 tensor with $O(N)^3$ symmetry was studied by Carrozza and Tanasa and shown to possess a melonic large N limit [31]. We study such a theory of one real rank- m fermionic tensor with interaction ψ^{m+1} . For $m = 3$, the interaction assumes the explicit form

$$V_{\text{uncolored}} = \frac{1}{4} g \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \psi^{a_2 b_2 c_1}. \quad (1.3)$$

¹The N^3 scaling of the degrees of freedom is also found for six-dimensional CFTs on N coincident M5-branes [14,15]. An interpretation of this scaling in terms of M2-branes with three holes attached to three different M5-branes, thus giving rise to tri-fundamental matter, was proposed in Sec. 5.2 of [16]. One may wonder if there is a precise connection between theories on M5-branes and tensor models.

²The random tensor models also have connections with the “holographic space-time” approach to quantum gravity [17,18].

The three indices are distinguishable, and the theory has $O(N)^3$ symmetry under

$$\begin{aligned} \psi^{abc} &\rightarrow M_1^{ad} M_2^{bb'} M_3^{cc'} \psi^{a'b'c'}, \\ M_1, M_2, M_3 &\in O(N). \end{aligned} \quad (1.4)$$

Thus, the real field ψ^{abc} transforms in the tri-fundamental representation of $O(N)^3$. Such an uncolored fermionic model does not work in $d = 0$ because the invariant quadratic term vanishes, $\psi^{abc} \psi^{abc} = 0$, but in $d = 1$ there is a nontrivial model with the kinetic term $\frac{i}{2} \psi^{abc} \partial_t \psi^{abc}$. We also consider analogous bosonic models where the anticommuting field in (1.3) is replaced by a commuting one, ϕ^{abc} . Then, in $d = 0$ we may add the quadratic term $\phi^{abc} \phi^{abc}$, while in $d > 0$ the standard kinetic term $\frac{1}{2} \partial_\mu \phi^{abc} \partial^\mu \phi^{abc}$. While the bosonic potential is generally not positive definite,³ the model may still be studied in perturbation theory. One may hope that, as in the matrix models, the restriction to leading large N limit can formally stabilize the theory.

In Sec. II we study the index structure of the expansion of the path integral in g and demonstrate that the large N limit is dominated by the melon diagrams.⁴ The argument, which applies to both the uncolored fermionic and bosonic models, contains a new ingredient compared to other models. In uncolored models with complex tensors, which were studied in [13], each index loop necessarily passed through an even number of vertices, but in models with real tensors a loop can also pass through an odd number of vertices. However, the diagrams dominant in the large N limit do not contain any index loops that pass through three vertices.

In Sec. III we show that the uncolored fermionic theory with interaction (1.3) is equivalent to the SYK model in the large N limit. We comment on the spectrum of operators in the gauged tensor models, pointing out that it appears to be vastly bigger than the “single Regge trajectory” which has been studied in the SYK model so far [32–35]. In Sec. III A we write down a $U(N)^2 \times O(N)$ symmetric quantum mechanical model with a complex fermionic 3-tensor. We study the large N limit of this model and derive the scaling dimensions of two-particle operators. We argue that

³We thank E. Witten for pointing this out to us.

⁴We constructed the argument before the existence of [31] was pointed out to us, so it may provide an independent perspective on the $O(N)^3$ theories.

this model is related in the large N limit to the generalization of SYK model which contains complex fermions [36,37]. In Sec. IV we study the large N limit of the uncolored bosonic model with $O(N)^3$ symmetry. We derive the Schwinger-Dyson equation for the two-point function and obtain its solution which is consistent with conformal invariance. It gives scaling dimension $\Delta_\phi = \frac{d}{4} + \mathcal{O}(1/N)$. We also derive the Schwinger-Dyson equation for the four-point function and find the scaling dimensions of two-particle operators. In Sec. IV we also mention models with only one $O(N)$ symmetry group and matter in the fully symmetric and traceless or antisymmetric representations. In these cases we have checked the melonic dominance at large N up to order g^7 , but a general proof seems harder to construct. In Sec. IVA we check this result by a perturbative calculation in $4 - \epsilon$ dimensions at large N . In Sec. V we discuss various possible extensions of our results, including supersymmetric models with quartic superpotentials for 3-tensor superfields.

II. MELONIC DOMINANCE IN THE $O(N)^3$ SYMMETRIC THEORIES

The arguments in this section, which are analogous to those in [31], apply to the uncolored models with $O(N)^3$ symmetry, both in the fermionic and bosonic cases and for any d . We ignore the coordinate dependence and just focus on the index structure.

The propagator of the ϕ^{abc} field has the index structure depicted in Fig. 2. The three colored wires (also called “strands” in the earlier literature) represent propagation of the three indices of the ϕ^{abc} field. In spite of this coloring, the model is “uncolored” in the standard terminology, since it contains only one tensor field. The vertex has the index structure depicted in Fig. 3. There are three equivalent ways to draw the vertex; for concreteness we use the first way. “Forgetting” the middle lines we obtain the standard matrix model vertex as in Fig. 4.

Let us consider the vacuum Feynman diagrams. Examples of melonic and nonmelonic diagrams with their resolved representations and fat (double-line) subgraphs are depicted in Figs. 5 and 6.



FIG. 2. The resolved propagator $\langle \phi^{abc} \phi^{a'b'c'} \rangle = \delta^{aa'} \delta^{bb'} \delta^{cc'}$.



FIG. 3. Three equivalent ways to represent the resolved vertex.

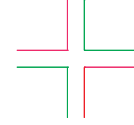


FIG. 4. The standard matrix model vertex obtained after “forgetting” the middle lines.

Each resolved Feynman diagram consists of loops of three different colors and is proportional to $N^{f_{\text{total}}}$, where f_{total} is the total number of index loops. Suppose we “forget” all wires of some particular color in our diagram, as in the Figs. 5 and 6. Then, we get a double-line fat graph (ribbon graph) of the kind one finds in matrix models. One can count the number of all index loops f in this fat graph using the Euler characteristic χ

$$f = \chi + e - v, \quad (2.1)$$

where e is the number of edges and v is the number of vertices. In our theory we obviously have $e = 2v$; therefore, $f = \chi + v$. We can forget red, blue or green wires, and in each case we get a fat graph made of the remaining two colors. If we forget, say, all red wires, then using the formula (2.1) we find $f_{bg} = \chi_{bg} + v$, where $f_{bg} = f_b + f_g$ is the number of blue and green loops and χ_{bg} is the Euler characteristic of this blue-green fat graph. Analogously, we get $f_{rg} = \chi_{rg} + v$ and $f_{br} = \chi_{br} + v$. Adding up all these formulas, we find

$$\begin{aligned} f_{bg} + f_{rg} + f_{br} &= 2(f_b + f_g + f_r) \\ &= \chi_{bg} + \chi_{br} + \chi_{rg} + 3v. \end{aligned} \quad (2.2)$$

Thus, the total number of loops is

$$f_{\text{total}} = f_b + f_g + f_r = \frac{3v}{2} + 3 - g_{bg} - g_{br} - g_{rg}, \quad (2.3)$$

where $g = 1 - \chi/2$ is the genus of a graph. Because $g \geq 0$, we obtain

$$f_{\text{total}} \leq 3 + \frac{3v}{2}. \quad (2.4)$$

Now the goal is to show that the equality $f_{\text{total}} = 3 + 3v/2$ is satisfied only for the melonic diagrams. We call the graphs which satisfy $f_{\text{total}} = 3 + 3v/2$ the maximal graphs. Thus, we should argue that maximal graphs are necessarily melonic. We note that, due to (2.3), each double-line fat subgraph of a maximal graph has genus zero.

Now, let us classify all loops in our graph according to how many vertices they pass through (a loop can pass the same vertex twice). Let us denote by $\mathcal{F}_s \geq 0$ the number of loops, which pass through s vertices. For a maximal graph,

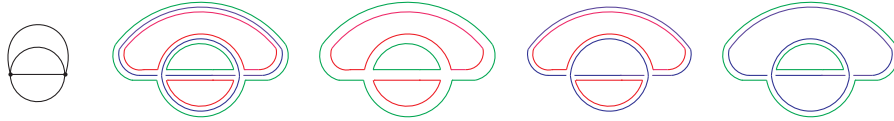


FIG. 5. A melonic second-order diagram and all its fat subgraphs.

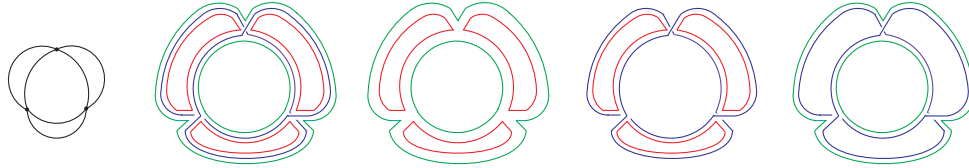


FIG. 6. A nonmelonic third-order diagram and all its fat subgraphs.

$$f_{\text{total}} = \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4 + \mathcal{F}_5 + \dots = 3 + \frac{3v}{2}, \quad (2.5)$$

where we set $\mathcal{F}_1 = 0$ because we assume that there are no tadpole diagrams. Since each vertex must be passed six times, we also get

$$2\mathcal{F}_2 + 3\mathcal{F}_3 + 4\mathcal{F}_4 + 5\mathcal{F}_5 + \dots = 6v. \quad (2.6)$$

Combining these two equations, we find

$$2\mathcal{F}_2 + \mathcal{F}_3 = 12 + \mathcal{F}_5 + 2\mathcal{F}_6 + \dots. \quad (2.7)$$

Now our goal is to show that $\mathcal{F}_2 > 0$ using this formula (in fact, $\mathcal{F}_2 \geq 6$, but all we need is that it is nonvanishing).

Let us first argue that a maximal graph must have $\mathcal{F}_3 = 0$. To have $\mathcal{F}_3 > 0$, we need a closed index loop passing through three vertices. Without a loss of generality, we can assume that this loop is formed by the middle lines in each vertex (blue lines). The only possibility with a closed loop of an internal (blue) index, which passes through three vertices, is shown in Fig. 7(a). After “forgetting” the color of this loop, we get a fat graph in Fig. 7(b), which is nonplanar due a twisted propagator. So, a graph with $\mathcal{F}_3 > 0$ cannot be maximal. Thus, setting $\mathcal{F}_3 = 0$ in (2.7), we deduce that a maximal graph should have $\mathcal{F}_2 > 0$.

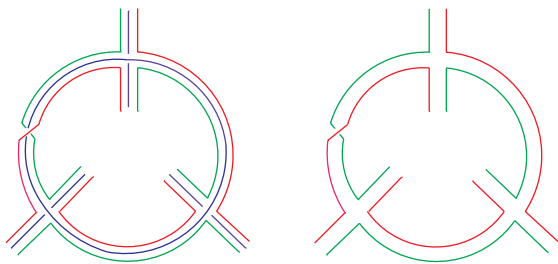


FIG. 7. (a) Local part of a graph with a middle index loop passing through three vertices. (b) The same figure where the middle index has been “forgotten.”

Finally, we need to show that the graphs with $\mathcal{F}_2 > 0$ are melonic. To do this, we follow Proposition 3 in [12]. Without a loss of generality, we assume that the loop passing through two vertices is formed by the middle lines in each vertex (blue lines). The only such possibility is shown in Fig. 8(a). After “forgetting” the color of this loop, we get a fat graph in Fig. 8(b).

Now we uncolor the lines in our fat graph and cut and sew two edges as in Fig. 9. We cut two edges but did not change the number of loops; therefore, the Euler characteristic of the new graph is $\chi = 4$. This is possible only if we separated our original graph into two genus zero parts. Therefore, our graph is two-particle reducible for the internal and external couples of lines. Thus, the whole unresolved graph looks like Fig. 10. Then, if graphs G' and



FIG. 8. (a) Local part of a graph with a middle index loop passing through two vertices v_1 and v_2 . (b) The same figure where the middle index has been “forgotten.”

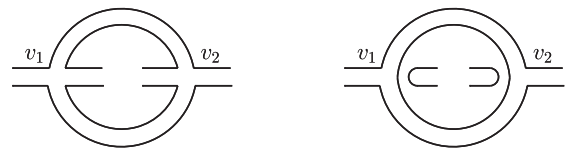


FIG. 9. Cutting and sewing lines.

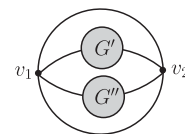


FIG. 10. General structure of the maximal graph.

G'' are empty, we get a second-order melon graph as in Fig. 5. If they are not empty, one can argue (see [12]) that they are also maximal graphs. So, we can recursively apply the same above argument to them, implying that the complete diagram is melonic.

III. UNCOLORED QUANTUM MECHANICS AND THE SYK MODEL

Using the interaction (1.3) we now consider an “uncolored” quantum mechanical model with real anti-commuting variables $\psi^{abc}(t)$ and the action

$$S = \int dt \left(\frac{i}{2} \psi^{abc} \partial_t \psi^{abc} + \frac{1}{4} g \psi^{a_1 b_1 c_1} \psi^{a_2 b_2 c_2} \psi^{a_2 b_1 c_2} \psi^{a_2 b_2 c_1} \right). \quad (3.1)$$

It has 1/4 of the degrees of freedom of the colored Gurau-Witten model (1.2). We argue that the uncolored model (3.1) is equivalent to the SYK model in the large N limit.

We recall that ψ^{abc} are the N^3 anticommuting fields and the indices, each of which runs from 1 to N , are treated as distinguishable. The Fermi statistics implies

$$\begin{aligned} \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \psi^{a_2 b_2 c_1} \\ = -\psi^{a_1 b_2 c_2} \psi^{a_1 b_1 c_1} \psi^{a_2 b_1 c_2} \psi^{a_2 b_2 c_1}. \end{aligned} \quad (3.2)$$

After relabeling $b_1 \leftrightarrow c_2$ and $b_2 \leftrightarrow c_1$, we get the relation

$$\begin{aligned} \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \psi^{a_2 b_2 c_1} \\ = -\psi^{a_1 c_1 b_1} \psi^{a_1 c_2 b_2} \psi^{a_2 c_2 b_1} \psi^{a_2 c_1 b_2}. \end{aligned} \quad (3.3)$$

This demonstrates the vanishing of the interaction term in the $O(N)$ symmetric theory with a fully symmetric or fully antisymmetric fermionic tensor. Fortunately, in the theory with general 3-index fermionic tensors the interaction is nontrivial.

Let us return, therefore, to the theory (3.1) with $O(N)^3$ symmetry, where the three indices are distinguishable. The symmetry may be gauged by the replacement

$$\begin{aligned} \partial_t \psi^{abc} \rightarrow (D_t \psi)^{abc} = \partial_t \psi^{abc} + A_1^{aa'} \psi^{a'bc} \\ + A_2^{bb'} \psi^{ab'c} + A_3^{cc'} \psi^{abc'}, \end{aligned} \quad (3.4)$$

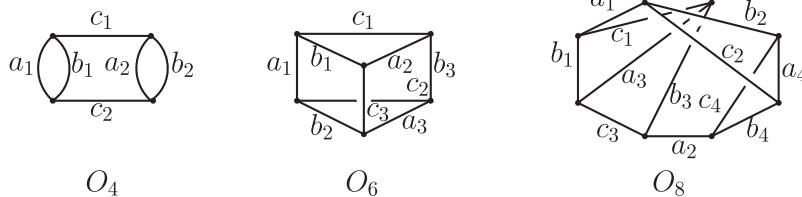


FIG. 11. Graphical representation of different operators.

where A_i is the gauge field corresponding to the i th $O(N)$ group. In $d = 1$ the gauge fields are nondynamical, and their only effect is to restrict the operators to be gauge singlets. There is a sequence of such operators of the form

$$O_2^n = \psi^{abc} (D_t^n \psi)^{abc}, \quad (3.5)$$

where n is odd. This set of operators is analogous to the “single Regge trajectory” [32,33,35] found in the Sachdev-Ye-Kitaev (SYK) model [20–23].

We should note, however, that theory (3.1) contains an abundance of additional “single-trace” $O(N)^3$ symmetric operators. A large class of them contains an even number of ψ fields without derivatives and with all indices contracted. One of such ψ^4 operators is the interaction term in the action, which is related by the equation of motion to $\psi^{abc} (D_t \psi)^{abc}$. Another type of ψ^4 operator is

$$O_4 = \psi^{a_1 b_1 c_1} \psi^{a_1 b_1 c_2} \psi^{a_2 b_2 c_1} \psi^{a_2 b_2 c_2}, \quad (3.6)$$

and there are similar operators where the second and third or the first and third indices have pairwise contractions (however, in the theory where the $O(N)^3$ symmetry is gauged such operators vanish because they are squares of the gauge symmetry generators). Moving on to the higher operators, we can try writing down the following ψ^6 operator:

$$O_6 = \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_1 c_3} \psi^{a_2 b_3 c_1} \psi^{a_3 b_2 c_3} \psi^{a_3 b_3 c_2}. \quad (3.7)$$

Due to the fermi statistics, this operator actually vanishes, but an operator with ψ fields replaced by scalars ϕ is present in the bosonic model that we study in Sec. IV. The following ψ^8 operator does not vanish in the fermionic model:

$$\begin{aligned} O_8 = \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_3 c_3} \psi^{a_2 b_4 c_4} \psi^{a_3 b_1 c_3} \\ \times \psi^{a_3 b_3 c_1} \psi^{a_4 b_2 c_4} \psi^{a_4 b_4 c_2}. \end{aligned} \quad (3.8)$$

All such operators can be represented graphically with ψ -fields corresponding to vertices and index contractions to edges (see Fig. 11). These representations are similar to the Feynman diagrams in ϕ^3 theory. A feature of the latter two operators is that each pair of ψ -fields has either one or no indices in common. We expect to find an infinite class of

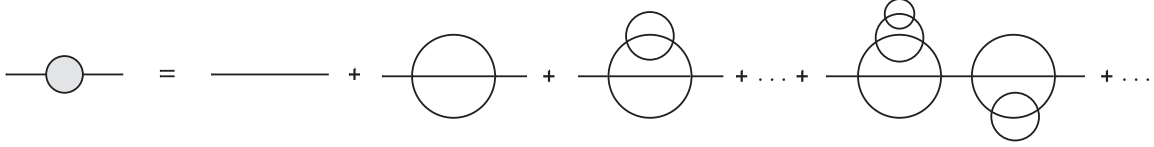


FIG. 12. Diagrams contributing to the two-point function in the leading large N order. The line with the gray circle represents the full two-point function. Each simple line is the bare propagator.

operators of this type—they should correspond to some number of tetrahedra glued together. Since there is no parametrically large dimension gap in the set of operator dimensions, the holographic dual of this theory should be highly curved.

Let us study some of the diagrammatics of the uncolored quantum mechanics model (3.1). We study the ungauged model; the effect of the gauging may be imposed later by restricting to the gauge invariant operators. The bare propagator is

$$\begin{aligned} \langle T(\psi^{abc}(t)\psi^{a'b'c'}(0)) \rangle_0 &= \delta^{aa'}\delta^{bb'}\delta^{cc'}G_0(t) \\ &= \delta^{aa'}\delta^{bb'}\delta^{cc'}\frac{1}{2}\text{sgn}(t). \end{aligned} \quad (3.9)$$

The full propagator in the large N limit receives corrections from the melonic diagrams represented in Fig. 12. Resummation of all melonic diagrams leads to the Schwinger-Dyson equation for the two-point function

$$\begin{aligned} G(t_1 - t_2) &= G_0(t_1 - t_2) \\ &+ g^2 N^3 \int dt dt' G_0(t_1 - t)G(t - t')^3 G(t' - t_2), \end{aligned} \quad (3.10)$$

represented graphically in Fig. 13. This is the same equation as derived in [32,33,35] for the large N SYK model. The solution to (3.10) in the IR limit is

$$G(t_1 - t_2) = -\left(\frac{1}{4\pi g^2 N^3}\right)^{1/4} \frac{\text{sgn}(t_1 - t_2)}{|t_1 - t_2|^{1/2}}. \quad (3.11)$$

To uncover the spectrum of the bilinear operators in the model, we need to study the four-point function $\langle \psi^{a_1 b_1 c_1}(t_1)\psi^{a_2 b_2 c_2}(t_2)\psi^{a_3 b_3 c_3}(t_3)\psi^{a_4 b_4 c_4}(t_4) \rangle$. Its structure is again the same as in the large N SYK model [32,33]:

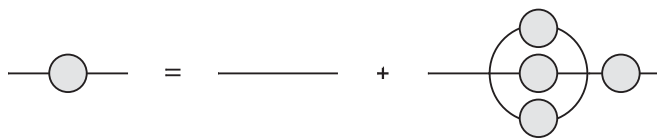


FIG. 13. The graphical representation of the Schwinger-Dyson equation for the two-point function.

$$\begin{aligned} &\langle \psi^{a_1 b_1 c_1}(t_1)\psi^{a_2 b_2 c_2}(t_2)\psi^{a_3 b_3 c_3}(t_3)\psi^{a_4 b_4 c_4}(t_4) \rangle \\ &= N^6 G(t_{12})G(t_{34}) + \Gamma(t_1, \dots, t_4), \end{aligned} \quad (3.12)$$

where $\Gamma(t_1, \dots, t_4)$ is given by a series of ladder diagrams depicted in Fig. 14.

Resumming the diagrams in Fig. 14, one finds a contribution to $\Gamma(t_1, \dots, t_4)$ as a series of diagrams in terms of the full propagators, see Fig. 15.

If we denote by Γ_n the ladder with n rungs, so $\Gamma = \sum_n \Gamma_n$, we have

$$\begin{aligned} \Gamma_0(t_1, \dots, t_4) &= N^3 (-G(t_{13})G(t_{24}) \\ &+ G(t_{14})G(t_{23})). \end{aligned} \quad (3.13)$$

For the next coefficient, one gets

$$\begin{aligned} \Gamma_1(t_1, \dots, t_4) &= 3g^2 N^6 \int dt dt' (G(t_1 - t)G(t_2 - t') \\ &\times G(t - t')^2 G(t - t_3)G(t - t_4) - (t_3 \leftrightarrow t_4)), \end{aligned} \quad (3.14)$$

and one can check further that

$$\begin{aligned} \Gamma_2(t_1, \dots, t_4) &= -3g^2 N^3 \int dt dt' (G(t_1 - t)G(t_2 - t') \\ &\times G(t - t')^2 \Gamma_1(t, t', t_3, t_4) - (t_3 \leftrightarrow t_4)). \end{aligned} \quad (3.15)$$

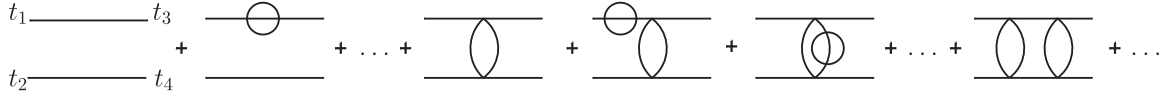
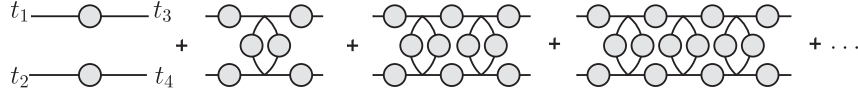
So, in general, one gets exactly the same recursion relation as in the SYK model,

$$\Gamma_{n+1}(t_1, \dots, t_4) = \int dt dt' K(t_1, t_2; t, t') \Gamma_n(t, t', t_3, t_4), \quad (3.16)$$

where the kernel is

$$K(t_1, t_2; t_3, t_4) = -3g^2 N^3 G(t_{13})G(t_{24})G(t_{34})^2. \quad (3.17)$$

In order to find the spectrum of the two-particle operators O_2^n , following [33,35] one has to solve the integral eigenvalue equation


 FIG. 14. Ladder diagrams contributing to $\Gamma(t_1, \dots, t_4)$.

 FIG. 15. Ladder diagrams contributing to $\Gamma(t_1, \dots, t_4)$.

$$v(t_0, t_1, t_2) = g(h) \int dt_3 dt_4 K(t_1, t_2; t_3, t_4) v(t_0, t_3, t_4), \quad (3.18)$$

where

$$v(t_0, t_1, t_2) = \langle O_2^n(t_0) \psi^{abc}(t_1) \psi^{abc}(t_2) \rangle = \frac{\text{sgn}(t_1 - t_2)}{|t_0 - t_1|^h |t_0 - t_2|^h |t_1 - t_2|^{1/2-h}}, \quad (3.19)$$

is the conformal three-point function. Then the scaling dimensions are determined by the equation $g(h) = 1$. To find $g(h)$ one can use the $SL(2)$ invariance to take t_0 to infinity and just consider eigenfunctions of the form

$$v(t_1, t_2) = \frac{\text{sgn}(t_1 - t_2)}{|t_1 - t_2|^{1/2-h}}. \quad (3.20)$$

It is not hard to find $g(h)$ using two basic integrals

$$\begin{aligned} \int_{-\infty}^{+\infty} du \frac{\text{sgn}(u - t_1) \text{sgn}(u - t_2)}{|u - t_1|^a |u - t_2|^b} &= l_{a,b}^+ \frac{1}{|t_1 - t_2|^{a+b-1}}, \\ \int_{-\infty}^{+\infty} du \frac{\text{sgn}(u - t_2)}{|u - t_1|^a |u - t_2|^b} &= l_{a,b}^- \frac{\text{sgn}(t_1 - t_2)}{|t_1 - t_2|^{a+b-1}}, \\ l_{a,b}^\pm &= \beta(1 - a, a + b - 1) \\ &\pm (\beta(1 - b, a + b - 1) - \beta(1 - a, 1 - b)), \end{aligned} \quad (3.21)$$

where $\beta(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Euler beta function. Plugging (3.20) into (3.18) and using (3.21), we find [33,35]

$$g(h) = -\frac{3}{4\pi} l_{\frac{3}{2}-h, \frac{1}{2}}^+ l_{1-h, \frac{1}{2}}^- = -\frac{3 \tan(\frac{\pi}{2}(h - \frac{1}{2}))}{2(h - 1/2)}. \quad (3.22)$$

The scaling dimensions are given by the solutions of $g(h) = 1$. The first solution is exact, $h = 2$; this is the important mode dual to gravity and responsible for the quantum chaos in the model [32–34,38–41]. The further solutions are $h \approx 3.77, 5.68, 7.63, 9.60$ corresponding to

the operators $\psi^{abc}(D_t^n \psi)^{abc}$ with $n = 3, 5, 7, 9$, respectively. In the limit of large n , $h_n \rightarrow n + \frac{1}{2}$. This is the expected limit $n + 2\Delta$, where $\Delta = \frac{1}{4}$ is the scaling dimension of the individual fermion.

A. Models with a complex fermion

Here, we consider two quantum mechanical models of a complex 3-tensor ψ^{abc} . One of them is an uncolored version of the colored quantum mechanical model recently studied by Gurau [24]:

$$S = \int dt \left(i\bar{\psi}^{abc} \partial_t \psi^{abc} + \frac{1}{4} g \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \psi^{a_2 b_2 c_1} + \frac{1}{4} \bar{g} \bar{\psi}^{a_1 b_1 c_1} \bar{\psi}^{a_1 b_2 c_2} \bar{\psi}^{a_2 b_1 c_2} \bar{\psi}^{a_2 b_2 c_1} \right), \quad (3.23)$$

which again has $O(N)^3$ symmetry. Another possibility is the model

$$S = \int dt \left(i\bar{\psi}^{abc} \partial_t \psi^{abc} + \frac{1}{2} g \psi^{a_1 b_1 c_1} \bar{\psi}^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \bar{\psi}^{a_2 b_2 c_1} \right), \quad (3.24)$$

where the symmetry is enhanced to $U(N) \times O(N) \times U(N)$ because the transformations on the first and the third indices of the tensor are allowed to be $U(N)$. Models of this type have been studied in $d = 0$ [28–30]. Gauging this symmetry in the quantum mechanical model restricts the operators to the singlet sector, allowing for the existence of a gravity dual. The gauge invariant two-particle operators have the form

$$\mathcal{O}_2^n = \bar{\psi}^{abc} (D_t^n \psi)^{abc} \quad n = 0, 1, \dots, \quad (3.25)$$

which includes $\bar{\psi}^{abc} \psi^{abc}$. There is also a variety of operators made out of the higher powers of the fermionic fields similarly to the operators (3.6), (3.7), (3.8) in the $O(N)^3$ symmetric model of real fermions. As established in [28–30], the large N limit of the complex uncolored model (3.24) is once again given by the melon diagrams

(the arguments are easier than in Sec. II since each index loop passes through an even number of vertices). The large N limit of this model appears to be related to the variant of SYK model where the real fermions are replaced by the complex ones [36,37].

Let us briefly discuss summing over melonic graphs in the model (3.24) at large N . The two-point function has the structure

$$\langle T(\bar{\psi}^{abc}(t)\psi^{a'b'c'}(0)) \rangle = \delta^{aa'}\delta^{bb'}\delta^{cc'}G(t), \quad (3.26)$$

and $G(t) = -G(-t)$. We find the same Schwinger-Dyson equation as (3.10); its solution is again (3.11) indicating that the fermion scaling dimension is $\Delta = 1/4$. Now, we need to study the four-point function $\langle \bar{\psi}^{a_1b_1c_1}(t_1) \times \psi^{a_1b_1c_1}(t_2)\bar{\psi}^{a_2b_2c_2}(t_3)\psi^{a_2b_2c_2}(t_4) \rangle$. It leads to the same integral eigenvalue equation as (3.18), but with kernel⁵

$$K(t_1, t_2; t_3, t_4) = -g^2N^3(2G(t_{13})G(t_{24})G(t_{34})^2 - G(t_{14})G(t_{23})G(t_{34})^2). \quad (3.27)$$

Now it is possible to have not only the antisymmetric eigenfunctions as in (3.20), but also the symmetric ones

$$v(t_1, t_2) = \frac{1}{|t_1 - t_2|^{1/2-h}}. \quad (3.28)$$

This can be justified by noticing that the three-point function now is $\langle \mathcal{O}_2^n(t_0)\psi^{abc}(t_1)\bar{\psi}^{abc}(t_2) \rangle$. We see that for odd n it is antisymmetric under $t_1 \leftrightarrow t_2$, while for even n it is symmetric. Substituting ansatz (3.28) into the integral equation (3.18), and using the integrals (3.21), we find

$$g_{\text{sym}}(h) = -\frac{1}{4\pi} \int_{\frac{3}{2}-h}^+ \int_{\frac{1}{2}}^+ = -\frac{1}{2} \frac{\tan(\frac{\pi}{2}(h + \frac{1}{2}))}{h - 1/2}. \quad (3.29)$$

The scaling dimensions of the operators \mathcal{O}_2^n with even n are given by the solutions of $g_{\text{sym}}(h) = 1$. The first eigenvalue is $h = 1$, corresponding to the conserved $U(1)$ charge $\bar{\psi}^{abc}\psi^{abc}$. The additional values are $h \approx 2.65, 4.58, 6.55, 8.54$ corresponding to the operators with $n = 2, 4, 6, 8$ respectively. For large n the scaling dimensions approach $n + 1/2$ as expected. The numerical results are in good agreement with the asymptotic formula [33]

$$h_n = n + \frac{1}{2} + \frac{1}{\pi n} + \mathcal{O}(n^{-3}) \quad (3.30)$$

for $n > 2$. For \mathcal{O}_2^n with odd n , the spectrum is the same as for the two-particle operators (3.5) in the model with $O(N)^3$ symmetry.

⁵We are grateful to J. Maldacena and D. Stanford for a very useful discussion about this, which helped us correct the normalization of (3.29).

IV. UNCOLORED BOSONIC TENSORS

In this section we consider the d -dimensional field theory of a real commuting tensor field $\phi^{abc}(x)$ with distinguishable indices $a, b, c = 1, \dots, N$:

$$S = \int d^d x \left(\frac{1}{2} \partial_\mu \phi^{abc} \partial^\mu \phi^{abc} + \frac{1}{4} g \phi^{a_1b_1c_1} \phi^{a_1b_2c_2} \phi^{a_2b_1c_2} \phi^{a_2b_2c_1} \right), \quad (4.1)$$

This is the bosonic analogue of the uncolored fermionic theory with interaction (1.3); it again has $O(N)^3$ symmetry. A feature of this theory is that the interaction potential is not bounded from below for $N > 2$. For $N = 2$, the potential may be written as a sum of squares, but for $N > 2$ we have explicitly checked that there is a negative direction. Nevertheless, we may consider formal perturbation theory in g .

The argument in Sec. II that the melonic diagrams dominate in the large N limit applies both to the fermionic and bosonic version of the theory in any dimension d . We may therefore resum all such diagrams and derive the exact Schwinger-Dyson equation similar to that in [8–11,42]. Let us explain this using a simple example of the two-point function in the theory (4.1).

We have for the bare propagator

$$\begin{aligned} \langle \phi^{abc}(p)\phi^{a'b'c'}(-p) \rangle_0 &= G_0(p)\delta^{aa'}\delta^{bb'}\delta^{cc'} \\ &= \frac{1}{p^2} \delta^{aa'}\delta^{bb'}\delta^{cc'}. \end{aligned} \quad (4.2)$$

In the large N limit one gets the same Schwinger-Dyson equation for the full two-point function as in (3.10), which we can write in the momentum space as

$$G(p) = G_0(p) + \lambda^2 G_0(p)\Sigma(p)G(p), \quad (4.3)$$

where we introduced the coupling $\lambda = gN^{3/2}$, which is held fixed in the large N limit and

$$\Sigma(p) = \int \frac{d^d k d^d q}{(2\pi)^{2d}} G(q)G(k)G(p+q+k). \quad (4.4)$$

One can rewrite (4.3) as

$$G^{-1}(p) = G_0^{-1}(p) - \lambda^2 \Sigma(p). \quad (4.5)$$

In the IR limit we can neglect the bare term $G_0(p)$ and get

$$G^{-1}(p) = -\lambda^2 \int \frac{d^d k d^d q}{(2\pi)^{2d}} G(q)G(k)G(p+q+k). \quad (4.6)$$

Using the integral

$$\begin{aligned}
 & \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^{2\alpha}(k+p)^{2\beta}} \\
 &= \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(d/2 - \alpha)\Gamma(d/2 - \beta)\Gamma(\alpha + \beta - d/2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d - \alpha - \beta)} \\
 & \quad \times \frac{1}{(p^2)^{\alpha+\beta-d/2}} \quad (4.7)
 \end{aligned}$$

it is not difficult to show that the solution to Eq. (4.6) is

$$G(p) = \lambda^{-1/2} \left(\frac{(4\pi)^d d \Gamma(\frac{3d}{4})}{4\Gamma(1 - \frac{d}{4})} \right)^{1/4} \frac{1}{(p^2)^{\frac{d}{4}}}. \quad (4.8)$$

Alternatively, one can work in the coordinate representation and use the Fourier transform

$$\int d^d x \frac{e^{ikx}}{(x^2)^\alpha} = \frac{\pi^{d/2} \Gamma(d/2 - \alpha)}{2^{2\alpha-d} \Gamma(\alpha)} \frac{1}{(k^2)^{d/2-\alpha}} \quad (4.9)$$

to find the solution of the equation $G^{-1}(x) = -\lambda^2 G^3(x)$:

$$G(x) = \lambda^{-1/2} \left(\frac{d \Gamma(\frac{3d}{4})}{4\pi^d \Gamma(1 - \frac{d}{4})} \right)^{1/4} \frac{1}{(x^2)^{\frac{d}{4}}}. \quad (4.10)$$

If one works with the cutoff regularization, then the UV divergence, which arises in the integrals can be absorbed into mass renormalization. Remarkably, the Schwinger-Dyson equation (4.6) was originally studied in 1964, and its $d = 3$ solution (4.8) was found [43].⁶

To find the spectrum of two-particle operators, we must solve for the eigenvalues $g_{\text{bos}}(h)$ and eigenvectors v_h of the kernel [32,33,35]

$$\int d^d x_3 d^d x_4 K(x_1, x_2; x_3, x_4) v_h(x_3, x_4) = g_{\text{bos}}(h) v_h(x_1, x_2), \quad (4.11)$$

where the kernel is⁷

$$K(x_1, x_2; x_3, x_4) = 3\lambda^2 G(x_{13})G(x_{24})G(x_{34})^2. \quad (4.12)$$

It is not hard to check using the integral (4.7) that there is a subset of spin-zero eigenvectors

$$v_h(x_1, x_2) = \frac{1}{[(x_1 - x_2)^2]^{\frac{d}{2}(h-1)}}, \quad (4.13)$$

and the corresponding eigenvalues are

$$g_{\text{bos}}(h) = -\frac{3\Gamma(\frac{3d}{4})\Gamma(\frac{d}{4} - \frac{h}{2})\Gamma(\frac{h}{2} - \frac{d}{4})}{\Gamma(-\frac{d}{4})\Gamma(\frac{3d}{4} - \frac{h}{2})\Gamma(\frac{d}{4} + \frac{h}{2})}. \quad (4.14)$$

⁶We thank A. Polyakov for pointing this out to us.

⁷We thank S. Giombi for correcting the sign error in the kernel that was present in an earlier version of this paper.

The scaling dimensions h of spin zero two-particle operators are then determined by solving $g_{\text{bos}}(h) = 1$. In $d = 1$ the smallest positive solution is $h = 2$, suggesting the existence of a gravity dual. We plan to study a more complete set of scaling dimensions in general d in future work.

We have also studied the effect of replacing in (4.1) the general 3-index tensor by the completely symmetric and traceless tensor field ϕ^{abc} . Such a theory would have a single $O(N)$ symmetry under

$$\phi^{abc} \rightarrow M^{ad} M^{bb'} M^{cc'} \phi^{a'b'c'}, \quad M \in O(N). \quad (4.15)$$

The corresponding fermionic model would be trivial due to the vanishing of the interaction, but the bosonic model is nontrivial. The key question is whether the leading contribution in N comes from the melonic diagrams only. We have checked all the vacuum diagrams up to order g^7 and did not find any violation of this rule (see p. 257 in [44] for pictures of all vacuum diagrams up to order g^7), but we have not constructed a proof to all orders yet. If the $O(N)$ symmetric theory of a symmetric traceless tensor is indeed melonic, then the derivation of the Schwinger-Dyson equation and its solution goes through just as for the $O(N)^3$ symmetric theory of a general tensor.

Finally, we may wonder if for theories with a single $O(N)$ group we may consider matter in other irreducible representations, such as completely antisymmetric or mixed symmetry. Such theories also appear to be melonic at low orders in perturbation theory, but a general proof to all orders has not been constructed.

A. ϵ -Expansion of a scaling dimension

Let us consider the uncolored bosonic model in $d = 4 - \epsilon$ dimension. Introducing renormalized fields and couplings and using an auxiliary scale μ , we can compactly write the action (4.1) in the form

$$S = \int d^d x \left(\frac{1}{2} (\partial_\mu \vec{\phi})^2 + \frac{1}{4} \mu^\epsilon g \vec{\phi}^4 + \frac{1}{2} \delta_\phi (\partial_\mu \vec{\phi})^2 + \frac{1}{4} \mu^\epsilon \delta_g \vec{\phi}^4 \right), \quad (4.16)$$

where $\vec{\phi} = \phi^{abc}$ and $\vec{\phi}^4 \equiv \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1}$. The latter is not the only quartic term allowed by the $O(N)^3$ symmetry. To renormalize the theory at finite N , we need to include two additional operators: the double-trace operator $O_{\text{double-trace}} = (\phi^{abc} \phi^{abc})^2$ and the ‘‘pillow operator’’ [31]:

$$\begin{aligned}
 O_{\text{pillow}} &= \phi^{a_1 b_1 c_1} \phi^{a_1 b_1 c_2} \phi^{a_2 b_2 c_1} \phi^{a_2 b_2 c_2} \\
 &+ \phi^{a_1 b_1 c_1} \phi^{a_2 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_2 c_2} \\
 &+ \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_1} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_2}. \quad (4.17)
 \end{aligned}$$

In this section we carry out just the leading large N analysis of operator dimension Δ_ϕ , where we believe the effects of these additional operators may be ignored.

The bare coupling is related to the renormalized one as

$$g_0 = \mu^\epsilon Z_g Z_\phi^{-2} g, \quad (4.18)$$

where $Z_g = 1 + \delta_g/g$ and $Z_\phi = 1 + \delta_\phi$. The bare propagator is

$$\langle \phi^{abc}(p) \phi^{a'b'c'}(-p) \rangle_0 = \delta^{aa'} \delta^{bb'} \delta^{cc'} \frac{1}{p^2}. \quad (4.19)$$

To compute the anomalous dimension of ϕ , we consider the diagram in Fig. 16.

We find at $d = 4 - \epsilon$ in the large N limit

$$\delta_\phi = -\frac{g^2 N^3}{2(4\pi)^4 \epsilon}, \quad \gamma_\phi = \frac{g^2 N^3}{2(4\pi)^4}. \quad (4.20)$$

On the other hand, for the four-point function all the one- and two-loop diagrams are subleading in N . Therefore, the beta-function is defined by the counter-term δ_ϕ

$$g_0 = \mu^\epsilon (g - 2\delta_\phi g) = \mu^\epsilon \left(g + \frac{g^3 N^3}{(4\pi)^4 \epsilon} + \dots \right). \quad (4.21)$$

In the large N limit where $g^2 N^3$ is held fixed, the beta-function is

$$\beta_g = -\epsilon g + \frac{2N^3 g^3}{(4\pi)^4} + \dots \quad (4.22)$$

The theory in $4 - \epsilon$ dimensions has a weakly coupled IR fixed point at

$$g_*^2 = \frac{(4\pi)^4 \epsilon}{2N^3}, \quad (4.23)$$

where the anomalous dimension at the critical point is

$$\gamma_\phi = \frac{\epsilon}{4}. \quad (4.24)$$

This leads to

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi = 1 - \frac{\epsilon}{4} + \mathcal{O}(\epsilon^2), \quad (4.25)$$

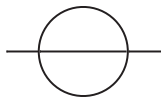


FIG. 16. Two-loop diagram contributing to the anomalous dimension of the ϕ field.

which agrees with the large N scaling dimension $\frac{d}{4}$ obtained in (4.8). It would be interesting to extend this perturbative analysis of the melonic ϕ^4 theory to higher orders in ϵ and to also include the $1/N$ corrections.

V. DISCUSSION

The existence of quantum mechanical models without disorder which admit a novel large N limit dominated by the melonic graphs, such as the colored models explored in [19,24] and the uncolored models in Sec. III, opens new avenues for further research. We have shown that various aspects of the $O(N)^3$ symmetric uncolored tensor model (3.1) agree in the large N limit with the SYK model [20–23]. Our uncolored tensor model is similar to the colored Gurau-Witten model (1.2). In particular, both models possess the same universal ‘‘Regge trajectory’’ of two-particle operators as has been uncovered in the SYK model [32,33,35]. In the uncolored model these are operators (3.5), while in the Gurau-Witten model they are $\psi_A^{abc}(D_i^n \psi_A)^{abc}$. It would be interesting to carry out a more detailed comparison between the colored and uncolored models. As we have discussed, the uncolored model has a tower of gauge invariant operators ψ^n . The same is true for the Gurau-Witten model; for example, at eighth order we find the operator

$$O_8 = \psi_0^{a_1 b_1 c_1} \psi_1^{a_1 d_1 e_1} \psi_0^{a_2 b_1 c_2} \psi_1^{a_2 d_1 e_1} \psi_0^{a_3 b_2 c_1} \times \psi_1^{a_3 d_2 e_2} \psi_2^{f_1 b_2 e_2} \psi_3^{f_1 d_2 c_2} \quad (5.1)$$

and similar operators with other choices of colors. The details of the operator spectra are not the same, however, due to the extra color label the Gurau-Witten model contains more gauge invariant operators than our uncolored model.

In the uncolored model, in addition to the quartic operator in the action (3.1) there are quartic operators of the form (3.6) [however, in the theory where the $O(N)^3$ symmetry is gauged such operators vanish]. These kinds of operators are also present in the colored model, such as $\psi_0^{abc} \psi_0^{fbc} \psi_1^{ade} \psi_1^{fde}$ and analogous operators with other choices of the color labels. In [31] such ‘‘pillow operators’’ were included in the action and shown not to destroy the melonic dominance in the large N limit. Thus, imposing the $O(N)^3$ invariance appears to produce a class of quartic quantum mechanical models rather than a unique model. This is reminiscent of the fact that, in the $SU(N)$ symmetric quantum mechanics of a Hermitian matrix Φ with potential $\text{tr}\Phi^4$, one can add a double-trace term $(\text{tr}\Phi^2)^2$, which can modify the free energy even in the leading large N limit [45–47]. The operators (3.6) in the tensor model seem analogous to the double-trace operators in the matrix model, and their effect needs to be studied carefully.

Some of the recent interest in the SYK model is related to the fact that it exhibits quantum chaos [32–34,39–41]. This was investigated numerically at finite N , providing further insights [48–50]. Since at large N the tensor quantum mechanical models become equivalent to the SYK model, one would expect them to be chaotic as well, at least for sufficiently large N . A numerical investigation of the energy levels and thermal partition functions in the finite N melonic quantum mechanical models should be possible. The procedure would be quite different from that in [48,50] because there is no averaging over disorder. This may facilitate such a numerical study in the context of tensor quantum mechanics.

It is also very interesting to ask if there exist quantum field theories in dimensions above 1, which possess such a melonic large N expansion. In Sec. IV we began to study a bosonic ϕ^4 tensor model which is renormalizable in $d = 4$ and by power counting may flow to a CFT in dimensions below 4. However, such a theory has the potential unbounded from below for $N > 2$, so it does not appear to be stable at finite N .

Another interesting possibility is to consider a supersymmetric theory with rank-3 tensor superfields Φ^{abc} and superpotential

$$W = \frac{1}{4} g \Phi^{a_1 b_1 c_1} \Phi^{a_1 b_2 c_2} \Phi^{a_2 b_1 c_2} \Phi^{a_2 b_2 c_1}. \quad (5.2)$$

A simple setting for such a superspace approach is the supersymmetric quantum mechanics [51,52], where (see, for example, [53])

$$\begin{aligned} \Phi^{a_1 b_1 c_1}(t, \theta, \bar{\theta}) &= \phi^{a_1 b_1 c_1}(t) + i\theta \psi^{a_1 b_1 c_1}(t) - i\bar{\psi}^{a_1 b_1 c_1}(t) \bar{\theta} \\ &+ \bar{\theta} \theta F^{a_1 b_1 c_1}(t), \end{aligned} \quad (5.3)$$

so that the degrees of freedom consist of a real bosonic 3-tensor and a complex fermionic one. The action may then be written as

$$S = \int dt d\theta d\bar{\theta} \left(\frac{1}{2} |D_\theta \Phi^{abc}|^2 + W \right), \quad (5.4)$$

and in terms of components it contains the two-boson two-fermion terms, such as

$$\int dt (g \bar{\psi}^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1} + \text{c.c.}). \quad (5.5)$$

It also contains the 6-boson interaction term $g^2 \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_3} \phi^{a_2 b_3 c_3} \phi^{a_3 b_2 c_3} \phi^{a_3 b_3 c_2}$, whose index structure is the same as that found in operator O_6 shown in (3.7); it can be represented graphically as the prism (see Fig. 11).

Such a construction may be viewed as a dimensional reduction of the $\mathcal{N} = 1$ supersymmetric theory in $d = 3$,

where it is renormalizable (the field content is a real scalar ϕ^{abc} and a two-component Majorana fermion χ^{abc}). The interaction becomes relevant in $d < 3$ so that the theory may flow to an interacting CFT. Alternatively, we could study a renormalizable $\mathcal{N} = 2$ supersymmetric theory in $d = 3$, whose field content is a complex scalar ϕ^{abc} and a two-component Dirac fermion. Since the R -charge of ϕ is fixed by the quartic superpotential (5.2) to be $1/2$, we know that its exact dimension is

$$\Delta_\phi = \frac{d-1}{2} R = \frac{d-1}{4}. \quad (5.6)$$

The dimension of the fermion superpartner is then $\Delta_\psi = \Delta_\phi + \frac{1}{2} = \frac{d+1}{4}$, so that the interaction term $\bar{\psi}^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1} + \text{c.c.}$ has dimension d . The scalar potential

$$V = |g|^2 \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \bar{\phi}^{a_3 b_3 c_1} \bar{\phi}^{a_3 b_2 c_4} \bar{\phi}^{a_2 b_3 c_4} \quad (5.7)$$

is, of course, non-negative. It would also be interesting to study a ‘‘colored’’ supersymmetric theory with superfields Φ_A^{abc} and quartic superpotential

$$W = g \Phi_0^{abc} \Phi_1^{ade} \Phi_2^{fbc} \Phi_3^{fdc}. \quad (5.8)$$

The existence of perturbative expansion using supergraphs suggests that the large N limit is dominated by melonic diagrams. The quantum properties of these supersymmetric theories in $d < 3$ may be studied using both the large N Schwinger-Dyson equations and the $3 - \epsilon$ expansion. We hope to address these problems in the future.

ACKNOWLEDGMENTS

We are very grateful to E. Witten for important advice on many aspects of this paper and for really useful comments on a draft. We thank R. Gurau for useful comments on a draft and especially for pointing out Ref. [31]. We also thank S. Giombi, D. Gross, J. Maldacena, J. Murugan, A. Polyakov, and D. Stanford for very useful discussions. The work of I. R. K. and G. T. was supported in part by the US NSF under Grant No. PHY-1620059. G. T. acknowledges the support of a Myhrvold-Havranek Innovative Thinking Fellowship.

Note added in the proof.—The construction of theories for a single rank 3 tensor field with the quartic interaction (1.3) may be generalized to a single rank $q - 1$ tensor with the $O(N)^{q-1}$ symmetric interaction of order q . Since the indices of each $O(N)$ group must be contracted pairwise, q has to be even. Every pair of tensors in the interaction term has one index in common. For example, for $q = 6$ the explicit form of the interaction of a real rank 5 tensor is

$$V_{\text{int}} = i \frac{g}{6} \psi^{a_1 b_1 c_1 d_1 e_1} \psi^{a_2 b_2 c_2 d_2 e_2} \psi^{a_3 b_3 c_3 d_3 e_3} \\ \times \psi^{a_2 b_3 c_2 d_1 e_3} \psi^{a_3 b_3 c_1 d_3 e_2} \psi^{a_3 b_1 c_3 d_2 e_3}.$$

Such rank $q - 1$ tensor theories have a large N limit with $g^2 N^{(q-1)(q-2)/2}$ held fixed, which is dominated by the

melonic diagrams. When ψ is taken to be a real anti-commuting rank $q - 1$ tensor in $0 + 1$ dimensions, we find an $O(N)^{q-1}$ symmetric quantum mechanical model. It provides a tensor implementation of the version of SYK model where the random interaction couples q fermions.

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