PHYSICAL REVIEW D **95,** 046001 (2017)

Fuzzy CP^2 spacetimes

A. Chaney[*] and A. Stern[†]

Department of Physics, University of Alabama, Tuscaloosa, Alabama 35487, USA

(Received 14 December 2016; published 1 February 2017)

Four-dimensional manifolds with changing signature are obtained by taking the large N limit of fuzzy CP^2 solutions to a Lorentzian matrix model. The regions of Lorentzian signature give toy models of closed universes which exhibit cosmological singularities. These singularities are resolved at finite N, as the underlying CP^2 solutions are expressed in terms of finite matrix elements.

DOI: 10.1103/PhysRevD.95.046001

I. INTRODUCTION

Fuzzy spheres are defined by $N \times N$ irreducible matrix representations of the $su(2)$ algebra [1–8]. In a previous work, [9], we showed that a fuzzy sphere embedded in a Minkowski background, which we denote by $S_F^{2,L}$, can serve as a two-dimensional toy model of a closed noncommutative cosmology. Noncommutative or matrix cosmologies have been of interest for some time, and they possess a limit, the "commutative" limit, where a spacetime manifold is recovered from the matrix configurations [10–18]. In [9], the commutative limit for $S_F^{2,L}$ corresponds to taking $N \to \infty$, which yields a sphere embedded in Minkowski space. This "sphere" had several novel features. The curvature computed from the induced metric is not constant and there are singularities at two fixed latitudes. Also, the induced metric has changing signature. Signature change is known to be a possible feature of both classical and quantum gravity [19–28]. The region bounded by the singular latitudes has Lorentzian signature, and describes a closed two-dimensional spacetime. The two singular latitudes behave as cosmological singularities, which get resolved at finite N.

The question naturally arises as to whether one can generalize $S_F^{2,L}$ to four-dimensional fuzzy cosmologies. Of course, a trivial generalization is obtained by taking the tensor product of two noncommutative spaces, for example $S_F^{2,L} \times S_F^2$, S_F^2 being a fuzzy sphere in a Euclidean background. Such tensor product spaces appear after extremizing the sum of two bosonic matrix actions, consisting of Yang-Mills terms, analogous to what appears in the Ishibashi, Kawai, Kitazawa, and Tsuchiya (IKKT) model [29], along with cubic terms and mass terms. The results of [9] can be straightforwardly repeated in this case.

Here, instead, we examine fuzzy CP^2 (CP_F^2) [7,30–35]. In order for time to emerge in the large N limit we embed CP_F^2 in a Lorentzian background. This CP_F^2 results from extremizing a matrix model action, again consisting of a Yang-Mills term, cubic term and mass term. The large N limit yields four dimensional manifolds which are, loosely speaking, embeddings of CP^2 in an eight (or greater)-dimensional Lorentzian target space with a flat metric tensor. Analogous to the two-dimensional model in [9], the induced metric tensor on the four-dimensional surface can have changing signature. Signature changes occur at two three-dimensional singular surfaces, which define the boundaries between regions of Euclidean and Lorentzian signature. The region of Lorentzian signature defines a closed spacetime, with the singular surfaces playing the role of cosmological singularities. A novel feature of these toy universes is that the cosmological singularities occur at nonzero distance scales, and that time cannot be defined for smaller distance scales. As with the two-dimensional models in [9], the singularities appear only after taking the large N limit, and so the finite N matrix description once again resolves the singularities of the continuum description.

The outline of this article is the following: We review CP^2 in Sec. II and CP_F^2 in Sec. III. In Sec. IV we show that CP_F^2 solutions result from both Euclidean and Lorentzian Yang-Mills type-matrix models. A one-parameter family of deformed CP_F^2 solutions (which contains the undeformed solution) is also found to Lorentzian Yang-Mills type-matrix model, which requires a mass term, along with a cubic term, in the action. The large N (or commutative) limit of these solutions is taken in Sec. V. There we plot the distance scale versus time in the comoving frame for the CP^2 universes. Concluding remarks are given in Sec. VI.

II. CP^2

To define CP^2 one starts with a three-dimensional complex vector space spanned by $z = (z_1, z_2, z_3)$, where $z_i \in C$ are not all zero, and then makes the identification of z with γz, for all complex nonzero values of γ. CP^2 can equivalently be defined as the space of $U(1)$ orbits on the unit 5—sphere S^5. The latter is spanned by z with $|z|^2 = z_i^* z_i = 1$, where i is summed from 1 to 3, and a point on the space of $U(1)$ orbits is $\{e^{i\beta} z, 0 \le \beta < 2\pi\}$. Upon introducing the following Poisson brackets

[*]adchaney@crimson.ua.edu
[†]astern@ua.edu

$$\{z_i, z_j^*\} = -i\delta_{ij} \qquad \{z_i, z_j\} = \{z_i^*, z_j^*\} = 0, \tag{2.1}$$

one can generate the $U(1)$ orbits from the 5–sphere constraint

$$\mathcal{C} = z_i^* z_i - 1 \approx 0 \tag{2.2}$$

Infinitesimal variations δ_ϵ along an orbit are then

$$\delta_\epsilon z_i = \{z_i, \mathcal{C}\}\epsilon = -i\epsilon z_i \qquad \delta_\epsilon z_i^* = \{z_i^*, \mathcal{C}\}\epsilon = i\epsilon z_i^*, \tag{2.3}$$

where ε is an infinitesimal parameter.

CP^2 is also defined as $SU(3)/U(2)$, i.e., the space of adjoint orbits of $SU(3)$ through λ^8, where $\lambda^\alpha, \alpha = 1, 2, \ldots, 8$ are the Gell-Mann matrices, $CP^2 = \{U\lambda^8 U^\dagger, U \in SU(3)\}$. Upon introducing

$$x^\alpha = \frac{\bar{z}\lambda^\alpha z}{|z|^2}, \tag{2.4}$$

one recovers the $su(3)$ Lie algebra from the Poisson bracket algebra on CP^2. Using the commutator $[\lambda^\alpha, \lambda^\beta] = 2if^{\alpha\beta\gamma}\lambda^\gamma$ we get from (2.1) that

$$\{x^\alpha, x^\beta\} = \frac{2}{|z|^2} f^{\alpha\beta\gamma} x^\gamma, \tag{2.5}$$

after imposing the constraint (2.2). x^α are functions of z and $\bar{z}$ which are invariant under $z \leftrightarrow \gamma z, \gamma \in C$, and so they span a four-dimensional constrained surface, i.e., CP^2, in $\mathbb{R}^8$. The constraints on x^α are

$$x^\alpha x^\alpha = \frac{4}{3} \qquad d^{\alpha\beta\gamma} x^\beta x^\gamma = \frac{1}{3} x^\alpha, \tag{2.6}$$

where $d^{\alpha\beta\gamma}$ are defined from the anticommutator of Gell-Mann matrices $[\lambda^\alpha, \lambda^\beta]_+ = \frac{4}{3}\delta_{\alpha,\beta}\mathbb{1}_3 + 2d^{\alpha\beta\gamma}\lambda^\gamma$, $\mathbb{1}_3$ being the 3×3 identity matrix. The constraints in (2.6) follow from the expression for x^α in (2.4).

The metric on CP^2 is given by

$$ds_E^2 = \frac{4}{|z|^4}(|z|^2|dz|^2 - |\bar{z}dz|^2), \tag{2.7}$$

where $|dz|^2 = dz_i^* dz_i$ and $\bar{z}dz = z_i^* dz_i$. It is known as the Fubini-Study metric and is invariant under: $z \to \gamma z$, $dz \to d\gamma z + \gamma dz$. The isometry of the metric is $SU(3)/Z_3$, with corresponding transformations: $z \to Uz$, $U \in SU(3)/Z_3$. The Fubini-Study metric is recovered from the embedding (2.4) of CP^2 in the $\mathbb{R}^8$ target space, where one assumes a flat Euclidean metric tensor. That is, starting with the $SO(8)$ invariant

$$ds_E^2 = dx^\alpha dx^\alpha, \tag{2.8}$$

and then substituting (2.4), one recovers (2.7).

The metric tensor in (2.7) can be reexpressed in terms of a pair of complex coordinates $\zeta_a = z_a/z_3$, $a = 1, 2$ (away from $z_3 = 0$), which are invariant under $z \to \gamma z$. Along with their complex conjugates, they span CP^2 when $z_3 \neq 0$. In terms of these coordinates, the invariant length (2.7) becomes

$$ds_E^2 = 2g_{a\bar{b}} d\zeta_a d\zeta_b^* = \frac{4}{(|\zeta|^2+1)^2}((|\zeta|^2+1)|d\zeta|^2 - |\bar{\zeta}d\zeta|^2), \tag{2.9}$$

where $|\zeta|^2 = \zeta_a^*\zeta_a$, $|d\zeta|^2 = d\zeta_a^* d\zeta_a$ and $\bar{\zeta}d\zeta = \zeta_a^* d\zeta_a$. It is well known to satisfy the sourceless Einstein equations with a positive cosmological constant, specifically $\Lambda = \frac{3}{2}$. From (2.1), the Poisson brackets are given by

$$\begin{aligned} &\{\zeta_a, \zeta_b^*\} = -i(|\zeta|^2+1)(\zeta_a\zeta_b^* + \delta_{ab}) \\ &\{\zeta_a, \zeta_b\} = \{\zeta_a^*, \zeta_b^*\} = 0. \end{aligned} \tag{2.10}$$

Their inverse gives the symplectic two-form

$$\Omega = -\frac{i}{2} g_{a\bar{b}} d\zeta_a \wedge d\zeta_b^*, \tag{2.11}$$

which is also the Kähler two-form.

The invariant length and Kähler two-form can furthermore be expressed in terms of left-invariant Maurer-Cartan one forms ω_i on $SU(2)$, satisfying $d\omega_i + \epsilon_{ijk}\omega_j \wedge \omega_k = 0$. For this take $\omega_i = \frac{i}{2}\mathrm{Tr}\sigma_i u^\dagger du$, where u is the $SU(2)$ matrix

$$u = \frac{1}{|\zeta|}\begin{pmatrix} \zeta_1^* & -i\zeta_2 \\ -i\zeta_2^* & \zeta_1 \end{pmatrix} \tag{2.12}$$

and σ_i are Pauli matrices. One can write

$$ds_E^2 = \frac{4(d|\zeta|)^2}{(|\zeta|^2+1)^2} + \frac{4|\zeta|^2}{(|\zeta|^2+1)}(\omega_1^2 + \omega_2^2) + \frac{4|\zeta|^2}{(|\zeta|^2+1)^2}\omega_3^2 \tag{2.13}$$

and

$$\Omega = -2d\left(\frac{|\zeta|^2}{(|\zeta|^2+1)}\omega_3\right). \tag{2.14}$$

The isometry of the metric tensor is $SU(3)/Z_3$. For any $|\zeta|(\neq 0)$-slice, the metric tensor and symplectic two-form are invariant under $SU(2) \times U(1)/Z_2$. The latter symmetry is also present for the manifolds we obtain in Sec. V. (On the other hand, the $SU(3)/Z_3$ isometry is broken for those manifolds.) The $SU(2) \times U(1)/Z_2$ transformations on u are of the form $u \to u' = vue^{i\lambda\sigma_3}$, $v \in SU(2)$ and $\lambda \in R$, which leave ω_3 and $\omega_1^2 + \omega_2^2$ invariant. We can parametrize the $SU(2)$ matrices in (2.12) by Euler angles (θ, ϕ, ψ) according to

$$\zeta_1 = e^{i\left(\frac{\psi+\phi}{2}\right)}\cos\frac{\theta}{2}|\zeta| \qquad \zeta_2 = e^{i\left(\frac{\psi-\phi}{2}\right)}\sin\frac{\theta}{2}|\zeta|, \tag{2.15}$$

where in order to span all of $SU(2)$, $0 \le \theta < \pi$, $0 \le \psi < 2\pi$ and $0 \le \phi \le 4\pi$. On the other hand, to parametrize the

Maurer-Cartan one forms, we only need ϕ to run from 0 to 2π. In terms of the Euler angles, the metric is given by

$$ds_E^2 = \frac{4(d|\zeta|)^2}{(|\zeta|^2+1)^2} + \frac{|\zeta|^2}{(|\zeta|^2+1)}(d\theta^2+\sin^2\theta d\phi^2) + \frac{|\zeta|^2}{(|\zeta|^2+1)^2}(d\psi+\cos\theta d\phi)^2. \quad (2.16)$$

The Killing vectors $k_a, a=1,\ldots,4$, generating the $SU(2)\times U(1)/Z_2$ isometry group on any $|\zeta|(\neq 0)$-slice are expressed in terms of the Euler angles according to

$$k_1 \pm ik_2 = e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\left(\cot\theta\frac{\partial}{\partial\phi} - \csc\theta\frac{\partial}{\partial\psi}\right)\right) \qquad k_3 = \frac{\partial}{\partial\phi} \qquad k_4 = \frac{\partial}{\partial\psi} \quad (2.17)$$

In Sec. V A we shall replace the eight-dimensional Euclidean target space by an eight-dimensional Minkowski space to get an alternative metric on "CP^2." Before doing this we first review CP_F^2, the fuzzy analogue of CP^2 in Sec. III.

III. CP_F^2

Loosely speaking, CP_F^2 is the quantization of CP^2. For this one replaces the complex coordinates z_i and z_i^*, $i=1$, 2, 3, by operators $a_i^\dagger$ and a_i, [7,36] satisfying the commutation relations of raising and lowering operators

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad (3.1)$$

acting on some Hilbert space $\mathcal{H}_n$. In analogy with the 5–sphere constraint (2.2), one fixes the eigenvalue of the total number operator $a_i^\dagger a_i$ to some positive integer value n. This restricts $\mathcal{H}_n$ to be spanned by the $N=\frac{(n+2)!}{2n!}$ harmonic oscillator states $\{|n_1,n_2,n_3>\}$, $n_i=0,1,2,\ldots$, where $n=n_1+n_2+n_3$. The action of the raising and lowering operators is incompatible with this restriction, so $a_i^\dagger$ and a_i cannot generate the algebra of CP_F^2. One can instead work with functions of the operators $a_i^\dagger a_j$, which do have a well-defined action on the N—dimensional Hilbert space $\mathcal{H}_n$. Of course, $a_i^\dagger a_i$ acts trivially on $\mathcal{H}_n$. The remaining operators, $a_i^\dagger a_j - \frac{1}{3}\delta_{ij}a_k^\dagger a_k$, generate $SU(3)$ and are the noncommutative analogues of (2.4), which we can also write as

$$X^\alpha = \frac{1}{n}a_i^\dagger \lambda_{ij}^\alpha a_j, \qquad \alpha = 1,2,\ldots,8. \quad (3.2)$$

From them we recover the $su(3)$ Lie-algebra

$$[X^\alpha, X^\beta] = \frac{2i}{n}f^{\alpha\beta\gamma}X^\gamma \quad (3.3)$$

X^α acting on $\mathcal{H}_n$ generate an irreducible representation of $SU(3)$, which is uniquely specified by the values of the quadratic and cubic Casimirs, $X^\alpha X^\alpha$ and $d^{\alpha\beta\gamma}X^\alpha X^\beta X^\gamma$. They are contained in the following fuzzy analogues of the quadratic CP^2 constraints (2.6),

$$X^\alpha X^\alpha|_{\mathcal{H}_n} = \frac{4}{3}+\frac{4}{n} \quad (3.4)$$

$$d^{\alpha\beta\gamma}X^\alpha X^\beta|_{\mathcal{H}_n} = \left(\frac{1}{3}+\frac{1}{2n}\right)X^\gamma|_{\mathcal{H}_n}, \quad (3.5)$$

in addition to

$$f^{\alpha\beta\gamma}X^\alpha X^\beta|_{\mathcal{H}_n} = \frac{6i}{n}X^\gamma|_{\mathcal{H}_n}. \quad (3.6)$$

The quadratic constraints (3.4)–(3.6) tend towards the commutative constraints (2.6) in the large n (or equivalently, large N) limit. (3.4) assigns a value to the quadratic Casimir, while for the cubic Casimir we then get

$$d^{\alpha\beta\gamma}X^\alpha X^\beta X^\gamma|_{\mathcal{H}_n} = \frac{4}{9}+\frac{2}{n}+\frac{2}{n^2}. \quad (3.7)$$

The CP_F^2 algebra is the algebra of $N\times N$ matrices which are polynomial functions of X^α, satisfying the constraints (3.4)–(3.6). The standard choice for the Laplace operator on CP_F^2 is $\Delta_E=[X^\alpha,[X^\alpha,\ldots]]$.

Star products for CP_F^2 are known [36,37]. Using a star product, denoted by $\star$, the CP_F^2 algebra is mapped to a noncommutative algebra of functions on CP^2. So for example, from (3.3), the images (or "symbols") $\mathcal{X}_\alpha$ of the operators X_α under the map satisfy the star commutator:

$$\mathcal{X}^\alpha\star\mathcal{X}^\beta - \mathcal{X}^\beta\star\mathcal{X}^\alpha = \frac{2i}{n}f^{\alpha\beta\gamma}\mathcal{X}^\gamma. \quad (3.8)$$

In the commutative limit $n\to\infty$, the star product of functions is required to reduce to the pointwise product (at zeroth order in $1/n$), and the star commutator of functions reduces to i times the Poisson bracket of functions (at first order in $1/n$). So for example, the left-hand side of (3.8) goes to $\frac{i}{n}\{\mathcal{X}^\alpha,\mathcal{X}^\beta\}$ as $n\to\infty$, and in that limit, $\mathcal{X}^\alpha$ satisfy the same Poisson bracket relations as x^α in (2.5). Therefore χ^α can be identified with the CP^2 embedding coordinates in the large n limit.

IV. CP_F^2 SOLUTIONS TO MATRIX MODELS

A. Euclidean background

CP_F^2 is easily seen to be a solution of a Yang-Mills matrix model with a Euclidean background metric. For this we introduce $M\times M$ matrices Y^α, $\alpha=1,\ldots,8$, whose dynamics is governed by the action [31]

$$S_E(Y) = \frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{4}[Y^\alpha,Y^\beta]^2 + \frac{2}{3}i\tilde{\alpha}f^{\alpha\beta\gamma}Y^\alpha Y^\beta Y^\gamma\right), \quad (4.1)$$

where $\tilde{\alpha}$ is a real coefficient. The first term in the trace defines the Yang-Mills matrix action (which can be trivially extended to ten dimensions) appears in the IKKT matrix

model [29]. It is invariant under rotations in the eight-dimensional Euclidean space. This $SO(8)$ symmetry is broken by the second term, which instead is invariant under the adjoint action of $SU(3)$, with infinitesimal variations $\delta Y^\alpha = 2if^{\alpha\beta\gamma}Y^\beta\epsilon^\gamma$, for infinitesimal parameters ϵ^α. Both terms are invariant under the common subgroup of rotations in the $\alpha = 1, 2, 3$ directions, as well as translations in the eight-dimensional Euclidean space.

The action (4.1) has extrema at

$$[[Y^\alpha, Y^\beta], Y^\beta] + i\tilde{\alpha}f^{\alpha\beta\gamma}[Y^\beta, Y^\gamma] = 0. \tag{4.2}$$

CP^2_F is a solution to (4.2). That means we identify Y^α with $N \times N$ matrix representations of the X^α, defined in the previous section. For this we also need to make the identification $\tilde{\alpha} = 2/n$, n being an integer such that $N = \frac{(n+2)!}{2n!} \leq M$.

B. Lorentzian background

The matrix action (4.1) was written in an eight-dimensional Euclidean ambient space. Here we change the ambient space to eight-dimensional Minkowski space, with metric tensor $\eta = \mathrm{diag}(1, 1, 1, 1, 1, 1, 1, -1)$. In order to find nontrivial solutions we also add a quadratic term to the action, which now reads

$$S_M(Y) = \frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{4}[Y^\alpha, Y^\beta][Y_\alpha, Y_\beta] + \frac{2}{3}i\tilde{\alpha}f^{\alpha\beta\gamma}Y_\alpha Y_\beta Y_\gamma + \frac{\beta}{2}Y^\alpha Y_\alpha\right), \tag{4.3}$$

where β is real and indices raised and lowered using η. The action is an extremum when

$$[[Y^\alpha, Y^\beta], Y_\beta] + i\tilde{\alpha}f^{\alpha\beta\gamma}[Y_\beta, Y_\gamma] + \beta Y^\alpha = 0. \tag{4.4}$$

A simple solution $Y^\alpha = \bar{Y}^\alpha$ to (4.4) is CP^2_F, now written in a Lorentzian background:

$$\bar{Y}^\alpha = n\tilde{\alpha}X^\alpha. \tag{4.5}$$

Here $\tilde{\alpha}$ and β are constrained by

$$\beta = -6\tilde{\alpha}^2. \tag{4.6}$$

For any fixed n, which defines a matrix representation, this solution is expressed in terms of only one free parameter, which sets an overall scale. This CP^2 solution is not invariant under all of $SU(3)$, since general transformations do not preserve the timelike direction of the background metric. On the other hand, the timelike direction is preserved under the adjoint action of the $SU(2) \times U(1)$ subgroup. In order for the Laplace operator associated with this solution to be consistent with the eight-dimensional Minkowski metric tensor η, we should take it to be $\Delta_M = [Y^\alpha, [Y_\alpha, \ldots]]$, rather than the standard Laplace operator on CP^2_F.

A more general solution to (4.4) which is also invariant under $SU(2) \times U(1)$ is

$$\begin{aligned} \bar{Y}^i &= \frac{n\rho}{2}X^i, \qquad i = 1, 2, 3 \\ \bar{Y}^a &= v\frac{n\rho}{2}X^a, \qquad a = 4, 5, 6, 7 \\ \bar{Y}^8 &= w\frac{n\rho}{2}X^8 \end{aligned} \tag{4.7}$$

where the parameters $v, w, \rho, \tilde{\alpha}$ and β are constrained by

$$\begin{aligned} v &= \frac{1}{2}\sqrt{\frac{\gamma + 5 + w - w^2 - w^3}{1 + w}} \\ \frac{\tilde{\alpha}}{\rho} &= \frac{5 + w + 7w^2 - w^3 - \gamma}{4(1 + 4w - w^2)} \\ \frac{\beta}{\rho^2} &= -\frac{3(1 + 15w - 8w^3 - w^4 + w^5 + (1 + 2w - w^2)\gamma)}{4(1 + w)(1 + 4w - w^2)} \end{aligned} \tag{4.8}$$

and

$$\gamma = \sqrt{25 - 6w + 7w^2 + 4w^3 - 17w^4 + 2w^5 + w^6}. \tag{4.9}$$

For any fixed n, this solution is determined by two parameters ρ and w, the former of which sets the overall scale. Again, here we assume the Laplace operator to be $\Delta_M = [Y^\alpha, [Y_\alpha, \ldots]]$. The solution is a one-parameter deformation of the previous CP^2_F solution, given by (4.5) and (4.6), and we can regard w as the deformation parameter. The previous solution is recovered for $w = 1$, since then (4.8) gives $v = 1$, $\tilde{\alpha} = \frac{\rho}{2}$ and $\beta = -\frac{3}{2}\rho^2$. v is real and finite for the domain $-1 < w \lesssim 1.32247$. v tends towards the lower bound $\approx .493295$ as w goes to the upper limit ≈ 1.32247, while v is singular in the limit $w \to -1$. v versus w is plotted for this domain in Fig. 1.

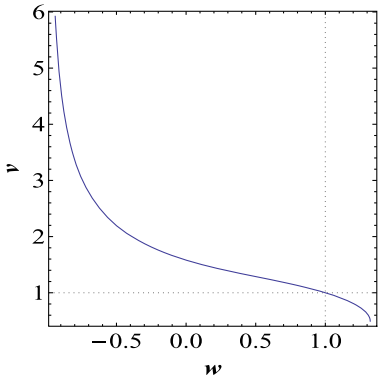

FIG. 1. v versus w is plotted for the one-parameter family of deformed CP^2_F solutions given in (4.7) and (4.8).

V. COMMUTATIVE LIMIT

The discussions in Sec. IV assume $N \times N$ matrix representations for the CP^2_F solution (4.5), (4.6), and the deformed CP^2_F solution (4.7)–(4.9). Here we take the $N \to \infty$ limit of these solutions to reveal different spacetime manifolds. We begin with the undeformed CP^2_F solution (4.5), (4.6).

A. CP^2 in a Lorentzian background

For convenience we first fix the scale of the solution (4.5), (4.6) by setting $\tilde{\alpha} = 1/n$ and $\beta = -1/n^2$. Thus $\bar{Y}^\alpha = X^\alpha$, for any n. For some star product we can introduce their corresponding symbols $\bar{\mathcal{Y}}^\alpha$, and so $\bar{\mathcal{Y}}^\alpha = \mathcal{X}^\alpha$, where $\mathcal{X}^\alpha$ satisfies the star commutator (3.8). Then in the $n \to \infty$ limit, $\bar{\mathcal{Y}}^\alpha$ obey the Poisson brackets (2.5), and constraints of the form (2.6). In the limit, $\bar{\mathcal{Y}}^\alpha$ can be expressed in terms of complex coordinates z as in (2.4), which once again span a four-dimensional manifold. However, now the manifold, strictly speaking, is not CP^2. While we recover the CP^2 constraints (2.6) and (2.5) in the commutative limit, the induced metric on the manifold cannot be the Fubini-Study metric (2.7). The latter followed from the Euclidean background metric tensor on $\mathbb{R}^8$, given in (2.8). Now the embedding matrices $\bar{Y}^\alpha$, and their symbols $\bar{\mathcal{Y}}^\alpha$, span eight-dimensional Minkowski space. Moreover, since the Laplace operator for the matrix solution (4.5) is constructed using the eight-dimensional Minkowski metric tensor η, the induced metric tensor on the surface that is recovered in the $n \to \infty$ limit of the solution should also be constructed using η. The induced metric tensor on the surface is thus computed from the invariant length for the eight-dimensional Minkowski space,

$$ds^2_M = d\bar{\mathcal{Y}}^\alpha d\bar{\mathcal{Y}}_\alpha = ds^2_E - 2(d\mathcal{X}^8)^2, \tag{5.1}$$

where we assume $\bar{\mathcal{Y}}^\alpha = \mathcal{X}^\alpha$. Then by writing $\mathcal{X}^\alpha = \frac{\bar{z}\lambda^\alpha z}{|z|^2}$, one gets corrections to the Fubini-Study metric

$$ds^2_M = ds^2_E - \frac{2(d(\bar{z}\lambda^8 z))^2}{|z|^4} - \frac{2(\bar{z}\lambda^8 z)^2(d|z|^2)^2 - d|z|^4 d(\bar{z}\lambda^8 z)^2}{|z|^8}. \tag{5.2}$$

In terms of the coordinates $\zeta_a = z_a/z_3$, $a = 1, 2$, which are invariant under $z \to \gamma z$, we get

$$\begin{aligned} ds^2_M &= ds^2_E - \frac{24|\zeta|^2}{(|\zeta|^2+1)^4}(d|\zeta|)^2 \\ &= 4\frac{(|\zeta|^2-1)^2 - 2|\zeta|^2}{(|\zeta|^2+1)^4}(d|\zeta|)^2 + \frac{4|\zeta|^2}{(|\zeta|^2+1)}(\omega_1^2+\omega_2^2) \\ &\quad + \frac{4|\zeta|^2}{(|\zeta|^2+1)^2}\omega_3^2, \end{aligned} \tag{5.3}$$

where the left-invariant one forms ω_i were defined previously in Sec. II.

The metric tensor obtained here differs from that on CP^2, and furthermore is not Kähler. On the other hand, the symplectic two-form remains unchanged, i.e. it is (2.14). $SU(3)/Z_3$ is no longer an isometry. Instead, the metric tensor (5.3) and symplectic two-form are invariant under $SU(2) \times U(1)/Z_2$, generated by the Killing vectors (2.17). A novel feature is that the metric tensor has variable signature. It has Euclidean signature for $0 < |\zeta|^2 < 2 - \sqrt{3}$ and $|\zeta|^2 > 2 + \sqrt{3}$, and Lorentzian signature for $2 - \sqrt{3} < |\zeta|^2 < 2 + \sqrt{3}$. The metric tensor, along with the Ricci scalar, is singular at the boundaries $|\zeta|^2 = 2 \pm \sqrt{3}$ between the regions, and so the boundaries define physical singularities. [There are also coordinate singularities located at $|\zeta| = 0$ and $|\zeta| \to \infty$, just as is the case with the CP^2 metric tensor given by (2.13).] Away from the singularities, the manifold is spatially homogeneous and axially symmetric at each point, and the invariant length (5.3) has a form which is similar to that of a Taub-NUT space (more specifically, the Taub region of Taub-NUT space since the coefficient of ω_3^2 is positive).

We now restrict to the Lorentzian region $2 - \sqrt{3} < |\zeta|^2 < 2 + \sqrt{3}$. $|\zeta|$ is a time parameter in this region, and one has the following properties:

(a) There are timelike geodesics which originate at the initial singularity, which we choose to be at $|\zeta| = \sqrt{2-\sqrt{3}}$, and terminate at the final singularity at $|\zeta| = \sqrt{2+\sqrt{3}}$. The elapsed proper time along a geodesic with $\omega_1 = \omega_2 = \omega_3 = 0$ can be written as a function of $|\zeta|$

$$\tau(|\zeta|) = 2\int_{\sqrt{2-\sqrt{3}}}^{|\zeta|} \frac{\sqrt{-r^4+4r^2-1}}{(r^2+1)^2}dr \tag{5.4}$$

The the total proper time from the initial singularity to the final singularity is $\tau(\sqrt{2+\sqrt{3}}) \approx .672$.

(b) From the volume of any time slice, which can be constructed from the determinant of the metric, ${}^3g|_{|\zeta|}$, on a time slice, one can assign a spatial distance scale a as a function of $|\zeta|$,

$$a(|\zeta|)^3 = \int \sqrt{{}^3g|_{|\zeta|}}\, d\theta d\phi d\psi = \frac{8\pi^2|\zeta|^3}{(|\zeta|^2+1)^2}, \tag{5.5}$$

where the integration is done on the time slice, which can be parametrized by the Euler angles in (2.12). A novel feature of this spacetime is that the distance scale is nonvanishing at the time of the initial and final singularities, corresponding to $|\zeta| = \sqrt{2-\sqrt{3}}$ and $|\zeta| = \sqrt{2+\sqrt{3}}$, respectively,

$$a(\sqrt{2-\sqrt{3}}) \approx 1.896 \qquad a(\sqrt{2+\sqrt{3}}) \approx 2.940.$$

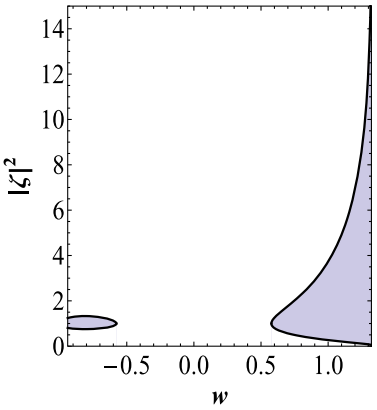

FIG. 2. Singularities at $|\zeta|^2 = |\zeta_\pm|^2$ are plotted as a function of w using (5.10). They define boundaries between regions of Lorentzian signature (shaded) and Euclidean signature (unshaded).

A plot of the normalized scale $a/a|_{\tau=0}$ as a function of the time τ from $\tau = 0$ (the time of the initial singularity) to the time of the final singularity appears in Fig. 3 (solid curve). It is seen to grow and deaccelerate.

B. Deformed CP^2 in a Lorentzian background

We can obtain a one-parameter family of spacetime manifolds, including the one obtained in the above subsection, by taking the commutative limit of the deformed CP^2_F solution (4.7)–(4.9). Here it is convenient to set $\rho = 2/n$. Then the symbols $\bar{\mathcal{Y}}^\alpha$ of the matrices $\bar{Y}^\alpha$ for the solution in (4.7) satisfy

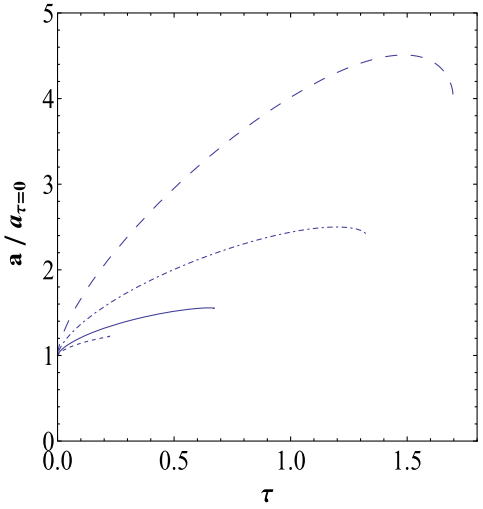

FIG. 3. $a/a|_{\tau=0}$ as a function of the time τ from $\tau = 0$ (the time of the initial singularity) to the time of the final singularity for four different choices for w (and hence v): $w \approx 1.3225$ (large dashed curve), $w = 1.25$ (dot-dashed curve), $w = 1$ (solid curve) and $w = .75$ (small dashed curve).

$$\begin{aligned}\bar{\mathcal{Y}}^i &= \mathcal{X}^i, \qquad i = 1,2,3\\ \bar{\mathcal{Y}}^a &= v\mathcal{X}^a, \qquad a = 4,5,6,7\\ \bar{\mathcal{Y}}^8 &= w\mathcal{X}^8,\end{aligned} \tag{5.6}$$

where $\mathcal{X}^\alpha$ again denote the symbols of the CP^2_F matrices. Recall v is real and finite for the domain $-1 < w \lesssim 1.32247$, while w is given in (4.8) and plotted in Fig. 1. In the $n \to \infty$ limit, we shall keep v and w fixed, which implies as before that $\tilde{\alpha}$ and β vanish in the limit, $\tilde{\alpha} \sim 1/n$ and $\beta \sim 1/n^2$. The invariant length in the eight-dimensional Minkowski space now reads

$$\begin{aligned}ds^2_M = d\bar{\mathcal{Y}}^\alpha d\bar{\mathcal{Y}}_\alpha &= v^2 ds^2_E + (1-v^2)(d\mathcal{X}^i)^2\\ &\quad - (w^2+v^2)(d\mathcal{X}^8)^2,\end{aligned} \tag{5.7}$$

where we substituted the commutative solution (5.6). Using the identities

$$\begin{aligned}(d\mathcal{X}^i)^2 &= \frac{4|\zeta|^4}{(1+|\zeta|^2)^2}(\omega_1^2+\omega_2^2) + \frac{4|\zeta|^2}{(1+|\zeta|^2)^4}(d|\zeta|)^2\\ (d\mathcal{X}^8)^2 &= \frac{12|\zeta|^2}{(1+|\zeta|^2)^4}(d|\zeta|)^2,\end{aligned} \tag{5.8}$$

which follows from $\mathcal{X}^\alpha = \frac{\bar{z}\lambda^\alpha z}{|z|^2}$ and the previous definition of the left-invariant one forms ω_i, we now get

$$\begin{aligned}ds^2_M &= 4\left(\frac{v^2(|\zeta|^2-1)^2 + (1-3w^2)|\zeta|^2}{(1+|\zeta|^2)^4}\right)(d|\zeta|)^2\\ &\quad + \frac{4|\zeta|^2(v^2+|\zeta|^2)}{(1+|\zeta|^2)^2}(\omega_1^2+\omega_2^2) + \frac{4v^2|\zeta|^2}{(|\zeta|^2+1)^2}\omega_3^2.\end{aligned} \tag{5.9}$$

This expression reduces to (5.3) when $w = v = 1$. The symplectic two-form is again given by (2.14).

As in the previous case, the metric tensor and symplectic two-form are invariant under $SU(2) \times U(1)/Z_2$, generated by the Killing vectors (2.17). The induced metric tensor now has physical singularities at $|\zeta| = |\zeta_\pm|$, where

$$|\zeta_\pm|^2 = \frac{2v^2+3w^2-1 \pm \sqrt{(3w^2-1)(4v^2+3w^2-1)}}{2v^2}, \tag{5.10}$$

which using (4.8) are functions of only w. The singularities are plotted as a function of w in Fig. 2. There are two singularities for the domains $-1 > w > -\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}} < w \lesssim 1.32247$, one singularity (at $|\zeta| = 1$) for $w = \pm\frac{1}{\sqrt{3}}$, and none for $-\frac{1}{\sqrt{3}} < w < \frac{1}{\sqrt{3}}$. As before, they define the boundaries between regions of Euclidean signature and Lorentzian signature. (The regions of Lorentzian signature are shaded in the figure.) For the domain

$-\frac{1}{\sqrt{3}} < w < \frac{1}{\sqrt{3}}$, the metric tensor in (5.9) has a Euclidean signature for all $|\zeta|^2$.

Once again there are timelike geodesics which originate at the initial singularity, which we choose to be at $|\zeta| = |\zeta_-|$, and terminate at the final singularity at $|\zeta| = |\zeta_+|$. The generalization of the expression (5.4) for the elapsed proper time along a geodesic with $\omega_1 = \omega_2 = \omega_3 = 0$ can be written as

$$\tau(|\zeta|) = 2\int_{|\zeta_-|}^{|\zeta|} \frac{\sqrt{(3w^2-1)r^2 - v^2(r^2-1)^2}}{(r^2+1)^2}\, dr. \tag{5.11}$$

The generalization of the expression (5.5) for the volume of a $|\zeta|$-slice, which we again denote by $a(|\zeta|)^3$, is

$$a(|\zeta|)^3 = \int \sqrt{{}^3g|_{|\zeta|}}\, d\theta d\phi d\psi = \frac{8\pi^2 v|\zeta|^3(|\zeta|^2+v^2)}{(|\zeta|^2+1)^3}. \tag{5.12}$$

We restrict to the region of Lorentzian signature for four different choices for w (and hence v), including the case $w = v = 1$ of the previous subsection, in Fig. 3. There we plot the normalized scale $a/a|_{\tau=0}$ as a function of the time τ, starting from $\tau = 0$ (the time of the initial singularity) to the time of the final singularity. In all cases the distance scale a is nonvanishing at the time of the initial and final singularities, and the scale grows and deaccelerates. The largest and longest expansion occurs when w takes its maximum value of ~ 1.3225, while the spacetime only exists for an instant for $w = \pm\frac{1}{\sqrt{3}}$.

VI. CONCLUDING REMARKS

We have constructed four-dimensional manifolds by taking the $N \to \infty$ of solutions to Lorentzian matrix equations (4.3). The metric tensor and symplectic two-form on the manifold are invariant under $SU(2) \times U(1)/Z_2$. The manifolds, in general, have changing signature. We get toy models of spacetime after restricting to regions with Lorentzian signature, complete with initial and final cosmological singularities. The metric tensor resembles that of the Taub region of Taub-NUT space. In all cases, the distance scale grows and deaccelerates as shown in Fig. 3, which clearly does not give a realistic picture of our universe.

Many other solutions of the Lorentzian matrix equations (4.3) are possible. On the other hand, not all solutions may have a well defined commutative (or large N) limit. One such example is

$$\begin{aligned} \bar{Y}^i &= -n(2+\sqrt{5})X^i, \qquad i = 1,2,3 \\ \bar{Y}^a &= -n\sqrt{29+13\sqrt{5}}X^a, \qquad a = 4,5,6,7 \\ \bar{Y}^8 &= nX^8 \end{aligned} \tag{6.1}$$

where again X^α is defined in (3.2). In this case both $\tilde{\alpha}$ and β are fixed, $\tilde{\alpha} = \frac{2}{19}(10-\sqrt{5})$, $\beta = -\frac{24}{19}(163+73\sqrt{5})$. Again, the dimension of the representation is $N = \frac{(n+2)!}{2n!}$. Now the only free parameter is n, and the solution is ill-defined when $n \to \infty$ and so there is no commutative limit. Upon modifying the matrix action (4.3), in particular the cubic term, it should be possible to find solutions associated with other noncommutative geometries, which may or may not have a commutative limit. One possibility is the fuzzy four-sphere embedded in a Lorentzian background.

Many other issues can be explored. Among them are: the question of stability for the various classical matrix solutions, the role played by the inclusion of fermionic degrees of freedom in the matrix model, and the computation of quantum effects. With regard to fermions, we note that supersymmetry, in addition to translation symmetry, is explicitly broken by the presence of the quadratic term in the action (4.3). Of course, it is also of interest to investigate whether a more physical cosmology can be found amongst the solutions of this, or related, matrix models. Since a compact coset space necessarily implies a closed spacetime cosmology, to get an open universe one proposal is to start with a noncompact noncommutative coset space. One expects matrix representations then to be infinite-dimensional, and although one cannot then take $N \to \infty$, it should be possible to define an alternative commutative limit in this case. A striking feature of the spacetimes recovered in Sec. V is that initial singularity occurs when the universe has a nonzero distance scale $a(|\zeta_-|)$. This distance scale should be greater than the Planck length since Planck scale effects are washed out in the continuum limit. Time cannot be defined for distance scales smaller than $a(|\zeta_-|)$. If this feature, i.e., that the universe begins with a nonzero spatial size, can be implemented in a realistic cosmology, then it may not be necessary to consider the very early universe, and perhaps, one can even avoid having an inflationary era.

ACKNOWLEDGMENTS

We are very grateful to A. Pinzul for valuable discussions.

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