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Aspects of general higher-order gravities

Pablo Bueno,[1,2,*] Pablo A. Cano,[3,†] Vincent S. Min,[1,‡] and Manus R. Visser[2,‡]

[1]*Instituut voor Theoretische Fysica, KU Leuven Celestijnenlaan 200D, B-3001 Leuven, Belgium*
[2]*Institute for Theoretical Physics, University of Amsterdam Science Park 904, 1090 GL Amsterdam, Netherlands*
[3]*Instituto de Física Teórica UAM/CSIC C/ Nicolás Cabrera, 13-15, C.U. Cantoblanco, 28049 Madrid, Spain*

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We study several aspects of higher-order gravities constructed from general contractions of the Riemann tensor and the metric in arbitrary dimensions. First, we use the fast-linearization procedure presented in [P. Bueno and P. A. Cano, arXiv:1607.06463] to obtain the equations satisfied by the metric perturbation modes on a maximally symmetric background in the presence of matter and to classify $\mathcal{L}$(Riemann) theories according to their spectrum. Then, we linearize all theories up to quartic order in curvature and use this result to construct quartic versions of *Einsteinian cubic gravity*. In addition, we show that the most general cubic gravity constructed in a dimension-independent way and which does not propagate the ghostlike spin-2 mode (but can propagate the scalar) is a linear combination of f(Lovelock) invariants, plus the Einsteinian cubic gravity term, plus a *new ghost-free gravity* term. Next, we construct the generalized Newton potential and the post-Newtonian parameter γ for general $\mathcal{L}$(Riemann) gravities in arbitrary dimensions, unveiling some interesting differences with respect to the four-dimensional case. We also study the emission and propagation of gravitational radiation from sources for these theories in four dimensions, providing a generalized formula for the power emitted. Finally, we review Wald's formalism for general $\mathcal{L}$(Riemann) theories and construct new explicit expressions for the relevant quantities involved. Many examples illustrate our calculations.

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I. INTRODUCTION AND SUMMARY OF RESULTS

Higher-order gravities have attracted a considerable amount of attention throughout the last few decades. The reasons for this interest are manifold. On the one hand, whatever the *right* ultraviolet completion of Einstein gravity might turn out to be, the effective action of the theory is expected to contain a series of higher-derivative terms involving different contractions of the Riemann tensor and its covariant derivatives. This is naturally what happens in String Theory, which generically predicts the appearance of infinitely many of these subleading terms.[1] correcting the Einstein-Hilbert (EH) action, e.g., Refs. [1–3].

Higher-curvature extensions of Einstein gravity have been extensively considered in the context of cosmology. In that case, the goal is going beyond the standard Lambda cold dark matter (Λ-CDM) model, e.g., providing explanations for late-time accelerated expansion, dark matter, or inflation—see, e.g., Refs. [4–7] for some reviews on the subject.

In the context of holography [8–10], higher-order gravities have also played a prominent role. In particular, they have been used as tools to characterize numerous properties of strongly coupled conformal field theories (CFTs), e.g., Refs. [11–19]. In some cases, they have even been essential in the discovery of new universal results valid for general CFTs—holographic or not [20–24].

Apart from these more or less well-delimited areas, another approach entails the identification and study of concrete classes of higher-order gravities which possess particularly interesting properties. In some cases, they mimic defining aspects of Einstein gravity [25–29]. In others, they improve problematic characteristics of the theory—e.g., by being renormalizable [30,31]. More generally, the systematic study of higher-order gravities provides a deeper understanding of Einstein gravity itself, since it helps unveil what features of the theory are generic and which ones are specific.

In this paper, we will explore several aspects of gravity theories of which the Lagrangian density is an arbitrary function of the Riemann tensor and the metric, i.e.,

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|}[\mathcal{L}(R_{\mu\nu\rho\sigma}, g^{\alpha\beta}) + L_{\text{matter}}], \quad (1.1)$$

where we have included an additional term L_{matter} to account for possible additional minimally coupled matter fields. Throughout the text, we shall refer to the class of theories defined by (1.1) as $\mathcal{L}$(Riemann) gravities. While

[*]pablo@itf.fys.kuleuven.be
[†]pablo.cano@uam.es
[‡]vincent.min@kuleuven.be
[§]m.r.visser@uva.nl

[1]Which terms appear depends on the particular setup considered.

(1.1) does not account for the most general higher-order gravity conceivable,[2] it does incorporate a broad class of theories exhibiting very different features. Many aspects of general $\mathcal{L}$(Riemann) theories have been previously developed in several contexts, including black-hole mechanics, linearized gravity, holography, and cosmology—see, e.g., Refs. [41–56] and references therein. We aim to develop some more here. In particular, we will perform a general and systematic study of the linearized spectrum of these theories, which we will use to compute relevant physical quantities such as the generalized Newtonian potential or the power radiated by sources. In addition, our classification will allow us to characterize some interesting previously unidentified theories. Finally, we will also study the Wald formalism for general $\mathcal{L}$(Riemann) providing new explicit formulas for some of the relevant quantities involved.[3]

A. Main results

The main results of the paper can be summarized as follows:

(i) In Sec. II, we start by reviewing the fast-linearization procedure on maximally symmetric backgrounds (msb) presented in Ref. [29] and valid for general theories of the form (1.1) in general dimensions. This reduces the problem to the evaluation of the corresponding Lagrangian density on a particular Riemann tensor—constructed from the metric and an auxiliary tensor—and the computation of two trivial derivatives. We use this result to identify the physical modes propagated by the metric and the corresponding dynamical equations satisfied by those modes in the presence of matter in (anti)-de Sitter and flat space. Finally, we construct an effective quadratic action from which the general linearized equations can be derived.

(ii) In Sec. III, we classify all theories of the form (1.1) according to the properties of their physical modes. The categories include theories which do not propagate an extra massive graviton but do incorporate a dynamical scalar; theories in which the extra graviton is present but the scalar is not; theories with two massless gravitons and a massive scalar, including generalized critical gravities, for which the scalar is absent; and Einstein-like theories, i.e., those that only propagate a massless graviton.

(iii) In Sec. IV, we use our method to linearize the equations of motion of all theories contained in (1.1) up to quartic order in curvature in arbitrary dimensions.

(iv) In Sec. V, we explain how to obtain the linearized equations of a theory defined as a function of arbitrary curvature invariants starting from the linearized equations of each invariant. In particular, we prove that theories constructed as general functions of scalars of which the linear combinations do not produce massive gravitons are also free of those modes.

(v) In Sec. VI, we extend the construction of *Einsteinian cubic gravity* (ECG) [29] to quartic order. The resulting theories only propagate a massless graviton on a msb in general dimensions, and they are defined in a dimension-independent manner; i.e., the relative couplings between the different invariants involved are the same in all dimensions.

(vi) In Sec. VII, we construct the most general dimension-independent cubic theory of the form (1.1) which is free of massive gravitons in general dimensions—without imposing conditions on the extra scalar mode. This theory, which we call *new ghost-free gravity*, includes all the terms appearing in the ECG action—see (6.1) below—plus all f(Lovelock) invariants up to cubic order, plus a previously unidentified term which reads $\mathcal{Y} \equiv R_{\mu\ \nu}^{\ \alpha\ \beta} R_{\alpha\ \beta}^{\ \rho\ \sigma} R_{\rho\ \sigma}^{\ \mu\ \nu} - 3R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + 2R_{\mu}^{\ \nu}R_{\nu}^{\ \rho}R_{\rho}^{\ \mu}$. Just like the ECG term, $\mathcal{Y}$ is nontrivial in four dimensions. As opposed to it, this new term does contribute to the denominator of the scalar mode mass[4] m_s.

(vii) In Sec. VIII, we use the results in Secs. II and III to compute the generalized Newton potential $U_D(r)$ and the *parametrized post-Newtonian* (PPN) parameter $\gamma(r)$ for a theory of the form (1.1) in general dimensions. We show that $U_D(r)$ takes the form of a combination of generalized Yukawa potentials which, for general D, we show to be given by $U_{D,\mathrm{Yukawa}}(r) \sim (m/r)^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(r)$, where $K_\ell(x)$ are modified Bessel functions of the second kind. We unveil interesting differences with respect to the four-dimensional case.

(viii) In Sec. IX, we use the results in Secs. II and III to study the emission and propagation of gravitational

[2]Indeed, note that we shall not consider terms involving covariant derivatives of the Riemann tensor here. In fact, even that case would not encapsulate the most general theory if one considers the affine connection $\Gamma^{\rho}_{\mu\nu}$ to be a dynamical field independent from the metric—*à la* Palatini—since that setup allows for even richer scenarios; see Refs. [32–37] and references therein. Of course, similar comments apply if we introduce extra fields besides the metric, as in the case of scalar-tensor gravities—see, e.g., Refs. [38–40].

[3]Our conventions throughout the paper are as follows. We use $(-,+,\cdots,+)$ signature for the metric and the usual conventions [57] for the Riemann and Einstein tensors. We set $\hbar = c = 1$ but keep the gravitational constant $\kappa \equiv 8\pi G$ explicit. Very often we consider $\kappa^{\frac{1}{D-2}}$ and $\kappa^{\frac{1}{2-D}}$ to be the natural length and mass scales, respectively.

[4]Recall that none of the terms in the ECG action contributes to the denominator of m_s, which explains why there is no extra scalar in ECG [29].

radiation from sources in general four-dimensional $\mathcal{L}$(Riemann) theories. We obtain general formulas for the radiative components of the different modes as well as for the total power emitted by a source in terms of the quadrupole moment and the scalar radiation. We apply these results to a binary system in a circular orbit.

(ix) In Sec. X, we give a detailed account of Wald's formalism and construct explicit expressions for the relevant quantities involved for general $\mathcal{L}$(Riemann) theories. New results are obtained for the symplectic structure $\boldsymbol{\omega}$ and the surface charge $\delta \mathbf{Q}_\xi - \xi \cdot \mathbf{\Theta}$.

(x) Finally, our Appendixes contain many examples which illustrate the results in Secs. II, III, V, and X.

II. LINEARIZED EQUATIONS OF $\mathcal{L}$(Riemann) THEORIES

In this section, we study the linearized equations of general $\mathcal{L}$(Riemann) theories on msb in arbitrary dimensions. The full nonlinear equations of this class of theories (1.1) read [45]

$$\mathcal{E}_{\mu\nu} \equiv P_{\mu}{}^{\sigma\rho\lambda} R_{\nu\sigma\rho\lambda} - \frac{1}{2} g_{\mu\nu} \mathcal{L} - 2\nabla^\alpha \nabla^\beta P_{\mu\alpha\beta\nu} = \frac{1}{2} T_{\mu\nu}, \quad (2.1)$$

where we defined the object

$$P^{\mu\nu\sigma\rho} \equiv \left[\frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\right]_{g^{\gamma\delta}} \quad \text{and} \quad T_{\mu\nu} \equiv -\frac{2}{\sqrt{|g|}} \frac{\delta(\sqrt{|g|} L_{\text{matter}})}{\delta g^{\mu\nu}} \quad (2.2)$$

is the usual matter stress-energy tensor.

Our goal in this section is to review the fast-linearization procedure presented in Ref. [29] and explain how it can be used to characterize the spectrum of these theories, which we will use in numerous applications throughout the paper. In the first subsection, we linearize (2.1) up to the identification of four constants a, b, c, and e. We argue that those constants can be easily obtained from the corresponding Lagrangian following some simple steps that we detail. Then, we show that the general linearized equations can in fact be written in terms of only three physical parameters which can be easily obtained from a, b, c, and e. These are nothing but the effective gravitational constant κ_{eff} and the masses of the two extra modes which appear in the linearized spectrum of generic $\mathcal{L}$(Riemann) theories, m_g^2 and m_s^2. As we show, both in (anti-)de Sitter and Minkowski backgrounds, the usual massless graviton is generically accompanied by a massive ghostlike graviton of mass m_g and a scalar mode of mass m_s. In Sec. II C, we obtain the matter-coupled wave equations satisfied by these modes. We close the section by constructing a quadratic effective action from which the linearized equations can be obtained from the variation of the metric perturbation.

A. Linearization procedure

Let us start giving a detailed account of the fast-linearization method for general $\mathcal{L}$(Riemann) theories presented in Ref. [29].

1. First-order variations on a general background metric

Consider a perturbed metric of the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (2.3)$$

where $h_{\mu\nu} \ll 1$ for all $\mu, \nu = 0, \ldots, D-1$ and where $\bar{g}_{\mu\nu}$ is any metric. Our goal is to expand the field equations (2.1) to linear order in $h_{\mu\nu}$ assuming that $\bar{g}_{\mu\nu}$ is a solution of the full nonlinear ones. For this purpose, it is useful to define the tensor

$$C^{\mu\gamma\sigma\nu}_{\sigma\rho\lambda\eta} \equiv g_{\sigma\alpha} g_{\rho\beta} g_{\lambda\chi} g_{\eta\xi} \frac{\partial P^{\mu\gamma\sigma\nu}}{\partial R_{\alpha\beta\chi\xi}}, \quad (2.4)$$

where $P^{\mu\nu\rho\sigma}$ was defined in (2.2). Now, using the identity [45]

$$\left[\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}}\right]_{R_{\rho\sigma\gamma\delta}} = 2P_{\mu}{}^{\rho\sigma\gamma} R_{\nu\rho\sigma\gamma}, \quad (2.5)$$

it is possible to prove that the variations of $\mathcal{L}$ and $P^{\mu\alpha\beta\nu}$ read, respectively,[5]

$$\delta \mathcal{L} = 2\delta g^{\mu\nu} \bar{P}_{\mu}{}^{\sigma\rho\lambda} \bar{R}_{\nu\sigma\rho\lambda} + \bar{P}^{\mu\sigma\rho\lambda} \delta R_{\mu\sigma\rho\lambda}, \quad (2.7)$$

$$\delta P^{\mu\alpha\beta\nu} = 2\delta g^{\lambda[\mu} \bar{P}_{\lambda}{}^{\alpha]\beta\nu} + 2\delta g^{\rho\eta} \bar{C}^{\mu\alpha\beta\nu}_{\lambda\eta\sigma\tau} \bar{R}^{\lambda}{}_{\rho}{}^{\sigma\tau} + \bar{C}^{\mu\alpha\beta\nu}_{\lambda\rho\sigma\tau} \bar{g}^{\lambda\eta} \bar{g}^{\rho\gamma} \bar{g}^{\sigma\kappa} \bar{g}^{\tau\upsilon} \delta R_{\eta\gamma\sigma\tau}, \quad (2.8)$$

where the bars mean evaluation on the background metric $\bar{g}_{\mu\nu}$.

[5]Observe that throughout the paper we choose $\{R_{\mu\nu\rho\sigma}, g^{\gamma\delta}\}$ to be the fundamental variables in $\mathcal{L}$. As explained in Refs. [45,58], all expressions obtained using these variables are consistent with alternative elections such as $\{R^{\mu}{}_{\nu\rho\sigma}, g^{\alpha\beta}\}$ or $\{R_{\mu\nu}^{\rho\sigma}\}$. In particular, using the identities analogous to (2.5) obtained in Ref. [45] for the different elections of variables, it is possible to show that (2.7) and (2.8) are correct independently of such election. For example, if we choose $\{R_{\mu\nu}^{\rho\sigma}\}$, Eqs. (2.7) and (2.8) can be written as

$$\delta \mathcal{L} = \bar{P}^{\rho\lambda}_{\mu\nu} \delta R^{\mu\nu}_{\rho\lambda}, \qquad \delta P^{\mu\alpha\beta\nu} = 2\delta g^{\lambda[\mu} \bar{P}_{\lambda}{}^{\alpha]\beta\nu} + \bar{C}^{\mu\alpha\beta\nu}_{\lambda\rho\sigma\tau} \bar{g}^{\lambda\eta} \bar{g}^{\rho\gamma} \delta R^{\kappa\upsilon}_{\eta\gamma}. \quad (2.6)$$

2. Maximally symmetric background

Since we are interested in the linearized version of (2.1) on an arbitrary msb $(\bar{\mathcal{M}}, \bar{g}_{\mu\nu})$, we will from now on assume that $\bar{g}_{\mu\nu}$ satisfies

$$\bar{R}_{\mu\nu\alpha\beta} = 2\Lambda \bar{g}_{\mu[\alpha}\bar{g}_{\beta]\nu} \tag{2.9}$$

for some constant Λ. Obviously, the explicit expressions of $\bar{P}^{\mu\alpha\beta\nu}$ and $\bar{C}^{\mu\alpha\beta\nu}_{\sigma\rho\lambda\eta}$ will depend on the particular Lagrangian $\mathcal{L}$ considered. Observe, however, that when these objects are evaluated on a msb, the resulting expressions can only contain terms involving combinations of $\bar{g}_{\mu\nu}$, $\bar{g}^{\mu\nu}$, and $\delta^{\nu}{}_{\mu}$. In addition, as it is clear from (2.2) and (2.4), $P^{\mu\nu\rho\sigma}$ and $C^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}$ inherit the symmetries of the Riemann tensors appearing in their definitions. This forces $\bar{P}^{\mu\alpha\beta\nu}$ to be given by

$$\bar{P}^{\mu\alpha\beta\nu} = 2e\bar{g}^{\mu[\beta}\bar{g}^{\nu]\alpha}, \tag{2.10}$$

where the value of the constant e depends on the theory. Similarly, $\bar{C}^{\mu\alpha\beta\nu}_{\sigma\rho\lambda\eta}$ is fully determined by three tensorial structures, namely,

$$\begin{aligned}\bar{C}^{\sigma\rho\lambda\eta}_{\mu\alpha\beta\nu} = {} & a[\delta^{[\sigma}_{\mu}\delta^{\rho]}_{\alpha}\delta^{[\lambda}_{\beta}\delta^{\eta]}_{\nu} + \delta^{[\lambda}_{\mu}\delta^{\eta]}_{\alpha}\delta^{[\sigma}_{\beta}\delta^{\rho]}_{\nu}] \\ & + b[\bar{g}_{\mu\beta}\bar{g}_{\alpha\nu} - \bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}][\bar{g}^{\sigma\lambda}\bar{g}^{\rho\eta} - \bar{g}^{\sigma\eta}\bar{g}^{\rho\lambda}] \\ & + 4c\delta^{[\sigma}{}_{(\tau}\bar{g}^{\rho][\lambda}\delta^{\eta]}{}_{\epsilon)}\delta^{\tau}{}_{[\mu}\bar{g}_{\alpha][\beta}\delta^{\epsilon}{}_{\nu]},\end{aligned} \tag{2.11}$$

where the only theory-dependent quantities are in turn the constants a, b, and c.

3. Background embedding equation

Imposing $\bar{g}_{\mu\nu}$ to solve the field equations (2.1) with $T_{\mu\nu} = 0$, one finds

$$\bar{\mathcal{L}}(\Lambda) = 4e(D-1)\Lambda. \tag{2.12}$$

This is a relation between the background scale Λ defined in (2.9) and all the possible couplings appearing in the higher-order Lagrangian $\mathcal{L}(\text{Riemann})$. Another equation relating e and Λ can be obtained using (2.9) and (2.10). This reads in turn

$$\frac{d\bar{\mathcal{L}}(\Lambda)}{d\Lambda} = \bar{P}^{\mu\nu\rho\sigma}2\bar{g}_{\mu[\rho}\bar{g}_{\sigma]\nu} = 2eD(D-1), \tag{2.13}$$

which, along with (2.12), produces the nice expression

$$\Lambda\frac{d\bar{\mathcal{L}}(\Lambda)}{d\Lambda} = \frac{D}{2}\bar{\mathcal{L}}(\Lambda). \tag{2.14}$$

This is the algebraic equation that needs to be solved in order to determine the possible vacua of the theory, i.e., the allowed values of Λ as functions of the scales and couplings appearing in $\mathcal{L}(\text{Riemann})$.[6] Remarkably, Eq. (2.14) is also valid for theories involving general covariant derivatives of the Riemann tensor. Indeed, the most general higher-order gravity can be written as $\mathcal{L}(R_{\mu\nu\rho\sigma}, \nabla_{\alpha}R_{\mu\nu\rho\sigma}, \nabla_{\beta}\nabla_{\alpha}R_{\mu\nu\rho\sigma}, \ldots)$. Now, maximally symmetric spaces have a covariantly constant Riemann tensor, so the derivatives of the Riemann do not have any effect on the background embedding equation. Therefore, Eq. (2.14) applies equally in such cases.

4. Linearization procedure

With the information from the previous items, we are ready to linearize (2.1). The result of a long computation in which we make use of (2.2)–(2.11) reads

$$\begin{aligned}\frac{1}{2}\mathcal{E}^{L}_{\mu\nu} = {} & +[e - 2\Lambda(a(D-1)+c) + (2a+c)\bar{\Box}]G^{L}_{\mu\nu} \\ & + [a+2b+c][\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}]R^{L} \\ & - \Lambda[a(D-3) - 2b(D-1) - c]\bar{g}_{\mu\nu}R^{L} = \frac{1}{4}T^{L}_{\mu\nu},\end{aligned} \tag{2.15}$$

where the linearized Einstein and Ricci tensors and the linearized Ricci scalar read, respectively,[7]

$$G^{L}_{\mu\nu} = R^{L}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{L} - (D-1)\Lambda h_{\mu\nu}, \tag{2.16}$$

$$R^{L}_{\mu\nu} = \bar{\nabla}_{(\mu|}\bar{\nabla}_{\sigma}h^{\sigma}{}_{|\nu)} - \frac{1}{2}\bar{\Box}h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}h + D\Lambda h_{\mu\nu} - \Lambda h\bar{g}_{\mu\nu}, \tag{2.17}$$

$$R^{L} = \bar{\nabla}^{\mu}\bar{\nabla}^{\nu}h_{\mu\nu} - \bar{\Box}h - (D-1)\Lambda h. \tag{2.18}$$

The above equations are quartic in derivatives of the perturbation for generic higher-derivative theories, as expected. The problem is hence reduced to the evaluation of a, b, c, and e for a given theory, something that can be done using (2.2), (2.4), (2.10), and (2.11). However, this is a very tedious procedure in general, which involves the computation of first and second derivatives of $\mathcal{L}(\text{Riemann})$ with respect to the Riemann tensor. The method presented in Ref. [29] allows for an important simplification of this problem. The procedure has several steps which we explain now:

[6]For example, for the Einstein-Hilbert action $\mathcal{L} = R - 2\Lambda_0$, (2.14) imposes $\Lambda_0 = (D-1)(D-2)\Lambda/2$. For Gauss-Bonnet with a negative cosmological constant $\mathcal{L} = R + (D-1)(D-2)/L^2 + L^2\lambda_{\text{GB}}/((D-3)(D-4))\mathcal{X}_4$, one finds the well-known relation $-L^2\Lambda = (1 \pm \sqrt{1-4\lambda_{\text{GB}}})/(2\lambda_{\text{GB}})$; see, e.g., Ref. [17].

[7]Here, we use the standard notation $h \equiv \bar{g}^{\mu\nu}h_{\mu\nu}$. Also, indices are raised and lowered with $\bar{g}^{\mu\nu}$ and $\bar{g}_{\mu\nu}$, respectively.

(1) Consider an auxiliary symmetric tensor $k_{\mu\nu}$ satisfying

$$k^{\mu}{}_{\mu} = \chi, \qquad k^{\mu}{}_{\alpha} k^{\alpha}{}_{\nu} = k^{\mu}{}_{\nu}, \tag{2.19}$$

where χ is an arbitrary integer constant smaller than D which we will leave undetermined throughout the calculation. Note that the indices of $k_{\mu\nu}$ are raised and lowered with $g^{\mu\nu}$ and $g_{\mu\nu}$, as usual.

(2) Define the following "Riemann tensor"[8],

$$\tilde{R}_{\mu\nu\sigma\rho}(\Lambda,\alpha) \equiv 2\Lambda g_{\mu[\sigma} g_{\rho]\nu} + 2\alpha k_{\mu[\sigma} k_{\rho]\nu}, \tag{2.20}$$

where α and Λ are two parameters. Observe that $\tilde{R}_{\mu\nu\sigma\rho}(\Lambda,\alpha)$ does not correspond—or more precisely, it does not need to correspond—to the Riemann tensor of any actual metric in general, even though it respects the symmetries of a true Riemann tensor. An exception occurs when $\alpha = 0$, as $\tilde{R}_{\mu\nu\sigma\rho}(\Lambda,0)$ becomes the Riemann tensor of a msb of curvature Λ associated to a metric $g_{\mu\nu} = \bar{g}_{\mu\nu}$ as defined in (2.9).

(3) Evaluate the higher-derivative Lagrangian (1.1) on $\tilde{R}_{\mu\nu\sigma\rho}(\Lambda,\alpha)$; i.e., replace all Riemann tensors appearing in $\mathcal{L}(\text{Riemann})$ by the object defined in (2.20). This gives rise to a function of Λ and α,[9]

$$\mathcal{L}(\Lambda,\alpha) \equiv \mathcal{L}(R_{\mu\nu\rho\sigma} = \tilde{R}_{\mu\nu\rho\sigma}(\Lambda,\alpha), g^{\gamma\delta}). \tag{2.21}$$

(4) The values of a, b, c, and e can be obtained from the expressions

$$\left.\frac{\partial\mathcal{L}}{\partial\alpha}\right|_{\alpha=0} = 2e\chi(\chi-1), \tag{2.22}$$

$$\left.\frac{\partial^2\mathcal{L}}{\partial\alpha^2}\right|_{\alpha=0} = 4\chi(\chi-1)(a + b\chi(\chi-1) + c(\chi-1)), \tag{2.23}$$

as can be proven using the chain rule along with Eqs. (2.2), (2.4), (2.10), and (2.11). Interestingly, since a, b, c, and e do not depend on χ and they appear multiplied by factors involving different combinations of this parameter, we can identify them unambiguously for any theory by simple inspection. Once $\mathcal{L}(\Lambda,\alpha)$ and its derivatives are computed, we just need to compare the resulting expressions with the rhs of (2.22) and (2.23) to obtain a, b, c, and e.[10,11]

(5) Replace the values of a, b, c, and e in the general expression (2.15).

This procedure is obviously simpler than computing $\bar{P}^{\mu\nu\rho\sigma}$ and $\bar{C}^{\mu\nu\alpha\beta}_{\lambda\eta\sigma\tau}$ explicitly using their definitions (2.2) and (2.4). Indeed, the most difficult step is the evaluation of $\mathcal{L}(\Lambda,\alpha)$, which simply involves trivial contractions of $g_{\mu\nu}$ and $k_{\mu\nu}$ for any theory. The function $\mathcal{L}(\Lambda,\alpha)$ is a sort of "prepotential" containing all the information needed for the linearization of a given higher-derivative theory of the form (1.1) on a msb.

We will apply this method in various sections of the paper—e.g., see Sec. IV for the linearization of general quartic theories and Sec. V for theories constructed as functions of curvature invariants. Appendix A contains a detailed application of our linearization procedure to quadratic theories and to a particular Born-Infeld-like theory.

Let us mention that in Refs. [51,52,59], a more refined method than the naive brute-force linearization of the full nonlinear equations was also introduced for general $\mathcal{L}(\text{Riemann})$ theories. This incorporates decompositions similar to the ones in (2.10) and (2.11) but still requires the somewhat tedious explicit evaluation of $\bar{P}^{\mu\nu\rho\sigma}$ and $\bar{C}^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}$ for each theory considered.

We close this subsection by mentioning that our linearization method reproduces all the particular cases previously studied in the literature. These include quadratic gravities [27,51,59–61], quasitopological gravity [19,62], $f(R)$ [23], and general $f(\text{Lovelock})$ theories [63].

B. Equivalent quadratic theory

The linearized equations (2.15) of any higher-order gravity of the form (1.1) characterized by some parameters a, b, c, and e can always be mapped to those of a quadratic theory of the form

[8]The associated "Ricci tensor" and "Ricci scalar" are $\tilde{R}_{\mu\nu} = \Lambda(D-1)g_{\mu\nu} + \alpha(\chi-1)k_{\mu\nu}$ and $\tilde{R} = \Lambda D(D-1) + \alpha\chi(\chi-1)$, respectively.

[9]Note that in this evaluation, indices are still lowered with $g_{\mu\nu}$, and not with some combination of $g_{\mu\nu}$ and $k_{\mu\nu}$.

[10]Observe that we only need $\mathcal{L}(\Lambda,\alpha)$ up to α^2 order; i.e., from $\mathcal{L}(\Lambda,\alpha) = \mathcal{L}(\Lambda) + [2\chi(\chi-1)e]\alpha + [2\chi(\chi-1)(a + b\chi(\chi-1) + c(\chi-1))]\alpha^2 + \mathcal{O}(\alpha^3)$, we can read off the values of all the relevant constants.

[11]Equivalently, they can be obtained through direct evaluation of the following formulas,

$$e = \frac{1}{2\chi(\chi-1)}\left.\frac{\partial\mathcal{L}}{\partial\alpha}\right|_{\alpha=0}, \qquad a = \left.\left[\frac{1}{4\chi(\chi-1)}\left.\frac{\partial^2\mathcal{L}}{\partial\alpha^2}\right|_{\alpha=0}\right]\right|_{\chi=1},$$

$$c = \left.\left[\frac{1}{(\chi-1)}\left[\frac{1}{4\chi(\chi-1)}\left.\frac{\partial^2\mathcal{L}}{\partial\alpha^2}\right|_{\alpha=0} - a\right]\right]\right|_{\chi=0},$$

$$b = \frac{1}{\chi(\chi-1)}\left[\frac{1}{4\chi(\chi-1)}\left.\frac{\partial^2\mathcal{L}}{\partial\alpha^2}\right|_{\alpha=0} - a - c(\chi-1)\right],$$

where $|_{\chi=1}$ means taking the limit $\lim_{\chi\to1}$ in the corresponding expression, etc.

$$\mathcal{L}_{\text{quadratic}} = \lambda(R - 2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma \mathcal{X}_4, \tag{2.24}$$

where $\mathcal{X}_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the dimensionally extended four-dimensional Euler density, also known as the Gauss-Bonnet term. Indeed, the parameters λ, α, β, and γ of the equivalent quadratic theory can be obtained in terms of a, b, c, and e through

$$\lambda = 2e - 4\Lambda[a + bD(D-1) + c(D-1)],$$
$$\alpha = 2b - a, \qquad \beta = 4a + 2c, \qquad \gamma = a. \tag{2.25}$$

Similarly, the cosmological constant Λ_0 can be trivially related to the parameters appearing in (1.1) through $\Lambda_0 = -\mathcal{L}(R_{\mu\nu\rho\sigma} = 0)/(2\lambda)$.

Notice that the mapping from (1.1) to (2.24) is surjective but not injective; i.e., all $\mathcal{L}$(Riemann) theories are mapped to some quadratic theory, but (infinitely) many of them are mapped to the same one. Observe also that the existence of this mapping is a consequence of the fact that the linearized equations of any theory come from its action expanded at quadratic order in $h_{\mu\nu}$—see Sec. II D. This means that the most general quadratic theory, namely, (2.24), already contains all the possible kinds of terms produced in the action at order $\mathcal{O}(h^2)$ of any $\mathcal{L}$(Riemann) theory. Observe, however, that the fact that the parameters a, b, c, and e for a given theory can be related to those appearing in (2.24) does not immediately help in identifying the values of those parameters for a given theory. The mapping was explicitly performed for general cubic theories in Ref. [59].

C. Physical modes

As we just reviewed, $\mathcal{E}^L_{\mu\nu}$ depends on four constants a, b, c, and e as well as on the background curvature Λ. For a given theory, the four constants can be computed using the procedure explained in Sec. II A, from which one can obtain the full linearized equations through (2.15). In this subsection, we will explore how (2.15) can be further simplified using the gauge freedom of the metric perturbation and used to characterize the additional physical modes propagated by the metric in a general theory of the form (1.1).

Let us start with the following observation. If we parametrize a, b, and c in terms of three new constants m_g^2, m_s^2, and κ_{eff} as

$$\begin{aligned} a &= [4e\kappa_{\text{eff}} - 1]/[8\Lambda(D-3)\kappa_{\text{eff}}], \\ b &= [(4e\kappa_{\text{eff}} - 1)(D-1)m_s^2 m_g^2 + 2(3 - 2D + 2(D-1)De\kappa_{\text{eff}})m_g^2\Lambda \\ &\quad + (D-3)\Lambda(Dm_s^2 + 4(D-1)\Lambda)]/[16\Lambda(D-3)\kappa_{\text{eff}}m_g^2(D-1)(m_s^2 + D\Lambda)], \\ c &= -[(4e\kappa_{\text{eff}} - 1)m_g^2 + (D-3)\Lambda]/[4\Lambda(D-3)\kappa_{\text{eff}}m_g^2], \end{aligned} \tag{2.26}$$

it is possible to rewrite (2.15) in terms of four different parameters, namely, κ_{eff}, m_s^2, m_g^2, and Λ. Indeed, one finds

$$\begin{aligned} \mathcal{E}^L_{\mu\nu} = \frac{1}{2\kappa_{\text{eff}}m_g^2}\Bigg\{&[m_g^2 + 2\Lambda - \bar{\Box}]G^L_{\mu\nu} + \left[\frac{(D-2)(m_g^2 + m_s^2 + 2\Lambda)}{2(m_s^2 + D\Lambda)}\right]\Lambda\bar{g}_{\mu\nu}R^L \\ &+ \left[\frac{(D-2)(m_g^2 - m_s^2 - 2(D-1)\Lambda)}{2(D-1)(m_s^2 + D\Lambda)}\right][\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu]R^L\Bigg\} = \frac{1}{2}T^L_{\mu\nu}, \end{aligned} \tag{2.27}$$

so the dependence on e disappears, while that on κ_{eff} gets factorized out from all terms. While (2.15) is more useful when computing the linearized equations of a particular theory—because we know a simple procedure to obtain a, b, c, and e—Eq. (2.27) is more illuminating from a physical point of view. Indeed, as we will see in a moment, κ_{eff} will be the effective Einstein constant[12] while m_g^2 and m_s^2 will correspond, respectively, to the squared masses of additional spin-2 and scalar modes.

It is straightforward to invert the relations (2.26) to obtain the values of such physical quantities in terms of a, b, c, and e. One finds

$$\kappa_{\text{eff}} = \frac{1}{4e - 8\Lambda(D-3)a}, \tag{2.28}$$

$$m_s^2 = \frac{e(D-2) - 4\Lambda(a + bD(D-1) + c(D-1))}{2a + Dc + 4b(D-1)}, \tag{2.29}$$

$$m_g^2 = \frac{-e + 2\Lambda(D-3)a}{2a + c}. \tag{2.30}$$

Let us stress that if we consider a theory consisting of a linear combination of invariants—like the one in (4.2) below—the values of a, b, c, and e of that theory can be simply computed as the analogous linear combination of the parameters for each of those terms. However, that is not

[12]Equivalently, $\kappa_{\text{eff}} \equiv 8\pi G_{\text{eff}}$ where G_{eff} is the effective Newton constant.

the case for κ_{eff}, m_s^2, and m_g^2, since they are not linear combinations of a, b, c, and e. Hence, in order to determine these quantities for a given linear combination of invariants, the natural procedure should be obtaining the total values of a, b, c, and e first and then using (2.28)–(2.30) to compute the corresponding values of κ_{eff}, m_s^2, and m_g^2. For example, for a general quadratic theory of the form

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|}\left[\frac{1}{2\kappa}(-2\Lambda_0 + R) + \kappa^{\frac{(4-D)}{D-2}}(\alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})\right], \tag{2.31}$$

the values of κ_{eff}, m_g, and m_s read, respectively,

$$\kappa_{\mathrm{eff}} = \frac{\kappa}{1 + 4\Lambda\kappa^{\frac{2}{D-2}}(\alpha_1 D(D-1) + \alpha_2(D-1) - 2\alpha_3(D-4))}, \tag{2.32}$$

$$m_s^2 = \frac{(D-2) + 4(D-4)\Lambda\kappa^{\frac{2}{D-2}}(\alpha_1 D(D-1) + \alpha_2(D-1) + 2\alpha_3)}{2\kappa^{\frac{2}{D-2}}(4\alpha_1(D-1) + \alpha_2 D + 4\alpha_3)}, \tag{2.33}$$

$$m_g^2 = \frac{-1 - 4\Lambda\kappa^{\frac{2}{D-2}}(\alpha_1 D(D-1) + \alpha_2(D-1) - 2\alpha_3(D-4))}{2\kappa^{\frac{2}{D-2}}(\alpha_2 + 4\alpha_3)}, \tag{2.34}$$

which we obtained using (2.28)–(2.30) and the values of a, b, c, and e which appear in Table II. During the remainder of this section, we will write all expressions in terms of κ_{eff}, m_s^2, and m_g^2, which will make the presentation clearer. Nonetheless, all equations can be converted back to the language of a, b, c, and e using the above relations.

The discussion proceeds slightly differently depending on whether we consider anti-de Sitter (AdS)/de Sitter (dS) or Minkowski as the background space-time, so we will consider the two cases separately. Let us start with the first.

1. (Anti-)de Sitter background

When studying the physical modes propagated by the metric perturbation on an AdS/dS background, it is customary and very convenient to work in the transverse gauge, in which[13]

$$\bar{\nabla}_\mu h^{\mu\nu} = \bar{\nabla}^\nu h. \tag{2.35}$$

Imposing this condition, many terms in (2.27) cancel out. Let us now expand the metric perturbation into its trace and traceless parts, which we denote by h and $h_{\langle\mu\nu\rangle}$, respectively,[14]

$$h_{\mu\nu} = h_{\langle\mu\nu\rangle} + \frac{1}{D}\bar{g}_{\mu\nu}h. \tag{2.36}$$

Doing the same with the field equations (2.27), we find

$$\mathcal{E}^L_{\langle\mu\nu\rangle} = +\frac{1}{2}T^L_{\langle\mu\nu\rangle} = \frac{1}{4m_g^2\kappa_{\mathrm{eff}}}\Bigg\{[\bar{\Box} - 2\Lambda][\bar{\Box} - 2\Lambda - m_g^2]h_{\langle\mu\nu\rangle} - \bar{\nabla}_{\langle\nu}\bar{\nabla}_{\mu\rangle}\bar{\Box}h + \left[\frac{m_g^2(m_s^2 + 2(D-1)\Lambda) + \Lambda((4-3D)m_s^2 - 4(D-1)^2\Lambda)}{(m_s^2 + D\Lambda)}\right]\bar{\nabla}_{\langle\nu}\bar{\nabla}_{\mu\rangle}h\Bigg\}, \tag{2.37}$$

$$\mathcal{E}^L = +\frac{1}{2}T^L = -\left[\frac{(D-1)(D-2)\Lambda(m_g^2 - (D-2)\Lambda)}{4\kappa_{\mathrm{eff}}m_g^2(m_s^2 + D\Lambda)}\right][\bar{\Box} - m_s^2]h. \tag{2.38}$$

The second is the equation of motion of a free scalar field of mass m_s, while the first is an inhomogeneous equation for $h_{\langle\mu\nu\rangle}$ as it involves also h. In order to obtain an independent equation for the traceless part, we define another traceless tensor,

[13]The metric decomposition performed in this section is similar to the one considered in Ref. [60].

[14]In this section, we denote the trace and traceless parts of rank-2 tensors $P_{\mu\nu}$ linear in $h_{\mu\nu}$ as $P \equiv \bar{g}^{\mu\nu}P_{\mu\nu}$ and $P_{\langle\mu\nu\rangle} \equiv P_{\mu\nu} - \frac{1}{D}\bar{g}_{\mu\nu}P$, respectively. In the case of the equations of motion, one can use the same notation, i.e., $\mathcal{E}^L \equiv \bar{g}^{\mu\nu}\mathcal{E}^L_{\mu\nu}$, $T^L \equiv \bar{g}^{\mu\nu}T^L_{\mu\nu}$—and similarly for the traceless part—because $\bar{\mathcal{E}}_{\mu\nu} = \bar{T}_{\mu\nu} = 0$. Observe, however, that $R^L = (g^{\mu\nu}R_{\mu\nu})^L$ is not the trace of $R^L_{\mu\nu}$, but rather $R^L = \bar{g}^{\mu\nu}R^L_{\mu\nu} - h^{\mu\nu}\bar{R}_{\mu\nu} = \bar{g}^{\mu\nu}R^L_{\mu\nu} - (D-1)h\Lambda$.

$$t_{\mu\nu} \equiv h_{\langle\mu\nu\rangle} - \frac{\bar{\nabla}_{\langle\mu}\bar{\nabla}_{\nu\rangle}h}{(m_s^2 + D\Lambda)}, \tag{2.39}$$

where we have implicitly assumed that $m_s^2 \neq -D\Lambda$. After some manipulations, it can be seen that $t_{\mu\nu}$ satisfies the equation

$$\frac{1}{2\kappa_{\text{eff}} m_g^2}(\bar{\Box} - 2\Lambda)(\bar{\Box} - 2\Lambda - m_g^2)t_{\mu\nu} = T^{L,\text{eff}}_{\langle\mu\nu\rangle}, \tag{2.40}$$

where we have defined the effective energy-momentum tensor

$$T^{L,\text{eff}}_{\langle\mu\nu\rangle} \equiv T^L_{\langle\mu\nu\rangle} + \frac{[\bar{\Box} + (D-4)\Lambda - m_g^2]\bar{\nabla}_{\langle\mu}\bar{\nabla}_{\nu\rangle}T^L}{\Lambda(D-1)(D-2)(m_g^2 - (D-2)\Lambda)}. \tag{2.41}$$

Now, observe that the object

$$t^{(m)}_{\mu\nu} \equiv -\frac{1}{m_g^2}(\bar{\Box} - 2\Lambda - m_g^2)t_{\mu\nu} \tag{2.42}$$

satisfies the equation of the usual massless graviton, namely,

$$-(\bar{\Box} - 2\Lambda)t^{(m)}_{\mu\nu} = 2\kappa_{\text{eff}}T^{L,\text{eff}}_{\langle\mu\nu\rangle}, \tag{2.43}$$

but with a nonstandard coupling to matter. On the other hand, using (2.42) and (2.43), it is easy to see that the tensor

$$t^{(M)}_{\mu\nu} \equiv t_{\mu\nu} - t^{(m)}_{\mu\nu} = \frac{1}{m_g^2}(\bar{\Box} - 2\Lambda)t_{\mu\nu} \tag{2.44}$$

satisfies instead

$$(\bar{\Box} - 2\Lambda - m_g^2)t^{(M)}_{\mu\nu} = 2\kappa_{\text{eff}}T^{L,\text{eff}}_{\langle\mu\nu\rangle}. \tag{2.45}$$

Hence, we identify $t^{(M)}_{\mu\nu}$ with a massive traceless spin-2 field with mass m_g. Observe that the coupling to matter of this mode has the wrong sign, which reflects its ghostlike behavior. Note that, apart from being a ghost, this mode is also tachyonic whenever $m_g^2 < 0$. The same occurs for the scalar when $m_s^2 < 0$.

In sum, using definitions (2.39), (2.42), and (2.44), we can decompose the metric perturbation $h_{\mu\nu}$ as

$$h_{\mu\nu} = t^{(m)}_{\mu\nu} + t^{(M)}_{\mu\nu} + \frac{\bar{\nabla}_{\langle\mu}\bar{\nabla}_{\nu\rangle}h}{(m_s^2 + D\Lambda)} + \frac{1}{D}\bar{g}_{\mu\nu}h, \tag{2.46}$$

where h, $t^{(M)}_{\mu\nu}$, and $t^{(m)}_{\mu\nu}$ satisfy (2.38), (2.45), and (2.43) and represent, respectively, a scalar mode of mass m_s; a ghostlike spin-2 mode of mass m_g, which we will often refer to as a "massive graviton" throughout the text; and a massless graviton.

2. Minkowski background

If we set $\Lambda = 0$ in (2.38), this equation would lead us to conclude that $T = 0$. This inconsistency is a reflection of the fact that the transverse gauge cannot be used in flat space-time. The usual choice is in this case the so-called *de Donder gauge*, given by

$$\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h. \tag{2.47}$$

In this gauge, the linearized field equations (2.27) in a Minkowski background can be written as

$$\mathcal{E}^L_{\mu\nu} = -\frac{1}{4\kappa_{\text{eff}}}\bar{\Box}\hat{h}_{\mu\nu} = \frac{1}{2}T^L_{\mu\nu}, \tag{2.48}$$

where we have defined

$$\hat{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h - \frac{1}{m_g^2}\left[\bar{\Box}h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h\right] + \left[\frac{m_g^2(D-2) + m_s^2}{2(D-1)m_g^2 m_s^2}\right][\eta_{\mu\nu}\bar{\Box} - \partial_\mu\partial_\nu]h. \tag{2.49}$$

Using the gauge condition (2.47), it is easy to see that $\hat{h}_{\mu\nu}$ is transverse, i.e.,

$$\partial_\mu \hat{h}^{\mu\nu} = 0. \tag{2.50}$$

Naturally, $\hat{h}_{\mu\nu}$ is the usual spin-2 massless graviton, as it satisfies the linearized Einstein equation (2.48). However, there are more degrees of freedom (dof). In particular, we find that the metric can be decomposed as

$$h_{\mu\nu} = \hat{h}_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}\hat{h} + \frac{1}{D-1}(m_g^{-2} - m_s^{-2})\partial_{\langle\mu}\partial_{\nu\rangle}\hat{h} + t_{\mu\nu} + \frac{2}{D(D-2)}\eta_{\mu\nu}\phi + \frac{1}{(D-1)m_s^2}\partial_{\langle\mu}\partial_{\nu\rangle}\phi, \tag{2.51}$$

where $t_{\mu\nu}$ is traceless and ϕ is a scalar field. These objects satisfy the equations

$$-(\bar{\Box} - m_s^2)\phi = 2\kappa_{\text{eff}}T^L, \tag{2.52}$$

$$(\bar{\Box} - m_g^2)t_{\mu\nu} = 2\kappa_{\text{eff}}\left[T^L_{\langle\mu\nu\rangle} + \frac{1}{(D-1)m_g^2}\partial_{\langle\mu}\partial_{\nu\rangle}T^L\right]. \tag{2.53}$$

Hence, even though we have proceeded in a different way as compared to the $\Lambda \neq 0$ case, we have found the same physical modes: we have a massless spin-2 graviton $\hat{h}_{\mu\nu}$, a massive one $t_{\mu\nu}$, and a scalar ϕ, the masses of the last two being the same as the ones we found for $t^{(M)}_{\mu\nu}$ and h in the (A)

TABLE I. Classification of theories according to their spectrum on a msb.

	$m_g^2 = 0$	$0 < m_g^2 < +\infty$	$m_g^2 = +\infty$
$0 \leq m_s^2 < +\infty$	Massless gravitons + scalar	General case	No massive graviton
$m_s^2 = +\infty$	Critical	No dynamical scalar	Einstein-like

dS case. Note, however, that, even though the dof and the masses are the same, the metric decomposition as well as the coupling of the fields to matter are different—compare (2.38) and (2.45) with (2.52) and (2.53) and (2.46) with (2.51). This can be understood as a consequence of the fact that the gauge which is convenient for (A)dS (2.35) differs from the de Donder one (2.47) utilized for Minkowski.

D. Quadratic action

As pointed out in Sec. II B, the linearized equations (2.27) come from terms of order $\mathcal{O}(h^2)$ in the action, which means that the structure of the linearized equations for the most general $\mathcal{L}$(Riemann) is already captured by the most general quadratic theory. Expanding the action of a higher-order gravity to $\mathcal{O}(h^2)$ is not trivial in general. However, we can use the expression for the linearized equations (2.27) to find an action that yields these equations when varied with respect to $h_{\mu\nu}$. The easiest possibility is

$$S_2 = -\frac{1}{2}\int_{\mathcal{M}} d^D x h^{\mu\nu}\mathcal{E}^L_{\mu\nu}. \tag{2.54}$$

Using (2.27) and integrating by parts several times, we find the effective action

$$S_2 = \int_{\mathcal{M}} \frac{d^D x}{4\kappa_{\rm eff}} \times \left[\frac{(D-2)[m_g^2 + (D-2)(m_s^2 + (D-1)\Lambda)]}{2(D-1)m_g^2(m_s^2 + D\Lambda)}(R^L)^2 - \left[h^{\mu\nu} + \frac{2G^{L\mu\nu}}{m_g^2}\right]G^L_{\mu\nu}\right]. \tag{2.55}$$

As pointed out in Ref. [51], where an analogous action was found, Eq. (2.55) is manifestly invariant under "gauge" transformations $h_{\mu\nu} \to h_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$ as follows from the invariance of the linearized Einstein tensor and Ricci scalar under such transformations.

III. CLASSIFICATION OF THEORIES

In this section, we will classify all gravity theories of the form (1.1) according to the properties of their physical modes. Indeed, depending on the values of the parameters a, b, c, and e, we will divide them into five classes[15]: 1) theories without massive gravitons, i.e., those for which the additional spin-2 mode is absent but the spin-0 one is dynamical; 2) theories without a dynamical scalar, i.e., those for which the additional graviton is dynamical but the spin-0 mode is absent; 3) theories with two massless gravitons and a massive scalar, i.e., those for which the extra graviton is massless—a property which to some extent cures its problematic behavior; 4) generalized *critical* gravities, i.e., those which belong to the previous category and, in addition, have no additional spin-0 mode; and, finally, 5) Einstein-like theories, i.e., theories for which the only mode is the usual massless graviton.[16] A summary of the different cases can be found in Table I, and various examples of particular theories belonging to each class are provided in Appendix B. Let us note in passing that boundary conditions can be sometimes used to remove spurious modes from the spectrum of certain higher-order gravities—see Ref. [65]. We shall not discuss this issue here. Finally, let us also mention that related analyses were previously performed in the absence of matter in Refs. [29,51,59].

A. Theories without massive graviton

The ghostlike massive spin-2 mode $t^{(M)}_{\mu\nu}$ found in the previous section can be removed from the linearized spectrum of the theory by imposing $m_g^2 = +\infty$. In terms of the parameters characterizing a given higher-derivative theory as described in Sec. II, such a condition will be satisfied whenever

$$2a + c = 0. \tag{3.1}$$

When this condition holds, the linearized equations (2.27) become

$$\mathcal{E}^L_{\mu\nu} = \frac{1}{2\kappa_{\rm eff}}\left\{G^L_{\mu\nu} + \left[\frac{(D-2)}{2(D-1)(m_s^2 + D\Lambda)}\right] \times [(D-1)\Lambda\bar{g}_{\mu\nu} + \bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu]R^L\right\}. \tag{3.2}$$

Observe that (3.1) has the effect of making the $\bar{\Box}G^L_{\mu\nu}$ term—responsible for the appearance of the extra spin-2 graviton—disappear. As a consequence, even though (3.2)

[15]Or six, if we count the general case in which m_g^2 is finite and different from zero, and $0 \leq m_s^2 < +\infty$.

[16]In principle, one could also impose more exotic conditions like $\kappa_{\rm eff} = 0$, which would remove all propagating modes; see, e.g., Ref. [64].

still contains quartic derivatives of $h_{\mu\nu}$, the equations do become second order when we choose the transverse gauge $\bar{\nabla}^{\mu} h_{\mu\nu} = \bar{\nabla}_{\nu} h$, as it can be immediately checked from (3.2) using (2.17)—or alternatively from (2.37) taking the limit $m_g^2 \to +\infty$ there.

On AdS/dS backgrounds—the extension to Minkowski is straightforward—Eq. (3.1) imposes $t^{(M)}_{\mu\nu} = 0$, so the metric decomposition becomes now

$$h_{\mu\nu} = t^{(m)}_{\mu\nu} + \frac{\bar{\nabla}_{\langle\mu}\bar{\nabla}_{\nu\rangle} h}{(m_s^2 + D\Lambda)} + \frac{1}{D}\bar{g}_{\mu\nu} h, \tag{3.3}$$

where h and $t^{(m)}_{\mu\nu}$ still satisfy (2.38) and (2.43), respectively. Observe that using (2.38) and (3.3) along with the transverse gauge condition (2.35), it is possible to show that $t^{(m)}_{\mu\nu}$ is transverse in the vacuum,

$$\bar{\nabla}^{\mu} t^{(m)}_{\mu\nu} = 0. \tag{3.4}$$

Notice also that after imposing (2.35), we still have some gauge freedom, because a gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + 2\bar{\nabla}_{(\mu}\xi_{\nu)}$ for any vector ξ_{μ} satisfying $\bar{\nabla}^{\mu}\bar{\nabla}_{(\mu}\xi_{\nu)} = \bar{\nabla}_{\nu}\bar{\nabla}_{\mu}\xi^{\mu}$ preserves (2.35). This allows us to impose additional conditions on $h_{\mu\nu}$. In particular, we can choose

$$t^{(m)}_{0\mu} = t^{(m)}_{\mu 0} = 0, \tag{3.5}$$

so that only the spatial components $t^{(m)}_{ij}$, $i, j = 1, \ldots, D-1$ are nonzero. Then, this tensor has $D(D-1)/2$ components, but we have also

$$\bar{\nabla}^{i} t^{(m)}_{ij} = 0, \qquad \bar{g}^{ij} t^{(m)}_{ij} = 0, \tag{3.6}$$

which follow from (3.4) and the tracelessness of $t^{(m)}_{\mu\nu}$, respectively. These are $(D-1) + 1 = D$ constraints, so the number of polarizations of $t^{(m)}_{\mu\nu}$ is $D(D-3)/2$, just like for the usual Einstein graviton. Of course, the trace h provides an additional degree of freedom, so these theories propagate $(D-1)(D-2)/2$ physical dof in the vacuum.

B. Theories without dynamical scalar

The condition for the absence of the scalar mode is naturally given by $m_s^2 = +\infty$. In terms of the parameters a, b, c, and e, this reads

$$2a + Dc + 4b(D-1) = 0. \tag{3.7}$$

The linearized equations of motion (2.27) become in that case

$$\mathcal{E}^{L}_{\mu\nu} = \frac{1}{2\kappa_{\rm eff} m_g^2}\left\{[m_g^2 + 2\Lambda - \bar{\Box}] G^{L}_{\mu\nu} + \frac{(D-2)}{2(D-1)}[(D-1)\Lambda\bar{g}_{\mu\nu} - \bar{g}_{\mu\nu}\bar{\Box} + \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}] R^{L}\right\}. \tag{3.8}$$

The metric decomposition simplifies to

$$h_{\mu\nu} = t^{(m)}_{\mu\nu} + t^{(M)}_{\mu\nu} + \frac{1}{D}\bar{g}_{\mu\nu} h, \tag{3.9}$$

where the trace of the metric perturbation is simply determined by the matter stress tensor through the expression

$$h = \frac{2\kappa_{\rm eff} m_g^2}{(D-1)(D-2)\Lambda(m_g^2 - (D-2)\Lambda)} T^{L}. \tag{3.10}$$

The massless and massive gravitons satisfy the same equations as in the general case, i.e., Eqs. (2.43) and (2.45), respectively.

C. Theories with two massless gravitons

As we saw, $t^{(M)}_{\mu\nu}$ is a ghost. In order to remove this instability, the simplest solution is to consider theories in which it is absent. Another possibility is to set $m_g = 0$, namely, impose its mass to be zero like for the usual graviton. The condition to be satisfied is in this case

$$-e + 2\Lambda(D-3)a = 0. \tag{3.11}$$

From (2.28), we learn that (3.11) also imposes the effective Einstein constant to diverge, $\kappa_{\rm eff} = +\infty$. This inconsistency is artificial and comes from a wrong identification of $\kappa_{\rm eff}$ in this case. In fact, the effective gravitational constant must be defined now as

$$\hat{\kappa}_{\rm eff} \equiv m_g^2 \kappa_{\rm eff} = -\frac{1}{4(2a + c)}, \tag{3.12}$$

which remains finite when we impose (3.11). Then, the equation for the trace reads

$$\left[\frac{(D-1)(D-2)^2\Lambda^2}{2\hat{\kappa}_{\rm eff}(m_s^2 + D\Lambda)}\right][\bar{\Box} - m_s^2] h = T^{L}. \tag{3.13}$$

On the other hand, we cannot decompose the traceless perturbation $t_{\mu\nu}$ into two independent fields. Instead, it fulfills the equation

$$\frac{1}{2\hat{\kappa}_{\rm eff}}(\bar{\Box} - 2\Lambda)^2 t_{\mu\nu} = T^{L,\rm eff}_{\langle\mu\nu\rangle}, \tag{3.14}$$

with a metric decomposition given now by

$$h_{\mu\nu} = t_{\mu\nu} + \frac{\bar{\nabla}_{\langle\mu}\bar{\nabla}_{\nu\rangle}h}{(m_s^2 + D\Lambda)} + \frac{1}{D}\bar{g}_{\mu\nu}h. \quad (3.15)$$

D. Critical gravities

Critical gravities [27] are theories in which the extra graviton is massless and, in addition, the scalar mode is absent, i.e., it satisfies $m_s^2 = +\infty$. As shown in Ref. [27] for the quadratic case in $D = 4$, the energies of both $t_{\mu\nu}^{(m)}$ and $t_{\mu\nu}^{(M)}$ become zero for this class of theories. We can easily check this statement from the quadratic action (2.55). Specifying for the critical gravity case, it reads

$$S_2 = \int_{\mathcal{M}} \frac{d^D x}{4\hat{\kappa}_{\text{eff}}}\left[\frac{(D-2)^2}{2(D-1)}(R^L)^2 - 2G^{L\mu\nu}G^L_{\mu\nu}\right]. \quad (3.16)$$

Now, in the vacuum, the field equations imply that $h = 0$, so that $R^L = 0$, and $(\Box - 2\Lambda)^2 h_{\langle\mu\nu\rangle} = 0$. There are solutions, corresponding to the usual massless graviton, which are annihilated by $(\Box - 2\Lambda)$, and they have $G^L_{\mu\nu} = 0$. Therefore, for these solutions, the Lagrangian as well as its derivatives vanish on shell. In particular, the Hamiltonian vanishes, since it is constructed from the Lagrangian and its first derivatives, so the gravitons have zero energy. However, there are additional logarithmic modes which are not annihilated by $(\Box - 2\Lambda)$, but by the full operator $(\Box - 2\Lambda)^2$ instead, and these modes do carry positive energy [27].

The conditions to be imposed for this class of theories are (3.11) and (3.7) as well as the redefinition of the Einstein constant in (3.12). Then, the traceless part of the metric satisfies

$$\frac{1}{2\hat{\kappa}_{\text{eff}}}[(\bar{\Box} - 2\Lambda)^2 h_{\langle\mu\nu\rangle} - \bar{\nabla}_{\langle\nu}\bar{\nabla}_{\mu\rangle}\bar{\Box}h] = T^L_{\langle\mu\nu\rangle}, \quad (3.17)$$

while the trace is determined by matter,

$$h = -\frac{2\hat{\kappa}_{\text{eff}}}{(D-1)(D-2)^2\Lambda^2}T^L. \quad (3.18)$$

E. Einstein-like theories

When both the massive graviton and the scalar mode are absent, we are left with a theory of which the only propagating degree of freedom is a massless graviton. The conditions $m_g^2 = m_s^2 = +\infty$ can be expressed as

$$2a + c = 4b + c = 0. \quad (3.19)$$

The linearized equations of motion drastically simplify and become identical to those of Einstein gravity with an effective Einstein constant,

$$\mathcal{E}^L_{\mu\nu} = \frac{1}{2\kappa_{\text{eff}}}G^L_{\mu\nu} = \frac{1}{2}T^L_{\mu\nu}. \quad (3.20)$$

The metric decomposition is very simple now,

$$h_{\mu\nu} = t^{(m)}_{\mu\nu} + \frac{1}{D}\bar{g}_{\mu\nu}h, \quad (3.21)$$

with $t^{(m)}_{\mu\nu}$ satisfying (2.43) and h being again completely determined by matter,

$$h = \frac{2\kappa_{\text{eff}}}{\Lambda(D-1)(D-2)}T^L. \quad (3.22)$$

Hence, according to the discussion in Sec. III A, the only propagating mode is the transverse and traceless part of the metric perturbation, which carries $D(D-3)/2$ dof, like in Einstein gravity. Let us stress at this point that throughout the text, we use the labels *Einstein-like* and *Einsteinian* with different meanings. By Einstein-like theories, we mean theories for which the extra modes are absent and the only dynamical field at the linearized level is the usual massless graviton of general relativity. By Einsteinian, we refer to those Einstein-like theories which are defined in a dimension-independent way—see Sec. VI.

IV. LINEARIZATION OF ALL THEORIES UP TO QUARTIC ORDER

Up to quartic order in curvature, the most general D-dimensional theory of the form (1.1) can be written as

$$S = \int_{\mathcal{M}} d^D x\sqrt{|g|}\left\{\frac{1}{2\kappa}(-2\Lambda_0 + R) + \kappa^{\frac{4-D}{D-2}}\sum_{i=1}^{3}\alpha_i\mathcal{L}_i^{(2)} + \kappa^{\frac{6-D}{D-2}}\sum_{i=1}^{8}\beta_i\mathcal{L}_i^{(3)} + \kappa^{\frac{8-D}{D-2}}\sum_{i=1}^{26}\gamma_i\mathcal{L}_i^{(4)}\right\}. \quad (4.1)$$

Here, $\mathcal{L}_i^{(2)}$, $\mathcal{L}_i^{(3)}$, and $\mathcal{L}_i^{(4)}$ represent, respectively, the quadratic, cubic, and quartic curvature invariants enumerated in Table II; α_i, β_i, and γ_i are dimensionless constants; and $\kappa = 8\pi G$ is again Einstein's constant. Also, Λ_0 is the cosmological constant, and we choose $\kappa^{\frac{1}{D-2}}$ to be the natural scale.[17] In general dimensions, there are 3 independent quadratic, 8 cubic, and 26 quartic invariants [66]. Naturally, these numbers get reduced as we consider small enough D. For example, in $D = 4$, there are only 2 quadratic, 6 cubic, and 13 quartic invariants.

[17] This election can be trivially changed by a rescaling of the couplings, e.g., $\alpha_i \to \alpha_i/(\Lambda_0\kappa^{\frac{2}{D-2}})^{\frac{4-D}{2}}$.

TABLE II. Parameters e, a, b, and c of the linearized equations for all Riemann curvature invariants up to fourth order. We have cross-checked all the terms independently for $D = 3$, 4, 5 using *Mathematica*.

Label	Term	e	a	b	c
$\mathcal{L}_1^{(1)}$	R	$\frac{1}{2}$	0	0	0
$\mathcal{L}_1^{(2)}$	R^2	$D(D-1)\Lambda$	0	$\frac{1}{2}$	0
$\mathcal{L}_2^{(2)}$	$R_{\mu\nu}R^{\mu\nu}$	$(D-1)\Lambda$	0	0	$\frac{1}{2}$
$\mathcal{L}_3^{(2)}$	$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$	2Λ	1	0	0
$\mathcal{L}_1^{(3)}$	$R_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \delta\ \gamma}R_{\delta\ \gamma}^{\ \mu\ \nu}$	$\frac{3}{2}(D-2)\Lambda^2$	$-\frac{3}{2}\Lambda$	0	$\frac{3}{2}\Lambda$
$\mathcal{L}_2^{(3)}$	$R_{\mu\nu}^{\ \ \rho\sigma}R_{\rho\sigma}^{\ \ \delta\gamma}R_{\delta\gamma}^{\ \ \mu\nu}$	$6\Lambda^2$	6Λ	0	0
$\mathcal{L}_3^{(3)}$	$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\delta}R^{\sigma\delta}$	$3(D-1)\Lambda^2$	$(D-1)\Lambda$	0	2Λ
$\mathcal{L}_4^{(3)}$	$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}R$	$3D(D-1)\Lambda^2$	$D(D-1)\Lambda$	2Λ	0
$\mathcal{L}_5^{(3)}$	$R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma}$	$\frac{3}{2}(D-1)^2\Lambda^2$	0	$\frac{1}{2}\Lambda$	$\frac{1}{2}(2D-3)\Lambda$
$\mathcal{L}_6^{(3)}$	$R_{\mu}^{\ \nu}R_{\nu}^{\ \rho}R_{\rho}^{\ \mu}$	$\frac{3}{2}(D-1)^2\Lambda^2$	0	0	$\frac{3}{2}(D-1)\Lambda$
$\mathcal{L}_7^{(3)}$	$R_{\mu\nu}R^{\mu\nu}R$	$\frac{3}{2}D(D-1)^2\Lambda^2$	0	$(D-1)\Lambda$	$\frac{1}{2}D(D-1)\Lambda$
$\mathcal{L}_8^{(3)}$	R^3	$\frac{3}{2}D^2(D-1)^2\Lambda^2$	0	$\frac{3}{2}D(D-1)\Lambda$	0
$\mathcal{L}_1^{(4)}$	$R^{\mu\nu\rho\sigma}R_{\mu\ \rho}^{\ \delta\ \gamma}R_{\delta\ \nu}^{\ \chi\ \xi}R_{\gamma\chi\sigma\xi}$	$2(3D-5)\Lambda^3$	$2(D-4)\Lambda^2$	0	$7\Lambda^2$
$\mathcal{L}_2^{(4)}$	$R^{\mu\nu\rho\sigma}R_{\mu\ \rho}^{\ \delta\ \gamma}R_{\delta\ \gamma}^{\ \chi\ \xi}R_{\nu\chi\sigma\xi}$	$2(D^2-3D+4)\Lambda^3$	$6\Lambda^2$	Λ^2	$2(D-3)\Lambda^2$
$\mathcal{L}_3^{(4)}$	$R^{\mu\nu\rho\sigma}R_{\mu\nu}^{\ \ \delta\gamma}R_{\rho\ \delta}^{\ \chi\ \xi}R_{\sigma\chi\gamma\xi}$	$4(D-2)\Lambda^3$	$(D-7)\Lambda^2$	0	$5\Lambda^2$
$\mathcal{L}_4^{(4)}$	$R^{\mu\nu\rho\sigma}R_{\mu\nu}^{\ \ \delta\gamma}R_{\rho\delta}^{\ \ \chi\xi}R_{\sigma\gamma\chi\xi}$	$8\Lambda^3$	$12\Lambda^2$	0	0
$\mathcal{L}_5^{(4)}$	$R^{\mu\nu\rho\sigma}R_{\mu\nu}^{\ \ \delta\gamma}R_{\delta\gamma}^{\ \ \chi\xi}R_{\rho\sigma\chi\xi}$	$16\Lambda^3$	$24\Lambda^2$	0	0
$\mathcal{L}_6^{(4)}$	$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho}^{\ \ \ \delta}R_{\gamma\xi\chi\sigma}R^{\gamma\xi\chi}{}_{\delta}$	$8(D-1)\Lambda^3$	$4(D-1)\Lambda^2$	0	$8\Lambda^2$
$\mathcal{L}_7^{(4)}$	$(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2$	$8D(D-1)\Lambda^3$	$4D(D-1)\Lambda^2$	$8\Lambda^2$	0
$\mathcal{L}_8^{(4)}$	$R^{\mu\nu}R^{\rho\sigma\delta\gamma}R_{\rho\ \delta\mu}^{\ \xi}R_{\sigma\xi\gamma\nu}$	$2(D-1)(D-2)\Lambda^3$	$-\frac{3}{2}(D-1)\Lambda^2$	$\frac{1}{2}\Lambda^2$	$\frac{1}{2}(5D-9)\Lambda^2$
$\mathcal{L}_9^{(4)}$	$R^{\mu\nu}R^{\rho\sigma\delta\gamma}R_{\rho\sigma\ \mu}^{\ \ \xi}R_{\delta\gamma\xi\nu}$	$8(D-1)\Lambda^3$	$6(D-1)\Lambda^2$	0	$6\Lambda^2$
$\mathcal{L}_{10}^{(4)}$	$R^{\mu\nu}R_{\mu\ \nu}^{\ \rho\ \sigma}R_{\delta\gamma\xi\rho}R^{\delta\gamma\xi}{}_{\sigma}$	$4(D-1)^2\Lambda^3$	$(D-1)^2\Lambda^2$	$2\Lambda^2$	$(3D-5)\Lambda^2$
$\mathcal{L}_{11}^{(4)}$	$RR_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \delta\ \gamma}R_{\delta\ \gamma}^{\ \mu\ \nu}$	$2D(D-1)(D-2)\Lambda^3$	$-\frac{3}{2}D(D-1)\Lambda^2$	$\frac{3}{2}(D-2)\Lambda^2$	$\frac{3}{2}D(D-1)\Lambda^2$
$\mathcal{L}_{12}^{(4)}$	$RR_{\mu\nu}^{\ \ \rho\sigma}R_{\rho\sigma}^{\ \ \delta\gamma}R_{\delta\gamma}^{\ \ \mu\nu}$	$8D(D-1)\Lambda^3$	$6D(D-1)\Lambda^2$	$6\Lambda^2$	0
$\mathcal{L}_{13}^{(4)}$	$R^{\mu\nu}R^{\rho\sigma}R^{\delta}{}_{\mu}{}^{\gamma}{}_{\rho}R_{\delta\nu\gamma\sigma}$	$4(D-1)^2\Lambda^3$	$(D-1)^2\Lambda^2$	$\frac{1}{2}\Lambda^2$	$\frac{1}{2}(9D-10)\Lambda^2$
$\mathcal{L}_{14}^{(4)}$	$R^{\mu\nu}R^{\rho\sigma}R^{\delta}{}_{\mu}{}^{\gamma}{}_{\nu}R_{\delta\rho\gamma\sigma}$	$2(D-1)^3\Lambda^3$	0	$\frac{1}{2}(3D-4)\Lambda^2$	$\frac{1}{2}(3D^2-8D+6)\Lambda^2$
$\mathcal{L}_{15}^{(4)}$	$R^{\mu\nu}R^{\rho\sigma}R^{\delta\gamma}{}_{\mu\rho}R_{\delta\gamma\nu\sigma}$	$4(D-1)^2\Lambda^3$	$(D-1)^2\Lambda^2$	Λ^2	$(4D-5)\Lambda^2$
$\mathcal{L}_{16}^{(4)}$	$R^{\mu\nu}R_{\nu}^{\ \rho}R^{\sigma\delta\gamma}{}_{\mu}R_{\sigma\delta\gamma\rho}$	$4(D-1)^2\Lambda^3$	$(D-1)^2\Lambda^2$	0	$5(D-1)\Lambda^2$
$\mathcal{L}_{17}^{(4)}$	$R_{\delta\gamma}R^{\delta\gamma}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$	$4D(D-1)^2\Lambda^3$	$D(D-1)^2\Lambda^2$	$4(D-1)\Lambda^2$	$D(D-1)\Lambda^2$
$\mathcal{L}_{18}^{(4)}$	$RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\delta}R^{\sigma\delta}$	$4D(D-1)^2\Lambda^3$	$D(D-1)^2\Lambda^2$	$3(D-1)\Lambda^2$	$2D(D-1)\Lambda^2$
$\mathcal{L}_{19}^{(4)}$	$R^2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$	$4D^2(D-1)^2\Lambda^3$	$D^2(D-1)^2\Lambda^2$	$5D(D-1)\Lambda^2$	0
$\mathcal{L}_{20}^{(4)}$	$R^{\mu\nu}R_{\mu\rho\nu\sigma}R^{\delta\rho}R_{\delta}^{\ \sigma}$	$2(D-1)^3\Lambda^3$	0	$(D-1)\Lambda^2$	$(D-1)(2D-3)\Lambda^2$
$\mathcal{L}_{21}^{(4)}$	$RR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma}$	$2D(D-1)^3\Lambda^3$	0	$\frac{1}{2}(D-1)(4D-3)\Lambda^2$	$\frac{1}{2}D(D-1)(2D-3)\Lambda^2$
$\mathcal{L}_{22}^{(4)}$	$R_{\mu}^{\ \nu}R_{\nu}^{\ \rho}R_{\rho}^{\ \sigma}R_{\sigma}^{\ \mu}$	$2(D-1)^3\Lambda^3$	0	0	$3(D-1)^2\Lambda^2$
$\mathcal{L}_{23}^{(4)}$	$(R_{\mu\nu}R^{\mu\nu})^2$	$2D(D-1)^3\Lambda^3$	0	$2(D-1)^2\Lambda^2$	$D(D-1)^2\Lambda^2$
$\mathcal{L}_{24}^{(4)}$	$RR_{\mu}^{\ \nu}R_{\nu}^{\ \rho}R_{\rho}^{\ \mu}$	$2D(D-1)^3\Lambda^3$	0	$\frac{3}{2}(D-1)^2\Lambda^2$	$\frac{3}{2}D(D-1)^2\Lambda^2$
$\mathcal{L}_{25}^{(4)}$	$R^2R_{\mu\nu}R^{\mu\nu}$	$2D^2(D-1)^3\Lambda^3$	0	$\frac{5}{2}D(D-1)^2\Lambda^2$	$\frac{1}{2}D^2(D-1)^2\Lambda^2$
$\mathcal{L}_{26}^{(4)}$	R^4	$2D^3(D-1)^3\Lambda^3$	0	$3D^2(D-1)^2\Lambda^2$	0

Using the procedure explained in Sec. II, we have linearized the quartic action (4.2); i.e., we have computed the quantity $\mathcal{L}(\Lambda, \alpha)$ defined in (2.21) at order $\mathcal{O}(\alpha^2)$ for every term in the action and obtained the values of a, b, c, and e from there. The results are shown in Table II. Finally, the parameters a, b, c, and e of the full theory (4.2) can be found by adding linearly the contribution of each term, with the corresponding coefficients in front in each case, namely,

$$e = \frac{1}{2\kappa} e[R] + \kappa^{\frac{4-D}{D-2}} \sum_{i=1}^{3} \alpha_i e[\mathcal{L}_i^{(2)}] + \kappa^{\frac{6-D}{D-2}} \sum_{i=1}^{8} \beta_i e[\mathcal{L}_i^{(3)}] + \kappa^{\frac{8-D}{D-2}} \sum_{i=1}^{26} \gamma_i e[\mathcal{L}_i^{(4)}], \tag{4.2}$$

where, e.g., $e[R] = 1/2$ is the value of e corresponding to the Einstein-Hilbert term R, and so on. Completely analogous expressions hold for a, b, and c.

Table II along with the results in Sec. III can be easily used to classify the different theories in (4.2) according to their spectrum.

V. f(scalars) THEORIES

In Sec. IV, we linearized all higher-derivative gravities of the form (1.1) up to quartic order. That class includes linear combinations of scalars $\mathcal{R}_i$ constructed from contractions of the Riemann tensor and the metric, but not theories constructed as arbitrary functions of those scalars, such as $f(R)$ gravity. In this section, we will consider the latter case; i.e., we will linearize the equations of motion of a theory of the form

$$\mathcal{L} = f(\mathcal{R}_1, \ldots, \mathcal{R}_m), \tag{5.1}$$

where the $\mathcal{R}_i$ are arbitrary scalars.

For a theory of this form, using the objects

$$P_i^{\mu\alpha\beta\nu} \equiv \frac{\partial \mathcal{R}_i}{\partial R_{\mu\alpha\beta\nu}}, \quad C_{i\ \sigma\rho\lambda\eta}^{\mu\gamma\sigma\nu} \equiv g_{\sigma\alpha} g_{\rho\beta} g_{\lambda\chi} g_{\eta\xi} \frac{\partial P_i^{\mu\gamma\sigma\nu}}{\partial R_{\alpha\beta\chi\xi}}, \tag{5.2}$$

we get the following result for the tensors defined in (2.2) and (2.4) evaluated on the background,

$$\begin{aligned} \bar{P}^{\mu\alpha\beta\nu} &= \partial_i f(\bar{\mathcal{R}}) \bar{P}_i^{\mu\alpha\beta\nu}, \\ \bar{C}_{\sigma\rho\lambda\eta}^{\mu\alpha\beta\nu} &= \partial_i f(\bar{\mathcal{R}}) \bar{C}_{i\ \sigma\rho\lambda\eta}^{\mu\alpha\beta\nu} + \partial_i \partial_j f(\bar{\mathcal{R}}) \bar{P}_i^{\mu\alpha\beta\nu} \bar{P}_{j\ \sigma\rho\lambda\eta}, \end{aligned} \tag{5.3}$$

where ∂_i denotes derivative with respect to $\mathcal{R}_i$ and $\bar{\mathcal{R}}$ means that we evaluate all the scalars on the background. Using these expressions, it is possible to obtain the values of the parameters a, b, c, and e defined in (2.11) and (2.10) for the theory (5.1). The result is

$$\begin{aligned} a &= \partial_i f(\bar{\mathcal{R}}) a_i, \qquad b = \partial_i f(\bar{\mathcal{R}}) b_i + \partial_i \partial_j f(\bar{\mathcal{R}}) e_i e_j, \\ c &= \partial_i f(\bar{\mathcal{R}}) c_i \qquad e = \partial_i f(\bar{\mathcal{R}}) e_i. \end{aligned} \tag{5.4}$$

Hence, once we have computed the parameters a_i, b_i, c_i, and e_i for the set of scalars $\mathcal{R}_i$, we can easily find the corresponding parameters for any other Lagrangian $\mathcal{L} = f(\mathcal{R}_1, \ldots, \mathcal{R}_m)$. Plugging the values (5.4) in (2.15), we obtain the linearized equations.

A. Theories without massive graviton

In Sec. III, we classified general $\mathcal{L}$(Riemann) theories according to their spectrum on a msb. One of the cases under consideration was that corresponding to theories for which $m_g^2 = +\infty$, i.e., those containing a single massless graviton plus an additional spin-0 mode. In terms of the parameters defined in the first section, this condition is $2a + c = 0$. Assume now that for certain scalars $\mathcal{R}_i$, the condition $2a_i + c_i = 0$ is satisfied for all i, so that a theory consisting of a linear combination of $\mathcal{R}_i$ would be free of massive gravitons. From (5.4), we learn that in fact this property is shared by any theory of the form $\mathcal{L} = f(\mathcal{R}_1, \ldots, \mathcal{R}_m)$ since in that case we find

$$2a + c = \partial_i f(\bar{\mathcal{R}})(2a_i + c_i) = 0. \tag{5.5}$$

Therefore, theories constructed as general functions of scalars of which the linear combinations do not produce massive gravitons are also free of those modes. This is a straightforward way of understanding why $f(R)$, or more generally f(Lovelock) theories—see Appendix B—inherit the property of Lovelock gravities [25,26] of not propagating the massive graviton [23,63].

Something similar happens for theories for which the extra graviton is massless. Assume now that the scalars $\mathcal{R}_i$ satisfy the condition $-e_i + 2\Lambda(D-3)a_i = 0$, so that $m_g = 0$ for a theory consisting of a linear combination of $\mathcal{R}_i$. Then, it is straightforward to prove that for a $f(\mathcal{R}_i)$ theory, the mass of the extra graviton is also zero:

$$-e + 2\Lambda(D-3)a = \partial_i f(\bar{\mathcal{R}})(-e_i + 2\Lambda(D-3)a_i) = 0. \tag{5.6}$$

Furthermore, note that the condition for the absence of scalar mode reads in turn

$$\begin{aligned} &2a + Dc + 4b(D-1) \\ &\quad = \partial_i f(\bar{\mathcal{R}})(2a_i + Dc_i + 4b_i(D-1)) \\ &\qquad + 4(D-1)\partial_i \partial_j f(\bar{\mathcal{R}}) e_i e_j = 0. \end{aligned} \tag{5.7}$$

This expression is more complicated than (5.5) since the expression for b in (5.4) contains a term involving the e_i. This is not surprising; $f(R)$ does propagate the additional scalar mode even though Einstein gravity does not.

VI. EINSTEINIAN QUARTIC GRAVITIES

In Ref. [29], we constructed a cubic theory which only propagates a massless graviton on msb. The theory was defined in a dimension-independent way, in the sense that the relative couplings between the different invariants involved in its definition were the same in all dimensions. In fact, we proved that up to cubic order in curvature, the most general theory satisfying those requirements reads

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \times \left\{ \frac{1}{2\kappa}(-2\Lambda_0 + R) + \kappa^{\frac{4-D}{D-2}} \alpha \mathcal{X}_4 + \kappa^{\frac{6-D}{D-2}} [\beta \mathcal{X}_6 + \lambda \mathcal{P}] \right\}, \tag{6.1}$$

where $\mathcal{X}_4$ and $\mathcal{X}_6$ are, respectively, the dimensionally extended Euler densities for $D = 4$ and $D = 6$ manifolds. $\mathcal{X}_4$ is defined below (2.24), and $\mathcal{X}_6$ is given in (B22). Hence, the only terms appearing in (6.1) are the Lovelock ones plus the new Einsteinian cubic gravity term $\mathcal{P}$, defined as

$$\mathcal{P} \equiv 12 R_{\mu\ \nu}^{\ \rho\ \sigma} R_{\rho\ \sigma}^{\ \gamma\ \delta} R_{\gamma\ \delta}^{\ \mu\ \nu} + R_{\mu\nu}^{\rho\sigma} R_{\rho\sigma}^{\gamma\delta} R_{\gamma\delta}^{\mu\nu} - 12 R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + 8 R_{\mu}^{\nu} R_{\nu}^{\rho} R_{\rho}^{\mu}. \tag{6.2}$$

The effective Einstein constant for the ECG theory (6.1) is

$$\kappa_{\rm eff} = \kappa [1 + 4\kappa^{\frac{2}{D-2}} \Lambda \alpha (D-4)(D-3) + 6\kappa^{\frac{4}{D-2}} \Lambda^2 (D-6)(D-3)((D-5)(D-4)\beta - 4\lambda)]^{-1}. \tag{6.3}$$

Interestingly, when restricted to $D = 4$, the above theory reduces to

$$S = \int_{\mathcal{M}} d^4 x \sqrt{|g|} \left\{ \frac{1}{2\kappa}(-2\Lambda_0 + R) + \kappa\lambda\mathcal{P} \right\}, \tag{6.4}$$

given that in that number of dimensions $\mathcal{X}_4$ is topological and $\mathcal{X}_6$ vanishes identically.

In this section, we will explain how to extend the above construction to quartic theories. We will take advantage of the results in Sec. IV to construct *Einsteinian quartic gravities* (EQGs).

As we have just reviewed, the construction of Einsteinian gravities requires the theories to be defined in a dimension-independent fashion. Apart from aesthetics, there are some practical reasons to consider theories satisfying this property. First, observe that this property is shared by all Lovelock gravities, which are the most general metric theories of gravity with divergence-free second-order equations of motion—at the full nonlinear level—in any number of dimensions [25,26].

In addition, theories defined in this way have the nice feature that they preserve the total number of dof under compactification, in the following sense. Consider, for example, the Kaluza-Klein reduction of the D-dimensional EH term along some direction x^0. The metric g_{MN}, which propagates $D(D-3)/2$ dof, gives rise to a $(D-1)$-dimensional metric $g_{\mu\nu}$ which contains $(D-1)(D-4)/2$ dof, plus a 1-form $A_\mu \equiv g_{\mu 0}$ with $(D-3)$ dof and a scalar field $\phi \equiv g_{00}$ with 1 dof. This property is shared by Einsteinian gravities, but not by theories which have a dimension-dependent definition. If a theory of that kind only propagates the $D(D-3)/2$ dof of the massless graviton in D dimensions, it will give rise to extra degrees of freedom when compactified, because the lower-dimensional metric will in general propagate the extra spin-2 and scalar modes in addition to the $(D-1)(D-4)/2 + (D-3) + 1 = D(D-3)/2$ dof of the massless graviton, the 1-form, and the scalar. From a similar perspective, if we consider some D-dimensional theory and assume some of the dimensions of our space-time to be compact, e.g., $\mathcal{M}^D = \mathcal{M}_{\rm nc}^{D'} \times \mathcal{M}_{\rm c}^{D-D'}$, where $\mathcal{M}_{\rm c}^{D-D'}$ is some compact manifold, then the resulting effective action on the noncompact dimensions will involve the same gravitational term only if this has been defined in a dimension-independent fashion—see, e.g., Refs. [67,68] for the Kaluza-Klein reduction of Gauss-Bonnet gravity. This is exactly what happens with the Einstein-Hilbert term in general String Theory compactifications.[18]

As explained in previous sections, the constraints required for a theory to share the spectrum of Einstein gravity at the linearized level can be written as $2a + c = 4b + c = 0$, which account for the conditions $m_g^2 = m_s^2 = +\infty$. Imposing those conditions at each order in curvature for the theory (4.2), one is left with six constraints on the coupling values, $F_g^{(2)}(\alpha_i) = F_s^{(2)}(\alpha_i) = F_g^{(3)}(\beta_i, D) = F_s^{(3)}(\beta_i, D) = F_g^{(4)}(\gamma_i, D) = F_s^{(4)}(\gamma_i, D) = 0$—see Appendix D for the explicit expressions. If these constraints are satisfied, the theory will only propagate a massless graviton on a msb. Imposing each constraint to be satisfied independently of the dimension multiplies the number of constraints. This is because, e.g., $F_{g,s}^{(3)}(\beta_i, D)$ is a polynomial of degree 2 in D, so we need to impose the coefficients of the D^0, D^1, and D^2 terms to vanish independently. More generally, at nth order in curvature, the corresponding constraints are polynomials of degree $2n - 4$ in D, and hence we will find $2n - 3$ constraints coming from the absence of the massive graviton and the same number from imposing the absence of scalar, which makes $2(2n - 3)$ in total. At the quartic level, this means ten constraints. Since in general dimensions there are up to 26 independent invariants at this order in curvature [66], see Table II, this means that there exists a 16-parameter family of EQGs. If we choose the 16 parameters to be $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{18}, \gamma_{20}, \gamma_{26}\}$, the rest of the couplings are given in terms of these as

[18]For example, the ten-dimensional type-IIA String Theory effective action reduces to a class of $D = 4$, $\mathcal{N} = 2$ supergravity theories when six of the dimensions are compact on a Calabi-Yau threefold—see, e.g., Ref. [69]. In the type-IIA action, the leading contribution from the metric is the ten-dimensional Einstein-Hilbert term $R^{(10)}$. Under compactification, this produces $R^{(4)}$—plus additional terms involving other fields.

$$\gamma_{11} = +\frac{1}{3}(12\gamma_{12} - 4\gamma_1 + 12\gamma_2 - 8\gamma_3 + 36\gamma_4 + 72\gamma_5 + 16\gamma_6 + 16\gamma_7 - 3\gamma_8 + 12\gamma_9), \tag{6.5}$$

$$\begin{aligned}
\gamma_{15} &= +\frac{1}{2}(-10\gamma_1 - 4\gamma_{10} - \gamma_{13} + \gamma_{14} + 16\gamma_2 - 14\gamma_3 + 48\gamma_4 + 96\gamma_5 + 16\gamma_6 - 4\gamma_8 + 12\gamma_9),\\
\gamma_{16} &= +\frac{1}{10}(36\gamma_1 + 10\gamma_{10} - 24\gamma_{12} - 5\gamma_{13} - 5\gamma_{14} - 74\gamma_2 - 2\gamma_{20} + 1140\gamma_{26} + 57\gamma_3 - 210\gamma_4\\
&\quad - 420\gamma_5 - 84\gamma_6 - 20\gamma_7 + 17\gamma_8 - 72\gamma_9),\\
\gamma_{17} &= -\gamma_{18} - 120\gamma_{26},\\
\gamma_{19} &= +6\gamma_{26},\\
\gamma_{21} &= +8\gamma_1 - 12\gamma_{12} - 3\gamma_{14} + 2\gamma_{18} - 18\gamma_2 - 2\gamma_{20} + 900\gamma_{26} + 13\gamma_3 - 54\gamma_4 - 108\gamma_5 - 20\gamma_6\\
&\quad - 20\gamma_7 + 3\gamma_8 - 12\gamma_9,\\
\gamma_{22} &= +\frac{1}{10}(16\gamma_1 - 24\gamma_{12} - 10\gamma_{14} - 14\gamma_2 - 2\gamma_{20} + 1140\gamma_{26} + 17\gamma_3 - 50\gamma_4 - 100\gamma_5 - 4\gamma_6\\
&\quad - 20\gamma_7 + 2\gamma_8 + 8\gamma_9),\\
\gamma_{23} &= +\frac{1}{20}(-154\gamma_1 + 216\gamma_{12} + 60\gamma_{14} - 40\gamma_{18} + 306\gamma_2 + 38\gamma_{20} - 22260\gamma_{26} - 233\gamma_3\\
&\quad + 930\gamma_4 + 1860\gamma_5 + 316\gamma_6 + 340\gamma_7 - 48\gamma_8 + 168\gamma_9),\\
\gamma_{24} &= +\frac{1}{30}(-6\gamma_1 + 24\gamma_{12} + 54\gamma_2 + 2\gamma_{20} + 9060\gamma_{26} - 27\gamma_3 + 150\gamma_4 + 300\gamma_5 + 84\gamma_6\\
&\quad + 60\gamma_7 - 12\gamma_8 + 72\gamma_9),\\
\gamma_{25} &= -24\gamma_{26}.
\end{aligned} \tag{6.6}$$

Plugging these back in the original quartic action, we obtain the family of 16 independent Einsteinian quartic gravities. In four dimensions, it can be seen that only 13 of the 26 invariants in Table II are nonvanishing and independent of each other [66]. We can use this fact to easily construct three Einsteinian quartic gravities. In particular, we can set $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_6 = \gamma_8 = \gamma_9 = \gamma_{10} = \gamma_{12} = \gamma_{13} = \gamma_{14} = \gamma_{18} = \gamma_{20} = 0$—the choice being nonunique. More explicitly, Eq. (6.6) becomes now

$$\begin{aligned}
&\gamma_{11} = +8/3(9\gamma_5 + 2\gamma_7), \qquad \gamma_{15} = +48\gamma_5,\\
&\gamma_{16} = +114\gamma_{26} - 42\gamma_5 - 2\gamma_7, \qquad \gamma_{17} = -120\gamma_{26},\\
&\gamma_{19} = +6\gamma_{26}, \qquad \gamma_{21} = +4(225\gamma_{26} - 27\gamma_5 - 5\gamma_7),\\
&\gamma_{22} = +2(57\gamma_{26} - 5\gamma_5 - \gamma_7), \qquad \gamma_{23} = -1113\gamma_{26} + 93\gamma_5 + 17\gamma_7,\\
&\gamma_{24} = +2(151\gamma_{26} + 5\gamma_5 + \gamma_7), \qquad \gamma_{25} = -24\gamma_{26},
\end{aligned} \tag{6.7}$$

where the three parameters are $\{\gamma_5, \gamma_7, \gamma_{26}\}$. Using these relations, we have constructed the following invariants:

$$\begin{aligned}
\mathcal{Q}_1 \equiv\ & +3R^{\mu\nu\rho\sigma}R_{\mu\nu}^{\gamma\delta}R_{\gamma\delta}^{\alpha\beta}R_{\rho\sigma\alpha\beta} - 15(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2 - 8RR_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \gamma\ \delta}R_{\gamma\ \delta}^{\ \mu\ \nu}\\
& + 144R^{\mu\nu}R^{\rho\sigma}R_{\mu\rho}^{\gamma\delta}R_{\gamma\delta\nu\sigma} - 96R^{\mu\nu}R_{\nu}^{\rho}R^{\alpha\beta\gamma}{}_{\mu}R_{\alpha\beta\gamma\rho} - 24RR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma}\\
& + 24(R_{\mu\nu}R^{\mu\nu})^2,\\
\mathcal{Q}_2 \equiv\ & +3(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2 + 16RR_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \gamma\ \delta}R_{\gamma\ \delta}^{\ \mu\ \nu} - 6R^{\mu\nu}R_{\nu}^{\rho}R^{\alpha\beta\gamma}{}_{\mu}R_{\alpha\beta\gamma\rho}\\
& - 60RR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} - 6R_{\mu}^{\nu}R_{\nu}^{\rho}R_{\rho}^{\sigma}R_{\sigma}^{\mu} + 51(R_{\mu\nu}R^{\mu\nu})^2 + 6RR_{\mu}^{\nu}R_{\nu}^{\rho}R_{\rho}^{\mu},\\
\mathcal{Q}_3 \equiv\ & +R^4 + 57(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2 - 120R_{\gamma\delta}R^{\gamma\delta}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 6R^2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\\
& - 240RR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} - 144(R_{\mu\nu}R^{\mu\nu})^2 + 416RR_{\mu}^{\nu}R_{\nu}^{\rho}R_{\rho}^{\mu} - 24R^2R_{\mu\nu}R^{\mu\nu}\\
& + 304RR_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \delta\ \gamma}R_{\delta\ \gamma}^{\ \mu\ \nu}.
\end{aligned} \tag{6.8}$$

Just like its cubic cousin $\mathcal{P}$ defined in (6.2), $\mathcal{Q}_1$, $\mathcal{Q}_2$, and $\mathcal{Q}_3$—or any linear combination of them—only propagate the usual massless graviton when linearized on a msb, not only in $D=4$ but in any number of dimensions.[19]

It is important to note that these three are not necessarily the only EQG theories in $D=4$. As we explained, there are 13 independent cubic invariants in that case, which means that there are 11 independent four-dimensional quartic Einstein-like invariants—because we have to impose two conditions on the couplings in that case, namely, $m_g^2 = m_s^2 = +\infty$. In order to determine all the possible theories, one should construct the 16 independent D-dimensional EQGs using (6.6) and then analyze how many of them are independent when $D=4$. Given that EQGs are particular cases of Einstein-like theories, we conclude that there could actually be up to eight additional EQG invariants.

VII. NEW GHOST-FREE GRAVITY

In the previous section, we reviewed ECG and extended the construction to quartic theories. As we explained, all those theories are free both of the ghostlike graviton and the scalar mode on a msb. In this section, we will relax the second condition to construct the most general cubic theory defined in a dimension-independent manner which does not propagate the massive graviton—but does in general include the scalar. As far as we know, the most general known theories which satisfy these requirements are those defined as functions of Lagrangian densities which, when considered as theories by themselves, do not propagate the massive graviton—a property first proven in Sec. V. All the known examples reduce to $f(\text{Lovelock})$ gravities and the more exotic case of $f(\text{ECG})$ or functions of the quartic theories studied in the previous section.

Recall that the condition for the absence of massive gravitons is $2a+c=0$. If we impose this on the general theory defined in (4.2) up to qubic order and ask it to be satisfied independently of the space-time dimension, we are left with the conditions

$$\frac{1}{2}\alpha_2 + 2\alpha_3 = 0, \tag{7.1}$$

$$-\beta_1 + 8\beta_2 - \beta_5 - \beta_6 = 0, \tag{7.2}$$

$$2\beta_3 - 2\beta_4 + \beta_5 + \frac{3}{2}\beta_6 - \frac{1}{2}\beta_7 = 0, \tag{7.3}$$

$$2\beta_4 + \frac{1}{2}\beta_7 = 0. \tag{7.4}$$

Hence, there are two independent quadratic terms and five cubic ones. They can all be written as

$$S = \int_{\mathcal{M}} d^Dx\sqrt{|g|}\left\{\frac{1}{2\kappa}(R-2\Lambda_0) + \kappa^{\frac{4-D}{D-2}}(\tilde{\alpha}_1 R^2 + \tilde{\alpha}_2 \mathcal{X}_4) + \kappa^{\frac{6-D}{D-2}}(\tilde{\beta}_1 R^3 + \tilde{\beta}_2 \mathcal{X}_6 + \tilde{\beta}_3 R\mathcal{X}_4 + \tilde{\beta}_4 \mathcal{P} + \tilde{\beta}_5 \mathcal{Y})\right\}. \tag{7.5}$$

In this action, we find all the $f(\text{Lovelock})$ terms up to this order in curvature, as well as two additional theories. The first, $\mathcal{P}$, is nothing but the Einsteinian cubic term defined in (6.2), while the second is a previously unidentified invariant which reads

$$\mathcal{Y} \equiv R_{\mu\ \nu}^{\ \alpha\ \beta} R_{\alpha\ \beta}^{\ \rho\ \sigma} R_{\rho\ \sigma}^{\ \mu\ \nu} - 3R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + 2R_\mu^{\ \nu}R_\nu^{\ \rho}R_\rho^{\ \mu}. \tag{7.6}$$

In the above expression, the pure Lovelock terms, R, $\mathcal{X}_4$, and $\mathcal{X}_6$ as well as $\mathcal{P}$ do not contribute to the denominator of the scalar mass—and hence any linear combination of those terms alone would yield $m_s^2 = +\infty$—while R^2, R^3, $R\mathcal{X}_4$, and $\mathcal{Y}$ do. Indeed, we obtain for this new ghost-free gravity (7.5)

$$\begin{aligned} m_s^2 = {} & [D-2+4(D-4)\kappa^{\frac{2}{D-2}}\Lambda(\tilde{\alpha}_1(D-1)D + \tilde{\alpha}_2(D-3)(D-2)) \\ & + 6(D-6)\kappa^{\frac{4}{D-2}}\Lambda^2(\tilde{\beta}_1(D-1)^2D^2 + \tilde{\beta}_2(D-5)(D-4)(D-3)(D-2) \\ & + \tilde{\beta}_3(D-3)(D-2)(D-1)D - 4\tilde{\beta}_4(D-3)(D-2) - \tilde{\beta}_5(D(D-3)+3))] \\ & \times [8(D-1)(\kappa^{\frac{2}{D-2}}\tilde{\alpha}_1 + \kappa^{\frac{4}{D-2}}\Lambda(3\tilde{\beta}_1(D-1)D + 2\tilde{\beta}_3(D-3)(D-2) - 3/2\tilde{\beta}_5))]^{-1}. \end{aligned} \tag{7.7}$$

Hence, setting $\tilde{\alpha}_1 = \tilde{\beta}_1 = \tilde{\beta}_3 = \tilde{\beta}_5 = 0$, one finds $m_s^2 = +\infty$, as expected. It is also worth pointing out that, just like ECG, $\mathcal{Y}$ is nontrivial in four dimensions. Moreover, the effective gravitational constant reads now

$$\begin{aligned} \kappa_{\text{eff}} = \kappa[& 1 + 4\kappa^{\frac{2}{D-2}}\Lambda(\tilde{\alpha}_1(D-1)D + \tilde{\alpha}_2(D-4)(D-3)) \\ & + 6\kappa^{\frac{4}{D-2}}\Lambda^2(\tilde{\beta}_1(D-1)^2D^2 + \tilde{\beta}_2(D-6)(D-5)(D-4)(D-3) \\ & + \tilde{\beta}_3(D-10/3)(D-3)(D-1)D - 4\tilde{\beta}_4(D-6)(D-3) - \tilde{\beta}_5((D-5)D+9))]^{-1}. \end{aligned} \tag{7.8}$$

[19]We have cross-checked the linearized equations of $\mathcal{P}$ and $\mathcal{Q}_i$, $i=1,2,3$, for $D=4,5,6$ using the *Mathematica* package xAct [70].

Let us stress that we have only proven this theory to be free of ghost modes at the linearized level. Hence, it is still possible that the theory develops instabilities beyond the linearized regime—e.g., the Boulware-Deser ghost [71]. We leave for future work exploring these potential issues and their possible solutions—e.g., using boundary conditions [65,72]. Note that an interesting property of f(Lovelock) gravities is that they are ghost free at the full nonlinear level, since they can be written as scalar-Lovelock theories with second-order equations of motion [63,73]. It is natural to wonder if $\mathcal{Y}$ has any chance of sharing this property. More generally, it would be interesting to explore further properties of this new cubic term.

VIII. GENERALIZED NEWTON POTENTIAL

In this section, we use the results of Sec. II to compute the Newton potential $U_D(r)$ and the PPN parameter γ for a general theory of the form (1.1) in general dimensions. We start reviewing the four-dimensional case, and then we extend our results to arbitrary D, pointing out interesting differences with respect to the $D=4$ case. Throughout this section and the following, we will tacitly assume that $m_s^2, m_g^2 \geq 0$.

A. Four dimensions

The analysis performed in Sec. II C 2 tells us that in order to obtain a solution of the linearized equations in a flat background, we must solve Eqs. (2.48), (2.52), and (2.53) and then reconstruct the metric perturbation (2.51). The same procedure can be naturally carried out for an (A)dS background using the expressions in Sec. II C 1. We find that the results are approximately the same, provided we consider distances shorter than the (A)dS scale $r \ll |\Lambda|^{-1/2}$ and $m_g^2 \gg |\Lambda|$. This is useful because in the flat case one cannot easily set the masses m_g and m_s to zero as only the Einstein-Hilbert term contributes to the numerator of those quantities when $\Lambda = 0$—see, e.g., Eqs. (2.32)–(2.34). In the (A)dS case, terms of all orders contribute, and it is in principle possible to set $m_s = 0$ or $m_g = 0$.

If we denote by $H_{\mu\nu}(x;m)$ the general solution of the Klein-Gordon equation

$$(\bar{\Box} - m^2)H_{\mu\nu}(x;m) = -4\pi T_{\mu\nu}(x) \tag{8.1}$$

and by $H(x;m)$ its trace, the solutions to (2.48), (2.52), and (2.53) can be written as

$$\hat{h}_{\mu\nu} = \frac{\kappa_{\rm eff}}{2\pi} H_{\mu\nu}(0), \qquad \phi = \frac{\kappa_{\rm eff}}{2\pi} H(m_s),$$
$$t_{\mu\nu} = -\frac{\kappa_{\rm eff}}{2\pi}\left[H_{\langle\mu\nu\rangle}(m_g) + \frac{1}{3m_g^2}\partial_{\langle\mu}\partial_{\nu\rangle}H(m_g)\right]. \tag{8.2}$$

Inserting this into the metric perturbation (2.51) and making the gauge transformation

$$h^N_{\mu\nu} \equiv h_{\mu\nu} - \partial_{(\mu}\xi_{\nu)}, \tag{8.3}$$

where N stands for "Newtonian gauge" and

$$\xi_\nu \equiv \frac{1}{3}\partial_\nu((m_g^{-2} - m_s^{-2})H(0) + m_s^{-2}H(m_s) - m_g^{-2}H(m_g)), \tag{8.4}$$

we obtain after some simplifications

$$h^N_{\mu\nu} = \frac{\kappa_{\rm eff}}{8\pi}\left[4H_{\mu\nu}(0) - 4H_{\mu\nu}(m_g) + \eta_{\mu\nu}\left(-2H(0) + \frac{4}{3}H(m_g) + \frac{2}{3}H(m_s)\right)\right]. \tag{8.5}$$

Now, if we restrict ourselves to static configurations, Eq. (8.1) reduces to the so-called screened Poisson equation, $(\triangle - m^2)H_{\mu\nu}(\vec{x};m) = -4\pi T_{\mu\nu}(\vec{x})$, the general solution of which reads

$$H_{\mu\nu}(\vec{x};m) = \int d^3\vec{x}' \frac{T_{\mu\nu}(\vec{x}')}{|\vec{x} - \vec{x}'|} e^{-m|\vec{x}-\vec{x}'|}. \tag{8.6}$$

This can be seen as a superposition of functions $1/|\vec{x} - \vec{x}'|$ weighted by the source $T_{\mu\nu}(\vec{x}')$ and with an exponential screening controlled by the mass m. Using this, we can rewrite (8.5) as

$$h^N_{\mu\nu}(x) = \frac{\kappa_{\rm eff}}{8\pi}\int d^3\vec{x}' T_{\alpha\beta}(\vec{x}')\Pi^{\alpha\beta}{}_{\mu\nu}(\vec{x} - \vec{x}'), \tag{8.7}$$

where the static propagator reads

$$\Pi^{\alpha\beta}{}_{\mu\nu}(\vec{x} - \vec{x}') = \frac{1}{|x - x'|}\left[4\delta^\alpha{}_{(\mu}\delta^\beta{}_{\nu)}(1 - e^{-m_g|\vec{x}-\vec{x}'|}) - 2\eta^{\alpha\beta}\eta_{\mu\nu}\left(1 - \frac{2}{3}e^{-m_g|\vec{x}-\vec{x}'|} - \frac{1}{3}e^{-m_s|\vec{x}-\vec{x}'|}\right)\right]. \tag{8.8}$$

Now, let us apply the previous expressions to the case of a solid and static sphere of radius R and mass M on a flat background. For this distribution of matter, the only nonvanishing component of the stress tensor reads

$$T_{00}(r) = \rho(r) = \rho_0\theta(R - r), \quad \text{with} \quad \rho_0 \equiv \frac{M}{4\pi R^3/3}, \tag{8.9}$$

where $\theta(x)$ is the Heaviside step function. For this configuration, the result for $H_{00}(r;m) = -H(r;m)$ in the outer region $r > R$ obtained from (8.6) reads

$$H(r;m) = -f(mR)\frac{M}{r}e^{-mr}, \tag{8.10}$$

where $f(mR)$ is a form factor given by

$$f(mR) = \frac{3}{(mR)^3}[mR\cosh(mR) - \sinh(mR)], \quad (8.11)$$

which behaves as $f(mR) \approx \frac{3}{2}\frac{1}{(mR)^2}e^{mR}$ if $mR \gg 1$ and as $f(mR) \approx 1$ in the pointlike limit, i.e., when $mR \ll 1$. Finally, inserting these results into the metric $h^N_{\mu\nu}$ in (8.5) and this in $g^N_{\mu\nu} = \eta_{\mu\nu} + h^N_{\mu\nu}$, we obtain

$$ds_N^2 = -(1+2U(r))dt^2 + (1-2\gamma(r)U(r))\delta_{ij}dx^i dx^j, \quad (8.12)$$

where $U(r)$ and $\gamma(r)$ are given by

$$U(r) = -\frac{G_{\rm eff}M}{r}\left[1 - \frac{4}{3}f(m_g R)e^{-m_g r} + \frac{1}{3}f(m_s R)e^{-m_s r}\right], \quad (8.13)$$

$$\gamma(r) = \frac{3 - 2f(m_g R)e^{-m_g r} - f(m_s R)e^{-m_s r}}{3 - 4f(m_g R)e^{-m_g r} + f(m_s R)e^{-m_s r}} \quad (8.14)$$

and $G_{\rm eff} \equiv \kappa_{\rm eff}/(8\pi)$. Evaluating these expressions in the pointlike limit of the sphere $f(mR) = 1$, we finally obtain the generalized Newtonian potential and the PPN parameter γ,

$$U(r) = -\frac{G_{\rm eff}M}{r}\left[1 - \frac{4}{3}e^{-m_g r} + \frac{1}{3}e^{-m_s r}\right],$$
$$\gamma(r) = \frac{3 - 2e^{-m_g r} - e^{-m_s r}}{3 - 4e^{-m_g r} + e^{-m_s r}}. \quad (8.15)$$

Let us make some comments about these results. First, observe that the usual Newton potential gets corrected by two Yukawa-like terms controlled by the masses of the two extra modes which can be computed for a given theory through (2.29) and (2.30). The above expression for $U(r)$ has been obtained before using different methods—see, e.g., Refs. [30,31,74].[20] Note that, while the contribution from the scalar has the usual sign for a Yukawa potential, the massive graviton one comes with the opposite sign, which is another manifestation of its ghost nature. Observe also that the whole contribution from the higher-derivative terms appears through m_g and m_s, the coefficients $-4/3$ and $1/3$ in front of the exponentials being common to all theories. In Table III, we present the values of $U(r)$ and γ for different limiting values of m_s and m_g. Naturally, when $m_g, m_s \gg 1$, one is left with the Einsteinian values of the Newton potential and γ, and the same happens if we go sufficiently far away from M for arbitrary values of the extra mode masses. It is also interesting that the only cases for which the potential is divergent as $r \to 0$ are those for which at least one of the extra modes is absent, i.e., when either $m_s = +\infty$ or $m_g = +\infty$ or both $m_g = m_s = +\infty$.

TABLE III. Newton's potential and $\gamma(r)$ for various values of the masses of the extra modes.

	$U(r)/G_{\rm eff}$	γ
$m_s = m_g = +\infty$	$-M/r$	1
$m_s = +\infty$, $\lvert m_g r\rvert \ll 1$	$+M/(3r)$	-1
$m_s = 0$, $m_g = +\infty$	$-4M/(3r)$	1/2
$m \equiv m_g = m_s$	$-M(1-e^{-mr})/r$	1

Indeed, $U(r)$ does not diverge as $r \to 0$ in the general case. In fact, one finds

$$U(r) = -G_{\rm eff}M\left[\frac{(4m_g - m_s)}{3} - \frac{(4m_g^2 - m_s^2)r}{6} + \mathcal{O}(r^2)\right], \quad (8.16)$$

which is a negative constant at $r = 0$ when $m_g > m_s/4$ (and vice versa). The potential grows linearly with r at first order for $m_g > m_s/2$, and in that case, it is monotonous in the whole range of r. When $m_g < m_s/2$ instead, $U(r)$ decreases linearly near $r = 0$, and it has a minimum at some intermediate value of r. Plots of $U(r)/G_{\rm eff}$ for various values of the masses satisfying the different situations can be found in Fig. 1.

B. Higher dimensions

The analysis of the previous section can be extended to general dimensions $D \geq 4$. The metric perturbation in the Newtonian gauge can be seen to be given by

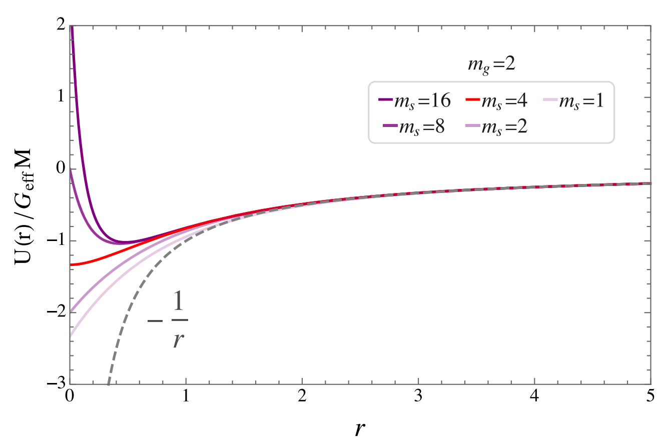

FIG. 1. $U(r)/(G_{\rm eff}M)$ for $m_g = 2$ and $m_s = 16, 8, 2, 1$ (purple curves) and $m_s = 4$ (red) and the usual Newton potential (dashed gray).

[20]See, e.g., Refs. [75,76] for results corresponding to higher-order gravities involving covariant derivatives of the Riemann tensor.

$$h^N_{\mu\nu} = 4G_{\rm eff}\Big[H_{\mu\nu}(0) - H_{\mu\nu}(m_g) + \frac{\eta_{\mu\nu}}{(D-1)(D-2)}(-(D-1)H(0) + (D-2)H(m_g) + H(m_s))\Big], \tag{8.17}$$

where again $H_{\mu\nu}(m)$ is a solution of (8.1). In the static case, we can write the solution explicitly as

$$H_{\mu\nu}(\vec{x};m) = 2\left(\frac{m}{2\pi}\right)^{\frac{D-3}{2}} \int d^{D-1}\vec{x}' \frac{T_{\mu\nu}(\vec{x}')}{|\vec{x}-\vec{x}'|^{\frac{D-3}{2}}} K_{\frac{D-3}{2}}(m|\vec{x}-\vec{x}'|), \tag{8.18}$$

where $K_\ell(x)$ is the modified Bessel function of the second kind. Now, specializing to a static pointlike particle of mass M, we can obtain the D-dimensional version of (8.12). The Newtonian potential and the gamma parameter read, respectively,

$$U_D(r) = -\mu(D)\frac{G_{\rm eff}M}{r^{D-3}}\left[1+\nu(D) r^{\frac{D-3}{2}}\left[-m_g^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(m_g r) + \frac{m_s^{\frac{D-3}{2}}}{(D-2)^2}K_{\frac{D-3}{2}}(m_s r)\right]\right],$$
$$\gamma_D(r) = \frac{1 - \frac{2}{(D-1)\Gamma(\frac{D-3}{2})}[(D-2)(\frac{m_g r}{2})^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(m_g r) + (\frac{m_s r}{2})^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(m_s r)]}{D-3-\frac{2}{(D-1)\Gamma(\frac{D-3}{2})}[(D-2)^2(\frac{m_g r}{2})^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(m_g r) - (\frac{m_s r}{2})^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(m_s r)]}, \tag{8.19}$$

with

$$\mu(D) \equiv \frac{8\pi}{(D-2)\Omega_{D-2}}, \quad \text{and} \quad \nu(D) \equiv \frac{(D-2)^2}{\Gamma[\frac{D+1}{2}]2^{\frac{D-1}{2}}}, \tag{8.20}$$

and where $\Omega_{D-2} \equiv 2\pi^{\frac{D-1}{2}}/\Gamma[\frac{D-1}{2}]$ is the volume of the $(D-2)$-dimensional unit sphere. When 2ℓ is odd, i.e., for even D, the Bessel functions $K_\ell(x)$ can be written explicitly in terms of elementary functions as

$$K_{\frac{D-3}{2}}(x) = e^{-x}\sqrt{\frac{\pi}{2x}}\sum_{j=1}^{\frac{D-2}{2}}\frac{(D-3-j)!}{(j-1)!(\frac{D-2}{2}-j)!(2x)^{\frac{D-2}{2}-j}} \quad (\text{even } D), \tag{8.21}$$

which allows for a simplification of (8.19) in those cases and from which it is easy to reproduce the $D=4$ results (8.15) presented in the previous section. From (8.19), we infer that the usual four-dimensional Yukawa potential for a force-mediating particle of mass m generalizes to higher dimensions as

$$U_{D,\rm Yukawa}(r) \sim \left(\frac{m}{r}\right)^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(mr). \tag{8.22}$$

Going back to higher-order gravities, observe that close to the origin, the generalized Newton potential $U_D(r)$ behaves for $D>5$ as

$$U_D(r\to 0) \sim -\frac{G_{\rm eff}M[(D-2)^2 m_g^2 - m_s^2]}{r^{D-5}} + \cdots, \tag{8.23}$$

up to a positive dimension-dependent constant for generic values of m_g and m_s. For $D=4$, we find a constant term (8.16), while for $D=5$, one finds a logarithmic divergence instead,

$$U_5(r\to 0) = \frac{G_{\rm eff}M}{12\pi}(9m_g^2 - m_s^2)\log r + \mathcal{O}(r^0). \tag{8.24}$$

This means that for generic values of the extra mode masses, $U_D(r)$ is divergent at $r=0$ in all dimensions higher than 4. In Fig. 2, we plot $U_5(r)$, which can be explicitly written as

$$U_5(r) = -\frac{G_{\rm eff}M}{6\pi r^2}[8 - 9m_g r K_1(m_g r) + m_s r K_1(m_s r)]. \tag{8.25}$$

As expected, most curves in Fig. 2 diverge at the origin. There is an exception (and only one), though, which corresponds to the case $m_g = m_s/3$, for which the potential is finite everywhere. The value $m_g = \frac{m_s}{(D-2)}$ is special in general dimensions, as it determines the transition between two kinds of potentials. In particular, when $m_g > \frac{m_s}{(D-2)}$, $U_D(r)$ is monotonous in the whole range of r and diverges to $-\infty$ at the origin, while for $m_g < \frac{m_s}{(D-2)}$, it has a minimum at some finite value of r and $U_D(r\to 0)\to +\infty$ instead—see Fig. 2 for an illustration of these features in the five-dimensional case. For the particular value $m_g = \frac{m_s}{(D-2)}$, the potential is also finite at the origin for $D=6$, but not for $D\geq 7$.

In Table IV, we present some particular cases for $U_D(r)$ and γ_D[21] corresponding to different limiting values of m_g and m_s. Once again, when $m_g, m_s \gg 1$, one is left with the Einsteinian values of the corresponding Newton potentials and γ_D, and the same happens at sufficiently large distances from M for general values of the extra mode masses. Just

[21]We use the following two limits of the modified Bessel functions:

$$\lim_{x\to\infty} x^\ell K_\ell(x) = 0, \quad \text{and} \quad \lim_{x\to 0} x^\ell K_\ell(x) = 2^{\ell-1}\Gamma(\ell). \tag{8.26}$$

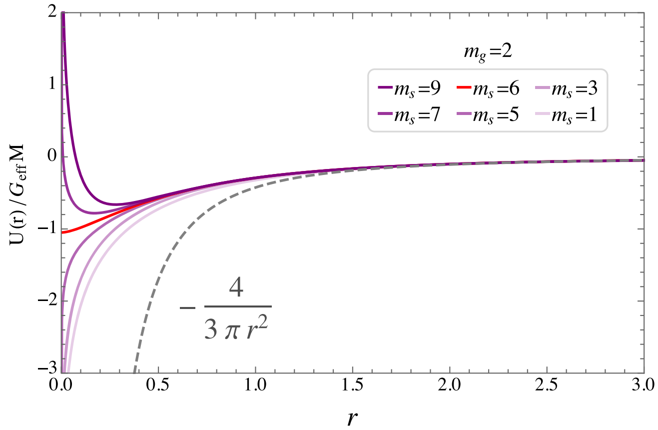

FIG. 2. $U(r)/(G_{\rm eff}M)$ in $D=5$ for $m_g=2$ and $m_s=1,3,5,7,9$ (purple curves) and $m_s=6$ (red) and the usual Newton potential in five dimensions (dashed gray).

like in four dimensions, when the masses of the extra modes are equal, $m_s=m_g$, the gamma parameter coincides with that of Einstein gravity, $\gamma_D=1/(D-3)$. Note also that when one of the modes is absent, the divergence of $U_D(r)$ at $r=0$ becomes stronger than in the generic case (8.23)—namely, of order $1/r^{D-3}$ instead of $1/r^{D-5}$.

IX. GRAVITATIONAL WAVES

In this section, we study the emission and propagation of gravitational radiation from sources in a general four-dimensional theory of the form (1.1) using the results of Sec. II. Our main result is a new formula for the power emitted by a source as a function of the quadrupole moment and the scalar radiation—see (9.40) below. This generalizes the Einstein gravity result to general $\mathcal{L}$(Riemann) theories. We point out that a previous expression obtained for $f(R)$ gravities in Ref. [77] is incorrect and provide the corrected expression, which is a particular case of our general result.

A. Polarization of gravitational waves

In the de Donder gauge (2.47), the relevant components of the metric perturbation decomposed as in (2.51) satisfy Eqs. (2.48), (2.52), and (2.53). In the vacuum, these reduce to

$$\bar{\Box}\hat{h}_{\mu\nu}=0,\qquad (\bar{\Box}-m_g^2)t_{\mu\nu}=0,\qquad (\bar{\Box}-m_s^2)\phi=0. \tag{9.1}$$

Using the tracelessness of $t_{\mu\nu}$, the gauge condition (2.47), and Eq. (9.1) along with (2.51), one can show that $\partial^\mu t_{\mu\nu}=0$. The gauge redundancy has not been completely exploited, as we still have the freedom to make gauge transformations $h_{\mu\nu}\to h_{\mu\nu}+2\partial_{(\mu}\xi_{\nu)}$ where ξ_μ satisfies $\bar{\Box}\xi_\mu=0$. This freedom can be used to impose four additional conditions on $\hat{h}_{\mu\nu}$. In particular, we can set $\hat{h}=0$ and $\hat{h}_{ti}=0$, which is called the *traceless-transverse gauge* (TT). Observe that we cannot impose similar conditions on $t_{\mu\nu}$ because we can only make transformations with a harmonic gauge parameter ξ_μ, but $t_{\mu\nu}$ is not harmonic because it is massive. Hence, no degrees of freedom in $t_{\mu\nu}$ can be removed with such a gauge transformation, and as a consequence, the massive particles conserve all their polarizations.

Let us now look for plane-wave solutions of frequency ω,

$$\hat{h}^{TT}_{\mu\nu}=A_{\mu\nu}e^{-ik_\mu x^\mu},\qquad t_{\mu\nu}=B_{\mu\nu}e^{-ip_\mu x^\mu},\qquad \phi=ce^{-iq_\mu x^\mu}, \tag{9.2}$$

where $k_\mu=(\omega,k_i)$, $p_\mu=(\omega,p_i)$, $q_\mu=(\omega,q_i)$. Equation (9.1) produces the following dispersion relations:

$$k^2=\omega^2,\qquad p^2=\omega^2-m_g^2,\qquad q^2=\omega^2-m_s^2. \tag{9.3}$$

Note that for the massive modes to propagate, the frequency must be greater than the corresponding mass, i.e., $\omega^2>m_g^2$ and $\omega^2>m_s^2$, respectively. Otherwise, the wave will be damped. Now, since we are working in the TT gauge, the polarization tensor $A_{\mu\nu}$ satisfies the following constraints,

$$A_{t\mu}=0,\qquad k^iA_{ij}=0,\qquad A_{ii}=0, \tag{9.4}$$

which leave us with only two independent polarizations $A^+_{\mu\nu}$ and $A^\times_{\mu\nu}$. On the other hand, $B_{\mu\nu}$ only satisfies the constraints

$$p^\mu B_{\mu\nu}=0,\qquad \eta^{\mu\nu}B_{\mu\nu}=0. \tag{9.5}$$

TABLE IV. Newton's potential and $\gamma(r)$ in higher dimensions $D\geq 4$ for various values of the masses of the extra modes.

	$U_D(r)/(\mu(D)G_{\rm eff}M)$	γ_D
$m_g=m_s=+\infty$	$-1/r^{D-3}$	$1/(D-3)$
$m_s=+\infty$, $\lvert m_gr\rvert\ll 1$	$+1/[(D-3)(D-1)r^{D-3}]$	-1
$m_s=0$, $m_g=+\infty$	$-(D-2)^2/[(D-3)(D-1)r^{D-3}]$	$1/(D-2)$
$m\equiv m_g=m_s$	$-\left[1-\frac{(D-3)\Omega_{D-2}}{(2\pi)^{(D-1)/2}}(mr)^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(mr)\right]/r^{D-3}$	$1/(D-3)$

There are 5 degrees of freedom which correspond to the choice of a spatial part of the polarization, B_{ij}, satisfying

$$p^i p^j B_{ij} = \omega^2 B_{ii}, \tag{9.6}$$

which include the + and × polarizations plus three additional ones. The time components are then given by

$$B_{tt} = B_{ii}, \qquad B_{ti} = \frac{p_j}{\omega} B_{ij}. \tag{9.7}$$

Finally, from (2.51), it follows that the contribution to the metric perturbation associated to the scalar mode is given by $\sim C_{\mu\nu} e^{-iq_\alpha x^\alpha}$ with polarization tensor

$$C_{\mu\nu} = \eta_{\mu\nu} - \frac{2q_\mu q_\nu}{m_s^2}, \tag{9.8}$$

which is linearly independent from $A_{\mu\nu}$ and $B_{\mu\nu}$ because it is not traceless.

In sum, gravitational waves in higher-order gravity can propagate up to six different polarizations—one for the scalar and five for the massive and massless gravitons. However, it is important to note that the massive modes do not propagate at lower frequencies, so the possible polarizations depend on the frequency.

B. Gravitational radiation from sources

Let us now consider a source $T_{\mu\nu}(t, \vec{x})$ concentrated in a region of which the diameter is much smaller than the distance r to the observer and which moves at a nonrelativistic characteristic speed. Under such approximations,

$$|\vec{x} - \vec{x}'| \approx r, \qquad \frac{d\vec{x}}{dt} \ll 1, \tag{9.9}$$

the solutions in (8.2) can be further simplified. In particular, for the massless graviton $\hat{h}_{\mu\nu}$, one finds

$$\hat{h}_{\mu\nu} = \frac{4G_{\rm eff}}{r} \int d^3\vec{x}' T_{\mu\nu}(t - r, \vec{x}'). \tag{9.10}$$

Our interest here is in the radiative contributions of the solutions, i.e., the ones which change with time. For gravitational waves, the time components $\hat{h}_{\mu 0}$ are determined by the purely spacelike ones, so we only need to compute those. The spatial components are radiative in general, and for them one finds the well-known quadrupole formula

$$\int d^3\vec{x}' T_{ij}(t - r, \vec{x}') = \frac{1}{2} \ddot{q}_{ij}(t - r), \tag{9.11}$$

where q_{ij} is the quadrupole moment of the source

$$q_{ij}(t - r) = \int d^3\vec{x} x^i x^j \rho(t - r, \vec{x}), \tag{9.12}$$

ρ is the energy density, and each dot denotes a time derivative. Therefore, the radiative part of $\hat{h}_{\mu\nu}$ is given by

$$\hat{h}_{ij} = \frac{2G_{\rm eff}}{r} \ddot{q}_{ij}(t - r). \tag{9.13}$$

Obviously, in the case of Einstein gravity—or for Einstein-like theories—this is the end of the story. However, in general $\mathcal{L}$(Riemann) theories, we also have to take into account the additional modes. For the scalar ϕ, one finds

$$\phi = \frac{4G_{\rm eff}}{r} \int d^3\vec{x}' T(t - r, \vec{x}') - 4G_{\rm eff} m_s \int_r^\infty dt' \frac{J_1(m_s \sqrt{t'^2 - r^2})}{\sqrt{t'^2 - r^2}} \int d^3\vec{x}' T(t - t', \vec{x}'), \tag{9.14}$$

where $J_1(x)$ is a Bessel function of the first kind. The integration of the trace yields

$$\int d^3\vec{x}' T(t - r, \vec{x}') = \int d^3\vec{x}' (-T_{00}(t - r, \vec{x}') + T_{ii}(t - r, \vec{x}')) = -M_0 - E_k(t - r) + \frac{1}{2} \ddot{q}_{ii}(t - r), \tag{9.15}$$

where M_0 is the rest mass and E_k is the kinetic energy of the source. Since the rest mass is constant, it does not source any radiation, and the radiative part of the field is

$$\phi = \frac{4G_{\rm eff}}{r} \left(\frac{1}{2} \ddot{q}_{ii}(t - r) - E_k(t - r) \right) - 4G_{\rm eff} m_s \int_r^\infty dt' \frac{J_1(m_s \sqrt{t'^2 - r^2})}{\sqrt{t'^2 - r^2}} \left(\frac{1}{2} \ddot{q}_{ii}(t - r) - E_k(t - r) \right). \tag{9.16}$$

It is important to note that this field does not always radiate. Indeed, if one considers the source to be a set of pointlike particles or a pressureless perfect fluid (dust), then one gets $\frac{1}{2}\ddot{q}_{ii}(t - r) - E_k(t - r) = \text{constant}$.[22]

[22] The energy-momentum tensor of a pressureless fluid has the form $T_{\mu\nu} = \rho u^\mu u^\nu$, where ρ is the energy density and u^μ is the 4-velocity field, satisfying $u^\mu u_\mu = -1$. Therefore, $T = -\rho$, and its integral yields the rest mass of the system. The same argument works for a set of pointlike particles. Also, an explicit computation in that case shows that—at least—when particles interact only gravitationally, then $\frac{1}{2}\ddot{q}_{ii} - E_k = E_k + E_p$, where E_p is the gravitational potential energy of the system, and the previous quantity is a constant of motion.

Finally, we have to determine the radiative part of $t_{\mu\nu}$. From (8.2), we can express this field as

$$t_{\mu\nu} = -H_{\langle\mu\nu\rangle} - \frac{1}{3m_g^2}\partial_{\langle\mu}\partial_{\nu\rangle}H, \tag{9.17}$$

where the purely spacelike components of $H_{\mu\nu}$ for far sources are given by

$$H_{ij} = -\frac{2G_{\rm eff}}{r}\ddot{q}_{ij}(t-r) + 2G_{\rm eff}m_g\int_r^\infty dt'\frac{J_1(m_g\sqrt{t'^2-r^2})}{\sqrt{t'^2-r^2}}\ddot{q}_{ij}(t-t'). \tag{9.18}$$

Moreover, in the vacuum, we get $0 = \partial_\mu t^{\mu\nu} = \partial_\mu H^{\mu\nu}$, so this allows us to characterize all the components of $H_{\mu\nu}$ and $t_{\mu\nu}$.[23] By using (2.51), (9.17), and the solutions for $\hat{h}_{ij}$, ϕ, and H_{ij} that we have just found, the full metric perturbation can be computed. Note that the perturbation at a distance r depends on the radiation emitted at all times previous to $t-r$ and not only on the radiation emitted at the time $t-r$. This is related to the fact that the massive graviton and the scalar do not propagate at the speed of light. Indeed, according to the dispersion relation $\omega = \sqrt{m_{g,s}^2 + k^2}$, a wave packet with a central frequency ω will travel at a velocity

$$v_{g,s} = \sqrt{1 - \frac{m_{g,s}^2}{\omega^2}}. \tag{9.19}$$

1. Harmonic source

Let us work out explicitly the case corresponding to a source with harmonic motion. Then, the quadrupole moment takes the form $q_{ij}(t) = a_{ij}e^{-i\omega t} + c_{ij}$, where a_{ij} is the polarization tensor and c_{ij} is some plausible constant term. We also assume that the kinetic energy can be expressed as $E_k = E_{k0}e^{-i\omega t}$, plus a possible constant term which does not produce radiation and which we neglect. For this kind of time dependence, the integrals above can be computed, and the fields take the following form:

$$\hat{h}_{ij} = -\frac{2G_{\rm eff}\omega^2}{r}e^{-i\omega(t-r)}a_{ij}, \tag{9.20}$$

$$H_{ij} = -\frac{2G_{\rm eff}\omega^2}{r}e^{-i\omega t + i\sqrt{\omega^2 - m_g^2}r}a_{ij}, \tag{9.21}$$

$$\phi = -\frac{4G_{\rm eff}\omega^2}{r}e^{-i\omega t + i\sqrt{\omega^2 - m_s^2}r}\left(\frac{1}{2}a_{ii} - E_{k0}\right). \tag{9.22}$$

Here, it is evident that the massive graviton (scalar) propagates only when $\omega^2 > m_g^2$ ($\omega^2 > m_s^2$). These expressions can be written in a more compact and suggestive way as

$$\hat{h}_{ij} = \frac{2G_{\rm eff}}{r}\ddot{q}_{ij}(t-r), \qquad H_{ij} = \frac{2G_{\rm eff}}{r}\ddot{q}_{ij}(t - v_g r),$$
$$\phi = \frac{4G_{\rm eff}}{r}\left(\frac{1}{2}\ddot{q}_{ii}(t - v_s r) - E_k(t - v_s r)\right), \tag{9.23}$$

where v_g and v_s are the group velocities of the massive graviton and the scalar, respectively[24] (9.19). Note that, while the expression for $\hat{h}_{ij}$ is actually valid in general, the formulas for H_{ij} and ϕ are only exact when the source is harmonic.

C. Power radiated by sources

In this subsection, we derive the formula for the power emitted by some system in the form of gravitational radiation for a general theory of the form (1.1). In order to do so, we need to find the energy carried by gravitational waves. There are several ways of doing this. For instance, one can interpret the gravitational equations (2.1) with its linear part in $h_{\mu\nu}$ subtracted—i.e., $\mathcal{E}_{\mu\nu} - \mathcal{E}^L_{\mu\nu}$—as the gravitational stress-energy tensor, for which one needs to compute the equations of motion up to quadratic order [78]. We will use a different approach here. As we saw in Sec. II D, it is possible to derive the linearized equations (2.27) from the quadratic action (2.55). From this, we can construct the canonical energy-momentum tensor $\tau_{\mu\nu}$ associated to $h_{\mu\nu}$ using the Noether prescription, e.g., [79]

$$\tau_{\mu\nu} = -\left[\frac{\partial\mathcal{L}}{\partial(\partial^\mu h_{\alpha\beta})} - \partial_\sigma\frac{\partial\mathcal{L}}{\partial(\partial^\mu\partial_\sigma h_{\alpha\beta})}\right]\partial_\nu h_{\alpha\beta} - \frac{\partial\mathcal{L}}{\partial(\partial^\mu\partial_\sigma h_{\alpha\beta})}\partial_\sigma\partial_\nu h_{\alpha\beta} + \eta_{\mu\nu}\mathcal{L}. \tag{9.24}$$

By construction, the total energy-momentum conservation law holds:

$$\partial_\mu(\tau^{\mu\nu} + T^{\mu\nu}) = 0. \tag{9.25}$$

[23]For example, for a plane-wave solution, we have $p^\mu H_{\mu\nu} = 0$, so we obtain the timelike components in terms of the purely spacelike ones: $H_{0i} = p^j H_{ij}/\omega$, $H_{00} = p^i p^j H_{ij}/\omega^2$. In the general case, the relations that we obtain are not algebraic but differential.

[24]Note that for this kind of dispersion relation, the group velocity (which is the physical one) is the inverse of the phase velocity, and that is why it seems that the velocity is in the wrong place.

Here, $T^{\mu\nu}$ is the stress tensor of matter (2.2), so $\tau_{\mu\nu}$ can be used to determine the gravitational energy flux from a source.[25] This tensor can be computed explicitly, but we will not need its general expression here. Instead, we will make the further assumption that the perturbation modes are plane waves (9.2). In that case, if the perturbations $\hat{h}_{\mu\nu}$, $t_{\mu\nu}$, and ϕ appeared separately in $\mathcal{L}$, the stress tensor for each of them would be given by

$$\tau_{\mu\nu}(\hat{h}_{\mu\nu}) = \frac{k_\mu k_\nu}{32\pi G_{\text{eff}}}\left\langle \hat{h}^{\alpha\beta}\hat{h}_{\alpha\beta} - \frac{1}{2}\hat{h}^2\right\rangle, \tag{9.26}$$

$$\tau_{\mu\nu}(t_{\mu\nu}) = -\frac{1}{32\pi G_{\text{eff}}} p_\mu p_\nu \langle t^{\alpha\beta}t_{\alpha\beta}\rangle, \tag{9.27}$$

$$\tau_{\mu\nu}(\phi) = \frac{1}{192\pi G_{\text{eff}}} q_\mu q_\nu \langle \phi^2\rangle, \tag{9.28}$$

where we have averaged the resulting expressions over space-time dimensions large compared with $1/\omega$, so that we are implicitly assuming $r \gg 1/\omega$. This averaging, which is the natural way of defining the energy and momentum of a wave, as it removes oscillations, e.g., Refs. [78,81], has the effect of killing crossed terms like $\hat{h}^{\alpha\beta}t_{\alpha\beta}$, $\hat{h}\phi$, as long as $0 \neq m_s \neq m_g \neq 0$. These terms would otherwise be present in the final expression of $\tau_{\mu\nu}$. In that case, one simply finds $\tau_{\mu\nu} = \tau_{\mu\nu}(\hat{h}) + \tau_{\mu\nu}(t) + \tau_{\mu\nu}(\phi)$. Note that, while $\hat{h}_{\mu\nu}$ and ϕ carry positive energy, the massive graviton $t_{\mu\nu}$ propagates negative energy, which is in agreement with its ghost behavior. Now, the total radiated power crossing a sphere of radius r is given by

$$P = \int d\Omega r^2 \tau_{0i} n^i, \tag{9.29}$$

where n^i is the unit vector normal to the sphere, and note that with this definition, a positive power means that the source loses energy. In order to perform the integration, we have to write the expressions above in terms of the spacelike components of the perturbations. In the case of $\hat{h}_{\mu\nu}$, we can write τ_{0i} for a harmonic wave as

$$\tau_{0i}(\hat{h}_{\mu\nu}) = \frac{n_i}{32\pi G_{\text{eff}}}\left\langle \dot{\hat{h}}^{\alpha\beta}\dot{\hat{h}}_{\alpha\beta} - \frac{1}{2}\dot{\hat{h}}^2\right\rangle, \tag{9.30}$$

where we used the relation $\omega^2\langle \hat{h}^{\alpha\beta}\hat{h}_{\alpha\beta}\rangle = \langle \dot{\hat{h}}^{\alpha\beta}\dot{\hat{h}}_{\alpha\beta}\rangle$. Now, since $\hat{h}_{\mu\nu}$ is transverse, $k^\mu \hat{h}_{\mu\nu} = 0$, we can write $\hat{h}_{00} = n^i n^j \hat{h}_{ij}$ and $\hat{h}_{0i} = n^j \hat{h}_{ij}$, so (9.30) takes the form

$$\tau_{0l}(\hat{h}_{\mu\nu}) = \frac{n_l}{32\pi G_{\text{eff}}}\left\langle \dot{\hat{h}}^{ij}\dot{\hat{h}}_{ij} - \frac{1}{2}\dot{\hat{h}}_{ii}^2 + n^i n^i(\dot{\hat{h}}_{ij}\dot{\hat{h}}_{kk} - 2\dot{\hat{h}}_{ik}\dot{\hat{h}}_{jk}) + \frac{1}{2}(n^i n^j \dot{\hat{h}}_{ij})^2\right\rangle. \tag{9.31}$$

Finally, using (9.13) and performing the integration over the solid angle yields the power radiated by the massless graviton $\hat{h}_{\mu\nu}$ in terms of the quadrupole moment

$$P(\hat{h}_{\mu\nu}) = \frac{G_{\text{eff}}}{5}\left\langle \dddot{q}^{ij}\dddot{q}_{ij} - \frac{1}{3}(\dddot{q}_{ii})^2\right\rangle. \tag{9.32}$$

This is the well-known result found for Einstein gravity [78,81] and the final answer for Einstein-like theories as defined in Sec. III.

For general theories, we need to compute the contributions from the extra modes, which we perform along the same lines. First, we note that it is convenient to write $t_{\alpha\beta}$ in terms of the auxiliary field $H_{\alpha\beta}$ (9.17), so we get, e.g., $t_{\alpha\beta}t^{\alpha\beta} = H_{\alpha\beta}H^{\alpha\beta} - \frac{1}{3}H^2$. Using this, we can write $\tau_{0i}(t_{\mu\nu})$ as

$$\tau_{0i}(t_{\mu\nu}) = -\frac{n_i v_g}{32\pi G_{\text{eff}}}\left\langle \dot{H}^{\alpha\beta}\dot{H}_{\alpha\beta} - \frac{1}{3}\dot{H}^2\right\rangle, \tag{9.33}$$

where we have taken into account that $p_i = v_g \omega n_i$, and again we have reabsorbed the ω factor in a time derivative. Since $H_{\alpha\beta}$ is also transverse, $p^\alpha H_{\alpha\beta} = 0$, we have $H_{00} = v_g^2 n^i n^j H_{ij}$, $H_{0i} = v_g n^j H_{ij}$, and hence we find

$$\tau_{0l}(t_{\mu\nu}) = -\frac{n_l v_g}{32\pi G_{\text{eff}}}\left\langle \dot{H}^{ij}\dot{H}_{ij} - \frac{1}{3}\dot{H}_{ii}^2 + v_g^2 n^i n^j\left(\frac{2}{3}\dot{H}_{ij}\dot{H}_{kk} - 2\dot{H}_{ik}\dot{H}_{jk}\right) + \frac{2}{3}v_g^4(n^i n^j \dot{H}_{ij})^2\right\rangle. \tag{9.34}$$

Now, we can already perform the integral over the solid angle, and by using (9.23), we get

$$P(t_{\mu\nu}) = -\frac{G_{\text{eff}}}{5}\left\langle \left(\frac{5}{2}v_g - \frac{5}{3}v_g^3 + \frac{2}{9}v_g^5\right)\dddot{q}^{ij}\dddot{q}_{ij} - \frac{1}{3}\left(\frac{5}{2}v_g - \frac{5}{3}v_g^3 - \frac{1}{3}v_g^5\right)(\dddot{q}_{ii})^2\right\rangle. \tag{9.35}$$

One can see that this flux is always negative, provided $0 \leq v_g \leq 1$. As a consequence, every time one of these

[25] In the nonlinear regime, one can construct a gravitational energy-momentum pseudotensor by using the same prescription as in (9.24), namely, $\tau_{\mu\nu}^{\text{nonlinear}} = -[\frac{\partial\mathcal{L}}{\partial(\partial^\mu g_{\alpha\beta})} - \partial_\sigma \frac{\partial\mathcal{L}}{\partial(\partial^\mu\partial_\sigma g_{\alpha\beta})}]\partial_\nu g_{\alpha\beta} - \frac{\partial\mathcal{L}}{\partial(\partial^\mu\partial_\sigma g_{\alpha\beta})}\partial_\sigma\partial_\nu g_{\alpha\beta} + \eta_{\mu\nu}\mathcal{L}$. Although this quantity is not a tensor, Noether's theorem ensures that $\partial^\mu[\sqrt{|g|}(\tau_{\mu\nu}^{\text{non-linear}} + T_{\mu\nu})] = 0$ [80]. In the linear regime, these expressions reduce to (9.24) and (9.25), respectively.

modes is emitted, some positive energy must be added to the source in order to keep the total energy constant. In other words, the massive graviton would have the effect of making moving sources soak up gravitational radiation from the environment instead of emitting it. This is yet another manifestation of the ghost nature of this mode.

Note also that this power does not cancel the one for the massless graviton, even if we set $v_g = 1$—corresponding to $m_g = 0$. There is no contradiction in this, since the polarization modes of $t_{\mu\nu}$ are different from those of $\hat{h}_{\mu\nu}$ and therefore the energy carried by these fields does not have to be necessarily opposite—and indeed, it is not. Observe that the same occurs for the generalized Newtonian potential; i.e., if we set $m_g = 0$ in (8.15), the contributions from the two gravitons do not cancel each other. This phenomenon is reminiscent of the so-called van Dam-Veltman-Zakharov *discontinuity* [82,83], which makes reference to the fact that the massless limit of a free massive graviton makes predictions different from the ones of linearized Einstein gravity.[26] We stress that (9.35) is valid only when the perturbation propagates, i.e., when $\omega^2 > m_g^2$. Otherwise, there is no emission of energy, and $P(t_{\mu\nu}) = 0$. Thus, we can always use the previous formula with the convention $v_g = 0$ if $\omega^2 < m_g^2$.

Finally, we can evaluate the power emitted by the scalar mode. The integral over the solid angle can be done straightforwardly, and the result is

$$P(\phi) = \frac{G_{\rm eff} v_s}{3} \left\langle \left(\frac{1}{2} \dddot{q}_{ii} - \dot{E}_k \right)^2 \right\rangle. \tag{9.36}$$

As stated previously, the scalar radiation vanishes as long as we consider our system to be composed of dust or non-interacting particles (without interactions different form gravity). For example, a binary—see the next epigraph—is very approximately a system of this kind, so there is no scalar radiation in that case. The scalar radiation only plays a role in systems where other interactions different from gravity are important, like in the explosion of a supernova [84]. Now, the final result for the power emitted in the form of gravitational waves in a theory of the form (1.1) reads

$$\begin{aligned} P = \frac{G_{\rm eff}}{5} \Bigg\langle & \left(1 - \frac{5}{2} v_g + \frac{5}{3} v_g^3 - \frac{2}{9} v_g^5 \right) \dddot{q}^{ij} \dddot{q}_{ij} \\ & - \frac{1}{3} \left(1 - \frac{5}{2} v_g + \frac{5}{3} v_g^3 + \frac{1}{3} v_g^5 \right) (\dddot{q}_{ii})^2 \\ & + \frac{5}{3} v_s \left(\frac{1}{2} \dddot{q}_{ii} - \dot{E}_k \right)^2 \Bigg\rangle, \end{aligned} \tag{9.37}$$

where

$$v_{g,s} = \begin{cases} \sqrt{1 - \frac{m_{g,s}^2}{\omega^2}} & \text{if } \omega^2 \geq m_{g,s}^2, \\ 0 & \text{if } \omega^2 < m_{g,s}^2. \end{cases} \tag{9.38}$$

If we decompose the quadrupole moment into its trace and traceless parts,

$$q_{ij} = Q_{ij} + \frac{1}{3} \delta_{ij} q_{kk}, \tag{9.39}$$

we can rewrite this expression as

$$\begin{aligned} P = \frac{G_{\rm eff}}{5} \Bigg\langle & \left(1 - \frac{5}{2} v_g + \frac{5}{3} v_g^3 - \frac{2}{9} v_g^5 \right) \dddot{Q}^{ij} \dddot{Q}_{ij} \\ & - \frac{5}{27} v_g^5 (\dddot{q}_{ii})^2 + \frac{5}{3} v_s \left(\frac{1}{2} \dddot{q}_{ii} - \dot{E}_k \right)^2 \Bigg\rangle. \end{aligned} \tag{9.40}$$

Note that in Einstein gravity, the result only involves the traceless part of q_{ij} [78,81], while here we also have contributions from its trace and from the variation of the source kinetic energy due to the presence of extra modes. Let us stress again that (9.40) is valid only for a harmonic source. In the case of a more general time dependence, q_{ij} and E_k can be Fourier expanded, and then the power of each Fourier mode can be extracted from (9.40). The total power would then be the sum of all of those contributions.

Equation (9.40) is the main result of this section. It generalizes the Einstein gravity formula (9.32) to general $\mathcal{L}$(Riemann) theories. A previous extension of (9.32) to $f(R)$ gravity was found in Ref. [77]. For $f(R)$, our formula above reduces to

$$P_{f(R)} = \frac{G}{5 f'(\bar{R})} \left\langle \dddot{Q}^{ij} \dddot{Q}_{ij} + \frac{5}{3} v_s \left(\frac{1}{2} \dddot{q}_{ii} - \dot{E}_k \right)^2 \right\rangle, \tag{9.41}$$

where v_s reads, see Appendix C,

$$v_s = \begin{cases} \sqrt{1 - \frac{m_s^2}{\omega^2}} & \text{if } \omega^2 \geq m_s^2, \\ 0 & \text{if } \omega^2 < m_s^2, \end{cases} \quad \text{where}$$
$$m_s^2 = \frac{f'(\bar{R}) - \bar{R} f''(\bar{R})}{3 f''(\bar{R})}. \tag{9.42}$$

This expression disagrees with the one found in Ref. [77]—see (82) in that paper. However, it is easy to see that the second term on the rhs of Eq. (43) in Ref. [77] is identically zero, so the second contribution on the rhs of (82) is absent. Similarly, the first term in their (82) is missing an overall[27]

[26]Note, however, that the situation considered here is slightly different from massive gravity. Indeed, in that case, the only field is a well-behaved massive graviton, while for linearized higher-order gravities, we deal with a massless graviton, a scalar mode, and a ghostlike massive graviton.

[27]This seems to arise from a wrong identification in (48). Note that Eqs. (46)–(50) in Ref. [77] are also inconsistent with each other.

$1/f'(\bar{R})^2$. And, finally, the authors seem to have ignored the contribution from the scalar mode, which explains why they do not find the term proportional to $(1/2\ddot{q}_{ii} - \dot{E}_k)^2$.

1. Binary system

As an application of (9.40), let us compute explicitly the power radiated by a system consisting of two masses m_1 and m_2 separated by a distance r in a circular orbit contained in the plane $z = 0$. For this kind of system, the position of the masses is given by

$$\vec{x}_1(t) = \frac{rm_2}{m_1 + m_2}(\cos(\Omega t), \sin(\Omega t), 0), \qquad (9.43)$$

$$\vec{x}_2(t) = -\frac{rm_1}{m_1 + m_2}(\cos(\Omega t), \sin(\Omega t), 0), \qquad (9.44)$$

where the orbital frequency Ω reads

$$\Omega^2 = \frac{G_{\rm eff}(m_1 + m_2)}{r^3}. \qquad (9.45)$$

Assuming the masses to be pointlike, the mass density can be written as $\rho(\vec{x}, t) = m_1\delta(\vec{x} - \vec{x}_1(t)) + m_2\delta(\vec{x} - \vec{x}_2(t))$. Then, the quadrupole moment (9.12) is

$$q_{ij}(t) = \frac{r^2 m_1 m_2}{2(m_1 + m_2)}\begin{pmatrix} 1 + \cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & 1 - \cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (9.46)$$

The trace and the kinetic energy are constant, $q_{ii} = (r^2 m_1 m_2)/(m_1 + m_2)$, $\dot{E}_k = 0$, so there is no scalar radiation in this case.[28] The traceless part of q_{ij} reads in turn

$$Q_{ij}(t) = \frac{r^2 m_1 m_2}{2(m_1 + m_2)} \times \begin{pmatrix} 1/3 + \cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & 1/3 - \cos(2\Omega t) & 0 \\ 0 & 0 & -2/3 \end{pmatrix}. \qquad (9.47)$$

Applying (9.40), we obtain the following result,

$$P = P_E\left(1 - \sqrt{1 - \frac{m_g^2}{4\Omega^2}}\left[\frac{19}{18} + \frac{11}{36}\frac{m_g^2}{\Omega^2} + \frac{1}{72}\frac{m_g^4}{\Omega^4}\right]\right), \qquad (9.48)$$

where

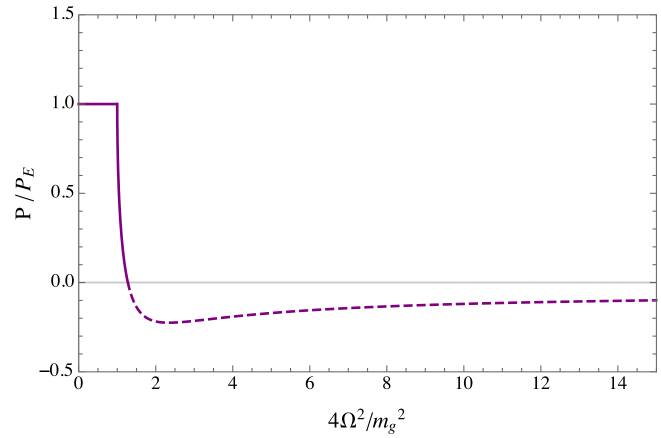

FIG. 3. Power emitted by a binary system for a theory of the form (1.1), P/P_E as a function of $4\Omega^2/m_g^2$.

$$P_E = \frac{32 G_{\rm eff}^4 m_1^2 m_2^2 (m_1 + m_2)}{5r^5} \qquad (9.49)$$

is the result corresponding to theories which do not propagate the massive graviton—see Sec. III. In particular, Eq. (9.49) is the Einstein gravity result when $G_{\rm eff} = G$. Expression (9.48) is valid for $4\Omega^2 > m_g^2$. When $4\Omega^2 < m_g^2$ instead, the result reduces to P_E—see Fig. 3. When $4\Omega^2 = m_g^2$, the effect of the massive graviton makes the power start to decrease. In particular, when $4\Omega^2/m_g^2 \simeq 1.2761$, the power emitted vanishes. For even smaller values of m_g^2 with respect to Ω^2, the power becomes negative acquiring its minimum value at $4\Omega^2/m_g^2 = 1 + 3/\sqrt{5}$, for which $P/P_E(1 + 3/\sqrt{5}) = 1 - \sqrt{3/2} \simeq -0.2247$. Finally, for $\Omega^2 \gg m_g^2$, the power tends to the constant value $P/P_E(\Omega^2 \gg m_g^2) = -1/18 \simeq -0.0556$. Given a theory with $m_g^2 < \infty$, there would exist a critical frequency $\Omega_c^2 \simeq 0.31903 m_g^2$ for which the source would stop emitting radiation and such that for greater frequencies the source would start absorbing radiation instead of emitting it. This exotic process should not be regarded as physical and illustrates the pathological character of the class of theories which propagate the additional spin-2 mode.

X. WALD FORMALISM FOR GENERAL $\mathcal{L}$(Riemann) THEORIES

In this section, we present a self-contained review of Wald's formalism [41] applied to general $\mathcal{L}$(Riemann) theories. Wald's formalism provides a systematic way of constructing conserved quantities in diffeomorphism invariant theories. It was originally developed to derive the first law of black-hole mechanics for generic theories of gravity [42,43], but it has led to many interesting applications, e.g., in holography [48,85,86]. Our discussion is mainly based on Refs. [42,87], where this formalism was

[28] In the case of a more general orbit, there is no scalar radiation either, because, as discussed earlier, $\frac{1}{2}\ddot{q}_{ii} - E_k = E_k + E_p =$ constant for that kind of system.

developed for higher-derivative theories of gravity. Here, we present new results for the symplectic structure $\boldsymbol{\omega}$ and the surface charge $\delta\mathbf{Q}_\xi - \xi\cdot\boldsymbol{\Theta}$ for $\mathcal{L}(\text{Riemann})$ theories. Throughout this section, we set $L_{\text{matter}} = 0$ in (1.1); i.e., we assume that the Lagrangian does not depend on any matter fields. In Appendix E, we provide explicit expressions for the quantities considered in this section for some relevant theories.

A. Lagrangian and symplectic potential

The starting point of the Wald formalism is a diffeomorphism covariant Lagrangian, which—in our case—is assumed to be a local functional of the metric and the Riemann tensor. The Lagrangian is treated as a D-form on the D-dimensional space-time manifold $\mathcal{M}$, namely,

$$\mathbf{L} = \mathcal{L}(R_{\mu\nu\rho\sigma}, g^{\alpha\beta})\boldsymbol{\epsilon}, \tag{10.1}$$

where $\mathcal{L}(R_{\mu\nu\rho\sigma}, g^{\alpha\beta})$ is the Lagrangian density and $\boldsymbol{\epsilon}$ is the volume form on $\mathcal{M}$. For future reference, we will be using the following shorthand notation for the volume form of any codimension-n submanifold:

$$\boldsymbol{\epsilon}_{\mu_1\ldots\mu_n} \equiv \frac{1}{(D-n)!}\sqrt{-g}\epsilon_{\mu_1\ldots\mu_n\nu_{n+1}\ldots\nu_D} dx^{\nu_{n+1}}\wedge\cdots\wedge dx^{\nu_D}. \tag{10.2}$$

Under a variation of the metric,[29] the first-order variation of the Lagrangian is given by

$$\delta\mathbf{L} = \boldsymbol{\epsilon}\mathcal{E}^{\mu\nu}\delta g_{\mu\nu} + d\boldsymbol{\Theta}(g,\delta g), \tag{10.3}$$

where $\mathcal{E}^{\mu\nu} = 0$ are the equations of motion for the theory, given by[30] (2.1) and $\boldsymbol{\Theta}$ is the boundary term that arises due to partial integration of terms involving derivatives of δg. The $(D-1)$-form $\boldsymbol{\Theta}$ is locally constructed from g and δg and is called the *symplectic potential form*. From (10.3), it is clear that $\boldsymbol{\Theta}$ is not uniquely defined, since one always has the freedom to add a closed—and hence locally exact [88]—form to it. However, as shown in Refs. [42,89], it is always possible to construct an explicit covariant formula for $\boldsymbol{\Theta}$ which fixes this ambiguity. For $\mathcal{L}(\text{Riemann})$ theories, this somewhat canonical formula reads [42,87]

$$\boldsymbol{\Theta} = \boldsymbol{\epsilon}_\mu(2P^{\mu\alpha\beta\nu}\nabla_\nu\delta g_{\alpha\beta} - 2\nabla_\nu P^{\mu\alpha\beta\nu}\delta g_{\alpha\beta}), \tag{10.4}$$

where $P^{\mu\beta\alpha\nu}$ is defined in (2.2). Furthermore, by employing the relation

$$\nabla_\nu\delta g_{\alpha\beta} = g_{\beta\rho}\delta\Gamma^\rho_{\nu\alpha} + g_{\alpha\rho}\delta\Gamma^\rho_{\nu\beta}, \tag{10.5}$$

the symplectic potential form can also be written as

$$\boldsymbol{\Theta} = \boldsymbol{\epsilon}_\mu(-2P^{\mu\alpha\beta}{}_\nu\delta\Gamma^\nu_{\alpha\beta} - 2\nabla_\nu P^{\mu\alpha\beta\nu}\delta g_{\alpha\beta}), \tag{10.6}$$

where we used that $P^{\mu\alpha\beta\nu}$ is antisymmetric in its last two indices, $P^{\mu\alpha\beta\nu} = -P^{\mu\alpha\nu\beta}$, which implies that $P^{\mu\alpha\beta\nu}\delta\Gamma^\rho_{\beta\nu} = 0$.

B. Symplectic form

The *symplectic current form* is defined as the antisymmetrized variation of $\boldsymbol{\Theta}$ [89],

$$\boldsymbol{\omega}(g,\delta_1 g,\delta_2 g) \equiv \delta_1\boldsymbol{\Theta}(g,\delta_2 g) - \delta_2\boldsymbol{\Theta}(g,\delta_1 g). \tag{10.7}$$

From (10.3) and (10.7), it follows that $\boldsymbol{\omega}$ obeys the relation

$$d\boldsymbol{\omega} = -\delta_1(\boldsymbol{\epsilon}\mathcal{E}^{\mu\nu})\delta_2 g_{\mu\nu} + \delta_2(\boldsymbol{\epsilon}\mathcal{E}^{\mu\nu})\delta_1 g_{\mu\nu}. \tag{10.8}$$

Here, it was used that the exterior derivative d commutes with the variation δ: $d(\delta\boldsymbol{\Theta}) = \delta(d\boldsymbol{\Theta})$. Therefore, if δg satisfies the linearized equations of motion $\delta(\boldsymbol{\epsilon}\mathcal{E}^{\mu\nu}) = 0$, then the symplectic current form is closed,

$$d\boldsymbol{\omega} = 0. \tag{10.9}$$

This relation implies—by Stokes's theorem—that the integral of $\boldsymbol{\omega}$ over a compact Cauchy surface $\mathcal{C}$ is independent of the choice of $\mathcal{C}$. For noncompact Cauchy surfaces, one has to impose appropriate boundary conditions on the metric and its perturbations on $\partial\mathcal{C}$ in order to assure convergence of the integral. Here, we just assume that such boundary conditions exist, so that the integral of $\boldsymbol{\omega}$ over a Cauchy surface $\mathcal{C}$ is a conserved quantity. This quantity is called the *symplectic 2-form* [42,89],

$$\Omega(g,\delta_1 g,\delta_2 g) \equiv \int_{\mathcal{C}}\boldsymbol{\omega}(g,\delta_1 g,\delta_2 g). \tag{10.10}$$

Let us explain the origin of its name. In fact, Ω can be regarded as a 2-form defined on the space of metric configurations $\mathcal{F}$. This is because Ω is a local functional of the linearized perturbations $\delta_1 g$ and $\delta_2 g$, where the variation δ can be viewed as the exterior derivative on this space. Moreover, from (10.7), it follows that Ω is closed, i.e., $\delta\Omega = 0$, due to the fact that the exterior derivative satisfies the relation $\delta^2 g = 0$. Now a proper symplectic form on phase space is both closed and nondegenerate. The form (10.10) is degenerate—and is hence sometimes called the *presymplectic form* instead—but one can construct a nondegenerate 2-form from (10.10) by modding out $\mathcal{F}$ by the degeneracy subspace of Ω. Then, the nondegenerate Ω and the solution submanifold of $\mathcal{F}$ constitute a well-defined *covariant phase space* [89].

[29]For convenience, we vary the Lagrangian with respect to the metric $g_{\mu\nu}$, although it was initially defined in terms of the inverse metric $g^{\mu\nu}$.

[30]Notice that the equations of motion with indices up and the one with indices down are related by a minus sign: $\mathcal{E}^{\mu\nu} = -g^{\mu\alpha}g^{\mu\nu}\mathcal{E}_{\alpha\beta}$.

Let us now compute the symplectic current form explicitly for $\mathcal{L}$(Riemann) theories. If we write the symplectic potential form as $\mathbf{\Theta} = \boldsymbol{\epsilon}_\mu \Theta^\mu$, then the definition of $\boldsymbol{\omega}$ (10.7) becomes

$$\boldsymbol{\omega}(g,\delta_1 g,\delta_2 g) = \boldsymbol{\epsilon}_\mu\left[(\delta_1\Theta^\mu(g,\delta_2 g)) + \frac{1}{2}g^{\mu\nu}\delta_1 g_{\mu\nu}\Theta^\mu(g,\delta_2 g)\right] - [1\leftrightarrow 2], \tag{10.11}$$

where we used $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}$. Next, one can insert two expressions for $\mathbf{\Theta}$ for $\mathcal{L}$(Riemann) theories. $\boldsymbol{\omega}$ simplifies immediately if one inserts the second expression (10.6), since in that case one can employ the relation $\delta_{[1}\delta_{2]}\Gamma^\nu_{\alpha\beta} = 0$. If one inserts the first expression (10.4) instead, one has to be careful with evaluating the term $4P^{\mu\alpha\beta\nu}\delta_{[1|}(\nabla_\nu\delta_{|2]}g_{\alpha\beta})$, because the variation and the covariant derivative do not commute. In the latter case, one can use the fact that the variation and the partial derivative commute, i.e., $[\delta,\partial_a]f = 0$. We checked that both procedures give the same answer. The result is

$$\boldsymbol{\omega} = \boldsymbol{\epsilon}_\mu[-(2\delta_1 P^{\mu\alpha\beta}{}_\nu + P^{\mu\alpha\beta}{}_\nu g^{\rho\sigma}\delta_1 g_{\rho\sigma})\delta_2\Gamma^\nu_{\alpha\beta} - (2\delta_1\nabla_\nu P^{\mu\alpha\beta\nu} + g^{\rho\sigma}\delta_1 g_{\rho\sigma}\nabla_\nu P^{\mu\alpha\beta\nu})\delta_2 g_{\alpha\beta}] - [1\leftrightarrow 2]. \tag{10.12}$$

By employing the formula for the variation of the Christoffel connection

$$\delta\Gamma^\nu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(\nabla_\alpha\delta g_{\beta\mu} + \nabla_\beta\delta g_{\alpha\mu} - \nabla_\mu\delta g_{\alpha\beta}), \tag{10.13}$$

the result (10.12) can also be written as

$$\boldsymbol{\omega} = \boldsymbol{\epsilon}_\mu[(2\delta_1 P^{\mu\alpha\beta\nu} + (P^{\mu\nu\rho\beta}g^{\alpha\sigma} + P^{\mu\alpha\rho\nu}g^{\beta\sigma} + P^{\mu\alpha\beta\rho}g^{\nu\sigma} + P^{\mu\alpha\beta\nu}g^{\rho\sigma})\delta_1 g_{\rho\sigma})\nabla_\nu\delta_2 g_{\alpha\beta} - (2\delta_1\nabla_\nu P^{\mu\alpha\beta\nu} + g^{\rho\sigma}\delta_1 g_{\rho\sigma}\nabla_\nu P^{\mu\alpha\beta\nu})\delta_2 g_{\alpha\beta}] - [1\leftrightarrow 2]. \tag{10.14}$$

Finally, by inserting a formula for the variation of $P^{\mu\alpha\beta\nu}$ that follows from (2.8),

$$\delta P^{\mu\alpha\beta\nu} = 2g^{\sigma[\mu}P^{\alpha]\rho\beta\nu}\delta g_{\rho\sigma} + g^{\lambda\gamma}g^{\eta\delta}C^{\mu\alpha\beta\nu}_{\rho\sigma\lambda\eta}\delta R^{\rho\sigma}{}_{\gamma\delta}, \tag{10.15}$$

where $C^{\mu\alpha\beta\nu}_{\rho\sigma\lambda\eta}$ is defined by (2.4), the symplectic current form can be written as

$$\boldsymbol{\omega} = \boldsymbol{\epsilon}_\mu[(S^{\mu\alpha\beta\nu\rho\sigma}\delta_1 g_{\rho\sigma} + 2g^{\lambda\gamma}g^{\eta\delta}C^{\mu\alpha\beta\nu}_{\rho\sigma\lambda\eta}\delta_1 R^{\rho\sigma}{}_{\gamma\delta})\nabla_\nu\delta_2 g_{\alpha\beta} - (2\delta_1\nabla_\nu P^{\mu\alpha\beta\nu} + g^{\rho\sigma}\delta_1 g_{\rho\sigma}\nabla_\nu P^{\mu\alpha\beta\nu})\delta_2 g_{\alpha\beta}] - [1\leftrightarrow 2], \tag{10.16}$$

with

$$S^{\mu\alpha\beta\nu\rho\sigma} \equiv -2P^{\nu(\alpha\beta)(\rho}g^{\sigma)\mu} + 2P^{\mu\nu(\rho|(\alpha}g^{\beta)|\sigma)} + P^{\mu(\rho|\nu(\alpha}g^{\beta)|\sigma)} + P^{\mu(\alpha\beta)(\rho}g^{\sigma)\nu} + P^{\mu(\alpha\beta)\nu}g^{\rho\sigma}. \tag{10.17}$$

To arrive at the expression for $S^{\mu\alpha\beta\nu\rho\sigma}$, we employed the first Bianchi identity for $P^{\mu\alpha\beta\nu}$: $P^{\mu\alpha\beta\nu} + P^{\mu\beta\nu\alpha} + P^{\mu\nu\alpha\beta} = 0$. This new formula for the symplectic current form applies to any higher-curvature gravity theory. Expressions for $\boldsymbol{\omega}$ were previously obtained for Einstein gravity [90–92] and $f(R)$ gravity [93]. It can be checked that this formula provides the same results in those cases, as we show in Appendix E.

C. Noether current and Noether charge

Next, let ξ be an arbitrary vector field on $\mathcal{M}$ which generates an infinitesimal diffeomorphism. Since the Lagrangian (10.1) is diffeomorphism invariant, it varies under a diffeomorphism as

$$\delta_\xi\mathbf{L} = \mathcal{L}_\xi\mathbf{L} = d(\xi\cdot\mathbf{L}), \tag{10.18}$$

where in the last equality Cartan's magic formula was used: $\mathcal{L}_\xi\mathbf{L} = \xi\cdot d\mathbf{L} + d(\xi\cdot\mathbf{L})$. The first term vanishes since $\mathbf{L}$ is a top form, and the dot in the second term denotes the interior product of the vector ξ with the form $\mathbf{L}$.

Since diffeomorphisms are local symmetries of the theory, one can associate a *Noether current*—represented as a $(D-1)$-form—to each vector field ξ [41,89],

$$\mathbf{J}_\xi \equiv \mathbf{\Theta}(g,\mathcal{L}_\xi g) - \xi\cdot\mathbf{L}. \tag{10.19}$$

It follows from (10.3) and (10.18) that the exterior derivative of $\mathbf{J}_\xi$ is

$$d\mathbf{J}_\xi = -\boldsymbol{\epsilon}\mathcal{E}^{\mu\nu}\mathcal{L}_\xi g_{\mu\nu}. \tag{10.20}$$

As a consequence, the Noether current form is closed if the equations of motion $\mathcal{E}^{\mu\nu} = 0$ are satisfied. In that case, Poincaré's lemma implies that it is locally exact [88]. What is more, in the Appendix of Ref. [94], it was shown that off-shell $\mathbf{J}_\xi$ can always be written in the form

$$\mathbf{J}_\xi = d\mathbf{Q}_\xi + \xi^\nu\mathbf{C}_\nu, \tag{10.21}$$

where $\mathbf{Q}_\xi$ is called the *Noether charge* $(D-2)$*-form* and $\mathbf{C}_\nu = 0$ are the constraint equations of the theory. For theories that only depend on the metric field, these equations are given by $\mathbf{C}_\nu = 2\boldsymbol{\epsilon}_\mu\mathcal{E}^\mu{}_\nu$ with $\mathcal{E}^\mu{}_\nu \equiv g^{\mu\alpha}\mathcal{E}_{\alpha\nu}$.

Although $\mathbf{Q}_\xi$ is not uniquely determined by Eq. (10.21), there exists an explicit algorithm by Ref. [88] to construct $\mathbf{Q}_\xi$ from $\mathbf{J}_\xi$. For $\mathcal{L}$(Riemann) theories of gravity, this construction yields [42,87]

$$\mathbf{Q}_\xi = \boldsymbol{\epsilon}_{\mu\nu}(-P^{\mu\nu\rho\sigma}\nabla_\rho\xi_\sigma - 2\xi_\rho\nabla_\sigma P^{\mu\nu\rho\sigma}). \tag{10.22}$$

Thus, by Eq. (10.21), the Noether current form is

$$\mathbf{J}_\xi = \boldsymbol{\epsilon}_\mu[-2\nabla_\nu(P^{\mu\nu\rho\sigma}\nabla_\rho\xi_\sigma) - 4\nabla_\nu(\xi_\rho\nabla_\sigma P^{\mu\nu\rho\sigma}) + 2\mathcal{E}^\mu{}_\nu\xi^\nu]. \tag{10.23}$$

D. Surface charge

From (10.3), (10.7), (10.19), and (10.21), one can obtain a fundamental identity,

$$\boldsymbol{\omega}(g,\delta g,\mathcal{L}_\xi g) = d\mathbf{k}_\xi(g,\delta g) + 2\delta(\boldsymbol{\epsilon}_\mu\mathcal{E}^\mu{}_\nu)\xi^\nu + \xi^\lambda\boldsymbol{\epsilon}_\lambda\mathcal{E}^{\mu\nu}\delta g_{\mu\nu}, \tag{10.24}$$

$$\text{where } \mathbf{k}_\xi(g,\delta g) \equiv \delta^{[g]}\mathbf{Q}_\xi(g) - \xi\cdot\boldsymbol{\Theta}(g,\delta g) \tag{10.25}$$

is known as the *Iyer-Wald surface charge* $(D-2)$*-form*. Notice that this relation applies to arbirary metrics g, metric pertubations δg, and vector fields ξ. This identity was first established off shell by Wald [41], and for field-dependent vector fields—e.g., vector fields that depend on the metric $\xi = \xi(g)$—a proof can be found in Refs. [85,95]. The variation $\delta^{[g]}\mathbf{Q}_\xi \equiv \delta\mathbf{Q}_\xi - \mathbf{Q}_{\delta\xi}$ acts only on the explicit dependence on the metric and its derivatives in $\mathbf{Q}_\xi$, and not on the implicit dependence on ξ.

A special case of the identity occurs when ξ is an exact Killing vector. In that case, the relation gives rise to the first law of black-hole mechanics [41,42]. Since $\mathcal{L}_\xi g = 0$, the left-hand side of (10.24) vanishes, and if g and δg satisfy, respectively, the full equations of motion and the linearized ones, one obtains

$$d\mathbf{k}_\xi = 0. \tag{10.26}$$

Therefore, the integral of $\mathbf{k}_\xi$ over a $(D-2)$-dimensional, spacelike compact surface S is "conserved," in the sense that it is independent of the choice of S. If the normal directions to S are the time and radial direction, then the integral is the same at every time and radial coordinate. In order for this integral to be the variation of a finite conserved charge, certain integrability conditions should be satisfied [89].

Let us now compute this quantity for general $\mathcal{L}$(Riemann) theories. Inserting the known expressions for $\mathbf{Q}_\xi$ (10.22) and $\boldsymbol{\Theta}$ (10.4) into the definition of $\mathbf{k}_\xi$ (10.25) yields

$$\begin{aligned}\mathbf{k}_\xi = {}& \delta^{[g]}[\boldsymbol{\epsilon}_{\mu\nu}(-P^{\mu\nu\rho}{}_\sigma\nabla_\rho\xi^\sigma - 2\xi_\rho\nabla_\sigma P^{\mu\nu\rho\sigma})] \\ & - \xi^\lambda\boldsymbol{\epsilon}_{\mu\lambda}(2P^{\mu\alpha\beta\nu}\nabla_\nu\delta g_{\alpha\beta} - 2\nabla_\nu P^{\mu\alpha\beta\nu}\delta g_{\alpha\beta}).\end{aligned} \tag{10.27}$$

By letting the variation act only on the explicit dependence on the metric, and collecting similar terms, we arrive at

$$\begin{aligned}\mathbf{k}_\xi = \boldsymbol{\epsilon}_{\mu\nu}\Bigg[&-\delta P^{\mu\nu\rho}{}_\sigma\nabla_\rho\xi^\sigma - 2\xi^\rho\delta(\nabla_\sigma P^{\mu\nu}{}_\rho{}^\sigma) \\ &+\left(-\frac{1}{2}P^{\mu\nu\rho\sigma}g^{\alpha\beta}\nabla_\rho\xi_\sigma + 2\xi^\nu\nabla_\lambda P^{\mu\alpha\beta\lambda}\right. \\ &\left.-\xi_\rho\nabla_\sigma P^{\mu\nu\rho\sigma}g^{\alpha\beta}\right)\delta g_{\alpha\beta} \\ &-(\xi^\alpha P^{\mu\nu\lambda\beta} + 2\xi^\nu P^{\mu\alpha\beta\lambda})\nabla_\lambda\delta g_{\alpha\beta}\Bigg].\end{aligned} \tag{10.28}$$

Here, we have defined the $\delta^{[g]}$ variation of the vector ξ^σ (with index up) to be zero, i.e., $\delta^{[g]}\xi^\sigma \equiv 0$, which implies that $\delta^{[g]}\xi_\sigma = \xi^\alpha\delta g_{\alpha\sigma}$ and $\delta^{[g]}(\nabla_\rho\xi^\sigma) = \xi^\alpha\delta\Gamma^\sigma_{\alpha\rho}$. Finally, introducing the variation of $P^{\mu\nu\rho\sigma}$ (10.15), we obtain the expression

$$\begin{aligned}\mathbf{k}_\xi = \boldsymbol{\epsilon}_{\mu\nu}\Bigg[&-g^{\gamma\lambda}g^{\delta\eta}C^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}\nabla_\rho\xi_\sigma\delta R^{\alpha\beta}{}_{\lambda\eta} - 2\xi^\rho\delta(\nabla_\sigma P^{\mu\nu}{}_\rho{}^\sigma) \\ &+\left(P^{\mu\nu\alpha\lambda}\nabla^\beta\xi_\lambda - \frac{1}{2}P^{\mu\nu\rho\sigma}g^{\alpha\beta}\nabla_\rho\xi_\sigma\right. \\ &\left.+2\xi^\nu\nabla_\lambda P^{\mu\alpha\beta\lambda} - \xi_\rho\nabla_\sigma P^{\mu\nu\rho\sigma}g^{\alpha\beta}\right)\delta g_{\alpha\beta} \\ &-(\xi^\alpha P^{\mu\nu\lambda\beta} + 2\xi^\nu P^{\mu\alpha\beta\lambda})\nabla_\lambda\delta g_{\alpha\beta}\Bigg].\end{aligned} \tag{10.29}$$

E. Barnich-Brandt-Compère definitions of $\boldsymbol{\omega}$ and $\mathbf{k}_\xi$

A different method for constructing a covariant phase space was developed by Barnich, Brandt, and Compère [96–98]. Their definitions of the relevant quantities are based on the equations of motion rather than the Lagrangian. Hence, their method is also universal, in the sense that it applies to any diffeomorphism invariant theory—in fact, their formalism is more general, since it holds for any theory with local gauge symmetries. Moreover, their definitions do not suffer from any ambiguities, as is the case for the Wald formalism—see the next epigraph. Most quantities agree with those defined by Lee, Wald, and Iyer, expect for the symplectic current $\boldsymbol{\omega}$ and the surface charge $\mathbf{k}_\xi$. For completeness, let us present here the Barnich-Brandt-Compère definitions of $\boldsymbol{\omega}$ and $\mathbf{k}_\xi$ for $\mathcal{L}$(Riemann) theories. A pedagogical review of this method can be found in Refs. [85,87].

First, the *Barnich-Compère symplectic current*—also known as the *invariant* symplectic current—differs from the Lee-Wald definition (10.7) by an exact form,

$$\boldsymbol{\omega}^{\rm BC}(g,\delta_1 g,\delta_2 g) = \boldsymbol{\omega}^{\rm LW}(g,\delta_1 g,\delta_2 g) - d\mathbf{E}(g,\delta_1 g,\delta_2 g), \tag{10.30}$$

where $\mathbf{E}$ was computed for arbitrary higher-derivative Lagrangians by Ref. [87]. We provide two equivalent expressions for $\mathbf{E}$:

$$\begin{aligned}\mathbf{E}(g,\delta_1 g,\delta_2 g) &= \boldsymbol{\epsilon}_{\mu\nu}\frac{1}{2}\left[-\frac{3}{2}P^{\mu\nu\rho\alpha}g^{\sigma\beta}+2P^{\mu\rho\sigma\alpha}g^{\nu\beta}\right]\\ &\quad\times\delta_1 g_{\rho\sigma}\delta_2 g_{\alpha\beta}-[1\leftrightarrow 2]\\ &= \boldsymbol{\epsilon}_{\mu\nu}\left[-\frac{3}{2}P^{\mu\nu\rho\alpha}g^{\sigma\beta}+P^{\mu\rho\sigma\alpha}g^{\nu\beta}-P^{\mu\alpha\beta\rho}g^{\nu\sigma}\right]\\ &\quad\times\delta_1 g_{\rho\sigma}\delta_2 g_{\alpha\beta}. \end{aligned} \tag{10.31}$$

Now, by adding the term "$-d\mathbf{E}(g,\delta g,\mathcal{L}_\xi g)$" on both sides of the equation (10.24), one can derive a new fundamental identity for the Barnich-Compère symplectic current (10.30), if one redefines the surface charge (10.25) as

$$\mathbf{k}_\xi^{\rm BB}(g,\delta g)\equiv\mathbf{k}_\xi^{\rm IW}(g,\delta g)-\mathbf{E}(g,\delta g,\mathcal{L}_\xi g), \tag{10.32}$$

where $\mathbf{k}_\xi^{\rm BB}$ is called the *Barnich-Brandt surface charge*. Notice that for exact Killing vectors, i.e., $\mathcal{L}_\xi g=0$, the Iyer-Wald and Barnich-Brandt definitions of the surface charge are equivalent. In the rest of the paper, especially in Appendix E, we restrict again to the Lee-Wald-Iyer proposals for $\boldsymbol{\omega}$ and $\mathbf{k}_\xi$.

F. List of ambiguities

In the previous epigraphs, we have given the "canonical" formulas for the relevant quantities in Wald's formalism. However, these quantities are not uniquely defined. Let us present here a list of all the corresponding ambiguities. The symplectic potential $\boldsymbol{\Theta}$ and the Noether charge $\mathbf{Q}_\xi$ are defined by (10.3) and (10.21), respectively, up to a closed —and hence locally exact—form, denoted by $d\mathbf{Y}$ and $d\mathbf{Z}$, respectively. Moreover, one can add a total derivative $d\boldsymbol{\mu}$ to the Lagrangian without changing the equations of motion. These ambiguities $\mathbf{Y}$, $\mathbf{Z}$, and $\boldsymbol{\mu}$ also give rise to ambiguities in the other relevant quantities. The full list reads [42]

$$\mathbf{L}\to\mathbf{L}+d\boldsymbol{\mu}, \tag{10.33}$$

$$\boldsymbol{\Theta}\to\boldsymbol{\Theta}+\delta\boldsymbol{\mu}+d\mathbf{Y}(g,\delta g), \tag{10.34}$$

$$\boldsymbol{\omega}\to\boldsymbol{\omega}+d(\delta_1\mathbf{Y}(g,\delta_2 g)-\delta_2\mathbf{Y}(g,\delta_1 g)), \tag{10.35}$$

$$\mathbf{J}_\xi\to\mathbf{J}_\xi+d(\xi\cdot\boldsymbol{\mu})+d\mathbf{Y}(g,\mathcal{L}_\xi g), \tag{10.36}$$

$$\mathbf{Q}_\xi\to\mathbf{Q}_\xi+\xi\cdot\boldsymbol{\mu}+\mathbf{Y}(g,\mathcal{L}_\xi g)+d\mathbf{Z}, \tag{10.37}$$

$$\begin{aligned}\mathbf{k}_\xi\to\mathbf{k}_\xi&+\delta^{[g]}\mathbf{Y}(g,\mathcal{L}_\xi g)-\mathcal{L}_\xi\mathbf{Y}(g,\delta g)\\ &+d(\delta^{[g]}\mathbf{Z}+\xi\cdot\mathbf{Y}(g,\delta g)),\end{aligned} \tag{10.38}$$

where the arrows mean that the expressions on the rhs are also compatible with the corresponding definitions. We have seen above that for exact Killing vectors, the integral of $\mathbf{k}_\xi$ is conserved. Moreover, here we observe that the integral of this form over a $(D-2)$-dimensional spacelike compact submanifold is unambiguous for Killing vectors, since in that case, the total derivative does not contribute and we have [42]

$$\delta^{[g]}\mathbf{Y}(g,\mathcal{L}_\xi g)=\mathbf{Y}(g,\mathcal{L}_\xi\delta g)=\mathcal{L}_\xi\mathbf{Y}(g,\delta g), \tag{10.39}$$

because $\mathcal{L}_\xi g=0$. Furthermore, we note that the Barnich-Compère symplectic current (10.30) and the Barnich-Brandt surface charge (10.32) do not fall within the class of ambiguities of the Wald definitions, Eqs. (10.35) and (10.38), respectively. This is because the form $\mathbf{E}(g,\delta_1 g,\delta_2 g)$ cannot be written as $\delta_1\mathbf{Y}(g,\delta_2 g)-\delta_2\mathbf{Y}(g,\delta_1 g)$, although it was previously suggested in Ref. [99] that this could be done. Thus, the proposals by Barnich-Brandt-Compère and Lee-Wald-Iyer for $\boldsymbol{\omega}$ and $\mathbf{k}_\xi$ are distinct. Which proposal is more appropriate seems to depend on the problem.[31]

XI. FINAL COMMENTS

In this paper, we have presented a collection of new results on $\mathcal{L}$(Riemann) theories of gravity. A summary of our findings can be found in Sec. I A.

Before closing, we would like to point out that one of our motivations to study the linearized spectrum of this class of theories came from the following observations. In Refs. [19,62], the authors constructed a cubic theory admitting analytic extensions of the Schwarzschild-AdS black hole characterized by a single function. Remarkably, they noticed that this theory—which was coined *quasitopological gravity*[32]—has the same linearized spectrum as Einstein gravity; i.e., it falls in the Einstein-like category considered in Sec. III—see Appendix B. In fact, as far as we know, all the known examples of higher-order gravities[33] for which nontrivial analytic black-hole solutions—generalizing the corresponding Einstein gravity ones—have been constructed for generic values of the coupling[34] fall into the Einstein-like category; this includes quasitopological gravity [19,62] and its generalizations to higher curvatures, e.g., Ref. [103], and Lovelock theories [104–109]. In all those cases, if we restrict to static and spherically symmetric solutions—and analogously for planar or hyperbolic horizons—a single function determines the corresponding metric—e.g., for Schwarzschild, $f(r)=1-2M/r$ in the usual coordinates. This is as opposed to black-hole solutions of theories which do not

[31] We thank Geoffrey Compère for clarifying this point.

[32] Note that, as opposed to, e.g., ECG, quasitopological gravity is defined in a dimension-dependent fashion.

[33] In this statement, we are referring to purely gravitational metric theories.

[34] The situation changes if one allows for fine-tuned couplings —see, e.g., Refs. [63,100,101]. Another possibility is considering theories which do not reduce to Einstein gravity when the corresponding couplings vanish, like pure R^2 gravity, e.g., Ref. [102]. We find these setups considerably less interesting.

belong to the Einstein-like class, e.g., Refs. [110,111], for which two independent functions are needed and generally can only be accessed numerically or in certain limits. This suggests the possibility of finding simple analytic extensions of Einstein's gravity black holes for that class of theories. Furthermore, it is natural to expect that only theories that do not propagate the extra scalar and the ghostlike graviton at the linearized level are susceptible to admitting extensions of Schwarzschild's solution with a single blackening factor. Additional evidence in favor of these claims coming from ECG was recently reported in Refs. [29,112–114]. A general study for arbitrary $\mathcal{L}$(Riemann) theories is also in progress.

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APPENDIX A: LINEARIZATION PROCEDURE: EXAMPLES

In this Appendix, we apply the linearization procedure explained in Sec. II to two instances. The first is a general quadratic theory in D dimensions, for which we give details of all the steps involved in the linearization process. The second is a Born-Infeld gravity. Our goal in that case is to illustrate that our method can be easily applied to theories of which the linearization would be difficult to achieve using different methods.

1. Quadratic gravity

Let us consider the most general quadratic gravity in general dimensions,

$$S = \int_{\mathcal{M}} d^D x\sqrt{|g|}\left\{\frac{1}{2\kappa}(-2\Lambda_0 + R) + \kappa^{\frac{4-D}{D-2}}(\alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho})\right\}. \tag{A1}$$

In order to obtain $\mathcal{L}(\Lambda,\alpha)$, we only have to substitute the Riemann tensors appearing in the above Lagrangian density by the expression (2.20) and use the algebraic properties of the auxiliary tensor $k_{\mu\nu}$ (2.19) to compute all the contractions. We find

$$\begin{aligned} R^2|_{(\Lambda,\alpha)} &= \Lambda^2 D^2(D-1)^2 + 2\Lambda\alpha D(D-1)\chi(\chi-1) + \alpha^2\chi^2(\chi-1)^2,\\ R_{\mu\nu}R^{\mu\nu}|_{(\Lambda,\alpha)} &= \Lambda^2 D(D-1)^2 + 2\Lambda\alpha(D-1)\chi(\chi-1) + \alpha^2\chi(\chi-1)^2,\\ R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}|_{(\Lambda,\alpha)} &= 2D(D-1)\Lambda^2 + 4\Lambda\alpha\chi(\chi-1) + 2\alpha^2\chi(\chi-1). \end{aligned} \tag{A2}$$

The final result for $\mathcal{L}(\Lambda,\alpha)$ reads

$$\begin{aligned}\mathcal{L}(\Lambda,\alpha) = &+\frac{1}{2\kappa}(-2\Lambda_0 + \Lambda D(D-1) + \alpha\chi(\chi-1))\\ &+ \kappa^{\frac{4-D}{D-2}}(\Lambda^2 D(D-1) + 2\Lambda\alpha\chi(\chi-1))\\ &\times(D(D-1)\alpha_1 + (D-1)\alpha_2 + 2\alpha_3)\\ &+ \kappa^{\frac{4-D}{D-2}}\alpha^2\chi(\chi-1)\\ &\times(\chi(\chi-1)\alpha_1 + (\chi-1)\alpha_2 + 2\alpha_3). \end{aligned} \tag{A3}$$

Then, applying (2.22), we get

$$e = \frac{1}{4\kappa} + \Lambda\kappa^{\frac{4-D}{D-2}}(D(D-1)\alpha_1 + (D-1)\alpha_2 + 2\alpha_3). \tag{A4}$$

The second derivative with respect to α yields

$$\frac{\partial^2\mathcal{L}}{\partial\alpha^2} = 2\chi(\chi-1)\kappa^{\frac{4-D}{D-2}}[\chi(\chi-1)\alpha_1 + (\chi-1)\alpha_2 + 2\alpha_3]. \tag{A5}$$

Hence, comparing with (2.23), we can easily obtain the values of a, b, and c. The result is

$$a = \kappa^{\frac{4-D}{D-2}}\alpha_3, \qquad b = \frac{\kappa^{\frac{4-D}{D-2}}\alpha_1}{2}, \qquad c = \frac{\kappa^{\frac{4-D}{D-2}}\alpha_2}{2}. \tag{A6}$$

Inserting the values of a, b, c, and e into (2.28)–(2.30) gives rise to Eqs. (2.32)–(2.34) for $\kappa_{\rm eff}$, m_s^2, and m_g^2.

Finally, from (2.14), we see that the cosmological constant is related to the background scale Λ and the couplings of the theory through

$$\Lambda_0 = \frac{(D-1)(D-2)\Lambda}{2} + \kappa^{\frac{2}{D-2}}\Lambda^2(D-4)(D-1) \times [D(D-1)\alpha_1 + (D-1)\alpha_2 + 2\alpha_3]. \quad \text{(A7)}$$

2. Born-Infeld gravity

Let us now consider the following theory, which has the form of a Born-Infeld model,

$$S = \frac{1}{\kappa^{\frac{D}{D-2}}(1+\lambda)^{\frac{D-2}{2}}} \times \int_{\mathcal{M}} d^D x\left[\sqrt{|g_{\mu\nu}(1+\lambda) + \kappa^{\frac{2}{D-2}}R_{\mu\nu}|} - \sqrt{|g_{\mu\nu}|}\right], \quad \text{(A8)}$$

where $|A_{\mu\nu}|$ stands for the absolute value of the determinant and λ is a dimensionless parameter—which we assume to be greater than -1. The normalization is chosen so that to leading order, the action becomes Einstein-Hilbert,

$$S = \frac{1}{2\kappa}\int_{\mathcal{M}} d^D x\sqrt{|g|}[-2\Lambda_0 + R + \cdots], \quad \text{(A9)}$$

where $\Lambda_0 = [(1+\lambda)^{1-D/2} - (1+\lambda)]\kappa^{\frac{2}{2-D}}$ and the ellipsis means an infinite series of higher-order terms in curvature. Linearizing this theory can be a nontrivial task, due to the presence of the determinant and the square root. Using our method, it becomes quite easy, though. First, extracting as a common factor the square root of the metric determinant,[35] we find the Lagrangian density

$$\kappa^{\frac{D}{D-2}}(1+\lambda)^{\frac{D-2}{2}}\mathcal{L} = \sqrt{|(1+\lambda)\delta^\mu{}_\nu + \kappa^{\frac{2}{D-2}}R^\mu{}_\nu|} - 1. \quad \text{(A10)}$$

Now, we follow our recipe and substitute the Riemann tensor (2.20) in this expression:

$$\kappa^{\frac{D}{D-2}}(1+\lambda)^{\frac{D-2}{2}}\mathcal{L}(\Lambda,\alpha) = \sqrt{|(1+\lambda+\kappa^{\frac{2}{D-2}}\Lambda(D-1))\delta^\mu{}_\nu + \alpha\kappa^{\frac{2}{D-2}}(\chi-1)k^\mu{}_\nu|} - 1. \quad \text{(A11)}$$

The determinant can be computed using (2.19) and the identity

$$|A| = e^{\mathrm{tr}(\log A)}. \quad \text{(A12)}$$

The result is

$$\kappa^{\frac{D}{D-2}}(1+\lambda)^{\frac{D-2}{2}}\mathcal{L}(\Lambda,\alpha) = (1+\lambda+\kappa^{\frac{2}{D-2}}\Lambda(D-1))^{D/2} \times \left(1 + \frac{\alpha\kappa^{\frac{2}{D-2}}(\chi-1)}{1+\lambda+\kappa^{\frac{2}{D-2}}\Lambda(D-1)}\right)^{\chi/2} - 1. \quad \text{(A13)}$$

This "prepotential" contains all the information about the linearized theory. Let us begin by determining Λ. The equation for the background curvature (2.14) becomes

$$[1+\lambda+\kappa^{\frac{2}{D-2}}\Lambda(D-1)]^{D/2} - 1 = \kappa^{\frac{2}{D-2}}\Lambda(D-1)[1+\lambda+\kappa^{\frac{2}{D-2}}\Lambda(D-1)]^{D/2-1}. \quad \text{(A14)}$$

A simple algebraic manipulation yields

$$1 = (1+\lambda)[1+\lambda+\kappa^{\frac{2}{D-2}}\Lambda(D-1)]^{D/2-1}. \quad \text{(A15)}$$

Thus, since we have assumed $\lambda > -1$, this equation has always one solution:

$$\Lambda = \frac{1}{\kappa^{\frac{2}{D-2}}(D-1)}[(1+\lambda)^{-2/(D-2)} - (1+\lambda)]. \quad \text{(A16)}$$

Now, we can compute the parameters a, b, c, and e. From (2.22), we get

$$e = \frac{1}{4\kappa}(1+\lambda)^{-D/2}, \quad \text{(A17)}$$

where we already evaluated the expression on the background. On the other hand, the second derivative of $\mathcal{L}(\Lambda,\alpha)$ with respect to α evaluated at $\alpha = 0$ yields

$$\frac{1}{4\chi(\chi-1)}\frac{\partial^2\mathcal{L}}{\partial\alpha^2}\bigg|_{\alpha=0} = \frac{1}{16}\kappa^{\frac{4-D}{D-2}}(\chi-1)(\chi-2)(1+\lambda)^{-\frac{D^2-2D-4}{2(D-2)}}, \quad \text{(A18)}$$

where we have also made use of (A16). Now, comparing this expression with (2.23), we find the values of the parameters, namely,

$$a = 0, \qquad b = \frac{1}{16}\kappa^{\frac{4-D}{D-2}}(1+\lambda)^{-\frac{D^2-2D-4}{2(D-2)}}, \qquad c = -\frac{1}{8}\kappa^{\frac{4-D}{D-2}}(1+\lambda)^{-\frac{D^2-2D-4}{2(D-2)}}. \quad \text{(A19)}$$

Finally, using (2.28)–(2.30), we can compute the physical parameters $\kappa_{\rm eff}$, m_s, and m_g,

$$\kappa_{\rm eff} = \kappa(1+\lambda)^{D/2}, \qquad m_s^2 = 2(1+\lambda)\kappa^{\frac{2}{2-D}}, \qquad m_g^2 = 2(1+\lambda)^{-2/(D-2)}\kappa^{\frac{2}{2-D}}. \quad \text{(A20)}$$

Therefore, we have completely characterized the linearized spectrum of this Born-Infeld model. Since we assumed that

[35]We use that $|A_{\mu\nu}| = |g_{\mu\alpha}A^\alpha{}_\nu| = |g_{\mu\nu}||A^\alpha{}_\beta|$.

$\lambda > -1$, all quantities are finite and real, and everything is well defined. For $D > 2$, the background (A16) is dS ($\Lambda > 0$) when $\lambda < 0$, AdS ($\Lambda < 0$) when $\lambda > 0$, and flat when $\lambda = 0$. In all cases, we have, apart from the massless graviton, a massive scalar and a massive spin-2 graviton. The masses squared and the effective gravitational constant are always positive.

APPENDIX B: CLASSIFICATION OF THEORIES: EXAMPLES

In this Appendix, we provide numerous examples of the different classes of theories characterized in Sec. III.

1. Theories without massive graviton

In Sec. VII, we characterized all theories being defined in a dimension-independent manner which do not propagate the extra massive graviton up to cubic order in curvature. The list of theories reduced to the particular f(Lovelock) terms, ECG (6.2) plus a new invariant, $\mathcal{Y}$, which we defined in (7.6). In this Appendix, we will study general f(Lovelock) theories, which—although not necessarily defined in a dimension-independent way—are a paradigmatic example of theories which only propagate the usual massless graviton plus the scalar at the linearized level [63].

2. f(Lovelock) gravities

The most general f(Lovelock) action can be written as

$$S = \frac{1}{2\kappa}\int_{\mathcal{M}} d^D x\sqrt{|g|}f(\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{\lfloor D/2 \rfloor}), \qquad \text{(B1)}$$

where f is some differentiable function of the dimensionally extended Euler densities[36]

$$\mathcal{L}_k \equiv \frac{1}{2^k}\delta^{\mu_1\nu_1\ldots\mu_k\nu_k}_{\alpha_1\beta_1\ldots\alpha_k\beta_k} R_{\mu_1\nu_1}{}^{\alpha_1\beta_1}\cdots R_{\mu_k\nu_k}{}^{\alpha_k\beta_k}, \qquad \text{(B2)}$$

where the generalized Kronecker symbol is defined as $\delta^{\mu_1\nu_1\ldots\mu_k\nu_k}_{\alpha_1\beta_1\ldots\alpha_k\beta_k} \equiv (2k)!\delta^{[\mu_1}_{\alpha_1}\delta^{\nu_1}_{\beta_1}\cdots\delta^{\mu_k}_{\alpha_k}\delta^{\nu_k]}_{\beta_k}$. Note that the first three densities are nothing but a constant that can be identified with the cosmological constant $\mathcal{L}_0 \equiv -2\Lambda_0$; the Einstein-Hilbert term, $\mathcal{L}_1 \equiv R$; and the Gauss-Bonnet gravity, $\mathcal{L}_2 \equiv \mathcal{X}_4$. A corollary from the results presented in Sec. V is that f(Lovelock) theories inherit the property of Lovelock gravities of not propagating the massive graviton[37] This means that the linearized equations of motion for f(Lovelock) gravities should not involve the $\bar{\Box}G^L_{\mu\nu}$ term. This is indeed the case. In particular, they read [63]

$$\mathcal{E}^L_{\mu\nu} = \alpha G^L_{\mu\nu} + \Lambda\beta\bar{g}_{\mu\nu}R^L + \frac{\beta}{D-1}(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\nu\bar{\nabla}_\mu)R^L = 0, \qquad \text{(B3)}$$

where α and β are the following constants[38]:

$$\alpha \equiv \frac{1}{2\kappa}\sum_{k=1}^{\lfloor D/2 \rfloor}\partial_k f(\bar{\mathcal{L}})\frac{k(D-3)!}{(D-2k-1)!}\Lambda^{k-1}, \qquad \text{(B4)}$$

$$\beta \equiv \frac{1}{2\kappa}\sum_{k,l=1}^{\lfloor D/2 \rfloor}\partial_k\partial_l f(\bar{\mathcal{L}})\frac{kl(D-2)!(D-1)!}{(D-2k)!(D-2l)!}\Lambda^{k+l-2}. \qquad \text{(B5)}$$

Here, $\partial_l f(\bar{\mathcal{L}})$ means that we should take a formal derivative of f with respect to the corresponding dimensionally extended Euler density and then evaluate the result in the background. Comparing with the linearized equations (3.2), we see that α determines the effective Einstein constant κ_{eff} and β is related to the mass of the scalar field

$$\kappa_{\text{eff}} = \frac{1}{2\alpha}, \qquad m_s^2 = \frac{D-2-2\beta D\Lambda}{2\beta}. \qquad \text{(B6)}$$

Note that for $\beta = 0$, the scalar mode is also absent, and the only physical field is the massless graviton. This applies, e.g., to pure Lovelock gravities and also to other nontrivial theories [63]—some of which we review in the last epigraph of this section. The parameters a, b, c, and e are given by

$$a = -\frac{1}{2}c = -\frac{\alpha-2e}{4(D-3)\Lambda}, \quad b = \frac{\beta}{4(D-1)} - \frac{\alpha-2e}{8(D-3)\Lambda},$$
$$e = \frac{f(\bar{\mathcal{L}})}{8\kappa\Lambda(D-1)}, \qquad \text{(B7)}$$

and the background embedding equation (2.14) reads in turn

$$f(\bar{\mathcal{L}}) = \sum_{k=1}^{\lfloor D/2 \rfloor}\frac{2k(D-1)!}{(D-2k)!}\Lambda^k\partial_k f(\bar{\mathcal{L}}). \qquad \text{(B8)}$$

An interesting subclass we shall not consider here is that of Lovelock-Chern-Simons theory [115,116], which is a particular case of the Lovelock theory. This is most naturally defined in general dimensions in terms of the tetrad and the spin connection. Their corresponding equations are first order, and when the torsion is set to zero, the metric field equations become second order, and the theory is a particular case of the Lovelock action considered in this paper, i.e., with a metric-compatible connection. In the latter case, the

[36]Namely, $\mathcal{L}_k$ becomes the Euler density when evaluated for a $2k$-dimensional manifold.

[37]In Appendix C, we show how the linearized equations of $f(R)$ can be obtained from those of Einstein gravity. The procedure can be naturally applied as well to f(Lovelock) theories starting from Lovelock, and the results will match the ones presented in this Appendix.

[38]Note that $\lfloor D/2 \rfloor$ stands for the largest integer smaller than or equal to $D/2$.

degrees of freedom propagated by the theory on a msb are of course the $D(D-3)/2$ of the usual massless graviton. Interestingly, if the torsionless condition is relaxed, the number of dynamical degrees of freedom is in fact greater—see, e.g., Ref. [117].

3. Theories without dynamical scalar

In the case of quadratic gravity, the most general theory which does not propagate a scalar field is [118]

$$S=\int_{\mathcal{M}}d^Dx\sqrt{|g|}\left\{\frac{1}{2\kappa}(-2\Lambda_0+R)+\kappa^{\frac{4-D}{D-2}}\left[\beta\left(R^2-\frac{4(D-1)}{D}R_{\mu\nu}R^{\mu\nu}\right)+\gamma\mathcal{X}_4\right]\right\},\tag{B9}$$

where $\mathcal{X}_4$ is again the Gauss-Bonnet term and β and γ are dimensionless constants. Observe that for $D=3$, this action is equivalent to *new massive gravity* [119]. There are two different interesting ways of writing this theory in terms of other well-known curvature tensors. First, it was observed in Ref. [120] that the contraction of the Einstein tensor $G_{\mu\nu}$ with the Schouten tensor[39] $S_{\mu\nu}$ is proportional to the curvature invariant in (B9) that multiplies β,

$$G_{\mu\nu}S^{\mu\nu}=-\frac{D}{4(D-2)(D-1)}\left(R^2-\frac{4(D-1)}{D}R_{\mu\nu}R^{\mu\nu}\right).\tag{B10}$$

Therefore, by rescaling β, we see that the theory is equivalent to

$$S=\int_{\mathcal{M}}d^Dx\sqrt{|g|}\left\{\frac{1}{2\kappa}(-2\Lambda_0+R)+\kappa^{\frac{4-D}{D-2}}(\bar{\beta}G_{\mu\nu}S^{\mu\nu}+\gamma\mathcal{X}_4)\right\}.\tag{B11}$$

Second, it turns out that the quadratic part of (B9) is equivalent to the higher-dimensional version of conformal gravity, consisting of the square of the Weyl tensor, together with a Gauss-Bonnet term. The square of the Weyl tensor is in fact equal to[40]

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}=\mathcal{X}_4-\frac{D(D-3)}{(D-2)(D-1)}\times\left(R^2-\frac{4(D-1)}{D}R_{\mu\nu}R^{\mu\nu}\right).\tag{B12}$$

By using this relation and redefining the couplings, the theory can be written as

$$S=\int_{\mathcal{M}}d^Dx\sqrt{|g|}\times\left\{\frac{1}{2\kappa}(-2\Lambda_0+R)+\kappa^{\frac{4-D}{D-2}}(\tilde{\beta}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}+\tilde{\gamma}\mathcal{X}_4)\right\}.\tag{B13}$$

Thus, we observe that conformal gravity in any dimension is free of the scalar mode and only propagates the two gravitons. Finally, for this theory, the effective gravitational constant and the mass of the extra graviton read, respectively,

$$\kappa_{\rm eff}=\frac{\kappa}{1-4\kappa^{\frac{2}{D-2}}\Lambda(D-3)(2\tilde{\beta}-\tilde{\gamma}(D-4))},\tag{B14}$$

$$m_g^2=\frac{2-D+4\kappa^{\frac{2}{D-2}}\Lambda(D-3)(D-2)(2\tilde{\beta}-\tilde{\gamma}(D-4))}{8\tilde{\beta}\kappa^{\frac{2}{D-2}}(D-3)}.\tag{B15}$$

If the numerator of (B15) becomes zero, then the extra graviton is massless. This particular case will be analyzed in the epigraph on critical gravities. Note finally that in $D=3$ both the Weyl tensor and the Gauss-Bonnet term vanish identically, so the theory reduces to Einstein gravity plus a cosmological constant.

4. Theories with two massless gravitons

The following is an example of a theory propagating two massless gravitons in addition to the scalar field:

$$S=\int_{\mathcal{M}}d^Dx\sqrt{|g|}\left\{\frac{1}{2\kappa}(-2\Lambda_0+R)+\kappa^{\frac{4-D}{D-2}}\alpha R^2-D\left(\kappa^{\frac{4-D}{D-2}}\alpha+\frac{1}{16\kappa\Lambda_0}\right)R_{\mu\nu}R^{\mu\nu}\right\}.\tag{B16}$$

Note that the $m_g^2=0$ condition has the unpleasant feature of mixing the couplings of terms of different order in curvature. In this case, we see that the $R_{\mu\nu}R^{\mu\nu}$ coupling depends on the combination $\kappa\Lambda_0$. For this theory, the background scale is related to the cosmological constant by

$$\Lambda=\frac{4\Lambda_0}{D(D-1)}.\tag{B17}$$

In addition, the effective gravitational constant and the mass of the scalar field read

$$\hat{\kappa}_{\rm eff}=\frac{2(D-1)\kappa\Lambda}{1+4\Lambda\kappa^{\frac{2}{D-2}}\alpha D(D-1)},\quad m_s^2=-\frac{4(D-1)\Lambda}{D+4\Lambda\kappa^{\frac{2}{D-2}}\alpha(D-1)(D-2)^2}.\tag{B18}$$

[39] The Schouten tensor is defined as $S_{\mu\nu}\equiv\frac{1}{D-2}(R_{\mu\nu}-\frac{1}{2(D-1)}Rg_{\mu\nu})$.

[40] The Weyl tensor is defined as $C_{\mu\nu\rho\sigma}\equiv R_{\mu\nu\rho\sigma}-\frac{2}{D-2}(g_{\mu[\rho}R_{\sigma]\nu}-g_{\nu[\rho}R_{\sigma]\mu})+\frac{2}{(D-1)(D-2)}Rg_{\mu[\rho}g_{\sigma]\nu}$.

As far as we know, this theory has not been considered before.

5. Critical gravities

Critical gravity was introduced in Ref. [27] as the four-dimensional quadratic theory for which the extra graviton is massless and the scalar mode is absent. Hence, it is a special case of the theories considered in the last two epigraphs—(B9) and (B16)—in the particular case of $D = 4$. The following action is a generalization of critical gravity to general dimensions [120]:

$$S = \int_{\mathcal{M}} d^D x\sqrt{|g|}\left\{\frac{1}{2\kappa}(-2\Lambda_0 + R) - \frac{D^2}{16\kappa\Lambda_0(D-2)^2} \times\left(R^2 - \frac{4(D-1)}{D}R_{\mu\nu}R^{\mu\nu}\right)\right\}. \tag{B19}$$

It can be obtained by setting $\beta = -D^2/(16\kappa^{2/(D-2)} \times \Lambda_0(D-2)^2)$ and $\gamma = 0$ in (B9) or, alternatively, from (B16) if we put $\alpha = -D^2/(16\kappa^{2/(D-2)}\Lambda_0(D-2)^2)$ there. In $D = 4$, this is the critical theory considered by Ref. [27], and for $D = 3$, it is equivalent to *critical new massive gravity* with a cosmological constant [121]. Furthermore, the effective gravitational constant of this theory is

$$\hat{\kappa}_{\rm eff} = -\frac{1}{2}(D-2)^2\kappa\Lambda, \tag{B20}$$

which is only positive for $\Lambda < 0$.

6. Einstein-like theories

In Sec. VI, we already constructed examples of Einstein-like theories in the sense defined in Sec. III, i.e., theories which only propagate a massless graviton on a msb. However, the theories considered in that section had the additional property of being defined in a dimension-independent manner, and we coined them Einsteinian. In this Appendix, we would like to present some more examples of Einstein-like theories of which the definition does, however, depend on the space-time dimension.

Quasitopological gravity.—The first example is *quasi-topological gravity* [18,19,62]. This is a cubic theory which has the nice property of admitting analytic black-hole solutions—which generalize Schwarzschild-AdS and its Gauss-Bonnet generalization [107]. It consists of a combination of all Lovelock gravities up to cubic order plus an additional quasitopological term:

$$S = \int_{\mathcal{M}} d^D x\sqrt{|g|}\left\{\frac{1}{2\kappa}(-2\Lambda_0 + R) + \kappa^{\frac{4-D}{D-2}}\alpha\mathcal{X}_4 + \kappa^{\frac{6-D}{D-2}}[\beta\mathcal{X}_6 + \gamma\mathcal{Z}]\right\}. \tag{B21}$$

Here, the cubic Lovelock term is given by

$$\begin{aligned}\mathcal{X}_6 \equiv{}& -8R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma}R_{\rho}{}^{\delta}{}_{\sigma}{}^{\gamma}R_{\delta}{}^{\mu}{}_{\gamma}{}^{\nu} + 4R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\delta\gamma}R_{\delta\gamma}{}^{\mu\nu} - 24R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\delta}R^{\sigma\delta}\\ &+ 3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}R + 24R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + 16R_{\mu}{}^{\nu}R_{\nu}{}^{\rho}R_{\rho}{}^{\mu} - 12R_{\mu\nu}R^{\mu\nu}R + R^3,\end{aligned} \tag{B22}$$

and the quasitopological one in general dimensions reads in turn [19,62]

$$\begin{aligned}\mathcal{Z} \equiv{}& R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma}R_{\rho}{}^{\delta}{}_{\sigma}{}^{\gamma}R_{\delta}{}^{\mu}{}_{\gamma}{}^{\nu} + \frac{1}{(2D-3)(D-4)}\left(-3(D-2)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\delta}R^{\sigma\delta} + \frac{3(3D-8)}{8}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}R + 3DR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma}\right.\\ &\left. + 6(D-2)R_{\mu}{}^{\nu}R_{\nu}{}^{\rho}R_{\rho}{}^{\mu} - \frac{3(3D-4)}{2}R_{\mu\nu}R^{\mu\nu}R + \frac{3D}{8}R^3\right).\end{aligned} \tag{B23}$$

The physical quantities for (B21) read

$$\kappa_{\rm eff} = \frac{\kappa}{f(\alpha,\beta,\gamma,\Lambda,\kappa)}, \qquad m_s = +\infty, \qquad m_g = +\infty, \tag{B24}$$

where

$$\begin{aligned}f(\alpha,\beta,\gamma,\Lambda,\kappa) \equiv{}& +1 + 4\Lambda\kappa^{\frac{2}{D-2}}\alpha(D-4)(D-3)\\ &+ 6\kappa^{\frac{4}{D-2}}\Lambda^2\beta(D-6)(D-5)(D-4)(D-3)\\ &+ \frac{3(D-6)(D-3)}{4(2D-3)}\kappa^{\frac{4}{D-2}}\Lambda^2\gamma(16 + 3D(D-5)).\end{aligned}$$

Hence, as explained in Ref. [19], this theory shares the linearized spectrum of Einstein gravity. Let us close this section by mentioning that a quartic version of quasitopological gravity was constructed in Ref. [103]. It would be interesting to use our results in Sec. IV to check that such theory also presents an Einstein-like spectrum.

Special $f(\text{Lovelock})$ *theories*.—The second example we would like to consider corresponds to a particular family of $f(\text{Lovelock})$ gravities. As we explained before, all $f(\text{Lovelock})$ theories are free of the massive graviton but do in general propagate the extra scalar. However, as pointed out in Ref. [63], it is possible to construct nontrivial theories—i.e., different from the pure Lovelock case—which are also free of the extra scalar and hence share the linearized spectrum of Einstein gravity.

Indeed, whenever β, as defined in (B5), vanishes, the mass of the scalar diverges—which is obvious from (B6). This is achieved whenever $\partial_k\partial_l f(\bar{\mathcal{L}})=0$ for all k, l, which leaves us with nothing but the usual Lovelock theory or, alternatively, if

$$\sum_{k,l=1}^{\lfloor D/2\rfloor}\partial_k\partial_l f(\bar{\mathcal{L}})\frac{kl(D-2)!(D-1)!}{(D-2k)!(D-2l)!}\Lambda^{k+l-2}=0,\qquad \partial_k\partial_l f(\bar{\mathcal{L}})\neq 0, \tag{B25}$$

for some k, l. This equation is, e.g., satisfied by all theories of the form [63]

$$S=\int_{\mathcal{M}}d^Dx\sqrt{|g|}\times\left\{\frac{1}{2\kappa}(-2\Lambda_0+R)+\kappa^{\frac{2(u+2s)-D}{D-2}}\lambda(R^u\mathcal{L}_2^s-\gamma R^{2s+u})\right\}, \tag{B26}$$

where γ is the dimension-dependent constant

$$\gamma\equiv\frac{u^2+4(s-1)s+u(4s-1)}{(u+2s)(u+2s-1)}\frac{(D-2)^s(D-3)^s}{D^s(D-1)^s}, \tag{B27}$$

for any $u,s\geq 0$. In particular, for $s=u=1$, one finds the cubic class of theories

$$S=\int_{\mathcal{M}}d^Dx\sqrt{|g|}\left\{\frac{1}{2\kappa}(-2\Lambda_0+R)+\kappa^{\frac{6-D}{D-2}}\lambda\left[R\mathcal{L}_2-\left(\frac{2(D-2)(D-3)}{3D(D-1)}\right)R^3\right]\right\}. \tag{B28}$$

The $D=4$ case of (B28) was also considered in Ref. [28] in a slightly different context. The effective gravitational constant of (B28) reads

$$\kappa_{\text{eff}}=\kappa[1+2(D-6)(D-3)(D-1)D\lambda\kappa^{\frac{4}{D-2}}\Lambda^2]^{-1}. \tag{B29}$$

APPENDIX C: f(scalars) THEORIES: EXAMPLES

Let us now illustrate how the expressions obtained in Sec. V can be used to easily compute the values of a, b, c, and e for theories consisting of functions of invariants, as long as we know the values of those parameters for the invariants themselves.

1. $f(R)$ gravity

Let us first consider $f(R)$ gravity, the Lagrangian in general dimensions of which we write as

$$S=\frac{1}{2\kappa}\int_{\mathcal{M}}d^Dx\sqrt{|g|}f(R). \tag{C1}$$

According to Table II, for R, we have $a=b=c=0$, $e=\frac{1}{2}$, and $\bar{R}=D(D-1)\Lambda$. Therefore, using the transformation rules (5.4) for the theory above, we have

$$a=c=0,\qquad b=\frac{1}{8\kappa}f''(\bar{R}),\qquad e=\frac{1}{4\kappa}f'(\bar{R}). \tag{C2}$$

Note that these expressions can also be easily obtained from the general $f(\text{Lovelock})$ ones (B7). Also, according to (2.14), the background curvature Λ is determined by the equation

$$f(\bar{R})=2(D-1)\Lambda f'(\bar{R}). \tag{C3}$$

If $f''(\bar{R})\neq 0$, we have a scalar mode with mass

$$m_s^2=\frac{(D-2)f'(\bar{R})-2\bar{R}f''(\bar{R})}{2(D-1)f''(\bar{R})}. \tag{C4}$$

The effective gravitational constant is in turn given by

$$\kappa_{\text{eff}}=\frac{\kappa}{f'(\bar{R})}. \tag{C5}$$

2. $f(R,R_{\mu\nu}^2,R_{\mu\nu\rho\sigma}^2)$ gravity

Let us now study all theories that can be constructed as functions of invariants up to quadratic order [122]. The independent scalars are R, $Q\equiv R_{\mu\nu}R^{\mu\nu}$, and $K\equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, so let us consider an action of the form

$$S=\frac{1}{2\kappa}\int_{\mathcal{M}}d^Dx\sqrt{|g|}f(R,Q,K). \tag{C6}$$

This theory includes, as particular cases, $f(R)$ and general quadratic gravities. In order to simplify the following expressions, let us write $\mathcal{R}\equiv(R,Q,K)$. Evaluated on the background, the invariants read

$$\bar{\mathcal{R}} = (D(D-1)\Lambda, D(D-1)^2\Lambda^2, 2D(D-1)\Lambda^2). \tag{C7}$$

Then, the background embedding equation (2.14) can be written in terms of these background scalars $\bar{R}$, $\bar{Q}$, $\bar{K}$ as

$$\bar{R}\partial_R f(\bar{\mathcal{R}}) + 2\bar{Q}\partial_Q f(\bar{\mathcal{R}}) + 2\bar{K}\partial_K f(\bar{\mathcal{R}}) = \frac{D}{2} f(\bar{\mathcal{R}}), \tag{C8}$$

which, in particular, generalizes (C3) for this theory. Finally, the parameters a, b, c, and e are given by

$$\begin{aligned}
a &= \frac{1}{2\kappa}\partial_K f(\bar{\mathcal{R}}),\\
b &= \frac{1}{2\kappa}\Big[\frac{1}{4}\partial_R\partial_R f(\bar{\mathcal{R}}) + (D-1)\Lambda\partial_R\partial_Q f(\bar{\mathcal{R}}) + 2\Lambda\partial_R\partial_K f(\bar{\mathcal{R}})\\
&\quad + (D-1)^2\Lambda^2\partial_Q\partial_Q f(\bar{\mathcal{R}}) + 4(D-1)\Lambda^2\partial_Q\partial_K f(\bar{\mathcal{R}}) + 4\Lambda^2\partial_K\partial_K f(\bar{\mathcal{R}})\Big],\\
c &= \frac{1}{4\kappa}\partial_Q f(\bar{\mathcal{R}}),\\
e &= \frac{1}{4\kappa}[\partial_R f(\bar{\mathcal{R}}) + 2(D-1)\Lambda\partial_Q f(\bar{\mathcal{R}}) + 4\Lambda\partial_K f(\bar{\mathcal{R}})],
\end{aligned} \tag{C9}$$

from which one can easily obtain the values of $\kappa_{\rm eff}$, m_s^2, and m_g^2.

APPENDIX D: EINSTEINIAN QUARTIC GRAVITIES

Here we provide the explicit expressions for the conditions $F_g^{(2)}(\alpha_i) = F_s^{(2)}(\alpha_i) = F_g^{(3)}(\beta_i, D) = F_s^{(3)}(\beta_i, D) = F_g^{(4)}(\gamma_i, D) = F_s^{(4)}(\gamma_i, D) = 0$ appearing in Sec. VI. These read

$$F_g^{(2)}(\alpha_i) \equiv +\frac{1}{2}\alpha_2 + 2\alpha_3 = 0, \tag{D1}$$

$$F_s^{(2)}(\alpha_i) \equiv +2\alpha_1 + \frac{1}{2}\alpha_2 = 0, \tag{D2}$$

$$F_g^{(3)}(\beta_i, D) \equiv -\frac{3}{2}\beta_1 + 12\beta_2 + 2D\beta_3 + 2D(D-1)\beta_4 + \left(D - \frac{3}{2}\right)\beta_5 + \frac{3}{2}(D-1)\beta_6 + \frac{1}{2}D(D-1)\beta_7 = 0, \tag{D3}$$

$$F_s^{(3)}(\beta_i, D) \equiv +\frac{3}{2}\beta_1 + 2\beta_3 + 8\beta_4 + \left(D + \frac{1}{2}\right)\beta_5 + \frac{3}{2}(D-1)\beta_6 + (D-1)\left(\frac{D}{2} + 4\right)\beta_7 + 6D(D-1)\beta_8 = 0, \tag{D4}$$

$$\begin{aligned}
F_g^{(4)}(\gamma_i, D) \equiv{}& +(4D-9)\gamma_1 + 2(D+3)\gamma_2 + (2D-9)\gamma_3 + 24\gamma_4 + 48\gamma_5 + 8\gamma_6\\
&+ 8D(D-1)\gamma_7 - \frac{1}{2}(D+3)\gamma_8 + 6(2D-1)\gamma_9 + (2D^2 - D - 3)\gamma_{10}\\
&- \frac{3}{2}D(D-1)\gamma_{11} + 12D(D-1)\gamma_{12} + \left(2D^2 + \frac{1}{2}D - 3\right)\gamma_{13}\\
&+ \frac{1}{2}(3D^2 - 8D + 6)\gamma_{14} + (2D^2 - 3)\gamma_{15} + (2D^2 + D - 3)\gamma_{16}\\
&+ D(D-1)(2D-1)\gamma_{17} + 2D^2(D-1)\gamma_{18} + 2D^2(D-1)^2\gamma_{19}\\
&+ (D-1)(2D-3)\gamma_{20} + \frac{1}{2}D(D-1)(2D-3)\gamma_{21} + 3(D-1)^2\gamma_{22}\\
&+ D(D-1)^2\gamma_{23} + \frac{3}{2}D(D-1)^2\gamma_{24} + \frac{1}{2}D^2(D-1)^2\gamma_{25} = 0.
\end{aligned} \tag{D5}$$

$$
\begin{aligned}
F_s^{(4)}(\gamma_i, D) \equiv\; & +7\gamma_1 + 2(D-1)\gamma_2 + 5\gamma_3 + 8\gamma_6 + 32\gamma_7 + \frac{5}{2}(D-1)\gamma_8 \\
& + 6\gamma_9 + 3(D-1)\gamma_{10} + \frac{3}{2}(D^2+3D-8)\gamma_{11} + 24\gamma_{12} + \frac{3}{2}(3D-2)\gamma_{13} \\
& + \frac{1}{2}(3D^2+4D-10)\gamma_{14} + (4D-1)\gamma_{15} + 5(D-1)\gamma_{16} \\
& + (D+16)(D-1)\gamma_{17} + 2(D+6)(D-1)\gamma_{18} + 20D(D-1)\gamma_{19} \\
& + (D-1)(2D+1)\gamma_{20} + \frac{1}{2}(D-1)(2D^2+13D-12)\gamma_{21} \\
& + 3(D-1)^2\gamma_{22} + (D-1)^2(D+8)\gamma_{23} + \frac{3}{2}(D-1)^2(D+4)\gamma_{24} \\
& + \frac{1}{2}D(D-1)^2(D+20)\gamma_{25} + 12D^2(D-1)^2\gamma_{26} = 0.
\end{aligned}
\tag{D6}
$$

Solving the last two equations order by order in D gives rise to the constraints which characterize Einsteinian quartic gravities (6.6).

APPENDIX E: WALD FORMALISM: EXAMPLES

In this Appendix, we evaluate explicitly the expressions found in Sec. X for some relevant theories, namely, Einstein gravity, $f(R)$ gravity, general quadratic gravities, and Lovelock theories. Note that the expressions below are valid for any background metric $g_{\mu\nu}$ and vector field ξ^μ. Some of these formulas—but not all of them—can also be found in Refs. [42,48,91–93,98,123,124]. The following identities are frequently used:

$$
\frac{\partial R_{\mu\alpha\beta\nu}}{\partial R_{\sigma\rho\lambda\eta}} = \frac{1}{2}[\delta_\mu^{[\sigma}\delta_\alpha^{\rho]}\delta_\beta^{[\lambda}\delta_\nu^{\eta]} + \delta_\mu^{[\lambda}\delta_\alpha^{\eta]}\delta_\beta^{[\sigma}\delta_\nu^{\rho]}],
\qquad
\frac{\partial R_{\rho\sigma}}{\partial R_{\mu\alpha\beta\nu}} = \delta^{[\alpha}_{(\rho} g^{\mu][\beta}\delta^{\nu]}_{\sigma)},
\tag{E1}
$$

$$
\frac{\partial R}{\partial R_{\mu\alpha\beta\nu}} = g^{\beta[\mu}g^{\alpha]\nu}, \qquad \delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta},
\qquad
\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}.
\tag{E2}
$$

1. Einstein gravity

$$
\mathbf{L} = \frac{1}{2\kappa}\boldsymbol{\epsilon}(-2\Lambda_0 + R),
\tag{E3}
$$

$$
P^{\mu\alpha\beta\nu} = \frac{1}{4\kappa}(g^{\mu\beta}g^{\alpha\nu} - g^{\mu\nu}g^{\alpha\beta}),
\tag{E4}
$$

$$
\mathcal{E}_{\mu\nu} = \frac{1}{2\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu}\right),
\tag{E5}
$$

$$
\mathbf{\Theta} = \frac{1}{2\kappa}\boldsymbol{\epsilon}_\mu(g^{\mu\beta}g^{\alpha\nu} - g^{\mu\nu}g^{\alpha\beta})\nabla_\nu\delta g_{\alpha\beta},
\tag{E6}
$$

$$
\boldsymbol{\omega} = \boldsymbol{\epsilon}_\mu S^{\mu\alpha\beta\nu\rho\sigma}(\delta_1 g_{\rho\sigma}\nabla_\nu\delta_2 g_{\alpha\beta} - \delta_2 g_{\rho\sigma}\nabla_\nu\delta_1 g_{\alpha\beta}),
\tag{E7}
$$

$$
S^{\mu\alpha\beta\nu\rho\sigma} = \frac{1}{2\kappa}\left[-g^{\mu(\alpha}g^{\beta)(\rho}g^{\sigma)\nu} + \frac{1}{2}g^{\mu(\alpha}g^{\beta)\nu}g^{\rho\sigma} + \frac{1}{2}g^{\alpha\beta}g^{\mu(\rho}g^{\sigma)\nu} + \frac{1}{2}g^{\mu\nu}g^{\alpha(\rho}g^{\sigma)\beta} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}g^{\rho\sigma}\right],
\tag{E8}
$$

$$
\mathbf{J}_\xi = \boldsymbol{\epsilon}_\mu\left[\frac{1}{\kappa}\nabla_\nu(\nabla^{[\nu}\xi^{\mu]}) + 2\mathcal{E}^\mu{}_\nu\xi^\nu\right],
\tag{E9}
$$

$$
\mathbf{Q}_\xi = -\frac{1}{2\kappa}\boldsymbol{\epsilon}_{\mu\nu}\nabla^\mu\xi^\nu,
\tag{E10}
$$

$$
\mathbf{k}_\xi = \frac{1}{2\kappa}\boldsymbol{\epsilon}_{\mu\nu}\left[\left(g^{\mu\alpha}\nabla^\beta\xi^\nu - \frac{1}{2}g^{\alpha\beta}\nabla^\mu\xi^\nu\right)\delta g_{\alpha\beta} + (g^{\mu\alpha}g^{\nu\lambda}\xi^\beta - g^{\mu\alpha}g^{\beta\lambda}\xi^\nu + g^{\alpha\beta}g^{\mu\lambda}\xi^\nu)\nabla_\lambda\delta g_{\alpha\beta}\right].
\tag{E11}
$$

2. $f(R)$ gravity

$$
\mathbf{L} = \frac{1}{2\kappa}\boldsymbol{\epsilon} f(R),
\tag{E12}
$$

$$
P^{\mu\alpha\beta\nu} = \frac{1}{4\kappa}f'(R)(g^{\mu\beta}g^{\alpha\nu} - g^{\mu\nu}g^{\alpha\beta}),
\tag{E13}
$$

$$
C^{\sigma\rho\lambda\eta}_{\mu\alpha\beta\nu} = \frac{1}{8\kappa}f''(R)(g_{\mu\beta}g_{\alpha\nu} - g_{\mu\nu}g_{\alpha\beta})(g^{\sigma\lambda}g^{\rho\eta} - g^{\sigma\eta}g^{\rho\lambda}),
\tag{E14}
$$

$$
\mathcal{E}_{\mu\nu} = \frac{1}{2\kappa}\left(f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f'(R)\right),
\tag{E15}
$$

$$
\mathbf{\Theta} = f'(R)\mathbf{\Theta}_{\text{Ein}} + \frac{1}{2\kappa}\boldsymbol{\epsilon}_\mu(g^{\alpha\beta}\nabla^\mu f'(R) - g^{\beta\mu}\nabla^\alpha f'(R))\delta g_{\alpha\beta},
\tag{E16}
$$

$$\boldsymbol{\omega} = f'(R)\boldsymbol{\omega}_{\rm Ein} + \frac{1}{2\kappa}\boldsymbol{\epsilon}_\mu\left[\frac{1}{2}g^{\mu\beta}g^{\alpha\nu}g^{\rho\sigma}\delta_1 g_{\rho\sigma}\delta_2 g_{\alpha\beta}\nabla_\nu f'(R) + (g^{\mu\beta}g^{\alpha\nu} - g^{\mu\nu}g^{\alpha\beta})(\delta_1(f'(R))\nabla_\nu\delta_2 g_{\alpha\beta} - \delta_1(\nabla_\nu f'(R))\delta_2 g_{\alpha\beta}) - [1\leftrightarrow 2]\right], \tag{E17}$$

$$\mathbf{J}_\xi = \boldsymbol{\epsilon}_\mu\left[\frac{1}{\kappa}\nabla_\nu(f'(R)\nabla^{[\nu}\xi^{\mu]} + 2\xi^{[\nu}\nabla^{\mu]}f'(R)) + 2\mathcal{E}^\mu{}_\nu\xi^\nu\right], \tag{E18}$$

$$\mathbf{Q}_\xi = -\frac{1}{2\kappa}\boldsymbol{\epsilon}_{\mu\nu}[f'(R)\nabla^\mu\xi^\nu + 2\xi^\mu\nabla^\nu f'(R)], \tag{E19}$$

$$\mathbf{k}_\xi = f'(R)\mathbf{k}_{\xi,\rm Ein} - \frac{1}{2\kappa}\boldsymbol{\epsilon}_{\mu\nu}[\nabla^\mu\xi^\nu\delta f'(R) - 2g^{\mu\alpha}\xi^\nu\delta(\nabla_\alpha f'(R)) + g^{\mu\alpha}\xi^\nu\nabla^\beta(f'(R))\delta g_{\alpha\beta}]. \tag{E20}$$

3. Quadratic gravity

$$\mathbf{L} = \boldsymbol{\epsilon}\left\{\frac{1}{2\kappa}(-2\Lambda_0 + R) + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right\}. \tag{E21}$$

Recall that Gauss-Bonnet gravity can be obtained by setting[41] $\alpha_1 = \alpha_3 = -\frac{1}{4}\alpha_2 = \alpha$. That theory has the interesting feature—shared by all Lovelock gravities—that $P^{\mu\alpha\beta\nu}$ is divergence free in all indices, e.g., $\nabla_\mu P^{\mu\alpha\beta\nu} = 0$. Hence, all derivatives of curvature tensors should cancel in that case for the forms below, which provides a simple check for our expressions.

The first derivative of the Lagrangian with respect to the Riemann tensor as defined in (2.2) is

$$P^{\mu\alpha\beta\nu} = \left(\frac{1}{4\kappa} + \alpha_1 R\right)(g^{\mu\beta}g^{\alpha\nu} - g^{\mu\nu}g^{\alpha\beta}) + \frac{1}{2}\alpha_2(R^{\mu\beta}g^{\alpha\nu} - R^{\alpha\beta}g^{\mu\nu} - R^{\mu\nu}g^{\alpha\beta} + R^{\alpha\nu}g^{\mu\beta}) + 2\alpha_3 R^{\mu\alpha\beta\nu}, \tag{E22}$$

and its divergence reads

$$\nabla_\mu P^{\mu\alpha\beta\nu} = \left(2\alpha_1 + \frac{1}{2}\alpha_2\right)g^{\alpha[\nu}\nabla^{\beta]}R + (\alpha_2 + 4\alpha_3)\nabla^{[\beta}R^{\nu]\alpha}, \tag{E23}$$

where we have used the following identities: $\nabla^\nu R_{\mu\nu} = \frac{1}{2}\nabla_\mu R$ and $\nabla^\rho R_{\mu\nu\sigma\rho} = -2\nabla_{[\mu}R_{\nu]\sigma}$. These can be derived from the second Bianchi identity and will be frequently employed to simplify our expressions below. Notice that the divergence indeed vanishes for Gauss-Bonnet gravity.

From this, we find for the tensor defined in (2.4)

$$C^{\sigma\rho\lambda\eta}_{\mu\alpha\beta\nu} = \frac{1}{2}\alpha_1(g_{\mu\beta}g_{\alpha\nu} - g_{\mu\nu}g_{\alpha\beta})(g^{\sigma\lambda}g^{\rho\eta} - g^{\sigma\eta}g^{\rho\lambda}) + 2\alpha_2\delta^{[\sigma}{}_{(\tau}g^{\rho][\lambda}\delta^{\eta]}{}_{\epsilon)}\delta^\tau{}_{[\mu}g_{\alpha][\beta}\delta^\epsilon{}_{\nu]} + \alpha_3(\delta^{[\sigma}_\mu\delta^{\rho]}_\alpha\delta^{[\lambda}_\beta\delta^{\eta]}_\nu + \delta^{[\lambda}_\mu\delta^{\eta]}_\alpha\delta^{[\sigma}_\beta\delta^{\rho]}_\nu). \tag{E24}$$

The equations of motion for quadratic gravity read

$$\mathcal{E}_{\mu\nu} = \frac{1}{2\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu}\right) + \alpha_1\left(2RR_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2 - 2\nabla_\mu\nabla_\nu R + 2g_{\mu\nu}\Box R\right) + \alpha_2\left(R_{\mu\rho}R_\nu{}^\rho + R_{\rho\sigma}R_\mu{}^\rho{}_\nu{}^\sigma + \frac{1}{2}g_{\mu\nu}(\Box R - R_{\rho\sigma}R^{\rho\sigma}) - \nabla_{(\mu}\nabla_{\nu)}R + \Box R_{\mu\nu}\right) + \alpha_3\left(2R_{\mu\rho\sigma\lambda}R_\nu{}^{\rho\sigma\lambda} - \frac{1}{2}g_{\mu\nu}R_{\rho\sigma\alpha\beta}R^{\rho\sigma\alpha\beta} - 4\nabla^\rho\nabla^\sigma R_{\mu\rho\sigma\nu}\right). \tag{E25}$$

The symplectic potential form (10.4) is

$$\boldsymbol{\Theta} = \boldsymbol{\epsilon}_\mu\left[\left(\frac{1}{2\kappa} + 2\alpha_1 R\right)(g^{\mu\beta}g^{\alpha\nu} - g^{\mu\nu}g^{\alpha\beta})\nabla_\nu\delta g_{\alpha\beta} + 4\alpha_1(g^{\beta[\alpha}\nabla^{\mu]}R)\delta g_{\alpha\beta} + 2\alpha_2\left(R^{\beta[\mu}\nabla^{\alpha]}\delta g_{\alpha\beta} + g^{\beta[\mu}R^{\alpha]\nu}\nabla_\nu\delta g_{\alpha\beta} + \nabla^{[\mu}R^{\alpha]\beta}\delta g_{\alpha\beta} - \frac{1}{2}g^{\beta[\mu}\nabla^{\alpha]}R\delta g_{\alpha\beta}\right) + 4\alpha_3(R^{\mu\alpha\beta\nu}\nabla_\nu\delta g_{\alpha\beta} + 2\nabla^{[\mu}R^{\alpha]\beta}\delta g_{\alpha\beta})\right]. \tag{E26}$$

[41]Note that in this section, the couplings are not assumed to be dimensionless. This avoids some clutter in the already messy expressions.

For Gauss-Bonnet gravity, this reduces to

$$\begin{aligned}\mathbf{\Theta}_{\mathrm{GB}} = \boldsymbol{\epsilon}_\mu \bigg[\left(\frac{1}{2\kappa} + 2\alpha R\right)(g^{\mu\beta}g^{\alpha\nu} - g^{\mu\nu}g^{\alpha\beta})\nabla_\nu \delta g_{\alpha\beta} \\ - 8\alpha(R^{\beta[\mu}\nabla^{\alpha]}\delta g_{\alpha\beta} + g^{\beta[\mu}R^{\alpha]\nu}\nabla_\nu \delta g_{\alpha\beta}) + 4\alpha R^{\mu\alpha\beta\nu}\nabla_\nu \delta g_{\alpha\beta}\bigg]. \end{aligned} \tag{E27}$$

Note that this object was previously computed in Eq. (70) of Ref. [42]. We observe that our expression above differs from their result by a total derivative

$$\mathbf{\Theta}_{\mathrm{IW}} - \mathbf{\Theta}_{\mathrm{GB}} = 8\alpha\boldsymbol{\epsilon}_\mu \nabla_\nu (R^{\alpha[\mu}g^{\nu]\beta}\delta g_{\alpha\beta}), \tag{E28}$$

but only if the sign of the second-to-last term in their formula (70) is modified—to be explicit, this term should be "$+4(\nabla^e R^{df})\delta g_{ef}$." Hence, we suspect there is a typo in their expression. This is consistent with Ref. [125], where the symplectic potential was also computed for quadratic gravity. Restricting their formula (3.7) for the symplectic potential to Gauss-Bonnet indeed produces the Iyer-Wald symplectic potential *with* the corrected sign.

The symplectic current form (10.7) reads

$$\begin{aligned}\boldsymbol{\omega} = \boldsymbol{\epsilon}_\mu \bigg[&(1 + 4\kappa\alpha_1 R) S^{\mu\alpha\beta\nu\rho\sigma}_{\mathrm{Ein}} \delta_1 g_{\rho\sigma} \nabla_\nu \delta_2 g_{\alpha\beta} \\ &+ \alpha_1 (g^{\mu\beta}g^{\alpha\nu}g^{\rho\sigma}\delta_1 g_{\rho\sigma}\delta_2 g_{\alpha\beta}\nabla_\nu R + 4g^{\mu[\beta}g^{\nu]\alpha}(\delta_1(R)\nabla_\nu \delta_2 g_{\alpha\beta} - \delta_1(\nabla_\nu R)\delta_2 g_{\alpha\beta})) \\ &+ \alpha_2 (2\delta_1(R^{\beta[\mu})g^{\alpha]\nu} - 2\delta_1(R^{\nu[\mu})g^{\alpha]\beta} + (R^{\rho[\mu}g^{\nu]\beta}g^{\alpha\sigma} - R^{\beta(\mu}g^{\nu)\rho}g^{\alpha\sigma} + R^{\rho[\mu}g^{\alpha]\nu}g^{\beta\sigma} \\ &+ R^{\nu(\mu}g^{\alpha)\rho}g^{\beta\sigma} + R^{\alpha[\beta}g^{\mu]\rho}g^{\nu\sigma} + R^{\rho[\alpha}g^{\mu]\beta}g^{\nu\sigma} + R^{\beta[\mu}g^{\alpha]\nu}g^{\rho\sigma} + R^{\nu[\alpha}g^{\mu]\beta}g^{\rho\sigma})\delta_1 g_{\rho\sigma})\nabla_\nu \delta_2 g_{\alpha\beta} \\ &- \alpha_2 \left(2\delta_1\left(\nabla^{[\alpha}R^{\mu]\beta} + \frac{1}{2}g^{\beta[\mu}\nabla^{\alpha]}R\right) + \left(\nabla^{[\alpha}R^{\mu]\beta} + \frac{1}{2}g^{\beta[\mu}\nabla^{\alpha]}R\right)g^{\rho\sigma}\delta_1 g_{\rho\sigma}\right)\delta_2 g_{\alpha\beta} \\ &+ 4\alpha_3(\delta_1 R^{\mu\alpha\beta\nu} + (R^{\mu\alpha\beta(\nu}g^{\rho)\sigma} - R^{\mu(\alpha\nu)\rho}g^{\beta\sigma})\delta_1 g_{\rho\sigma})\nabla_\nu \delta_2 g_{\alpha\beta} \\ &+ 4\alpha_3(2\delta_1\nabla^{[\mu}R^{\alpha]\beta} + g^{\rho\sigma}\delta_1 g_{\rho\sigma}\nabla^{[\mu}R^{\alpha]\beta})\delta_2 g_{\alpha\beta}\bigg] - [1 \leftrightarrow 2]. \end{aligned} \tag{E29}$$

One can check that all terms involving derivatives acting on curvature tensors cancel for Gauss-Bonnet gravity.

The Noether current (10.23) and charge (10.22) are given by

$$\begin{aligned}\mathbf{J}_\xi = \boldsymbol{\epsilon}_\mu \bigg[\nabla_\nu \bigg[\left(\frac{1}{\kappa} + 4\alpha_1 R\right)\nabla^{[\nu}\xi^{\mu]} + 8\alpha_1 \xi^{[\nu}\nabla^{\mu]}R + 4\alpha_2(R^{\rho[\nu}\nabla_\rho \xi^{\mu]} + 2\xi^{[\nu}\nabla_\rho R^{\mu]\rho}) \\ - 4\alpha_3(R^{\mu\nu\rho\sigma}\nabla_\rho \xi_\sigma + 2\xi_\rho \nabla_\sigma R^{\mu\nu\rho\sigma})\bigg] + 2\mathcal{E}^\mu{}_\nu \xi^\nu\bigg], \end{aligned} \tag{E30}$$

$$\begin{aligned}\mathbf{Q}_\xi = -\boldsymbol{\epsilon}_{\mu\nu}\bigg[\left(\frac{1}{2\kappa} + 2\alpha_1 R\right)\nabla^\mu \xi^\nu + 4\alpha_1 \xi^\mu \nabla^\nu R \\ + 2\alpha_2(R^\mu{}_\rho \nabla^{[\rho}\xi^{\nu]} + 2\xi^{[\mu}\nabla^{\rho]}R^\nu{}_\rho) + 2\alpha_3(R^{\mu\nu\rho\sigma}\nabla_\rho \xi_\sigma + 2\xi_\rho \nabla_\sigma R^{\mu\nu\rho\sigma})\bigg]. \end{aligned} \tag{E31}$$

For Gauss-Bonnet gravity, we find the same expression as in Ref. [42], namely,

$$\mathbf{Q}_{\xi,\mathrm{GB}} = -\boldsymbol{\epsilon}_{\mu\nu}\left[\left(\frac{1}{2\kappa} + 2\alpha R\right)\nabla^\mu \xi^\nu - 8\alpha R^\mu{}_\rho \nabla^{[\rho}\xi^{\nu]} + 2\alpha R^{\mu\nu\rho\sigma}\nabla_\rho \xi_\sigma\right]. \tag{E32}$$

Finally, the Iyer-Wald surface charge (10.29) is

$$\begin{aligned}\mathbf{k}_\xi &= (1+4\kappa\alpha_1)\mathbf{k}_{\xi,\mathrm{Ein}}\\ &+\epsilon_{\mu\nu}\Big[2\alpha_1(-\nabla^\mu\xi^\nu\delta R+2\xi^\nu\delta(\nabla^\mu R)+g^{\mu\alpha}\xi^\nu\nabla^\beta R\delta g_{\alpha\beta})\\ &+\alpha_2(\nabla^{[\nu}\xi^{\lambda]}\delta R^\mu{}_\lambda+g^{\mu\lambda}g_{\alpha\beta}\nabla^{[\nu}\xi^{\alpha]}\delta R^\beta{}_\lambda+2\xi^\lambda\delta(\nabla^\mu R^\nu{}_\lambda)+\xi^\nu\delta(\nabla^\mu R))\\ &+\alpha_2(2R^{\mu[\alpha}g^{\lambda]\nu}\nabla^\beta\xi_\lambda-g^{\alpha\beta}R^{\mu[\rho}g^{\sigma]\nu}\nabla_\rho\xi_\sigma+2\xi^\nu\nabla^{[\alpha}R^{\mu]\beta}+\xi^\nu\nabla^{[\alpha}Rg^{\mu]\beta}\\ &-2\xi_\rho g^{\alpha\beta}\nabla_\sigma R^{\mu[\rho}g^{\sigma]\nu})\delta g_{\alpha\beta}-2\alpha_2(2\xi^\nu R^{\alpha(\mu}g^{\lambda)\beta}+\xi^\alpha R^{\nu[\beta}g^{\lambda]\mu})\nabla_\lambda\delta g_{\alpha\beta}\\ &-\alpha_3(\nabla^\alpha\xi^\beta\delta R^{\mu\nu}{}_{\alpha\beta}+g^{\mu\alpha}g^{\nu\beta}\nabla_\rho\xi_\sigma\delta R^{\rho\sigma}{}_{\alpha\beta}-8\xi^\lambda\delta(\nabla^\mu R^\nu{}_\lambda))\\ &+2\alpha_3\left(R^{\mu\nu\alpha\lambda}\nabla^\beta\xi_\lambda-\frac{1}{2}R^{\mu\nu\rho\sigma}g^{\alpha\beta}\nabla_\rho\xi_\sigma-4\xi^\nu\nabla^{[\mu}R^{\alpha]\beta}+2\xi_\lambda\nabla^\mu R^{\nu\lambda}g^{\alpha\beta}\right)\delta g_{\alpha\beta}\\ &-2\alpha_3(\xi^\alpha R^{\mu\nu\lambda\beta}+2\xi^\nu R^{\mu\alpha\beta\lambda})\nabla_\lambda\delta g_{\alpha\beta}\Big].\end{aligned}\tag{E33}$$

Again, it is straightforward to verify that all terms involving derivatives of curvature tensors cancel for Gauss-Bonnet gravity.

4. Lovelock gravity

The Lagrangian of Lovelock gravity is

$$\mathbf{L}=\frac{1}{2\kappa}\epsilon\sum_{k=0}^{\lfloor D/2\rfloor}c_k\mathcal{L}_k\quad\text{with}\quad\mathcal{L}_k=\frac{1}{2^k}\delta^{\mu_1\nu_1\ldots\mu_k\nu_k}_{\alpha_1\beta_1\ldots\alpha_k\beta_k}R_{\mu_1\nu_1}{}^{\alpha_1\beta_1}\cdots R_{\mu_k\nu_k}{}^{\alpha_k\beta_k},\tag{E34}$$

where the c_k are arbitrary constants. The objects defined in (2.2) and (2.4) read, respectively,

$$\begin{aligned}P^{\mu\alpha\beta\nu}&=\frac{1}{2\kappa}\sum_{k=0}^{\lfloor D/2\rfloor}c_kP^{\mu\alpha\beta\nu}_{(k)},\qquad P^{\mu\nu}_{(k)\alpha\beta}=\frac{k}{2^k}\delta^{\mu\nu\mu_2\nu_2\ldots\mu_k\nu_k}_{\alpha\beta\alpha_2\beta_2\ldots\alpha_k\beta_k}R_{\mu_2\nu_2}{}^{\alpha_2\beta_2}\cdots R_{\mu_k\nu_k}{}^{\alpha_k\beta_k},\\ C^{\sigma\rho\lambda\eta}_{\mu\alpha\beta\nu}&=\frac{1}{2\kappa}\sum_{k=0}^{\lfloor D/2\rfloor}\frac{k(k-1)c_k}{2^k}g_{\beta\gamma}g_{\nu\delta}g^{\lambda\chi}g^{\eta\xi}\delta^{\sigma\rho\gamma\delta\mu_3\nu_3\ldots\mu_k\nu_k}_{\mu\alpha\chi\xi\alpha_3\beta_3\ldots\alpha_k\beta_k}R_{\mu_3\nu_3}{}^{\alpha_3\beta_3}\cdots R_{\mu_k\nu_k}{}^{\alpha_k\beta_k}.\end{aligned}\tag{E35}$$

The equations of motion are

$$\mathcal{E}_{\mu\nu}=\frac{1}{2\kappa}\sum_{k=0}^{\lfloor D/2\rfloor}c_k\mathcal{E}^{(k)}_{\mu\nu}\quad\text{with}\quad\mathcal{E}^{(k)\mu}{}_\nu=\frac{-1}{2^{k+1}}\delta^{\mu\rho_1\sigma_1\ldots\rho_k\sigma_k}_{\nu\alpha_1\beta_1\ldots\alpha_k\beta_k}R_{\rho_1\sigma_1}{}^{\alpha_1\beta_1}\cdots R_{\rho_k\sigma_k}{}^{\alpha_k\beta_k}.\tag{E36}$$

Both tensors (E35) and (E36) are divergence free in all indices, e.g., $\nabla_\mu P^{\mu\alpha\beta\nu}=0$, $\nabla^\mu\mathcal{E}_{\mu\nu}=0$. Note that the equations of motion are second order in the metric, as is well known for Lovelock gravity.

The rest of the relevant quantities read

$$\mathbf{\Theta}=\frac{1}{2\kappa}\epsilon_\mu\sum_{k=0}^{\lfloor D/2\rfloor}\frac{kc_k}{2^{k-1}}\delta^{\mu\nu_1\mu_2\nu_2\ldots\mu_k\nu_k}_{\alpha_1\beta_1\alpha_2\beta_2\ldots\alpha_k\beta_k}R_{\mu_2\nu_2}{}^{\alpha_2\beta_2}\cdots R_{\mu_k\nu_k}{}^{\alpha_k\beta_k}g^{\alpha_1\lambda}\nabla^{\beta_1}\delta g_{\nu_1\lambda},\tag{E37}$$

$$\begin{aligned}\boldsymbol{\omega}&=\frac{1}{2\kappa}\epsilon_\mu\sum_{k=0}^{\lfloor D/2\rfloor}\frac{kc_k}{2^k}[\delta^{\mu\alpha\mu_2\nu_2\ldots\mu_k\nu_k}_{\gamma\delta\alpha_2\beta_2\ldots\alpha_k\beta_k}(2(k-1)g^{\beta\gamma}g^{\nu\delta}\delta_1R_{\mu_2\nu_2}{}^{\alpha_2\beta_2}\\ &+(g^{\beta\gamma}g^{\rho\sigma}g^{\nu\delta}+g^{\beta\delta}g^{\rho\gamma}g^{\nu\sigma}-g^{\beta\sigma}g^{\rho\gamma}g^{\nu\delta})\delta_1g_{\rho\sigma}R_{\mu_2\nu_2}{}^{\alpha_2\beta_2})\\ &+g^{\beta\sigma}g^{\rho\gamma}g^{\alpha\delta}\delta_1g_{\rho\sigma}\delta^{\mu\nu\mu_2\nu_2\ldots\mu_k\nu_k}_{\gamma\delta\alpha_2\beta_2\ldots\alpha_k\beta_k}R_{\mu_2\nu_2}{}^{\alpha_2\beta_2}]R_{\mu_3\nu_3}{}^{\alpha_3\beta_3}\cdots R_{\mu_k\nu_k}{}^{\alpha_k\beta_k}\nabla_\nu\delta_2g_{\alpha\beta}-[1\leftrightarrow2],\end{aligned}\tag{E38}$$

$$\mathbf{J}_\xi=\frac{1}{2\kappa}\epsilon_\mu\sum_{k=0}^{\lfloor D/2\rfloor}c_k\left[\nabla_\nu\left(\frac{-k}{2^{k-1}}\delta^{\mu\nu\mu_2\nu_2\ldots\mu_k\nu_k}_{\alpha_1\beta_1\alpha_2\beta_2\ldots\alpha_k\beta_k}R_{\mu_2\nu_2}{}^{\alpha_2\beta_2}\cdots R_{\mu_k\nu_k}{}^{\alpha_k\beta_k}\nabla^{\alpha_1}\xi^{\beta_1}\right)+2\mathcal{E}^{(k)\mu}{}_\nu\xi^\nu\right],\tag{E39}$$

$$\mathbf{Q}_\xi = -\frac{1}{2\kappa}\boldsymbol{\epsilon}_{\mu\nu}\sum_{k=0}^{\lfloor D/2\rfloor}\frac{kc_k}{2^k}\delta^{\mu\nu\mu_2\nu_2\ldots\mu_k\nu_k}_{\alpha_1\beta_1\alpha_2\beta_2\ldots\alpha_k\beta_k}R_{\mu_2\nu_2}{}^{\alpha_2\beta_2}\cdots R_{\mu_k\nu_k}{}^{\alpha_k\beta_k}\nabla^{\alpha_1}\xi^{\beta_1}, \tag{E40}$$

$$\begin{aligned}\mathbf{k}_\xi = \frac{1}{2\kappa}\boldsymbol{\epsilon}_{\mu\nu}\sum_{k=0}^{\lfloor D/2\rfloor}\frac{kc_k}{2^k}\Bigg[&\delta^{\mu\nu\mu_2\nu_2\ldots\mu_k\nu_k}_{\gamma\delta\alpha_2\beta_2\ldots\alpha_k\beta_k}\Bigg(-(k-1)\nabla^\gamma\xi^\delta\delta R_{\mu_2\nu_2}{}^{\alpha_2\beta_2}\\ &+\left(g^{\rho\gamma}\nabla^\sigma\xi^\delta\delta g_{\rho\sigma}-\frac{1}{2}g^{\rho\sigma}\nabla^\gamma\xi^\delta\delta g_{\rho\sigma}-\xi^\rho g^{\sigma\delta}\nabla^\gamma\delta g_{\rho\sigma}\right)R_{\mu_2\nu_2}{}^{\alpha_2\beta_2}\Bigg)\\ &-2\xi^\alpha g^{\gamma\sigma}\nabla^\delta\delta g_{\rho\sigma}\delta^{\mu\rho\mu_2\nu_2\ldots\mu_k\nu_k}_{\gamma\delta\alpha_2\beta_2\ldots\alpha_k\beta_k}R_{\mu_2\nu_2}{}^{\alpha_2\beta_2}\Bigg]R_{\mu_3\nu_3}{}^{\alpha_3\beta_3}\cdots R_{\mu_k\nu_k}{}^{\alpha_k\beta_k}.\end{aligned} \tag{E41}$$

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