

**Conserved charges of the extended Bondi-Metzner-Sachs algebra**Éanna É. Flanagan<sup>1,\*</sup> and David A. Nichols<sup>2,†</sup><sup>1</sup>*Department of Physics, Cornell University, Ithaca, New York 14853, USA*<sup>2</sup>*Cornell Center for Astrophysics and Planetary Science (CCAPS), Cornell University, Ithaca, New York 14853, USA*

(Received 28 October 2015; published 2 February 2017)

Isolated objects in asymptotically flat spacetimes in general relativity are characterized by their conserved charges associated with the Bondi-Metzner-Sachs (BMS) group. These charges include total energy, linear momentum, intrinsic angular momentum and center-of-mass location, and, in addition, an infinite number of supermomentum charges associated with supertranslations. Recently, it has been suggested that the BMS symmetry algebra should be enlarged to include an infinite number of additional symmetries known as super-rotations. We show that the corresponding charges are finite and well defined, and can be divided into electric parity “super center-of-mass” charges and magnetic parity “superspin” charges. The supermomentum charges are associated with ordinary gravitational-wave memory, and the super center-of-mass charges are associated with total (ordinary plus null) gravitational-wave memory, in the terminology of Bieri and Garfinkle. Superspin charges are associated with the ordinary piece of spin memory. Some of these charges can give rise to black hole hair, as described by Strominger and Zhiboedov. We clarify how this hair evades the no-hair theorems.

DOI: [10.1103/PhysRevD.95.044002](https://doi.org/10.1103/PhysRevD.95.044002)**I. INTRODUCTION**

Spacetimes which are asymptotically flat at future null infinity in general relativity have a group of asymptotic symmetries known as the Bondi-Metzner-Sachs (BMS) group [1–3]. Associated with each generator  $\vec{\xi}$  of this group and each cross section of future null infinity, there is a conserved charge<sup>1</sup>  $Q$  [4–6]. These charges include all the charges associated with the Poincaré group and, in addition, an infinite number of new supermomentum charges associated with supertranslations. The generators  $\vec{\xi}$  of the BMS group are smooth vector fields on future null infinity  $\mathcal{I}^+$ .

Recently Banks [7] and Barnich and Troessaert [8–10] have suggested that a larger symmetry algebra might be physically relevant. In particular, they suggested including vector fields called “super-rotations” which contain analytic singularities as functions on the conformal 2-sphere at infinity. These are formally infinitesimal symmetries of the theory, but they cannot be exponentiated to yield smooth finite diffeomorphisms, unlike the generators of the standard BMS group. There is not yet a general theory for understanding when singular vector fields of this type can be used to construct conserved charges and fluxes at future null infinity, and there has been some debate in the literature on the physical relevance or utility of the extended algebra.

One approach to this question is to determine whether the new symmetries give rise to relations between  $S$ -matrix elements, like the standard symmetries do [11,12]. This was shown to be the case for the tree level  $S$ -matrix by Kapec *et al.* [13].

Another approach is to determine whether the new symmetries give rise to well-defined and finite classical conserved quantities. Here we adopt this approach, following Barnich and Troessaert [10], who showed that the charges associated with super-rotations vanish in the Kerr spacetime. We show that the super-rotation charges are in general finite. There are two pieces of these charges, an electric parity piece and a magnetic parity piece. We call the electric parity charges “super center-of-mass” charges, since they are generalizations of the center-of-mass piece of special-relativistic angular momentum. We call the magnetic parity charges “superspin” charges, since they are generalizations of the intrinsic angular momentum piece of special-relativistic angular momentum. In addition, we show that supermomentum charges are associated with ordinary gravitational-wave memory and super center-of-mass charges are associated with total (ordinary plus null) gravitational-wave memory, in the terminology of Bieri and Garfinkle [14]. The superspin charges are associated with the ordinary piece of “spin memory,” the new type of gravitational-wave memory discovered by Pasterski *et al.* [15].

The paper is organized as follows. Section II reviews the standard BMS group and algebra and also the extended BMS algebra. In Sec. III, we compute the conserved charges. In Sec. IV, we make some remarks about the physical significance of BMS and extended BMS charges and how they can be measured.

\*eef3@cornell.edu

†dnichols@astro.cornell.edu

<sup>1</sup>By conserved charge we mean a charge that would be conserved in the absence of fluxes of radiation to null infinity.

## II. THE BMS SYMMETRY GROUP AND THE EXTENDED BMS ALGEBRA

### A. Asymptotically flat spacetimes in retarded Bondi coordinates

We start by reviewing the definition of the BMS symmetry group and its action on solutions of Einstein's equations near future null infinity. We closely follow the exposition of Barnich and Troessaert [8,16] as simplified and specialized by Strominger and collaborators [11–13,15,17]. Our notation, however, will follow Barnich and Troessaert in maintaining covariance with respect to the 2-sphere coordinates instead of using the complex-coordinate convention used in Refs. [11–13,15,17].

Following Refs. [1,8,11–13,15–17] we use retarded Bondi coordinates  $(u, r, \theta^1, \theta^2)$  near future null infinity. The metric has the form

$$ds^2 = -Ue^{2\beta}du^2 - 2e^{2\beta}dudr + r^2\gamma_{AB}(d\theta^A - \mathcal{U}^A du)(d\theta^B - \mathcal{U}^B du), \quad (2.1)$$

where  $A, B = 1, 2$ , and  $U, \beta, \mathcal{U}^A$ , and  $\gamma_{AB}$  are functions of  $u, r$ , and  $\theta^A$ . The four gauge conditions that are imposed are  $g_{rr} = 0$ ,  $g_{rA} = 0$ , and<sup>2</sup>

$$\partial_r \det(\gamma_{AB}) = 0. \quad (2.2)$$

We now expand the metric functions as series in  $1/r$ . The order in  $1/r$  at which the various expansions start can be deduced from the covariant definition of asymptotic flatness at future null infinity [18]. The expansions are<sup>3</sup> [1,8,11–13,15–17]

$$\beta = \frac{\beta_0}{r} + \frac{\beta_1}{r^2} + \frac{\beta_2}{r^3} + O(r^{-4}), \quad (2.3a)$$

$$U = 1 - \frac{2m}{r} - \frac{2\mathcal{M}}{r^2} + O(r^{-3}), \quad (2.3b)$$

$$\gamma_{AB} = h_{AB} + \frac{1}{r}C_{AB} + \frac{1}{r^2}D_{AB} + \frac{1}{r^3}E_{AB} + O(r^{-4}), \quad (2.3c)$$

$$\mathcal{U}^A = \frac{1}{r^2}U^A + \frac{1}{r^3}\left[-\frac{2}{3}N^A + \frac{1}{16}D^A(C_{BC}C^{BC}) + \frac{1}{2}C^{AB}D^C C_{BC}\right] + O(r^{-4}). \quad (2.3d)$$

Here the various coefficients on the right-hand sides are functions of  $(u, \theta^A)$  only. The metric  $h_{AB}(\theta^C)$  is the fixed

<sup>2</sup>One could further specialize the gauge by imposing  $\partial_u \det(\gamma_{AB}) = 0$ , but in the context of the expansion in powers of  $1/r$  this condition follows from Eqs. (2.2) and (2.3).

<sup>3</sup>Some of the higher-order terms in these expansions are not needed in this section but will be needed in Sec. IID below.

round metric on the unit 2-sphere. In adapted coordinates  $(\theta, \varphi)$ , it is  $d\theta^2 + \sin^2\theta d\varphi^2$ , but we will use general coordinates  $\theta^A$  and retain two-dimensional covariance throughout. We adopt the convention that capital latin indices (e.g.,  $A, B$ ) are raised and lowered with  $h_{AB}$ , and we denote by  $D_A$  the covariant derivative associated with  $h_{AB}$ . There are three important, leading-order functions in the metric's expansion coefficients [1,8,11–13,15–17]: the Bondi mass aspect  $m(u, \theta^A)$ ; the angular-momentum aspect<sup>4</sup>  $N^A(u, \theta^A)$ ; and the symmetric tensor  $C_{AB}(u, \theta^A)$  whose derivative

$$N_{AB} = \partial_u C_{AB} \quad (2.4)$$

is the Bondi news tensor.

Imposing the gauge condition (2.2) now yields the constraints

$$h^{AB}C_{AB} = 0, \quad (2.5a)$$

$$D_{AB} = C_{CD}C^{CD}h_{AB}/4 + \mathcal{D}_{AB}, \quad (2.5b)$$

$$E_{AB} = C_{CD}\mathcal{D}^{CD}h_{AB}/2 + \mathcal{E}_{AB}, \quad (2.5c)$$

where the tensors  $\mathcal{D}_{AB}$  and  $\mathcal{E}_{AB}$  are traceless.

We assume the following behavior of the stress-energy tensor as  $r \rightarrow \infty$ :

$$T_{uu} = \frac{1}{r^2}\hat{T}_{uu}(u, \theta^A) + O(r^{-3}), \quad (2.6a)$$

$$T_{rr} = \frac{1}{r^4}\hat{T}_{rr}(u, \theta^A) + \frac{1}{r^5}\tilde{T}_{rr}(u, \theta^A) + O(r^{-6}), \quad (2.6b)$$

$$T_{uA} = \frac{1}{r^2}\hat{T}_{uA}(u, \theta^A) + O(r^{-3}), \quad (2.6c)$$

$$T_{rA} = \frac{1}{r^3}\hat{T}_{rA}(u, \theta^A) + O(r^{-4}), \quad (2.6d)$$

$$T_{AB} = \frac{1}{r}\hat{T}(u, \theta^A)h_{AB} + O(r^{-2}), \quad (2.6e)$$

together with  $T_{ur} = O(r^{-4})$ . These assumptions are motivated by the behavior of radiative scalar-field solutions in Minkowski spacetime.<sup>5</sup> Imposing stress-energy conservation yields

<sup>4</sup>Our definition of the angular-momentum aspect  $N_A$  [cf. Eq. (2.3d)] follows Ref. [15] rather than the one used in Ref. [16]. Our definition also coincides with that used in Sec. V.6 of the book by Chrusciel *et al.* [19], up to a factor of  $-3$ .

<sup>5</sup>The case considered in Refs. [11–13,15,17] corresponds to  $\hat{T} = 0$ , which applies for example to conformally invariant fields.

$$\partial_u \hat{T}_{rA} = D_A \hat{T}, \quad (2.7a)$$

$$\partial_u \hat{T}_{rr} = -2\hat{T}, \quad (2.7b)$$

from  $O(A, 5)$  and  $O(r, 3)$ , respectively, where  $O(\alpha, n)$  means the  $O(r^{-n})$  piece of the  $\alpha$  component of  $\nabla^\beta T_{\alpha\beta} = 0$ . Combining Eqs. (2.7) we can write

$$\hat{T}_{rA}(u, \theta^A) = \check{T}_{rA}(\theta^A) - \frac{1}{2} D_A \hat{T}_{rr}(u, \theta^A), \quad (2.8)$$

for some function  $\check{T}_{rA}(\theta^A)$ .

We now impose Einstein's equations  $G_{ab} = 8\pi T_{ab}$ , in units with  $G = 1$ . We adopt the shorthand notation that  $O(\alpha\beta, n)$  means the  $O(r^{-n})$  piece of the  $(\alpha\beta)$  component of Einstein's equations. We obtain

$$U_A = -D^B C_{AB}/2, \quad (2.9a)$$

$$\beta_0 = 0, \quad (2.9b)$$

$$\beta_1 = -\frac{1}{32} C_{AB} C^{AB} - \pi \hat{T}_{rr}, \quad (2.9c)$$

$$\beta_2 = -\frac{1}{12} C_{AB} D^{AB} - \frac{2\pi}{3} \check{T}_{rr}, \quad (2.9d)$$

$$D^A \mathcal{D}_{AB} = -8\pi \check{T}_{rB}, \quad (2.9e)$$

$$\partial_u \mathcal{D}_{AB} = 0, \quad (2.9f)$$

from  $O(rA, 2)$ ,  $O(rr, 3)$ ,  $O(rr, 4)$ ,  $O(rr, 5)$ ,  $O(rA, 3)$ , and  $O(AB, 1)$ , respectively. Note that it follows from Eq. (2.9e) that

$$\mathcal{D}_{AB} = 0 \quad (2.10)$$

in vacuum. We also obtain from  $O(uu, 2)$  and  $O(uA, 2)$  evolution equations for the Bondi mass aspect  $m$  and the angular-momentum aspect  $N_A$  [11,15,16]:

$$\dot{m} = -4\pi \hat{T}_{uu} - \frac{1}{8} N_{AB} N^{AB} + \frac{1}{4} D_A D_B N^{AB}, \quad (2.11a)$$

$$\begin{aligned} \dot{N}_A = & -8\pi \hat{T}_{uA} + \pi D_A \partial_u \hat{T}_{rr} + D_A m + \frac{1}{4} D_B D_A D_C C^{BC} \\ & - \frac{1}{4} D_B D^B D^C C_{CA} + \frac{1}{4} D_B (N^{BC} C_{CA}) \\ & + \frac{1}{2} D_B N^{BC} C_{CA}. \end{aligned} \quad (2.11b)$$

Here dots denote derivatives with respect to  $u$ . The leading-order components of the Weyl tensor for these solutions are listed in Appendix A.

## B. BMS symmetry group

BMS symmetries are diffeomorphisms of future null infinity,  $\mathcal{I}^+$ , to itself that preserve its intrinsic geometric properties [18,20]. Explicitly, in Bondi coordinates, the diffeomorphism  $\psi$  takes the point  $(u, \theta^A)$  on  $\mathcal{I}^+$  to  $(\bar{u}, \bar{\theta}^A)$ , where

$$\bar{u} = \frac{1}{w(\theta^A)} [u + \alpha(\theta^A)], \quad (2.12a)$$

$$\bar{\theta}^A = \bar{\theta}^A(\theta^B). \quad (2.12b)$$

Here the mapping  $\theta^A \rightarrow \bar{\theta}^A(\theta^B)$  must be a conformal isometry of the 2-sphere, of which there is a six-parameter group, and the corresponding function  $\omega$  is defined by  $\psi_* h_{AB} = \omega^{-2} h_{AB}$ . The function  $\alpha$  can be freely chosen. The corresponding infinitesimal symmetries are  $\bar{u} = u + \xi^u$ ,  $\bar{\theta}^A = \theta^A + \xi^A$ , where the vector field  $\vec{\xi}$  on  $\mathcal{I}^+$  is

$$\begin{aligned} \vec{\xi} &= \xi^u \partial_u + \xi^A \partial_A \\ &= \left[ \alpha(\theta^A) + \frac{1}{2} u D_A Y^A(\theta^B) \right] \partial_u + Y^A(\theta_B) \partial_A. \end{aligned} \quad (2.13)$$

Here  $Y^A(\theta^B)$  must be a conformal Killing vector on the 2-sphere—i.e., be a solution of

$$2D_{(A} Y_{B)} - D_C Y^C h_{AB} = 0. \quad (2.14)$$

The general solution can be written as

$$Y^A = D^A \chi + \epsilon^{AB} D_B \kappa, \quad (2.15)$$

where  $\chi$  and  $\kappa$  are  $l = 1$  spherical harmonics, that is, solutions of  $(D^2 + 2)\chi = 0$  and  $(D^2 + 2)\kappa = 0$ , where  $D^2 = D_A D^A$ . These solutions comprise the Lorentz algebra, with the three electric parity solutions  $D^A \chi$  corresponding to boosts, while the three magnetic parity solutions  $\epsilon^{AB} D_B \kappa$  correspond to rotations.

The symmetry vector fields  $\vec{\xi}$  can be extended from future null infinity  $\mathcal{I}^+$  into the interior of the spacetime to give approximate asymptotic Killing vectors by demanding that they maintain the retarded Bondi coordinate conditions and the assumed scalings with  $r$  of the metric components. This gives [11,16]

$$\begin{aligned} \bar{\xi} = f\partial_u + \left[ Y^A - \frac{1}{r}D^A f + \frac{1}{2r^2}C^{AB}D_B f + O(r^{-3}) \right] \partial_A \\ - \left[ \frac{1}{2}rD_A Y^A - \frac{1}{2}D^2 f - \frac{1}{2r}U^A D_A f \right. \\ \left. + \frac{1}{4r}D_A(D_B f C^{AB}) + O(r^{-2}) \right] \partial_r, \end{aligned} \quad (2.16)$$

where

$$f(u, \theta^A) = \alpha(\theta^A) + \frac{1}{2}u D_B Y^B(\theta^A). \quad (2.17)$$

Under these transformations, the metric transforms via pullback as  $g_{ab} \rightarrow \psi_* g_{ab} = g_{ab} + \mathcal{L}_{\bar{\xi}} g_{ab}$ . This yields the following transformations of the metric functions [11,16],

$$\begin{aligned} \delta m = f\dot{m} + \frac{1}{4}N^{AB}D_A D_B f + \frac{1}{2}D_A f D_B N^{AB} + \frac{3}{2}m\psi \\ + Y^A D_A m + \frac{1}{8}C^{AB}D_A D_B \psi, \end{aligned} \quad (2.18a)$$

$$\delta C_{AB} = fN_{AB} - 2D_A D_B f + h_{AB}D^2 f - \frac{1}{2}\psi C_{AB} + \mathcal{L}_{\bar{\xi}} C_{AB}, \quad (2.18b)$$

$$\begin{aligned} \delta N_A = f\dot{N}_A + 3mD_A f + \frac{1}{4}C_{AB}D^B D^2 f \\ - \frac{3}{4}D_B f (D^B D_C C^C_A - D_A D_C C^{BC}) \\ + \frac{3}{8}D_A (C^{BC}D_B D_C f) + \frac{1}{4}(2D_A D_B f - h_{AB}D^2 f)D_C C^{BC} \\ + \mathcal{L}_{\bar{\xi}} N_A + \psi N_A - \frac{1}{2}D_B (\psi D^{BA}) - \frac{1}{32}D_A \psi C_{BC} C^{BC} \\ + \dots, \end{aligned} \quad (2.18c)$$

where on the right-hand sides  $N_{AB}$  is the Bondi news tensor (2.4), overdots denote derivatives with respect to  $u$ ,  $\mathcal{L}$  is a Lie derivative,  $f$  is defined by Eq. (2.17),  $\psi$  is defined by

$$\psi \equiv D_A Y^A, \quad (2.19)$$

and the ellipsis (...) in Eq. (2.18c) indicates terms involving the news tensor  $N_{AB}$  that we will not need in what follows.

### C. Terminology for regions of future null infinity

If the Bondi news tensor  $N_{AB}$  vanishes in a region of future null infinity in a given BMS frame, then it will vanish in that region in all BMS frames. The region is then called “nonradiative.” We call a region of  $\mathcal{I}^+$  “stationary” if the spacetime is stationary in a neighborhood of that region. If a region is stationary then it must be nonradiative [20]. However the converse is not true; for example, a linear superposition of the linearized gravitational fields of two

point particles with a relative boost is nonradiative but nonstationary.

When computing charges later in this paper we will specialize, for simplicity, to nonradiative regions of  $\mathcal{I}^+$ , and sometimes in addition specialize further to stationary regions. We will consider nonradiative-to-nonradiative transitions, that is, spacetimes which possess a nonradiative region of  $\mathcal{I}^+$ , followed by a radiative region, followed by another nonradiative region. We will also consider stationary-to-stationary transitions.

### D. Canonical Bondi frame for stationary vacuum regions

Consider a stationary region of  $\mathcal{I}^+$  in which the leading-order stress-energy components (2.6) as well as the subleading components vanish. Then, there exists a preferred, canonical Bondi frame in which the metric takes a simple form, as we now review.

First, it is known that the news tensor (2.4) must vanish in stationary regions [20], so that  $C_{AB}$  is independent of  $u$ . Next, it follows from the evolution equation (2.11a) for the Bondi mass aspect that  $m$  is also independent of  $u$ . From the  $O(ur, 4)$  component of Einstein’s equation we now obtain

$$6\mathcal{M} + D_A N^A + \frac{3}{16}C_{AB}C^{AB} + \frac{3}{4}D_A C^{AB} D^C C_{CB} = 0, \quad (2.20)$$

which will be useful below.

We now specialize to Bondi frames in which the angular momentum aspect is independent of  $u$ , so that

$$\partial_u N_A = 0. \quad (2.21)$$

The existence of such frames is established in Sec. VI. 7 of Ref. [19]. We will show that it is possible to further specialize the frame to the preferred, canonical one.

We first derive some properties of the Bondi mass aspect  $m$  and angular-momentum aspect  $N_A$  under the above assumptions. From the evolution equation (2.11b) with  $N_{AB} = \dot{N}_A = 0$ , we obtain  $4D_A m - \epsilon_{AB} D^B \gamma = 0$ , where  $\gamma = h^{BD} \epsilon^{AC} D_C D_B C_{AD}$ . It follows that

$$m(\theta^A) = m_0, \quad (2.22)$$

a constant. Taking next the subleading  $O(uA, 3)$  component and using Eq. (2.10) as well as Eq. (2.20) to eliminate the subleading mass function  $\mathcal{M}$  yields

$$D^2 N_A + N_A = -3D^B (m C_{AB}) + w_A. \quad (2.23)$$

Here  $w_A$  is an expression quadratic and cubic in  $C_{AB}$  and its derivatives, whose precise form will not be needed. We have also assumed that the subleading angular momentum

aspect is independent of  $u$ . Equation (2.23) will be useful below.

We now derive the transformation to the preferred, canonical Bondi frame. The tensor  $C_{AB}$  can be decomposed into electric and magnetic parity pieces,

$$C_{AB} = \left( D_A D_B - \frac{1}{2} h_{AB} D^2 \right) \Phi + \epsilon_{C(A} D_{B)} D^C \Psi, \quad (2.24)$$

where, without loss of generality,  $\Phi$  and  $\Psi$  have no  $l = 0, 1$  components. Taking the magnetic parity part of the time-evolution equation (2.11b) for the angular-momentum aspect by contracting it with  $\epsilon^{AB} D_B$ , using  $N_{AB} = \dot{N}_A = 0$ , and commuting indices using  $R_{ABCD} = h_{AC} h_{BD} - h_{AD} h_{BC}$  gives  $(D^2 + D^4/2)\Psi = 0$ . This forces<sup>6</sup> the magnetic part  $\Psi$  to vanish. Next, by using a BMS transformation with  $Y^A = 0$  and  $\alpha = \Phi/2$ , we see from Eq. (2.18b) that we can also make the electric part  $\Phi$  vanish, so that  $C_{AB} = 0$ .

Equation (2.23) for the angular-momentum aspect now reduces to

$$D^2 N_A + N_A = 0. \quad (2.25)$$

We decompose  $N_A$  into electric parity and magnetic parity pieces,

$$N_A = D_A \Upsilon + \epsilon_{AB} D^B \Theta, \quad (2.26)$$

where  $D^2 \Upsilon = D_A N^A$  and  $D^2 \Theta = -\epsilon^{AB} D_A N_B$ . Substituting into Eq. (2.25) now shows that  $\Upsilon$  and  $\Theta$  satisfy

$$(D^2 + 2)\Upsilon = 0, \quad (D^2 + 2)\Theta = 0 \quad (2.27)$$

(i.e., they are both  $l = 1$  spherical harmonics). Thus, the solutions of Eq. (2.25) for the angular-momentum aspect coincide with the solutions discussed after Eq. (2.14) of the conformal Killing vector equation: three electric parity conformal Killing vectors and three magnetic parity Killing vectors.

We now perform a BMS transformation with  $\alpha = -\Upsilon/(3m_0)$  and  $Y^A = 0$ . From the transformation law (2.18b) for  $C_{AB}$ , we find that this transformation does not alter the gauge specialization  $C_{AB} = 0$  that we have already achieved, because the differential operator  $2D_A D_B - h_{AB} D^2$  on the right-hand side annihilates  $l = 1$  spherical harmonics. The effect of the transformation is to set the electric parity piece of  $N_A$  to zero, from Eq. (2.18c). Since the electric parity piece of  $N_A$  encodes information

about the center of mass, this transformation corresponds roughly to translating to the center-of-mass frame. The remaining magnetic parity piece of  $N_A$  encodes the intrinsic angular momentum.

To summarize, we have achieved a Bondi frame in which

$$m(\theta^A) = m_0 = \text{constant}, \quad (2.28a)$$

$$C_{AB}(\theta^A) = 0, \quad (2.28b)$$

$$N_A(\theta^A) = \text{magnetic parity}, \quad l = 1. \quad (2.28c)$$

We call the frame which satisfies these conditions the canonical frame. The explicit construction of this frame in the Kerr spacetime can be found in Appendix C.7 of Ref. [19].

## E. Extended BMS algebra

The BMS algebra of approximate Killing vector fields  $\vec{\xi}$  described above consists of vector fields that are smooth and finite on future null infinity  $\mathcal{I}^+$ . Relaxing this requirement, Banks [7] and Barnich and Troessaert [8–10] suggested instead that a larger algebra might be relevant. In particular, they suggested adding to the algebra more general solutions of the conformal Killing equation (2.14), in addition to the six smooth solutions discussed above. In complex stereographic coordinates  $(z, \bar{z})$  with  $z = \cot(\theta/2)e^{i\varphi}$ , the conformal Killing equation reduces to

$$\partial_z Y^{\bar{z}} = 0, \quad \partial_{\bar{z}} Y^z = 0, \quad (2.29)$$

and one can consider solutions  $Y^z = Y^z(z)$ ,  $Y^{\bar{z}} = Y^{\bar{z}}(\bar{z})$ , where  $Y^z$  and  $Y^{\bar{z}}$  are meromorphic functions of their arguments. A basis<sup>7</sup> of this set of vector fields is

$$l_m = -z^{m+1} \partial_z, \quad (2.30a)$$

$$\bar{l}_m = -\bar{z}^{m+1} \partial_{\bar{z}}, \quad (2.30b)$$

for  $m \in \mathbf{Z}$ . Of this infinite basis, the six vector fields  $l_{-1}, l_0, l_1, \bar{l}_{-1}, \bar{l}_0, \bar{l}_1$  are those discussed after Eq. (2.14) above that occur in the usual BMS algebra. The remaining new vector fields are singular and cannot be used to define smooth, finite diffeomorphisms of the 2-sphere to itself.

The new vector fields have been called ‘‘super-rotations’’ in the literature [8–10,13], since they are generalizations of the six generators of the Lorentz group. They might also be called ‘‘superboosts,’’ since they are conformal Killing vectors but not Killing vectors on the 2-sphere, like normal boosts but unlike normal rotations.

<sup>6</sup>The inverse of the angular differential operator  $D^2 + D^4/2$  on the space of functions with no  $l = 0, 1$  pieces is given explicitly in Eq. (2.17) of Ref. [15] and also in Appendix C of Ref. [14].

<sup>7</sup>Although only real vector fields  $Y^A(\theta^B)$  are physical, for convenience, we use a complex basis in what follows. A real basis can be obtained by taking linear combinations.

What is the physical relevance or utility of the extended<sup>8</sup> BMS algebra? There has been some debate in the literature on this issue. One approach to answering this question is to check whether there are constraints on the quantum gravity  $S$ -matrix associated with the additional symmetries. Kapec *et al.* [13] showed that this is indeed the case at tree level. Strominger and Zhibodov [21] showed that finite super-rotations map asymptotically flat spacetimes into a larger class of spacetimes which are asymptotically flat except at isolated points which have cosmic string defects. Finally, another approach is to determine whether there are well-defined classical conserved quantities for the new symmetries, just as there are for the standard BMS symmetries. Barnich and Troessaert followed this approach in Ref. [10], where they computed charge integrals associated with the new generators (2.30) with  $|m| > 1$  in the Kerr spacetime. They found that these charges vanish. In the next section, we will extend their analysis to more general situations to show that the charges are finite and to clarify their physical interpretation.

### III. BMS CONSERVED CHARGES

#### A. Charges and conservation laws

We first review the charges and conservation laws associated with the standard BMS group. There are two types of BMS conservation laws: (i) laws that relate quantities at one cut or cross section of future null infinity  $\mathcal{I}^+$  to another [4–6], and (ii) laws that relate quantities at past null infinity  $\mathcal{I}^-$  to quantities at future null infinity  $\mathcal{I}^+$  [11,15,22].

Consider the first type of conservation law. Normally the charges associated with conservation laws can be derived from Noether's theorem. However, this does not apply to charges associated with BMS generators at future null infinity, since the associated charges are not actually conserved because of fluxes of gravitational radiation. Wald and Zoupas have derived a generalization of Noether's theorem that allows one to define conserved charges and fluxes in very general situations of this kind

<sup>8</sup>One might think it necessary to include in the algebra all the vector fields generated by taking Lie brackets of BMS generators and/or super-rotations. This would yield the algebra summarized in Eqs. (4.17) of Ref. [16], which contains “extended super-translation” generators of the form (2.13), but where the function  $\alpha$  can contain singularities of the form  $z^p \bar{z}^q$  with  $p, q$  negative integers. These charges associated with these generators are ill defined [10]. However, one could imagine generalizing the definition of asymptotic flatness by defining a class of solutions which are locally asymptotically flat on  $\mathcal{I}^+$  except for a finite number of points on the 2-sphere with meromorphic singularities of the kind generated by acting with a finite super-rotation [21]. A diffeomorphism that maps one such solution onto another has weaker singularities with finite charges, since one is in effect forbidden from performing two successive super-rotations with the same singular point on the 2-sphere which would change the nature of the singularity. Thus, the divergent charge integrals computed in [10] might not be a sign of a fatal inconsistency.

[6]. One obtains for each generator  $\vec{\xi}$  a 2-form  $\Xi$  on  $\mathcal{I}^+$ , which depends linearly on  $\vec{\xi}$ , and whose integral over any cut (cross section)  $\mathcal{C}$  gives the charge

$$Q(\mathcal{C}, \vec{\xi}) = \int_{\mathcal{C}} \Xi \quad (3.1)$$

associated with that cut. In addition, the exterior derivative  $d\Xi$  of the 2-form can be interpreted as a flux that can be integrated over a region  $\mathcal{R}$  of  $\mathcal{I}^+$  between two cuts  $\mathcal{C}_1$  and  $\mathcal{C}_2$  to give the change in the charge between two cuts:

$$\int_{\mathcal{R}} d\Xi = Q(\mathcal{C}_2, \vec{\xi}) - Q(\mathcal{C}_1, \vec{\xi}). \quad (3.2)$$

For general relativity, the flux formula had previously been obtained by Ashtekar and Streubel [4], and the charge associated with a cut had been obtained using a different method by Dray and Streubel [5].

The second type of conservation law is as follows. Suppose that for a given generator  $\vec{\xi}$  of the BMS group acting on  $\mathcal{I}^+$  one can identify an associated generator  $\vec{\xi}'$  of the BMS group acting on  $\mathcal{I}^-$ , with associated 2-form  $\Xi'$ . Then one might anticipate a conservation law of the form

$$\lim_{\mathcal{D} \rightarrow i^0} \int_{\mathcal{D}} \Xi' = \lim_{\mathcal{C} \rightarrow i^0} \int_{\mathcal{C}} \Xi, \quad (3.3)$$

where the first limit to spacelike infinity  $i^0$  is taken from the past along cuts  $\mathcal{D}$  of  $\mathcal{I}^-$ , and the second limit to  $i^0$  is taken from the future along cuts  $\mathcal{C}$  of  $\mathcal{I}^+$ . Using relations of the form (3.2) on both  $\mathcal{I}^-$  and  $\mathcal{I}^+$ , the conservation law (3.3) is equivalent to

$$\lim_{\mathcal{D} \rightarrow i^-} \int_{\mathcal{D}} \Xi' + \int_{\mathcal{I}^-} d\Xi' = \lim_{\mathcal{C} \rightarrow i^+} \int_{\mathcal{C}} \Xi + \int_{\mathcal{I}^+} d\Xi, \quad (3.4)$$

assuming that the relevant limits exist at future timelike infinity  $i^+$  and past timelike infinity  $i^-$ .

For the translation subgroup of the BMS group, a method of identifying the subgroups at  $\mathcal{I}^-$  and  $\mathcal{I}^+$  was found by Ashtekar and Magnon-Ashtekar [22], together with an associated conservation law of the form (3.3) for the 4-momentum. More recently, for the special class of spacetimes studied by Christodoulou and Klainerman [23], Strominger found a method of identifying the two BMS groups and derived an associated conservation law<sup>9</sup>

<sup>9</sup>Ashtekar [24] has pointed out that in the Christodoulou-Klainerman spacetimes, this conservation law (3.3) for super-momentum does not yield any information beyond the conservation of 4-momentum, since the additional charges all vanish. This can be seen from Eqs. (2.26) and (2.29) of [11] and Eq. (B4) below. If the conservation law extends to more general spacetimes it would yield nontrivial constraints.

of the form (3.4) for general generators, in which the boundary terms at  $i^-$  and  $i^+$  vanish [11].

### B. Charges for standard BMS algebra

We now turn to the derivation of an explicit expression for general BMS charges  $Q$  in the retarded Bondi coordinates used here. For simplicity, we specialize to regions of  $\mathcal{I}^+$  which are nonradiative, and we assume that the leading and subleading stress-energy components vanish. It follows that the metric functions  $m$  and  $C_{AB}$  are independent of  $u$ , as discussed in Sec. II D above. Then, for the generator  $\vec{\xi}$  given in terms of  $\alpha(\theta^A)$  and  $Y^A(\theta^B)$  by Eq. (2.16), and for the cut  $\mathcal{C}$  given by  $u = u_0$ , the charge is

$$Q = \frac{1}{16\pi} \int d^2\Omega \left[ 4\alpha m - 2u_0 Y^A D_A m + 2Y^A N_A - \frac{1}{8} Y^A D_A (C_{BC} C^{BC}) - \frac{1}{2} Y^A C_{AB} D_C C^{BC} \right]. \quad (3.5)$$

Note that this charge is independent of  $u_0$ , i.e., independent of the cut  $\mathcal{C}$ .<sup>10</sup> This is as expected since the flux  $d\Xi$  for all BMS generators vanishes in nonradiative vacuum regions [see Eq. (C1) below].

The formula (3.5) can be obtained from the prescription given after Eq. (83) of Wald and Zoupas [6]. We decompose the generator  $\vec{\xi}$  uniquely as the sum  $\vec{\xi} = \vec{\xi}_1 + \vec{\xi}_2$  of two generators, where  $\vec{\xi}_1$  is tangent to  $\mathcal{C}$  and  $\vec{\xi}_2$  is a supertranslation. Explicit expressions for the two generators are

$$\vec{\xi}_1 = \frac{1}{2} (u - u_0) D_A Y^A \partial_u + Y^A \partial_A \quad (3.6)$$

and  $\vec{\xi}_2 = (\alpha + \frac{1}{2} u_0 D_A Y^A) \partial_u$ . Next, we use the linear dependence of the charge on  $\vec{\xi}$  to evaluate the charge as  $Q = Q[\vec{\xi}_1] + Q[\vec{\xi}_2]$ . The contribution to the charge from  $\vec{\xi}_1$  is given by the integral of the Noether charge 2-form given by Eq. (44) of [6] (i.e., the Komar formula). To get a unique result from this prescription, Wald explains that from the equivalence class of vector fields on spacetime that corresponds to the desired BMS generator, one should choose a representative  $\vec{\xi}$  that satisfies  $\nabla_a \xi^a = 0$ , as proved in Ref. [25]. In fact, an examination of the argument in Ref. [25] shows that a sufficient condition for uniqueness is  $\nabla_a \xi^a = O(1/r^2)$ , which is satisfied by the representatives (2.16) used here. Using a vector field of the form (2.16)

<sup>10</sup>This can be seen by differentiating both sides with respect to  $u_0$ , and using the evolution equation (2.11b) for the derivative of  $N_A$ . The integral of  $Y^A$  against the third and fourth terms on the right-hand side of Eq. (2.11b) vanishes since it is proportional to  $\int d^2\Omega \kappa (D^4 + D^6/2) \Psi$ , where  $\kappa$  is defined by Eq. (2.15) and  $\Psi$  by Eq. (2.24). This expression vanishes since  $\kappa$  is purely  $l = 1$  while  $\Psi$  is purely  $l \geq 2$ .

associated with the generator (3.6) together with the metric (2.1) and computing the integral (92) of Ref. [6] over the surface  $u = u_0$  and  $r = r_0$  with  $r_0 \rightarrow \infty$ , we find the third, fourth, and fifth terms in Eq. (3.5) above. The remaining first and second terms are obtained by inserting the generator  $\vec{\xi}_2$  into the integral (98) of Ref. [6]. The formula (3.5) was derived by a different method<sup>11</sup> by Barnich and Troessaert in Ref. [10].

We next introduce some notation to describe the different charges. The quantity (3.5) is a linear function of  $\alpha$  and  $Y^A$ , and we choose a basis of this vector space as follows. We parametrize the function  $\alpha$  as

$$\alpha = t^0 - t^i n_i + \sum_{l=2}^{\infty} \sum_{m=-l}^l \alpha_{lm} Y_{lm}, \quad (3.7)$$

where  $n_i = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$  and  $t^\mu = (t^0, t^i)$  are real parameters. Similarly we write  $Y^A$  as a linear combination of conformal Killing vectors as

$$Y^A = \omega^{0i} e^A_i + \omega^{ij} e^A_{[i} n_{j]}, \quad (3.8)$$

where  $e^A_i = D^A n_i$ ; in Minkowski spacetime, this corresponds via Eq. (2.16) to the limiting form of the Killing vector  $\omega^{\alpha\beta} x_{[\beta} \partial_{\alpha]}$ . We now define the quantities  $P^\mu$ ,  $J^{\mu\nu}$ , and  $\mathcal{P}_{lm}$  by

$$Q = -P^\mu t_\mu + \frac{1}{2} J^{\mu\nu} \omega_{\mu\nu} + \frac{1}{4\pi} \sum_{l=2}^{\infty} \sum_{m=-l}^l \mathcal{P}_{lm}^* \alpha_{lm}. \quad (3.9)$$

Here the 4-momentum  $P^\mu$  and angular momentum  $J^{\mu\nu}$  transform in the normal way under the Lorentz transformation subgroup of the BMS group given by taking  $\alpha = 0$  in Eq. (2.12). The quantities  $\mathcal{P}_{lm}$  are usually called ‘‘supermomentum’’ [4,27,28], although they have also been considered to be generalizations of angular momentum [29]. We will use the terminology supermomentum since they are conjugate to supertranslations, they have the same physical dimension as momentum, and they are invariant under translations and supertranslations (see Appendix B), like normal momentum.

By comparing Eqs. (3.5) and (3.9) we see that the Bondi mass  $P^0$  is given by the  $l = 0$  component of the Bondi mass aspect  $m(\theta^A)$ , the linear momentum  $P^i$  is given by the  $l = 1$

<sup>11</sup>The method used by Barnich and Troessaert is based on a formula for a variation of the charge which is not integrable. This nonintegrability issue is resolved in Ref. [6]; however, it does not affect the nonradiative case considered here. Formulas similar to Eq. (3.5) were derived in Ref. [19] [their Eq. (6.14)] and Ref. [26], but those authors found the results  $\alpha_1 = 0$ ,  $\alpha_2 = -1/2$  and  $\alpha_1 = -1/8$ ,  $\alpha_2 = -2$ , respectively, where  $\alpha_1$  and  $\alpha_2$  are the coefficients of the last two terms in Eq. (3.5), whose values here are  $\alpha_1 = -1/8$ ,  $\alpha_2 = -1/2$ . We were unable to uncover the source of the discrepancies.

component, and the supermomentum by the higher  $l$  components. Similarly, when  $C_{AB} = 0$ , the angular momentum is given by the  $l = 1$  component of the angular-momentum aspect  $N_A(\theta^B)$ , with the intrinsic angular momentum being encoded in the magnetic parity piece, and the center-of-mass information being encoded in the electric parity piece, as discussed in Sec. II D above.

The definitions (3.9) are of course dependent on the choice of Bondi frame: the 4-momentum  $P^\mu$ , angular momentum  $J^{\mu\nu}$ , and supermomentum  $\mathcal{P}_{lm}$  transform into one another under BMS transformations, just as energy and 3-momentum transform into one another under Lorentz transformations. See Appendix B for details.

As a check of the charge formula (3.5), in Appendix C we compute the flux  $d\Xi$  for each generator  $\vec{\xi}$ , and verify the expected relation (3.2) between the charges on two cuts  $\mathcal{C}_1$  and  $\mathcal{C}_2$  and the integral of the flux over the intervening region  $\mathcal{R}$  of  $\mathcal{I}^+$  when the cuts are in nonradiative vacuum regions.

### C. Charges for extended BMS algebra

We now consider the additional symmetries of the extended BMS algebra discussed in Sec. II E above. Should one expect the existence of conserved quantities for singular symmetry generators  $\vec{\xi}$  like super-rotations? The generalization of Noether's theorem derived in Ref. [6] remains formally valid, but it is possible that some of the steps in the argument are invalidated when the vector field  $\vec{\xi}$  is not smooth. Ideally, one would like to generalize the derivation given there to the present context. A simpler alternative, as a first step, is to simply evaluate the final expression (3.5) for the conserved charges for the super-rotations and see if one obtains a finite result. This is the approach we will follow here, following Barnich and Troessaert [10]. Clearly a more fundamental investigation of the applicability of the generalized Noether's theorem to singular symmetry vector fields is warranted.

As before, we specialize to nonradiative vacuum regions of  $\mathcal{I}^+$ . The super-rotation charges are obtained from the general BMS charge integral (3.5), with  $\alpha = 0$  and with  $Y^A$  taken to be the super-rotation generator  $l_m$  given by Eq. (2.30a) with  $|m| > 1$ . This charge can be written as

$$Q = \frac{1}{8\pi} \int d^2\Omega Y^A \hat{N}_A, \quad (3.10)$$

where

$$\hat{N}_A = N_A - u D_A m - \frac{1}{16} D_A (C_{BC} C^{BC}) - \frac{1}{4} C_{AB} D_C C^{BC}. \quad (3.11)$$

We next explain why the integral (3.10) is finite, despite the fact that  $Y^A$  is singular. The integrand in Eq. (3.5) is a

product of a meromorphic function of  $z$  times a smooth function, and it is a well-known property of meromorphic functions that such integrals are locally finite. This result can be understood by expanding the smooth function as a sum of terms of the form  $z^p \bar{z}^q$  with  $p, q$  non-negative. Integrating against a singularity  $z^{-m}$  with  $m > 0$  yields

$$\int dz d\bar{z} z^{-m} z^p \bar{z}^q = \int d\theta \int d\rho \rho^{1-m+p+q} e^{i\theta(-m+p-q)},$$

where  $z = \rho e^{i\theta}$ . The angular integral vanishes unless  $m = p - q$ , and then the remaining factor in the integrand is proportional to  $\rho^{1+2q}$ , which is nonsingular.<sup>12</sup>

Next, we note that we can decompose  $\hat{N}_A$  uniquely into electric parity and magnetic parity pieces, as in Eq. (2.26) above. This gives rise to a decomposition of the charge (3.10) into two pieces, which we write as

$$Q = Q_e + Q_b. \quad (3.12)$$

For the super-rotations (2.30), we will call the charges  $Q_e$  ‘‘super center-of-mass charges’’. The motivation for this terminology is as follows. As discussed in Sec. II E above, super-rotations are generalizations of Lorentz transformations. For Lorentz transformations, boosts have electric type parity, while rotations have magnetic type parity [cf. Eq. (2.15)]. So it is natural to consider the electric parity pieces of super-rotations to be generalizations of boost symmetries. Finally, boost symmetries are conjugate to the center-of-mass piece of angular momentum.

Similarly, we will call the charges  $Q_b$  ‘‘superspin charges,’’ since the magnetic parity pieces of super-rotations can be thought of as generalizations of rotation symmetries, which are conjugate to intrinsic angular momentum. For  $m = 0, \pm 1$ , the charges  $Q_e$  and  $Q_b$  reduce to the normal center-of-mass and spin charges discussed in Sec. III B above.

### D. Consistency of charges of extended algebra with fluxes

In Appendix C we compute the fluxes associated with BMS generators  $\vec{\xi}$  and show that they are consistent with the charge expression (3.5) in the sense that the conservation law (3.2) is satisfied for cuts  $\mathcal{C}_1$  and  $\mathcal{C}_2$  in nonradiative regions of  $\mathcal{I}^+$ . We now consider the consistency issue for the generators  $\vec{\xi}$  of the extended algebra. All of the computations of Appendix C continue to apply in this more general context. We find from Eq. (C8) that the conservation law is now not satisfied; there is a discrepancy proportional to

<sup>12</sup>A similar computation shows that for integrals on the 2-sphere involving  $Y^A$  we can freely integrate by parts despite the singularity,  $\int d^2\Omega D_A(\varphi Y^A) = 0$  for any smooth function  $\varphi$ .



$$\int du \int d^2\Omega Y^A \epsilon_{AB} D^B \left( D^2 + \frac{1}{2} D^4 \right) \Psi, \quad (3.13)$$

where  $\Psi$  is the magnetic parity piece of  $C_{AB}$ , given by Eq. (2.24). This vanishes for BMS generators, for which  $Y^A$  is constructed from  $l = 1$  harmonics, but not for general super-rotations.

What is the explanation for this discrepancy? The general consistency between the flux and charge formulas derived by Wald and Zoupas [6] assumes that the vector fields  $\vec{\xi}$  are asymptotic symmetry vector fields, that is, vector fields which preserve asymptotic flatness. This condition is violated by the super-rotation generators used here. Consistency is restored if we add to the standard BMS flux formula (C1) the quantity<sup>13</sup>

$$\frac{1}{32\pi} \int du \int d^2\Omega C_{CE} \epsilon_{AB} \epsilon^{CD} D^E D_D D^B Y^A, \quad (3.14)$$

from Eqs. (C8) and (C9). The integrand here vanishes identically for standard BMS generators, but not for super-rotations. We conjecture that the correction (3.14) gives the correct flux formula for the extended BMS algebra. It would be useful to derive this modified flux formula from a version of the Wald-Zoupas formalism, generalized to accommodate vector fields of the form used here. A key element of such a generalization would be an enlarged space of solutions which are not asymptotically flat but are nevertheless physically relevant, as described by Strominger and Zhiboedov [21].

### E. Super center-of-mass and superspin charges in stationary regions

We now turn to deriving an explicit expression for the charge  $Q[l_m]$  associated with the super-rotation  $l_m$ . We specialize for simplicity to stationary vacuum regions of  $\mathcal{I}^+$ , and to Bondi frames satisfying the constraint (2.21) that the angular momentum aspect be nonevolving (the latter assumption will be relaxed in Appendix D). To simplify the computation, we also restrict attention to Bondi frames<sup>14</sup> which are close to the canonical Bondi frame (2.28), so that we can linearize in the deviation. In particular, we can neglect the terms quadratic in  $C_{AB}$  in Eq. (3.5), yielding from Eq. (2.22) that

<sup>13</sup>Alternatively one could restore consistency by using the standard flux (C1) and modifying the formula (3.5) for the charge integral by a quantity proportional to the  $u$  integral of the magnetic piece of  $C_{AB}$ . However this modification would be nonlocal in time.

<sup>14</sup>It might seem that the quantity we are computing is a pure gauge effect, since it vanishes in the canonical Bondi gauge. This is true locally in time, but as discussed in Sec. IV C below, the change in the charge between two successive stationary regions encodes physical, nongauge information.

$$Q = \frac{1}{8\pi} \int d^2\Omega Y^A N_A. \quad (3.15)$$

Also, Eq. (2.23) reduces to

$$D^2 N_A + N_A = -3m_0 D^B C_{AB}, \quad (3.16)$$

where we have used  $m(\theta^A) = m_0$ , a constant, from Eq. (2.22). We now use the decomposition (2.24) of  $C_{AB}$  into electric and magnetic parity pieces, and use the fact that the magnetic parity piece vanishes in stationary regions, as shown in Appendix E. This allows us to solve Eq. (3.16) to obtain

$$N_A = -3m_0 D_A \Phi / 2 + N_A^{l=1}, \quad (3.17)$$

where  $N_A^{l=1}$  is a  $l = 1$ , homogeneous solution of Eq. (3.16) of the type given by Eqs. (2.26) and (2.27). Next, we expand  $\Phi$  as

$$\Phi = \sum_{l \geq 2} \Phi_{lm} Y_{lm}, \quad (3.18)$$

and combine Eqs. (2.30a), (3.15) and (3.17) to yield

$$Q[l_m] = m_0 \sum_{l \geq |m|} \kappa_{lm} \Phi_{l,-m}, \quad \text{if } |m| > 1, \quad (3.19)$$

where the constants  $\kappa_{lm}$  are given by

$$\kappa_{lm} = -\frac{3}{8} \int_0^\pi d\theta \sin\theta (m - \cos\theta) \cot^m(\theta/2) Y_{l,-m}(\theta, 0). \quad (3.20)$$

This integral evaluates to  $\hat{\kappa}_{lm}$  for  $m \geq 2$ , and to  $(-1)^{l+1} \hat{\kappa}_{l(-m)}$  for  $m \leq -2$ , where

$$\hat{\kappa}_{lm} = -\frac{3}{8} \sqrt{\frac{(2l+1)(l+m)!}{\pi(l-m)!}} \frac{1}{(l^2+l-2)\Gamma(m-1)}. \quad (3.21)$$

These constants are finite even for  $m < 0$ . The charge  $Q[\bar{l}_m]$  is given by a similar expression but with  $\Phi_{l,-m}$  replaced by  $\Phi_{lm}$  in Eq. (3.19) and with  $Y_{l,-m}$  replaced by  $Y_{lm}$  in Eq. (3.20).

We note that the final result (3.19) comes purely from the electric parity piece of the expression (3.17), since the second term in that expression does not contribute. Hence the charge (3.19) is a super center-of-mass charge,  $Q = Q_e$ , while the superspin charge  $Q_b$  is vanishing.

The final result is that the super center-of-mass charges  $Q_e[l_m]$  and  $Q_e[\bar{l}_m]$  give information about the tensor  $C_{AB}$ ,

via Eqs. (2.24), (3.18), and (3.19). We note that the information is incomplete, since one cannot reconstruct  $C_{AB}$  from these charges.

The computations in this subsection of the charges (3.19) were specialized to Bondi frames obeying the constraint (2.21) that the angular momentum aspect be nonevolving. While such frames always exist in stationary regions, they are not the most general frames. In Appendix D we generalize the computations to remove this constraint, while retaining the assumption of linearization about the canonical Bondi frame. The result is that there are modifications to the charges for  $|m| \leq 2$  but not for higher values of  $|m|$ . In particular the superspin charges are no longer vanishing, for  $|m| \leq 2$ .

For more general Bondi frames in stationary regions, the superspin charges  $Q_b$  need not vanish. However, these charges do not contain any information not already contained in the standard BMS charges and the super center-of-mass charges  $Q_e$ . This follows from the fact that all the super-rotation charges vanish in the canonical Bondi frame in stationary regions, from Eqs. (2.28) and (3.5), and from considering the number of free functions in the general BMS transformation (2.12) to an arbitrary Bondi frame.

### F. Changes in super center-of-mass and superspin charges in nonradiative-to-nonradiative transitions

We now turn to considering the super center-of-mass and superspin charges in more general, nonstationary but still nonradiative situations. In these situations, both types of charge are finite, by the argument given in Sec. III C above. They are also independent, and it is not possible to compute them in terms of the strain tensor  $C_{AB}$ .

For the superspin charges  $Q_b$ , we now derive a formula for the change in the charges in a transition from an early nonradiative region at  $u = u_1$  to a later nonradiative region at  $u = u_2$ . Taking the magnetic part of Eq. (C6) and integrating with respect to  $u$  [or equivalently using the corrected flux given by Eqs. (3.14) and (C1)] and combining with Eq. (3.10) gives the total change

$$\begin{aligned} Q_b(u_2) - Q_b(u_1) &= - \int d^2\Omega \int du Y^A (\hat{T}_{uA}^m + \mathcal{T}_{uA}^m) \\ &\quad + \frac{1}{64\pi} \int d^2\Omega \int du Y^A \epsilon_{AB} D^B D^2 (D^2 + 2) \Psi. \end{aligned} \quad (3.22)$$

Here the superscript  $m$  means ‘‘magnetic part of,’’ and the quantity  $\mathcal{T}_{uA}$  is given by

$$\begin{aligned} \mathcal{T}_{uA} &= \frac{1}{64\pi} [3N_{AB} D_C C^{BC} - 3C_{AB} D_C N^{BC} \\ &\quad - D_B C_{AC} N^{BC} + D_B N_{AC} C^{BC}], \end{aligned} \quad (3.23)$$

a kind of gravitational-wave angular momentum flux.<sup>15</sup> This result is consistent with Pasterski *et al.* [15], who argued that the conserved quantities<sup>16</sup> associated with the super-rotation symmetries are of the form<sup>17</sup> (3.22). Here we extend their arguments to also include the electric pieces (3.19) of the super-rotation charges.

## IV. PHYSICAL INTERPRETATION OF BMS AND EXTENDED BMS CHARGES

In this final section, we make some remarks about the physical significance of BMS charges and, in particular, about how they can be measured.

### A. General considerations

For all symmetry generators  $\vec{\xi}$  and cuts  $\mathcal{C}$  of future null infinity  $\mathcal{I}^+$ , the charges  $Q[\vec{\xi}, \mathcal{C}]$  are defined in terms of integrals over  $\mathcal{C}$ , which can be evaluated in terms of a limit of integrals over finite 2-surfaces that tend to  $\mathcal{C}$ . It follows that the charges can in principle be measured by a collection of observers distributed over a 2-surface near future null infinity, who each make local measurements of the spacetime geometry in their vicinity and of their motion and orientation relative to their neighbors, who then communicate this information to one another, and who finally process it in a suitable way. Thus, in principle, the charges are measurable quantities. Of course, it would be useful to understand in more detail how to specify a local<sup>18</sup> operational prescription<sup>19</sup> for such measurements. The formalism of rigid quasilocal frames of Refs. [32–35] might be useful for this purpose, as well as the covariant-conformal-completion formalism for asymptotically flat spacetimes [18].

Specifying an asymptotic Bondi frame allows observers to establish a convention for labeling the various charges. By an asymptotic Bondi frame, we mean a choice of coordinates  $(u, \theta^A)$  on  $\mathcal{I}^+$  for which the spacetime metric takes the form (2.1), which is unique up to BMS transformations. A Bondi frame is determined up to an overall

<sup>15</sup>Note that this flux differs from the gravitational-wave angular momentum flux defined in Eq. (2.3) of Ref. [15]. That flux characterizes the evolution of the angular momentum aspect, but not the radiated angular momentum. The two fluxes differ by a total derivative with respect to  $u$ .

<sup>16</sup>They derived a conservation law of the form (3.4) for gravitational scattering from past null infinity to future null infinity for Christodoulou-Klainerman spacetimes.

<sup>17</sup>The difference between the angular momentum flux definition (3.23) and that used in [15] implies that the charges here and there do not coincide; however the two conservation laws are equivalent in the sense that each can be derived from the other.

<sup>18</sup>By this we mean to exclude, for example, demanding that observers be stationary with respect to retarded Bondi coordinates, which would be a nonlocal requirement.

<sup>19</sup>Such a prescription can be given for the Poincaré charges in stationary situations [30,31].

$SO(3)$  rotation by specifying a single cross section or cut of  $\mathcal{I}^+$ . One can equivalently think of a Bondi frame as an equivalence class of coordinate systems on spacetime whose asymptotic limits coincide in a suitable way, or as a class of asymptotic observers who adjust their relative motions, clocks and orientations in such a way as to establish an approximate consistent convention for specifying the results of asymptotic measurements.<sup>20</sup>

### B. Nonradiative vacuum regions of $\mathcal{I}^+$

In nonradiative vacuum regions of  $\mathcal{I}^+$ , the Bondi mass aspect  $m = m(\theta^A)$  and shear  $C_{AB}(\theta^C)$  are independent of  $u$ , as argued in Sec. IID above. Moreover the angular momentum aspect has the form  $N_A(u, \theta^B) = {}^0N_A(\theta^B) + {}^1N_A(\theta^B)u$ , from Eq. (2.11b). These functions can in principle be extracted from measurements of the asymptotic components of the Weyl tensor, listed in Appendix A; see Ref. [36] for details.

The Bondi mass aspect  $m(\theta^A)$  encodes the Bondi 4-momentum and supermomentum, as in Eq. (B4) below. In stationary regions, the supermomentum does not contain any additional information aside from the Bondi 4-momentum, from Eqs. (2.28a) and (B5). However in more general nonradiative regions it does. For example, for the linearized gravitational field of two point particles which have a relative boost, one can extract from the supermomentum the individual 4-momenta of the two particles. The conservation laws associated with supermomentum can be described as a separate conservation law for energy at every angle, as explained by Strominger [11].

Similarly the combination  $\hat{N}_A$  of the angular momentum aspect and strain tensor given by Eq. (3.11) encodes the super center-of-mass and superspin charges via Eq. (3.10). The super center-of-mass charges encode the supertranslation that relates the given BMS frame to the BMS center-of-mass frame; they can be set to zero using a supertranslation. In stationary regions the superspin charges encode just the spin of the spacetime. However in more general, nonradiative regions they contain more information, like the supermomentum. For example, for the linearized gravitational field of two spinning point particles which have a relative boost, one can extract from the superspin charges the individual spins of the two particles. The conservation laws associated with superspin could be described as a separate conservation law for angular momentum at every angle [cf. Eq. (4.8) below].

<sup>20</sup>There is a close analogy to local Lorentz frames, which can be thought of as a specification of a set of orthonormal basis vectors at a point in spacetime; an equivalence class of local coordinate systems; or a class of observers who adjust their motions, clocks and orientations in order to establish an approximate convention for specifying the results of measurements of components of tensors near that point.

### C. Relation to gravitational-wave memory

Gravitational-wave memory is the relative displacement of initially comoving observers caused by the passage of a burst of gravitational waves [37–39]. There is a well-known close relation between gravitational-wave memory for observers near future null infinity and the BMS group: the supertranslation that relates the canonical Bondi frame of an initially nonradiative region to that of a final nonradiative region encodes the observed memory effect [25,29,40]. See Strominger *et al.* [17] for a recent clear exposition of this relation in the retarded Bondi coordinates used here. The memory/supertranslation effect can also be characterized in a gauge-invariant but nonlocal way in terms of a generalized holonomy around a suitable closed loop in spacetime near  $\mathcal{I}^+$  [30].

Here we point out a new aspect of this story: a close correspondence between the two different infinite families of extended BMS charges (supermomentum and super center-of-mass) and the two different types of memory (ordinary and null [14]).

Consider a spacetime in which the flux of energy to future null infinity vanishes in the vicinity of some early retarded time  $u_1$ , so that the news tensor  $N_{AB}$  and stress-energy tensor vanish there. Suppose that there is subsequently a burst of gravitational waves and/or matter energy flux to infinity, and that the fluxes vanish again in the vicinity of some later retarded time  $u_2$ . Freely falling, initially comoving adjacent observers near infinity can measure their net relative displacement, and as shown in Ref. [17], to leading order in  $1/r$  this displacement is encoded in the change

$$\Delta C_{AB} = C_{AB}(u_2) - C_{AB}(u_1) \quad (4.1)$$

of the tensor  $C_{AB}$ . Thus, we will identify the change (4.1) as the gravitational-wave-memory observable.<sup>21</sup>

The observable change (4.1) can be decomposed into electric parity and magnetic parity pieces, as in Eq. (2.24):

$$\Delta C_{AB} = \left( D_A D_B - \frac{1}{2} h_{AB} D^2 \right) \Delta \Phi + \epsilon_{C(A} D_{B)} D^C \Delta \Psi, \quad (4.2)$$

where  $\Delta \Phi = \Phi(u_2) - \Phi(u_1)$  and  $\Delta \Psi = \Psi(u_2) - \Psi(u_1)$ . These two pieces can in principle be measured by surrounding a source of gravitational waves with a collection of observers distributed on a 2-sphere, having them each measure the gravitational-wave memory, and then decomposing the resulting function on the 2-sphere into electric and magnetic pieces, as discussed by Winicour [41,42].

<sup>21</sup>From Eq. (2.18b) the change  $\Delta C_{AB}$  is invariant under supertranslations and transforms just under the Lorentz group.

This would be analogous to measurements of E and B modes of the cosmic-microwave-background polarization. We now discuss these two pieces separately.

### 1. Electric parity piece of shear

We can compute the electric parity piece  $\Delta\Phi$  as follows. Following Ref. [17], we substitute the decomposition (2.24) into the evolution equation (2.11a) for the Bondi mass aspect and integrate from  $u_1$  to  $u_2$ . The result is

$$\Delta m = -4\pi\Delta\mathcal{E} + \mathcal{D}\Delta\Phi. \quad (4.3)$$

Here  $\Delta m = m(u_2) - m(u_1)$  is the change in the Bondi mass aspect,  $\mathcal{D}$  is the angular differential operator

$$\mathcal{D} = D^2/4 + D^4/8, \quad (4.4)$$

and

$$\Delta\mathcal{E} = \int_{u_1}^{u_2} du \left[ \hat{T}_{uu} + \frac{1}{32\pi} N_{AB}N^{AB} \right] \quad (4.5)$$

is the total energy radiated per unit solid angle in either matter or gravitational waves. Next, we act on both sides of Eq. (4.3) with the projection operator  $\mathcal{P}$  that sets to zero the  $l=0, 1$  pieces of functions on the sphere, and by the inverse<sup>22</sup> of the operator  $\mathcal{D}$ . Using  $\mathcal{P}\mathcal{D} = \mathcal{D}$  the result is [17]

$$\Delta\Phi = \mathcal{D}^{-1}\mathcal{P}\Delta m + 4\pi\mathcal{D}^{-1}\mathcal{P}\Delta\mathcal{E}. \quad (4.6)$$

The left-hand side of this equation is the observable, the (electric parity piece of) the gravitational-wave memory. The second term on the right-hand side is what Bieri and Garfinkle called the “null memory,” the piece of the memory that is computable directly in terms of fluxes of energy<sup>23</sup> to future null infinity. The first term is what Bieri and Garfinkle called “ordinary memory,” the kind originally discussed by Zel’dovich [37], which is computable from the change in the asymptotic component (A4) of the Weyl curvature tensor between early and late times.

We see from Eq. (4.6) that the ordinary memory is reflected in the  $l \geq 2$  components of the Bondi mass aspect  $m(\theta^A)$ , or equivalently the supermomenta  $\mathcal{P}_{lm}$ , from Eq. (B4). The total, ordinary plus null memory is encoded

<sup>22</sup>See footnote 6 above.

<sup>23</sup>Bieri and Garfinkle worked in the context of linearized gravity, so they did not have the gravitational-wave energy-flux term in Eq. (4.5). This term was originally computed in vacuum by Christodoulou [43] who called the effect “nonlinear memory.” The formula with both matter and gravitational-wave fluxes was derived in [17].

in the shear tensor, or  $\Delta\Phi$ . In the special case of stationary-to-stationary transitions this is in turn (partially) encoded in the super center-of-mass charges, from Eq. (3.19).

### 2. Magnetic parity piece of shear and spin memory

Turn now to the magnetic parity piece  $\Delta\Psi$  of the shear. For stationary-to-stationary transitions, it follows from the result of Appendix E that  $\Delta\Psi$  vanishes. For more general nonradiative-to-nonradiative transitions, it is known that  $\Delta\Psi$  vanishes in the context of linearized gravity [14,42]. We conjecture that  $\Delta\Psi$  also vanishes in full general relativity for such transitions. If this conjecture is true, then there is no magnetic piece of normal gravitational-wave memory.

However, there is another observable that will be generically nonvanishing, the time integral over the burst of gravitational waves of the magnetic piece of the shear, or

$$\int du\Psi. \quad (4.7)$$

This constitutes a new type of gravitational-wave memory, *spin memory*, discovered by Pasterski *et al.* [15]. It can be measured by observers who monitor the time-dependent gravitational-wave strain, integrate that quantity with respect to time, and decompose on a 2-sphere to extract the magnetic parity part. The time integral of the shear can alternatively be measured in principle by measuring the mapping between initial relative displacement and velocity, and final relative displacement and velocity for a pair of adjacent freely falling test masses [44]. Finally, the spin memory observable (4.7) can also be measured using Sagnac interferometers by a certain class of accelerated observers [15].

Just as for normal (electric parity) memory, spin memory can be decomposed into null and ordinary pieces. Following [15] we integrate the magnetic piece of Eq. (C6) with respect to  $u$  and contract with  $\epsilon^{AC}D_C$  to obtain

$$\Delta(\epsilon^{AC}D_C\hat{N}_A) = -8\pi\epsilon^{AC}D_C\Delta\mathcal{E}_A + D^2\mathcal{D} \int du\Psi. \quad (4.8)$$

Here  $\mathcal{D}$  is given by Eq. (4.4),

$$\Delta\mathcal{E}_A = \int_{u_1}^{u_2} du [\hat{T}_{uA} + \mathcal{T}_{uA}] \quad (4.9)$$

is the total angular momentum radiated per unit solid angle in either matter or gravitational waves, and  $\mathcal{T}_{uA}$  is given by Eq. (3.23). It follows that

$$\begin{aligned} \int du\Psi &= \mathcal{D}^{-1}D^{-2}\mathcal{P}\Delta(\epsilon^{AC}D_C\hat{N}_A) \\ &\quad + 8\pi\mathcal{D}^{-1}D^{-2}\mathcal{P}\epsilon^{AC}D_C\Delta\mathcal{E}_A. \end{aligned} \quad (4.10)$$

The second term here is null spin memory, computable in terms of the flux of angular momentum to null infinity. The first term is ordinary spin memory, computable from the changes in the asymptotic components of the Weyl tensor, or, equivalently, from changes in the superspin charges.

#### D. BMS charges as black hole hair

The BMS charges we have been discussing are universal, applying to any kind of isolated object in an asymptotically flat spacetime. In particular, they apply to black holes. In this context, they can be thought of as a kind of black hole hair, as pointed out by Strominger and Zhiboedov (SZ) [17], who dubbed them “soft hair.” SZ discussed how this hair could be measured in terms of a gravitational-wave-memory observation. Here we expand on that description to clarify how the no-hair theorems [45] are evaded.

The supermomentum charges  $\mathcal{P}_{lm}$  characterize departures from stationarity [this follows from Eqs. (2.28a), (3.5), and (3.9) above], so they vanish<sup>24</sup> for black holes once they settle down to a stationary state. Thus, black holes do not admit supermomentum hair. The same is true for superspin charges.

The super center-of-mass charges (i.e. the electric parity piece of the shear tensor  $C_{AB}$ ) are closely analogous to the angles  $(\theta, \varphi)$  that specify the direction of the black hole’s intrinsic angular momentum  $\mathbf{S}$ . Do those angles constitute black hole hair? Clearly, they do not give information about intrinsic properties of the black hole, since they merely reflect an orientation with respect to an arbitrarily chosen asymptotic reference frame. On the other hand, if one considers observations at more than one time, and if the black hole accretes some angular momentum, then angle  $\Delta\Theta$  by which the orientation changes between early and late times is a physical property of the black hole, independent of any choice of reference frame. In addition, even if one restricts attention to one instant of time, in quantum mechanics one can have superpositions of different angular-momentum eigenstates, and the existence of a nontrivial superposition is again a physical property of the black hole, independent of any choice of reference frame.

The super center-of-mass hair is exactly analogous. In the classical theory, at one instant of time, they do not give any information about intrinsic properties of the black hole. Instead, they give information about properties of the black hole relative to an arbitrarily chosen asymptotic Bondi frame, and those properties can be

<sup>24</sup>More precisely, in a stationary region of  $\mathcal{I}^+$  the supermomentum components  $\mathcal{P}_{lm}$  need not vanish, depending on the choice of Bondi frame. However, they are determined by the Bondi 4-momentum, so they contain no additional information.

made to vanish with a suitable choice of Bondi frame in stationary situations [cf. Eq. (2.28b) above]. Hence the no-hair theorems are not violated. On the other hand, if one considers measurements made at two different times at which the black hole is stationary, then the changes (4.1) in the charges give nontrivial physical information, independent of any choice of reference frame. This information is the gravitational-wave memory/supertranslation/generalized holonomy, as explained by SZ. Finally, if one restricts attention to one instant of time, one can have nontrivial superpositions of super center-of-mass eigenstates, and the existence of a nontrivial superposition is a physical property of the black hole, independent of any choice of reference frame. To produce such superpositions one can throw into a black hole matter that is in a superposition of two states, one state for which the gravitational-wave emission associated with the accretion produces a net gravitational-wave memory, and one state for which the net memory is zero. Thus in quantum gravity, the set of quantum states associated with low energy, asymptotic degrees of freedom of the black hole is richer than what would be expected from the classical theory locally in time.

#### V. CONCLUSIONS

In this paper, we have investigated the suggestion of Refs. [7–10] that the BMS symmetry algebra be extended. While we have found that some of the symmetry generators of the extended algebra have conserved charges that are finite and are associated with gravitational-wave memory, there are several outstanding puzzles and open issues:

- (i) Consistency of the superspin charges with fluxes requires a correction to the standard formula for the flux associated with the BMS generator. It would be useful to derive this correction from first principles.
- (ii) We computed the charges only in a certain regime where some nonlinearities could be neglected [cf. the discussion before Eq. (3.15) above] and only in stationary, vacuum regions of  $\mathcal{I}^+$ . It might be interesting to investigate the properties of the charges more generally.
- (iii) The new charges capture some but not all of the information associated with the observable gravitational-wave memory—cf. the discussion after Eq. (3.20) above. This suggests that yet larger symmetry algebras might be relevant.

A summary of the status of the various charges and results discussed in this paper is given in Table I.

#### ACKNOWLEDGMENTS

We thank David Garfinkle, Abhay Ashtekar and Blagoje Oblak for helpful conversations, and Leo Stein for helpful

TABLE I. A summary of the charges discussed in this paper, the corresponding symmetries, the status of the corresponding conservation laws, and the relations to gravitational-wave memory.

Charge	Symmetry	Metric function in which encoded	Interpretation of conservation law	Status of conservation law from one cut of $\mathcal{I}^+$ to another	Status of conservation law for gravitational scattering from $\mathcal{I}^-$ to $\mathcal{I}^+$	Relation to gravitational-wave memory
Supermomentum	Supertranslations (standard BMS algebra)	Bondi mass aspect $m(\theta^A)$	Energy conservation at every angle [11]	Established in [4–6]	Established in [11] for Christodoulou-Klainerman spacetimes [23]. In that context supermomentum conservation contains no information beyond 4-momentum conservation. <sup>a</sup> Not yet established more generally.	Changes in supermomentum encode ordinary [14] piece of normal gravitational-wave memory (Sec. IV C 1)
Superspin	Magnetic parity piece of super-rotations (extended BMS algebra)	Magnetic parity piece of angular momentum aspect $N_A(\theta^B)$	Angular momentum conservation at every angle [15]	Established in [15]	Established in [15] for Christodoulou-Klainerman spacetimes. Not yet established more generally.	Changes in superspin encode ordinary [14] piece of gravitational-wave spin memory [15] (Sec. IV C 2)
Super center-of-mass (also called “soft hair” in the context of black holes [46–49])	Electric parity piece of super-rotations (extended BMS algebra)	Electric parity piece of angular momentum aspect $N_A(\theta^B)$	Center-of-mass conservation at every angle (Sec. III C)	Established here (Sec. III D)	Not yet established	Changes in super center-of-mass charges encode the total (ordinary plus null) normal gravitational-wave memory for stationary-to-stationary transitions (Sec. IV C 1)

<sup>a</sup>See footnote 9.

discussions and for evaluating the integral (3.20). We thank Andy Strominger, Sabrina Pasterski and Thomas Maedler for helpful correspondence, and an anonymous referee for insightful comments. This research was supported in part by National Science Foundation Grants No. PHY-1404105 and No. PHY-1068541.

## APPENDIX A: WEYL TENSOR

For vacuum solutions, the leading-order components of the Weyl tensor in the retarded Bondi coordinates are as follows. We define the basis of vector fields

$$\vec{e}_{\hat{u}} = \partial_u, \quad \vec{e}_{\hat{r}} = \partial_r, \quad \vec{e}_{\hat{A}} = \frac{1}{r}\partial_A, \quad (\text{A1})$$

which is asymptotically orthonormal as  $r \rightarrow \infty$ . The leading-order,  $O(1/r)$  components of the Weyl tensor on this basis are Petrov type IV from the peeling theorem. They are given by

$$C_{\hat{u}\hat{A}\hat{u}\hat{B}} = -\frac{1}{2r}\ddot{C}_{AB} \quad (\text{A2})$$

with all other components (except those related to these by symmetries) vanishing at this order. The subleading-order,  $O(1/r^2)$  components are given by

$$C_{\hat{u}\hat{r}\hat{u}\hat{A}} = -\frac{1}{2r^2}D^B\dot{C}_{AB}, \quad (\text{A3a})$$

$$C_{\hat{u}\hat{A}\hat{u}\hat{B}} = \frac{1}{4r^2}[D^2\dot{C}_{AB} - 2\dot{C}_{AB} - h_{AB}C_{CD}\ddot{C}^{CD}] \quad (\text{A3b})$$

$$C_{\hat{u}\hat{A}\hat{B}\hat{C}} = -\frac{1}{r^2}D_{[B}\dot{C}_{C]A}. \quad (\text{A3c})$$

At order  $O(1/r^2)$ , the Weyl tensor is Petrov type III. The remaining components scale as  $C_{\hat{u}\hat{r}\hat{u}\hat{r}} \sim C_{\hat{u}\hat{r}\hat{A}\hat{B}} \sim C_{\hat{A}\hat{B}\hat{C}\hat{D}} \sim r^{-3}$  and  $C_{\hat{u}\hat{r}\hat{r}\hat{A}} \sim C_{\hat{r}\hat{A}\hat{r}\hat{B}} \sim C_{\hat{r}\hat{A}\hat{B}\hat{C}} \sim r^{-4}$ . In particular we have

$$C_{\hat{u}\hat{r}\hat{u}\hat{r}} = -\frac{1}{r^3}\left[2m + \frac{1}{4}C_{AB}N^{AB}\right] + O\left(\frac{1}{r^4}\right) \quad (\text{A4})$$

and

$$C_{\hat{u}\hat{r}\hat{r}\hat{A}} = \frac{N_A}{r^4} + O\left(\frac{1}{r^5}\right). \quad (\text{A5})$$

## APPENDIX B: TRANSFORMATION PROPERTIES OF CHARGES UNDER FINITE BMS TRANSFORMATIONS

The transformation properties of the charges under finite BMS transformations can be derived from the definitions (2.12), (2.13), and (3.1); see, for example, Appendixes C.5

and C.6 of Chrusciel *et al.* [19]. Here we restrict attention to vacuum nonradiative regions of  $\mathcal{I}^+$  so that we can neglect the cut dependence of the charges. Consider a finite BMS transformation  $\psi: \mathcal{I}^+ \rightarrow \mathcal{I}^+$  of the form (2.12). It can be parametrized in terms of a map  $\varphi: S^2 \rightarrow S^2$  of the 2-sphere to itself and a function  $\beta$  on the 2-sphere [denoted by  $\alpha$  in Eq. (2.12)]. The BMS generator  $\vec{\xi} = (\alpha, Y^A)$  given by Eq. (2.13) is mapped by the pullback to  $\psi_*\vec{\xi} = (\hat{\alpha}, \hat{Y}^A)$ , where  $\hat{Y}^A = \varphi_*Y^A$ ,

$$\hat{\alpha} = \omega_\varphi\varphi_*\alpha + \frac{1}{2}\beta D_A\hat{Y}^A - \hat{Y}^A D_A\beta, \quad (\text{B1})$$

and  $\omega_\varphi$  is defined by  $\varphi_*h_{AB} = \omega_\varphi^{-2}h_{AB}$ . For rotations  $\omega_\varphi = 1$ , while for a boost with rapidity parameter  $\eta$  in the direction  $\mathbf{m}$ ,  $\omega_\varphi(\mathbf{n}) = \cosh\eta + \sinh\eta\mathbf{n} \cdot \mathbf{m}$ . The charges transform according to<sup>25</sup>

$$Q[\vec{\xi}] \rightarrow \bar{Q}[\vec{\xi}] = Q[\psi_*^{-1}\vec{\xi}]. \quad (\text{B2})$$

For boosts and rotations the results are as follows. The 4-momentum and angular momentum transform in the standard way:

$$\bar{P}^\alpha = \Lambda^\alpha_\beta P^\beta, \quad \bar{J}^{\alpha\beta} = \Lambda^\alpha_\mu \Lambda^\beta_\nu J^{\mu\nu}. \quad (\text{B3})$$

Here  $\Lambda^\alpha_\beta$  is the Lorentz transformation that is naturally associated with  $\varphi$ , which can be obtained by demanding that the action of the Lorentz transformation on the set of null directions  $(1, \mathbf{n})$  coincide with that of  $\varphi^{-1}$ .<sup>26</sup> For the supermomentum, it is more convenient to give the transformation law in terms of the Bondi mass aspect, which encodes the supermomenta according to [cf. Eqs. (3.5), (3.7), and (3.9)]

$$m(\theta^A) = P^0 + 3P_i n^i + \sum_{l \geq 2} \sum_m \mathcal{P}_{lm} Y_{lm}. \quad (\text{B4})$$

The transformation law is

$$\bar{m}(\theta^A) = m[\varphi(\theta^A)]\omega_\varphi(\theta^A)^{-3}. \quad (\text{B5})$$

Finally the tensor  $C_{AB}$  which encodes the super center-of-mass charges transforms as

<sup>25</sup>We use  $\psi^{-1}$  instead of  $\psi$  in this transformation law in order to agree with the convention of the linearized analysis of Sec. II B above. The charges are invariant under  $\vec{\xi} \rightarrow \psi_*\vec{\xi}$ ,  $g_{ab} \rightarrow \psi_*g_{ab}$ , so the convention  $g_{ab} \rightarrow \psi_*g_{ab}$  used there is equivalent to the convention  $\vec{\xi} \rightarrow \psi_*^{-1}\vec{\xi}$  used here.

<sup>26</sup>Formally the Lorentz transformation  $\Lambda^\alpha_\beta$  as well as  $P^\alpha$  and  $J^{\alpha\beta}$  are tensors over the four-parameter translation subgroup of the BMS group, which has a flat  $(-, +, +, +)$  metric [20].

$$\bar{C}_{AB} = \omega_\varphi \varphi_* C_{AB}. \quad (\text{B6})$$

Consider next translations and supertranslations, which are parametrized by the function  $\beta$ . The 4-momentum and supermomentum are invariant under these transformations. The angular momentum transforms as  $\bar{J}_{\mu\nu} = J_{\mu\nu} + \delta J_{\mu\nu}$  with

$$\delta J_{ij} = \frac{1}{2\pi} \int d^2\Omega m e^A_{[i} n_{j]} D_A \beta, \quad (\text{B7a})$$

$$\delta J_{0i} = -\frac{1}{4\pi} \int d^2\Omega \beta (e_i^A D_A m - 3n_i m). \quad (\text{B7b})$$

By using Eq. (B4) these angular momentum changes can be expressed in terms of the 4-momentum and supermomentum. For a normal translation with  $\beta = t^0 - t^i n_i$  we recover the standard transformation law  $\delta J_{\mu\nu} = -t_\mu P_\nu + t_\nu P_\mu$ . Finally the tensor  $C_{AB}$  transforms according to the same transformation law as in the linearized case:

$$\bar{C}_{AB} = C_{AB} - 2D_A D_B \beta + h_{AB} D^2 \beta. \quad (\text{B8})$$

### APPENDIX C: VERIFICATION OF FLUX CONSERVATION LAW FOR STANDARD BMS ALGEBRA

In this appendix we compute the flux  $d\Xi$  for each generator  $\vec{\xi}$  of the standard BMS algebra, and verify the expected relation (3.2) between the charges on two cuts  $\mathcal{C}_1$  and  $\mathcal{C}_2$  and the integral of the flux over the intervening region  $\mathcal{R}$  of  $\mathcal{I}^+$ . This computation serves as a check of the charge formula (3.5). We assume that the cuts  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are in nonradiative vacuum regions of  $\mathcal{I}^+$ .

The flux in vacuum is proportional to the Bondi news tensor  $N_{AB}$  and is given by Eq. (82) of Wald and Zoupas [6]. Translating this formula to Bondi coordinates and adding the appropriate stress-energy flux gives for the total flux

$$\int_{\mathcal{R}} d\Xi = - \int_{\mathcal{R}} \left( \frac{1}{32\pi} N^{AB} \delta C_{AB} + \hat{T}_{ua} \xi^a \right) du d^2\Omega. \quad (\text{C1})$$

Here the quantity  $\delta C_{AB}$  is the change in  $C_{AB}$  under the BMS transformation  $\vec{\xi}$ , given by Eq. (2.18b).

Since the conservation law (3.2) is linear in the generator  $\vec{\xi}$ , it is sufficient to verify the law separately for the translation/supertranslation piece of  $\vec{\xi}$ , parametrized by  $\alpha$ , and the remaining piece, parametrized by  $Y^A$ . We first consider the translation/supertranslation piece. Using the expressions (2.18b) and (2.17) for  $\delta C_{AB}$  and the

formula (2.13) for the generator, specialized to  $Y^A = 0$ , and integrating by parts, we obtain

$$\int_{\mathcal{R}} d\Xi = -\frac{1}{32\pi} \int_{\mathcal{I}^+} \alpha (N^{AB} N_{AB} - 2D_A D_B N^{AB} + 32\pi \hat{T}_{uu}) du d^2\Omega. \quad (\text{C2})$$

Using the evolution equation (2.11a) for the Bondi mass aspect and assuming that the cuts  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are of the form  $u = u_1$  and  $u = u_2$  gives

$$\int_{\mathcal{R}} d\Xi = \frac{1}{4\pi} \int_{\mathcal{C}_2} \alpha m d^2\Omega - \frac{1}{4\pi} \int_{\mathcal{C}_1} \alpha m d^2\Omega, \quad (\text{C3})$$

which coincides with the required form (3.2) by Eq. (3.5).

Turn now to the Lorentz transformations parametrized by  $Y^A$ . Inserting the expressions (2.18b) and (2.17) for  $\delta C_{AB}$  and the formula (2.13) for the generator into Eq. (C1), specializing  $\alpha = 0$ , and integrating by parts, we obtain

$$\int_{\mathcal{R}} d\Xi = -\frac{1}{32\pi} \int d^2\Omega \int_{u_1}^{u_2} du Y^A \mathcal{H}_A \quad (\text{C4})$$

where

$$\begin{aligned} \mathcal{H}_A = & -\frac{1}{2} u D_A (N_{BC} N^{BC}) + u D_A D_B D_C N^{BC} \\ & + \frac{1}{2} D_A (C_{BC} N^{BC}) + N^{BC} D_A C_{BC} \\ & - 2D_B (N^{BC} C_{AC}) + 32\pi \hat{T}_{uA} - 16\pi u D_A \hat{T}_{uu}. \end{aligned} \quad (\text{C5})$$

We next compute the change in the charge, given by the right-hand side of (3.2), to compare with (C4). Differentiating the definition (3.11) of  $\hat{N}_A$  with respect to  $u$  and using the evolution equations (2.11) together with (2.7) gives

$$\begin{aligned} \partial_u \hat{N}_A = & -8\pi \hat{T}_{uA} + 4\pi u D_A \hat{T}_{uu} + \frac{u}{8} D_A (N_{BC} N^{BC}) \\ & - \frac{3}{8} N_{AB} D_C C^{BC} + \frac{3}{8} C_{AB} D_C N^{BC} + \frac{1}{8} D_B C_{AC} N^{BC} \\ & - \frac{1}{8} D_B N_{AC} C^{BC} + \frac{1}{4} D_B D_A D_C C^{BC} - \frac{1}{4} D_B D^B D^C C_{AC} \\ & - \frac{1}{4} u D_A D_B D_C N^{BC} - 2\pi \partial_u \hat{T}_{rA}. \end{aligned} \quad (\text{C6})$$

Here we have also used the identities

$$D_A C_{BC} N^{BC} = D_B C_{CA} N^{BC} + N_{AB} D_C C^{BC}, \quad (\text{C7a})$$

$$D_A N_{BC} C^{BC} = D_B N_{CA} C^{BC} + C_{AB} D_C N^{BC}, \quad (\text{C7b})$$

which can be verified by evaluating both sides in complex stereographic coordinates  $(z, \bar{z})$ .



We now integrate Eq. (C6) between  $u_1$  and  $u_2$ , use the expression (3.10) for the charge  $Q$ , use the fact that  $\hat{T}_{rA}$  vanishes at  $u_1$  and  $u_2$ , and compare with the flux (C4). The result is

$$\int_{\mathcal{R}} d\Xi = Q(\mathcal{C}_2, \vec{\xi}) - Q(\mathcal{C}_1, \vec{\xi}) + \Delta\mathcal{F}, \quad (\text{C8})$$

where the anomalous term is

$$\Delta\mathcal{F} = \frac{1}{32\pi} \int du \int d^2\Omega Y^A \epsilon_{AB} \epsilon^{CD} D^B D_D D^E C_{CE}. \quad (\text{C9})$$

If we now decompose  $C_{AB}$  into electric and magnetic parity pieces according to Eq. (2.24), we can rewrite this as

$$\Delta\mathcal{F} = -\frac{1}{32\pi} \int du \int d^2\Omega Y^A \epsilon_{AB} D^B \left( D^2 + \frac{1}{2} D^4 \right) \Psi. \quad (\text{C10})$$

For BMS transformations,  $Y^A$  is of the form (2.15), where  $\chi$  and  $\kappa$  are purely  $l = 1$ . Integrating by parts, we see that the expression (C10) vanishes, since  $l = 1$  harmonics are annihilated by the operator  $D^2 + 2$ . Hence we have verified the conservation law (3.2).

#### APPENDIX D: COMPUTATION OF SUPER-ROTATION CHARGES IN STATIONARY REGIONS IN A MORE GENERAL CLASS OF BONDI FRAMES

The computations of the super-rotation charges in Sec. III C were specialized to Bondi frames obeying the constraint (2.21) that the angular momentum aspect be nonevolving. While such frames always exist in stationary regions, they are not the most general frames. We now extend those computations to remove this constraint, by using a different computational method.

As before, we restrict attention to a region of future null infinity in which the spacetime is stationary, and in which the leading and subleading stress-energy tensor components vanish. We start from the canonical Bondi frame (2.28), and write the angular momentum aspect in that frame as

$$N_A = \epsilon_{AB} D^B \Theta, \quad (\text{D1})$$

where  $\Theta$  is purely  $l = 1$  and independent of  $u$  and encodes the intrinsic angular momentum. We now perform a general linearized BMS transformation parametrized by the vector field on future null infinity of the form [cf. Eq. (2.13) above, with some changes of notation]

$$\left[ -\frac{1}{2} \Phi + \frac{1}{2} u \lambda \right] \partial_u + \left[ -\frac{1}{2} D^A \lambda + \epsilon^{AB} D_B \kappa \right] \partial_A. \quad (\text{D2})$$

Here  $\Phi$ ,  $\lambda$  and  $\kappa$  are functions on the 2-sphere which are independent of  $u$ . The supertranslation piece  $\Phi$  is arbitrary, while the boost piece  $\lambda$  and the rotation piece  $\kappa$  are  $l = 1$  harmonics. We now combine this with the transformation laws (2.18), the canonical form (2.28) and (D1) of the metric functions, the result (2.25), and use  $D_A D_B \lambda = -h_{AB} \lambda$  and similarly for  $\kappa$ . Working to linear order in  $\Phi$ ,  $\lambda$  and  $\kappa$ , this yields for the metric functions in the new frame

$$m = m_0 + \frac{3}{2} m_0 \lambda, \quad (\text{D3a})$$

$$C_{AB} = D_A D_B \Phi - \frac{1}{2} h_{AB} D^2 \Phi, \quad (\text{D3b})$$

$$N_A = \epsilon_{AB} D^B [\tilde{\Theta}(1 + \lambda/2)] - \frac{3}{2} m_0 D_A \Phi + \frac{3}{2} u m_0 D_A \lambda. \quad (\text{D3c})$$

Here  $\tilde{\Theta}$  is given by

$$\tilde{\Theta} = \Theta + \epsilon^{AB} D_B \kappa D_A \Theta, \quad (\text{D4})$$

is purely  $l = 1$ , and represents the intrinsic angular momentum in the rotated frame. The results (D3) agree with the expressions (3.17) derived in Sec. III C above, except for the new terms involving the rotation  $\kappa$  and the boost  $\lambda$ .

Next, we insert the results (D3) into the formula (3.5) for the BMS charge, specialized to  $\alpha = 0$ , neglecting terms quadratic in  $C_{AB}$  as before. This gives

$$Q = \frac{1}{8\pi} \int d^2\Omega Y^A \hat{N}_A, \quad (\text{D5})$$

where

$$\hat{N}_A = \epsilon_{AB} D^B [\tilde{\Theta}(1 + \lambda/2)] - \frac{3}{2} m_0 D_A \Phi. \quad (\text{D6})$$

Comparing this with Eq. (3.17) we see that there is an extra contribution to the magnetic parity piece of the super-rotation charges  $Q[l_m]$  for  $|m| > 1$  associated with the boost piece  $\lambda$  of the BMS transformation. It gives a nonzero contribution only for  $|m| = 2$ .

#### APPENDIX E: MAGNETIC PARITY PIECE OF SHEAR VANISHES IN STATIONARY VACUUM REGIONS

In this appendix we show that the magnetic parity piece of the shear tensor  $C_{AB}$  vanishes in stationary vacuum regions of future null infinity  $\mathcal{I}^+$ , in arbitrary Bondi frames. Closely related results have been derived by Winicour [41,42], and by Bieri and Garfinkle [14] in linearized gravity.

In the canonical Bondi frame discussed in Sec. II D above, the metric functions take the simple form (2.28), and in particular  $C_{AB} = 0$ . Consider now making a transformation to an arbitrary Bondi frame, using the general nonlinear BMS transformation discussed in Appendix B. We can decompose

such a transformation into a rotation, followed by a boost, followed by a supertranslation. The rotation and boost maintain  $C_{AB} = 0$ , by Eq. (B6). Finally, under the supertranslation the shear tensor undergoes the transformation (B8), which generates only an electric parity piece of  $C_{AB}$ .

- 
- [1] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Gravitational waves in general relativity. VII. Waves from axi-symmetric isolated systems, *Proc. R. Soc. A* **269**, 21 (1962).
- [2] R. K. Sachs, Gravitational waves in general relativity. VIII. Waves in asymptotically flat space-time, *Proc. R. Soc. A* **270**, 103 (1962).
- [3] R. K. Sachs, Asymptotic symmetries in gravitational theory, *Phys. Rev.* **128**, 2851 (1962).
- [4] A. Ashtekar and M. Streubel, Symplectic geometry of radiative modes and conserved quantities at null infinity, *Proc. R. Soc. A* **376**, 585 (1981).
- [5] T. Dray and M. Streubel, Angular momentum at null infinity, *Classical Quantum Gravity* **1**, 15 (1984).
- [6] R. M. Wald and A. Zoupas, A general definition of “conserved quantities” in general relativity and other theories of gravity, *Phys. Rev. D* **61**, 084027 (2000).
- [7] T. Banks, A critique of pure string theory: Heterodox opinions of diverse dimensions, [arXiv:hep-th/0306074](https://arxiv.org/abs/hep-th/0306074) (see footnote 17).
- [8] G. Barnich and C. Troessaert, Symmetries of Asymptotically Flat 4 Dimensional Spacetimes at Null Infinity Revisited, *Phys. Rev. Lett.* **105**, 111103 (2010).
- [9] G. Barnich and C. Troessaert, Supertranslations call for superrotations, *Ann. U. Craiova Phys.* **21**, S11 (2011).
- [10] G. Barnich and C. Troessaert, BMS charge algebra, *J. High Energy Phys.* **12** (2011) 105.
- [11] A. Strominger, On BMS invariance of gravitational scattering, *J. High Energy Phys.* **07** (2014) 152.
- [12] T. He, V. Lysov, P. Mitra, and A. Strominger, BMS supertranslations and Weinberg’s soft graviton theorem, *J. High Energy Phys.* **05** (2015) 151.
- [13] D. Kapec, V. Lysov, S. Pasterski, and A. Strominger, Semiclassical Virasoro symmetry of the quantum gravity  $S$ -matrix, *J. High Energy Phys.* **08** (2014) 058.
- [14] L. Bieri and D. Garfinkle, Perturbative and gauge invariant treatment of gravitational wave memory, *Phys. Rev. D* **89**, 084039 (2014).
- [15] S. Pasterski, A. Strominger, and A. Zhiboedov, New gravitational memories, [arXiv:1502.06120](https://arxiv.org/abs/1502.06120).
- [16] G. Barnich and C. Troessaert, Aspects of the BMS/CFT correspondence, *J. High Energy Phys.* **05** (2010) 062.
- [17] A. Strominger and A. Zhiboedov, Gravitational memory, BMS supertranslations and soft theorems, *J. High Energy Phys.* **01** (2016) 086.
- [18] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, Illinois, USA, 1984).
- [19] P. T. Chrusciel, J. Jezierski, and J. Kijowski, *Hamiltonian Field Theory in the Radiating Regime* (Springer-Verlag, New York, 2002).
- [20] R. Geroch, Asymptotic structure of space-time, in *Asymptotic Structure of Space-Time*, edited by F. P. Esposito and L. Witten (Plenum Press, New York, 1977).
- [21] A. Strominger and A. Zhiboedov, Superrotations and black hole pair creation, [arXiv:1610.00639](https://arxiv.org/abs/1610.00639).
- [22] A. Ashtekar and A. Magnon-Ashtekar, Energy-Momentum in General Relativity, *Phys. Rev. Lett.* **43**, 181 (1979).
- [23] D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of the Minkowski Space* (Princeton University Press, Princeton, NJ, 1993).
- [24] A. Ashtekar, The BMS Group, Conservation Laws, and Soft Gravitons, Perimeter Institute, Nov. 8, 2016, recording available online at <http://pirsa.org/16080055/>.
- [25] R. Geroch and J. Winicour, Linkages in general relativity, *J. Math. Phys. (N.Y.)* **22**, 803 (1981).
- [26] I. Fujisawa and R. Nakayama, Bondi-Metzner-Sachs surface-charge algebra via a Hamiltonian framework, *Phys. Rev. D* **91**, 126005 (2015).
- [27] P. J. McCarthy, The Bondi-Metzner-Sachs group in the nuclear topology, *Proc. R. Soc. A* **343**, 489 (1975).
- [28] G. Barnich and B. Oblak, Notes on the BMS group in three dimensions: II. Coadjoint representation, *J. High Energy Phys.* **3** (2015) 33.
- [29] A. D. Helfer, Angular momentum of isolated systems, *Gen. Relativ. Gravit.* **39**, 2125 (2007).
- [30] É. É. Flanagan and D. A. Nichols, Observer dependence of angular momentum in general relativity and its relationship to the gravitational-wave memory effect, *Phys. Rev. D* **92**, 084057 (2015).
- [31] É. É. Flanagan, D. A. Nichols, L. C. Stein, and J. Vines, Prescriptions for measuring and transporting local angular momenta in general relativity, *Phys. Rev. D* **93**, 104007 (2016).
- [32] R. J. Epp, R. B. Mann, and P. L. McGrath, Rigid motion revisited: Rigid quasilocal frames, *Classical Quantum Gravity* **26**, 035015 (2009).
- [33] R. J. Epp, R. B. Mann, and P. L. McGrath, On the existence and utility of rigid quasilocal frames, [arXiv:1307.1914](https://arxiv.org/abs/1307.1914).
- [34] R. J. Epp, P. L. McGrath, and R. B. Mann, Momentum in general relativity: Local versus quasilocal conservation laws, *Classical Quantum Gravity* **30**, 195019 (2013).
- [35] P. L. McGrath, Rigid quasilocal frames, [arXiv:1402.1443](https://arxiv.org/abs/1402.1443).
- [36] R. Bousso *et al.* (to be published).

- [37] Y. B. Zel'dovich and A. G. Polnarev, Radiation of gravitational waves by a cluster of superdense stars, *Sov. Astron.* **18**, 17 (1974).
- [38] V. B. Braginskii and K. S. Thorne, Gravitational-wave bursts with memory and experimental prospects, *Nature (London)* **327**, 123 (1987).
- [39] V. B. Braginskii and L. P. Grishchuk, Kinematic resonance and memory effect in free-mass gravitational antennas, *Sov. Phys. JETP* **62**, 427 (1985).
- [40] E. T. Newman and R. Penrose, Note on the Bondi-Metzner-Sachs group, *J. Math. Phys. (N.Y.)* **7**, 863 (1966).
- [41] J. Winicour, Global aspects of radiation memory, *Classical Quantum Gravity* **31**, 205003 (2014).
- [42] T. Mädler and J. Winicour, The sky pattern of the linearized gravitational memory effect, *Classical Quantum Gravity* **33**, 175006 (2016).
- [43] D. Christodoulou, Nonlinear Nature of Gravitation and Gravitational-Wave Experiments, *Phys. Rev. Lett.* **67**, 1486 (1991).
- [44] E. E. Flanagan, A. I. Harte, and D. A. Nichols, Gravitational-wave memory observables (to be published).
- [45] M. Heusler, *Black Hole Uniqueness Theorems* (Cambridge University Press, Cambridge, UK, 1996).
- [46] S. W. Hawking, M. J. Perry, and A. Strominger, Soft Hair on Black Holes, *Phys. Rev. Lett.* **116**, 231301 (2016).
- [47] S. W. Hawking, M. J. Perry, and A. Strominger, Superrotation charge and supertranslation hair on black holes, [arXiv:1611.09175](https://arxiv.org/abs/1611.09175).
- [48] G. Compère and J. Long, Vacua of the gravitational field, *J. High Energy Phys.* **07** (2016) 137.
- [49] G. Compère and J. Long, Classical static final state of collapse with supertranslation memory, *Classical Quantum Gravity* **33**, 195001 (2016).