Consistent higher derivative gravitational theories with stable de Sitter and anti–de Sitter backgrounds

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In this paper we provide the criteria for any generally covariant, parity preserving, and torsion-free theory of gravity to possess a stable de Sitter (dS) or anti–de Sitter (AdS) background. By stability we mean the absence of tachyonic or ghostlike states in the perturbative spectrum that can lead to classical instabilities and violation of quantum unitarity. While we find that the usual suspects, the $F(R)$ and $F(G)$ theories, can indeed possess consistent (A)dS backgrounds, G being the Gauss-Bonnet term, another interesting class of theories, string-inspired infinite derivative gravitational theories, can also be consistent around such curved vacuum solutions. Our study should not only be relevant for quantum gravity and early universe cosmology involving ultraviolet physics, but also for modifications of gravity in the infrared sector vying to replace dark energy.

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I. INTRODUCTION

Einstein's general relativity (GR) is an extremely successful theory in the infrared (IR), which matches a plethora of predictions and observations, including various solar system tests and cosmological predictions [1]. However, as it stands it has shortcomings in the ultraviolet (UV); it is incomplete classically as well as quantum mechanically; general relativity admits black hole and cosmological singularities, while the quantum loops render the theory nonrenormalizable beyond one loop [2]. In the case of a black hole, at least the singularity is covered by a horizon, but the cosmological singularity is "naked" where the energy density of the Universe and all the curvatures blow up for physical time, $t \to 0$ [3]. At the quantum level there have been many attempts to formulate a finite theory of gravity [4–6] such as string theory (ST) [7], loop quantum gravity (LQG) [8], causal set [9], dynamical triangulation [10], and asymptotic safety [11] with varying degrees of success. Intriguingly, most of these approaches to gravity have lead to nonlocal phenomena.

For instance in ST, strings and branes are nonlocal objects with interactions spread over a region of space-time. Nonlocal structures also appear in noncommutative geometry and string field theory (SFT) [12], p-adic strings [13], zeta strings [14], and strings quantized on a random lattice [15,16]; for a review, see [17]. LQG and causal set approaches are primarily based on nonlocal Wilsonian operators, while nonlocality in the form of an infinite set of derivatives has been discussed in the context of renormalization group arguments within the context of asymptotic safety [18]. It turns out that this appearance of an infinite series of higher derivative terms incorporating the nonlocality, often in the form of an exponential kinetic correction, is also a key feature of many of the stringy constructions [19–21]. Thus, one of the main focuses of our paper is to continue to investigate the consistency and viability of these infinite-derivative models and their implications for fundamental physics.

One of the typical challenges of higher derivative theories is that they suffer from Ostrogradsky instabilities at a classical level; see [22]. They also appear while canonically quantizing the theory; see [23]. The Ostrogradsky argument relies on having a highest "momentum" associated with the "highest derivative" in the theory in which the energy is seen to be linear, as opposed to quadratic. This makes the energy of the system unbounded from below and signals the presence of instability in the spectrum of the theory, which leads to lack of unitarity, predictability, and stability of the vacuum.

In gravity, a classic example of a higher derivative theory that has ghosts is Stelle's fourth derivative theory of gravity [24] (see also [25]), which is renormalizable, but unfortunately contains a massive spin-two ghost. In the path-integral approach, the presence of ghosts can be identified from the extra poles/roots that arise in the propagator with wrong sign residues. As the Ostrogradski argument suggests, the issue of ghosts is hard to tame order by order; one is invariably left with a highest momentum operator. In the language of the propagator, a finite fourth or higher order polynomial in momentum causes trouble for the stability of the action as in such a case it is easy to prove that the residue at one or more of the poles will inevitably have the wrong sign. In order to make sure that there exist no extra poles in the propagator, one is required to modify the propagator by an *entire function*,[1] which contains no poles in the finite domain, and essential singularities only at the boundary, i.e. $\pm\infty$ [19–21]. However, such a modification of a propagator also demands that the theory must contain infinite derivatives. Since in this case there is no highest momentum associated with the highest derivative, the Ostrogradsky problem can be avoided and one is forced to work with the path-integral formulation. Perhaps not surprisingly, the stringy higher derivative modifications that we alluded to before are precisely of this form. While our paper mainly focuses on viability of such infinite-derivative modifications in the context of gravity, our discussions and results are equally valid for most covariant higher derivative theories of gravity, including those that may be relevant for inflation; for a review see [26], or the dark energy problem [27].

In [28,29], consistency of gravitational theories that could contain any combination of the Reimann tensor around the Minkowski space-time in four space-time dimensions was investigated and concrete criteria were established to ensure the absence of any ghosts and tachyons in the perturbative spectrum. The analysis generalized similar criteria that were earlier found in [21] (see also [30,31] for robustness and perturbative stability of these models) for theories that only contained terms involving the scalar curvature. In particular, the obtained criteria reiterated the consistency of the widely popular $F(R)$ and $F(G)$ models where R and G are the Ricci scalar and Gauss-Bonnet scalars (see also [32]). It also corroborated the consistency of the class of infinite-derivative gravitational (IDG) theories involving the Ricci scalar considered in [21] while demonstrating that these theories can also be viewed as an infinite-derivative p-adic/SFT-type scalar field nonminimally coupled to general relativity; please see [33] for a more detailed account. Most interestingly however, the authors Biswas, Gerwick, Koivisto and Mazumdar (BGKM) also found a class of consistent IDG theories comprising terms at most quadratic in the Riemann tensor (not just the Ricci scalar) that contained no extra poles in the propagator other than the one corresponding to the massless graviton and no extra scalar degree of freedom. We should mention that the idea of evading ghosts by means of infinitely many derivatives was introduced in the second paper of [34], and later taken up in [21,28,35].

In the new class of theories introduced in [28,29], the only modification was in the form of a multiplicative *entire function* to the graviton propagator. In particular, in the UV the propagator could now become more convergent than the usual inverse square dependence of the momentum. Thus these theories can be thought of as ghost-free infinite-derivative extensions of Stelle's fourth derivative theory of gravity. In fact, it was shown in [36] that although softening of the propagator by an exponential inevitably implies an exponential enhancement in the interactions in the UV, the superficial degree of divergence, D, which comes from a combination of vertex operators and propagators, reads an encouraging $D = 1 - L$, where L is the number of loops. One can therefore hope that for $L > 1$, the theory becomes super-renormalizable; similar results also hold for other forms of entire functions which are not exponentially suppressed [34,35]. The idea was tested and verified in a scalar-toy model of gravity up to two loops explicitly for an exponential propagator [36]. For the same setup, high energy scatterings of gravitons were also analyzed, and it was found that vertices can overcome the propagator contributions at finite loop levels to make the scattering diagrams finite for given external momenta [37].[2]

While the quantum nature of IDG theories is encouraging, perhaps the most striking feature of IDG theories is their classical behavior at the UV; the same propagator which softens quantum aspects of higher loops also seems to be able to avoid classical singularity for a point source—as long as the mass of the source $m \lesssim \sqrt{M^2/M_p^2}$, where M_p is the four-dimensional Planck mass and M is the scale of nonlocality [28]. The classical avoidance of singularity was tested vigorously in the linearized limit for both static [40], and time dependent cases [41]. The avoidance of cosmological singularity has, in fact, been tested beyond linear level. First, an ansatz was recognized that resolved the cosmological big bang singularity problem by replacing it with a big bounce that conjoins the expanding Universe with a previous contraction [21]; see also [42,43]. Secondly, these background solutions were perturbed on sub- [44] and super-Hubble scales [30] to seek any unstable mode, but no instability has been observed yet; see also [45–47] for general features of perturbative evolutions that can also be applied to these bouncing scenarios and that further corroborate these findings. Finally, the avoidance of cosmological singularity has been tested at a nonlinear

[1]Often it is more convenient to impose restriction on the inverse momentum operator rather than the propagator. Since we do not want the propagators to have extra poles, this means the inverse propagator cannot have extra 0's. This can be ensured if the inverse propagator is the exponential of an entire function which can never vanish in the finite complex plane.

[2]Infinite derivatives with Gaussian kinetic term also have many applications in field theory [38], and particle physics model building [39].

level by understanding the geodesics of null rays to see whether they diverge or converge to test the focusing theorem of Penrose and Hawking [48]. It was observed that IDG theories indeed give rise to defocusing of null rays without violating any of the energy conditions required upon matter [49]; for more details on the computations, see [50].

Last but not the least, an intriguing connection has been established between the gravitational entropy and the propagating degrees of freedom in the space-time. The gravitational entropy for ghost-free IDG does not get a contribution from the UV, but only from the Einstein-Hilbert action and follows strictly the area law for entropy for a static spherically symmetric black hole [51].

Given all the encouraging results that have emerged in the IDG theories, it stands to reason that we investigate the viability of these theories further. An obvious choice is to look at the consistency of other backgrounds that these theories may admit; after all a theory of quantum gravity should enable us to compute quantum amplitudes around any classical background, not just the Minkowski vacuum. The situation is similar to particle theories; while calculations around Minkowski space-time are the most important, field theories can be consistently expanded around solitonic backgrounds and provide sensible answers. Due to their simplicity as well as importance in cosmology and fundamental physics, looking at perturbations around de Sitter (dS) and anti–de Sitter (AdS) space-times seems the natural choice to make progress in this direction. On one hand, our hope is that the requirement of consistency around these curved backgrounds provides us with additional constraints on IDG theories shedding light on what a fundamental theory of gravity may look like. On the other hand, it is known that for several applications of gravity, ranging from testing gravity in our Solar System to understanding cosmological phenomena, often one only needs to understand the dynamics of the relevant space-time background and linearized perturbations around it. We hope that our results for the dS background not only aid inflationary or dark energy related cosmological model building efforts,[3] but also that the techniques we have developed to study curved backgrounds pave the way to investigate more nontrivial backgrounds, such as the Friedmann-Lemaître-Robertson-Walker (FLRW) space-times, and their perturbations.

We begin by writing down the general form of an action that consists of terms that are at most quadric in curvatures. In the companion book chapter [31] it was shown that given any arbitrary covariant action of gravity that is parity preserving, torsion free and admits a well-defined Minkowski limit, one can always find such a "quadratic curvature" action that is "equivalent" as far as the physics of the linearized fluctuations around dS/AdS/Minkowski are concerned. For details on how to obtain the equivalent quadratic action for any higher derivative action potentially containing arbitrary high powers of curvatures, we again refer the readers to [31]. In this paper, our main goal is to obtain consistency conditions for the quadratic curvature action. We first vary the quadratic curvature action of gravity around dS and AdS keeping up to second order terms in fluctuations of the metric.[4] We decompose the ten metric components in four space-time dimensions into the transverse and traceless spin-two graviton field containing five degrees of freedom, the three transverse vector degrees of freedom, and two scalar degrees of freedom. There are, of course, four gauge degrees of freedom, three of which reduce the spin-two field to the two helicity states of the graviton, while the remaining gauge freedom is used to cancel the longitudinal vector mode reducing the vector degrees of freedom to the two helicity states as well. This decomposition explicitly demonstrates that just as in GR, the vector and one of the scalars vanish from the action even for the higher derivative action. This can also be seen from the Bianchi identities that the field equations must satisfy. To summarize, we are left with just the spin-two graviton and one scalar physical degree of freedom; indeed the latter is the familiar Brans-Dicke scalar that popularly arises in $F(R)$ theories. The final aim of this paper is to write down explicitly the action for the graviton and the scalar mode in order to determine when these fields can propagate without encountering ghostlike or tachyonic instabilities.

Our paper is organized as follows: In Sec. II, we expand the quadratic curvature action around (A)dS background keeping terms that are quadratic in metric fluctuations. In Sec. III, we decompose the action into two parts corresponding to the physically surviving scalar and tensor modes of the metric. We then proceed to obtain the consistency conditions for the theory to be free from ghostlike and tachyonic instabilities around (A)dS backgrounds in Sec. IV. Apart from the usual local theories that are known to be consistent, we see how consistent nonlocal IDG theories can also emerge. In particular, we provide examples of IDG theories that provide consistent theories

[3]The dS space-time that our Universe seems to be heading towards "presently" has a very small cosmological constant, and thus the Minkowski space-time analysis suffices if the scale of gravitational modification is close to Planckian. However, our results are completely general and should be useful for understanding classical stability of modifications of gravity at much smaller scales that people have been pursuing in the context of the cosmic speed-up problem, and can be thought of as a generalization of the constraints obtained in [52,53] for a more limited class of gravitational actions. More pertinently however, there has been a lot of recent interest in inflationary cosmology from gravitational modifications. The inflationary scale can certainly be high enough to be relevant for our study.

[4]Previous studies have concentrated on finding the graviton propagator around dS and AdS backgrounds in the context of Einstein-Hilbert action; see [54–59]. Here we generalize to IDG.

in the presence of an arbitrary cosmological constant, thus generalizing previous constructions of viable theories around Minkowski background. We have three appendixes: In Appendix A we discuss various notations and identities, in Appendix B we enumerate the commutation relations involving covariant derivatives that we need in our computations, and in Appendix C we provide the details of the cancellation of the vector and scalar modes in covariant gravitational actions.

II. LINEARIZED NONLOCAL GRAVITY ON DS AND ADS BACKGROUNDS

A. Equivalent action around constant curvature backgrounds

One can start from the most general covariant, torsion-free, parity preserving quadratic action of gravity with a well-defined Minkowski limit and obtain a simpler equivalent action that reproduces the same quadratic action for fluctuations around a constant curvature background, such as dS, AdS and Minkowski. We consider actions expandable in Taylor series in curvatures and derivatives such that terms like $1/R$ or $1/(\Box R)$ are not present. In future, one may be able to relax some of these conditions, but with these restrictions, one can easily see that the most general action can be written in the form[5]

$$S = \int d^4x\sqrt{-g}\left[\mathcal{P}_0 + \sum_i \mathcal{P}_i \prod_I (\hat{\mathcal{O}}_{iI}\mathcal{Q}_{iI})\right], \tag{2.1}$$

where $\mathcal{P}$'s and $\mathcal{Q}$'s are quantities composed only of the Riemann and the metric tensor and $\hat{\mathcal{O}}$'s are differential operators *solely* constructed from covariant derivatives.

It is clear that the theory admits dS, AdS and/or Minkowski vacuum solutions depending on its algebraic properties. These backgrounds are of natural interest to cosmology, and AdS/CFT correspondence (for a review see, for instance, [60]) among other topics in high energy physics. The AdS and dS backgrounds are maximally symmetric space-times (MSS), where we have

$$R = \bar{R} = \text{const}, \qquad R_{\mu\nu} = \frac{\bar{R}}{4}\bar{g}_{\mu\nu},$$
$$R^{\rho}_{\mu\sigma\nu} = \frac{\bar{R}}{12}(\delta^{\rho}_{\sigma}\bar{g}_{\mu\nu} - \delta^{\rho}_{\nu}\bar{g}_{\mu\sigma}). \tag{2.2}$$

Hereafter "$^{-}$" designates the background quantity, and $\bar{R} = 0$ in the above formulas yields the Minkowski space-time.[6] In [31], it was demonstrated that in order to study the perturbative properties of the action Eq. (2.1) around dS, AdS or Minkowski backgrounds, it is sufficient to look at a simpler equivalent action that just contains terms that are quadratic in curvature, but potentially an infinite set of covariant derivatives. We introduce the metric fluctuations as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{2.3}$$

where $\bar{g}_{\mu\nu}$ is the background dS/AdS/Minkowski metric. The corresponding equivalent action reproducing the $\mathcal{O}(h^2)$ variation of Eq. (2.1) around a MSS is

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \Lambda + \frac{\lambda}{2}(R\mathcal{F}_1(\Box)R + S_{\mu\nu}\mathcal{F}_2(\Box)S^{\mu\nu} + C_{\mu\nu\lambda\sigma}\mathcal{F}_3(\Box)C^{\mu\nu\lambda\sigma})\right], \tag{2.4}$$

where S and W are the traceless-Ricci (TR) and Weyl tensors respectively defined by

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} \tag{2.5}$$

and

$$C^{\mu}_{\alpha\nu\beta} = R^{\mu}_{\alpha\nu\beta} - \frac{1}{2}(\delta^{\mu}_{\nu}R_{\alpha\beta} - \delta^{\mu}_{\beta}R_{\alpha\nu} + R^{\mu}_{\nu}g_{\alpha\beta} - R^{\mu}_{\beta}g_{\alpha\nu}) + \frac{R}{6}(\delta^{\mu}_{\nu}g_{\alpha\beta} - \delta^{\mu}_{\beta}g_{\alpha\nu}). \tag{2.6}$$

For future convenience, we also write this action as

$$S = S_{\mathrm{EH}+\Lambda} + S_{R^2} + S_{S^2} + S_{C^2}. \tag{2.7}$$

The $\mathcal{F}$'s are assumed to be expandable as a Taylor series,

$$\mathcal{F}_i = \sum_{n=0} c_{in}\Box^n/M^{2n}.$$

The dimensionless coupling λ is introduced to control the higher derivative terms. For $\lambda \to 0$, we recover the Einstein-Hilbert action, while as $M \to \infty$, we recover purely a local action, quadratic in curvature. The reduction to this latter action is described in detail in [31].

One can show that the equations of motion upon substitution of values from Eq. (2.2) imply

$$M_P^2\bar{R} = 4\Lambda, \tag{2.8}$$

which is identical to what one obtains from just the $S_{\mathrm{EH}+\Lambda}$ action. We also note that the Gauss-Bonnet scalar

$$G = R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\alpha\beta}^2 = \frac{1}{6}R^2 - 2S_{\mu\nu}^2 + C_{\mu\nu\alpha\beta}^2, \tag{2.9}$$

being a topological invariant in four dimensions, allows us to set one of the coefficients among $c_{1,0}, c_{2,0}, c_{3,0}$ to 0, if we wanted to. Unfortunately, no such simplification is possible for higher derivative terms.

[5]We are using $(-,+,+,+)$ signature of the metric.
[6]The greek indices run as μ, $\nu = 0, 1, 2, 3$.

We proceed by obtaining the $\mathcal{O}(h^2)$ part of the action (2.7) which is all that we need to study perturbative consistency of theories (2.1). Unfortunately, the calculations are rather technical, so we have summarized the main results in our concluding section, in case the reader wants to skip these details. Also, in our derivations, we very much rely on the formulas collected in the appendixes, and wherever appropriate direct the reader to the relevant sections in them.

B. Quadratic action

1. The Einstein-Hilbert term and Λ

Let us start with the pure Einstein-Hilbert action with a cosmological term

$$S_{\mathrm{EH}+\Lambda} = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \Lambda\right]. \tag{2.10}$$

Its quadratic variation is obviously very well known, see for instance [52,53,61], but for the sake of completeness we include it here. Using Eq. (A6) from Appendix A 3, and assuming the background configuration Eq. (2.2), together with the relation Eq. (2.8), we obtain the following quadratic variation for action Eq. (2.10):

$$\delta^2 S_{\mathrm{EH}+\Lambda} = \int dx^4\sqrt{-\bar{g}}\frac{M_P^2}{2}\delta_0, \tag{2.11}$$

$$\delta_0 \equiv \left(\frac{1}{4}h_{\mu\nu}\bar{\Box}h^{\mu\nu} - \frac{1}{4}h\bar{\Box}h + \frac{1}{2}h\bar{\nabla}_\mu\bar{\nabla}_\rho h^{\mu\rho} + \frac{1}{2}\bar{\nabla}_\mu h^{\mu\rho}\bar{\nabla}_\nu h^\nu_\rho\right) - \frac{1}{48}\bar{R}(h^2 + 2h^\mu_\nu h^\nu_\mu). \tag{2.12}$$

2. The quadratic terms involving Ricci scalar

Again, utilizing Eq. (A3) from Appendix A 3 for notations and actual computations, and using heavily that $\bar{R} = \mathrm{const}$, we obtain

$$\delta^2 S_{R^2} = \frac{\lambda}{2}\int d^4x\sqrt{-\bar{g}}\left[2\left(\frac{h}{2}r + \frac{1}{2}\left(\frac{h^2}{8} - \frac{h_{\mu\nu}h^{\mu\nu}}{4}\right)\bar{R} + \delta^2(R)\right)c_{1,0}\bar{R} + r\mathcal{F}_1(\bar{\Box})r + \left(\frac{h}{2}\bar{R} + r\right)\delta(\mathcal{F}_1(\Box))\bar{R} + \bar{R}\delta^2(\mathcal{F}_1(\Box))\bar{R} + \frac{h}{2}\bar{R}(\mathcal{F}_1(\bar{\Box}) - c_{1,0})r + \bar{R}\delta(\mathcal{F}_1(\Box))r\right], \tag{2.13}$$

where we have already integrated by parts some terms. By inspection the terms in the first line can be rewritten as

$$\frac{\lambda}{2}\int d^4x\sqrt{-\bar{g}}[2c_{1,0}\bar{R}\delta_0 + r\mathcal{F}_1(\bar{\Box})r].$$

Now, turning to the second line of Eq. (2.13), we first notice that the first two terms are actually 0. This is because $\bar{R}$ is a constant and a scalar, and therefore annihilated by $\Box = \nabla^\mu\partial_\mu$ as well as by $\delta\Box$ [see Eq. (A5)]. For the last term, the variation of $\delta\mathcal{F}_1(\Box)$ must appear in the $\Box$ appearing at the extreme left; otherwise, the term becomes a total derivative. In other words, the last two terms in the second line of Eq. (2.13) can be combined as

$$\frac{\lambda}{2}\sum_{n=1}^{\infty}c_{1,n}\bar{R}\int d^4x\sqrt{-\bar{g}}\left[\frac{h}{2}\bar{\Box} + \delta(\Box)\right]\bar{\Box}^{n-1}r,$$

using the Taylor series representation for function $\mathcal{F}_1(\Box)$. Now, reciting explicitly Eq. (A5),

$$\delta(\Box)\varphi = (-h^{\mu\nu}\nabla_\mu\partial_\nu - g^{\mu\nu}\gamma^\rho_{\mu\nu}\partial_\rho)\varphi,$$

which is valid for any scalar field, φ, and integrating by parts one can explicitly show that under the integral, $\delta(\Box)\varphi$ is equivalent to $-\frac{1}{2}(\bar{\Box}h)\varphi$, and therefore the two terms actually cancel. In other words, we have just proved

$$\delta^2 S_{R^2} = \frac{\lambda}{2}\int d^4x\sqrt{-\bar{g}}[2c_{1,0}\bar{R}\delta_0 + r\mathcal{F}_1(\bar{\Box})r]. \tag{2.14}$$

3. Terms involving the TR and Weyl tensors and the complete quadratic action

Variations of the terms containing the Weyl or TR ternsors are extremely simple as both these tensors are 0 on constant curvature background, see Eq. (2.2), and they enter the action quadratically. This means that the only terms which survive are

$$\delta^2 S_{S^2} = \frac{\lambda}{2}\int d^4x\sqrt{-\bar{g}}\delta(S^\mu_\nu)\mathcal{F}_2(\bar{\Box})\delta(S^\nu_\mu) \quad \text{and}$$
$$\delta^2 S_{C^2} = \frac{\lambda}{2}\int d^4x\sqrt{-\bar{g}}\delta(C^{\mu\alpha}{}_{\nu\beta})\mathcal{F}_3(\bar{\Box})\delta(C_{\mu\alpha}{}^{\nu\beta}). \tag{2.15}$$

Respective variations can be easily written in terms of r, r^μ_ρ, $r^{\mu\nu}_{\rho\sigma}$, these quantities being defined and computed in Eqs. (A3) and (A4); however one has to perform some algebraic manipulations to account properly for all the contractions of the Kronecker symbols. A simplifying point is that r, r^μ_ρ, $r^{\mu\nu}_{\rho\sigma}$ terms do not mix, thanks to the symmetry properties of the Riemann tensor. We leave the explicit algebraic manipulations to the reader, and here just present the final result for the action containing quadratic fluctuations.

Summing all the individual contributions, Eqs. (2.12), (2.14), and (2.15), we get

$$\delta^2 S = \int dx^4 \sqrt{-\bar{g}} \left[\left(\frac{M_P^2}{2} + \lambda c_{1,0} \bar{R} \right) \delta_0 + \frac{\lambda}{2} (r \hat{\mathcal{F}}_1(\bar{\Box}) r + r^\mu_\nu \hat{\mathcal{F}}_2(\bar{\Box}) r^\nu_\mu + r^{\mu\alpha}_{\nu\beta} \hat{\mathcal{F}}_3(\bar{\Box}) r^{\nu\beta}_{\mu\alpha}) \right] = s_0 + s_1 + s_2 + s_3, \quad (2.16)$$

where we have introduced the following functions:

$$\hat{\mathcal{F}}_1(\bar{\Box}) = \mathcal{F}_1(\bar{\Box}) - \frac{1}{4}\mathcal{F}_2(\bar{\Box}) + \frac{1}{3}\mathcal{F}_3(\bar{\Box}), \quad (2.17)$$

$$\hat{\mathcal{F}}_2(\bar{\Box}) = \mathcal{F}_2(\bar{\Box}) - 2\mathcal{F}_3(\bar{\Box}), \quad (2.18)$$

$$\hat{\mathcal{F}}_3(\bar{\Box}) = \mathcal{F}_3(\bar{\Box}). \quad (2.19)$$

III. QUADRATIC ACTION FOR THE GRAVITON AND THE BRANS-DICKE SCALAR

A. Decoupling tensor, vector, and scalar modes

Even though technically we have derived the second order action, to understand the dynamical properties we need to identify the physical excitations, or the correct propagating degrees of freedom. Now, any second rank tensor can be decomposed as, see for instance [56],

$$h_{\mu\nu} = h^\perp_{\mu\nu} + \bar{\nabla}_\mu A^\perp_\nu + \bar{\nabla}_\nu A^\perp_\mu + \left(\bar{\nabla}_\mu \bar{\nabla}_\nu - \frac{1}{4} \bar{g}_{\mu\nu} \bar{\Box} \right) B + \frac{1}{4} \bar{g}_{\mu\nu} h, \quad (3.1)$$

where the factor 4 comes from dimensionality. In four dimensions, the metric tensor contains ten degrees of freedom: $h^\perp_{\mu\nu}$, the transverse and traceless massless spin-two graviton,

$$\bar{\nabla}^\mu h^\perp_{\mu\nu} = \bar{g}^{\mu\nu} h^\perp_{\mu\nu} = 0, \quad (3.2)$$

represents five degrees of freedom, $A^\perp_\mu$ the transverse vector field,

$$\bar{\nabla}^\mu A^\perp_\mu = 0, \quad (3.3)$$

accounts for three degrees of freedom, and the two scalars, B and h, make up the remaining two degrees of freedom. *A priori*, these fields represent six physical fields, since three gauge degrees reduce the spin-two field to the two spin-two helicity states of a graviton, and one gauge freedom can be used to reduce the vector field to its two transversal spin-one helicity states as well. The aim of this section is to write down explicitly the action in terms of the tensor, vector and scalar components in order to analyze their respective properties. To achieve this we use a variety of identities collected in the appendix in order to commute various derivative operators.

To begin with, we claim that once we directly substitute Eq. (3.1) in Eq. (2.16) all terms involving the vector field, A_μ, vanish, and so do all the terms involving the $\bar{\nabla}_\mu \bar{\nabla}_\nu B$ piece. In other words, the quadratic action only contains $h^\perp_{\mu\nu}$, and a single scalar field combination

$$\phi \equiv \bar{\Box} B - h. \quad (3.4)$$

So, effectively Eq. (3.1) is reduced to

$$h_{\mu\nu} = h^\perp_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \phi. \quad (3.5)$$

This result is identical to what happens in Einstein's gravity, but the algebraic computations needed to verify it for our general case are quite tedious, so we have briefly outlined it in Appendixes C 1 and C 2.

Next, we note that group representation theory dictates that at the linearized level, the tensor, vector and scalar degrees should decouple from one another. Although it is a well-known fact for pure GR, it may be not so transparent in a more general setting. The suspicion comes from the presence of higher rank tensorial structures in the action. To understand this more deeply, let us look at the GR terms in Eq. (2.12) first, as a warm-up exercise. Note that the tensor modes can in principle enter only in the very first and very last terms in δ_0. At any other place, they cancel due to the transverse and traceless properties of $h^\perp_{\mu\nu}$ [see Eq. (3.2)]. Thus if a mixing were to occur it can only be in the first or the last term in δ_0. The relevant expression reads

$$\delta_{0,\mathrm{mix}} = -\frac{1}{8} h^\perp_{\mu\nu} \bar{\Box} \bar{g}^{\mu\nu} \phi + \frac{\bar{R}}{48} h^{\perp\mu}{}_\nu \delta^\nu_\mu \phi = 0, \quad (3.6)$$

as $h^\perp_{\mu\nu}$ is traceless and $\bar{\Box}$ commutes with the metric tensor.

The higher derivative terms are less trivial. Essentially we need to analyze expressions for $r^{\mu\nu}_{\rho\sigma}$. Regarding the tensor modes, the structure of indices in the expression for $r^{\sigma\mu}_{\nu\rho}$ and the transverse, traceless properties of $h^\perp_{\mu\nu}$ suggest that one gains the expression for $r^{\sigma\mu}_{\nu\rho}(h^\perp_{\mu\nu})$ by just replacing $h_{\mu\nu} \to h^\perp_{\mu\nu}$. This trivial procedure yields

$$r^{\sigma\mu}_{\nu\rho}(h^\perp_{\mu\nu}) = \frac{\bar{R}}{24} (\delta^\mu_\nu h^{\perp\sigma}{}_\rho - \delta^\mu_\rho h^{\perp\sigma}{}_\nu - \delta^\sigma_\nu h^{\perp\mu}{}_\rho + \delta^\sigma_\rho h^{\perp\mu}{}_\nu) + \frac{1}{2} (\bar{\nabla}_\nu \bar{\nabla}^\mu h^{\perp\sigma}{}_\rho - \bar{\nabla}_\nu \bar{\nabla}^\sigma h^{\perp\mu}{}_\rho - \bar{\nabla}_\rho \bar{\nabla}^\mu h^{\perp\sigma}{}_\nu + \bar{\nabla}_\rho \bar{\nabla}^\sigma h^{\perp\mu}{}_\nu). \quad (3.7)$$

The reason why nothing more can be simplified at this stage is because $h^\perp_{\mu\nu}$'s are being acted by covariant derivatives with indices different from those in $h^\perp_{\mu\nu}$. Therefore, no symmetry property can be utilized yet.

The scalar part of $r^{\sigma\mu}_{\nu\rho}(\phi)$ allows some tinkering. The simplification comes from the fact that no more than two derivatives appear and they act on a scalar. In this case these derivatives commute, and we should not worry about their order. The direct substitution gives

$$r^{\sigma\mu}_{\nu\rho}(\phi) = \frac{\bar{R}}{48}(\delta^\sigma_\nu\delta^\mu_\rho - \delta^\sigma_\rho\delta^\mu_\nu)\phi - \frac{1}{8}(\bar{\nabla}_\nu\bar{\nabla}^\mu\delta^\sigma_\rho - \bar{\nabla}_\nu\bar{\nabla}^\sigma\delta^\mu_\rho - \bar{\nabla}_\rho\bar{\nabla}^\mu\delta^\sigma_\nu + \bar{\nabla}_\rho\bar{\nabla}^\sigma\delta^\mu_\nu)\phi. \quad (3.8)$$

For future convenience, we rewrite the latter expression as follows,

$$r^{\sigma\mu}_{\nu\rho}(\phi) = \frac{1}{8}(\bar{\mathcal{D}}^\sigma_\nu\delta^\mu_\rho + \bar{\mathcal{D}}^\mu_\rho\delta^\sigma_\nu - \bar{\mathcal{D}}^\mu_\nu\delta^\sigma_\rho - \bar{\mathcal{D}}^\sigma_\rho\delta^\mu_\nu)\phi + \frac{3\bar{\Box}+\bar{R}}{48}(\delta^\sigma_\nu\delta^\mu_\rho - \delta^\sigma_\rho\delta^\mu_\nu)\phi, \quad (3.9)$$

where

$$\bar{\mathcal{D}}^\mu_\nu = \bar{\nabla}_\nu\bar{\nabla}^\mu - \delta^\mu_\nu\frac{\bar{\Box}}{4}, \qquad \bar{\mathcal{D}}^\mu_\mu = 0,$$

i.e. it is a traceless operator, which is extremely useful when evaluating the action later.

So, the question is whether any term survives in the following combination:

$$\int dx^4\sqrt{-g}[r_{\sigma\mu}{}^{\nu\rho}(\phi)\mathcal{F}(\bar{\Box})r^{\sigma\mu}_{\nu\rho}(h^\perp_{\mu\nu})]. \quad (3.10)$$

Schematically, four structures may arise,

$$\delta{:}\delta{:}\mathcal{F}\delta{:}h^\perp{:}, \qquad \delta{:}\delta{:}\mathcal{F}\bar{\nabla}.\bar{\nabla}^{\cdot}h^\perp{:},$$
$$\delta{:}\bar{\nabla}.\bar{\nabla}^{\cdot}\mathcal{F}\delta{:}h^\perp{:}, \qquad \delta{:}\bar{\nabla}.\bar{\nabla}^{\cdot}\mathcal{F}\bar{\nabla}.\bar{\nabla}^{\cdot}h^\perp{:}.$$

Dots denote some indices, and each term is a scalar, i.e., indices are fully contracted. Then, we see that the first term goes away as finally you have to contract the indices of $h^\perp_{\mu\nu}$ with a δ. In the second term, in order to avoid the appearance of the trace of $h^\perp_{\mu\nu}$, after the δ contractions we must be left with $\bar{\nabla}^\mu\bar{\nabla}^\nu h^\perp_{\mu\nu}$, but this is 0 as $h^\perp_{\mu\nu}$ is transverse. Similarly, after the delta contractions the third term looks like $\bar{\nabla}^\mu\bar{\nabla}^\nu\mathcal{F}(\bar{\Box})h^\perp_{\mu\nu}$. In Appendix B we have proved that $\bar{\Box}$ acting on the transverse and traceless symmetric second rank tensor gives again a transverse and traceless symmetric second rank tensor, and therefore $\mathcal{F}(\bar{\Box})h^\perp_{\mu\nu}$ must be transverse and traceless ensuring that the third term also vanishes.

The fourth term generates four possibilities upon contraction with the δ-symbol,

$$\bar{\nabla}^\mu\bar{\nabla}^\nu\mathcal{F}\bar{\nabla}_\mu\bar{\nabla}_\nu h^{\perp\rho}{}_\rho, \qquad \bar{\nabla}^\mu\bar{\nabla}^\nu\mathcal{F}\bar{\nabla}_\mu\bar{\nabla}_\rho h^{\perp\rho}{}_\nu,$$
$$\bar{\nabla}^\mu\bar{\nabla}^\nu\mathcal{F}\bar{\Box}h^\perp{}_{\mu\nu}, \qquad \bar{\nabla}^\mu\bar{\nabla}^\nu\mathcal{F}\bar{\nabla}_\rho\bar{\nabla}_\nu h^{\perp\rho}{}_\mu.$$

It is easy to see that the first three terms vanish due to arguments similar to what we presented above. The last term is not that obvious, however, but one finds that

$$\bar{\nabla}_\rho\bar{\nabla}_\nu h^{\perp\rho}{}_\mu = \frac{\bar{R}}{3}h^\perp{}_{\nu\mu},$$

which reduces it to the form of the third term.

To summarize, we have shown that in the quadratic action Eq. (2.16), only terms involving the transverse and traceless graviton field, $h^\perp{}_{\nu\mu}$, and a particular scalar field combination, ϕ, survive. Further we have argued that the scalar and the tensor mode must decouple, and therefore it is sufficient to calculate the actions for scalar and tensor fields separately.

B. Scalar modes

Let us start with the scalar mode. Contracting Eq. (3.9) with the Kronecker delta over the first and third indices, we immediately get

$$r_{\mu\nu}(\phi) = \frac{1}{4}\bar{\mathcal{D}}_{\mu\nu}\phi + \bar{g}_{\mu\nu}\left(\frac{3\bar{\Box}+\bar{R}}{16}\right)\phi, \quad (3.11)$$

and then

$$r(\phi) = \left(\frac{3\bar{\Box}+\bar{R}}{4}\right)\phi. \quad (3.12)$$

Also, quite a short computation is needed to get

$$\delta_0(\phi) = -\frac{1}{32}\phi(3\bar{\Box}+\bar{R})\phi. \quad (3.13)$$

We are now ready to look at the different terms in Eq. (2.16). The pure GR part and the r-part require no simplifications, and one simply obtains

$$s_0 + s_1 \xrightarrow{\phi} \int dx^4\sqrt{-\bar{g}}\phi\left[-\frac{1}{32}\left(\frac{M_P^2}{2} + \lambda c_{1,0}\bar{R}\right) + \frac{\lambda}{32}\hat{\mathcal{F}}_1(\bar{\Box})(3\bar{\Box}+\bar{R})\right](3\bar{\Box}+\bar{R})\phi. \quad (3.14)$$

Next, let us look at the term, s_2, involving $r_{\mu\nu}$. By inspection, it is clear that there are three possible terms, terms containing two $\bar{\mathcal{D}}^{\mu\nu}$'s, one $\bar{\mathcal{D}}^{\mu\nu}$, and no $\bar{\mathcal{D}}^{\mu\nu}$. The last one again does not require any simplification except for a trivial trace of the metric and the second term actually vanishes as $\bar{\mathcal{D}}_{\mu\nu}$ is traceless. Thus we are really left to evaluate terms such as

$$\int dx^4\sqrt{-\bar{g}}[\phi\bar{\mathcal{D}}_{\mu\nu}\mathcal{F}(\bar{\Box})\bar{\mathcal{D}}^{\mu\nu}\phi]. \quad (3.15)$$

As it will become progressively clear, in order to understand the dynamic properties of the fields, we need to express the kinetic operators as functions of the $\Box$ operator. To achieve this we have to commute covariant derivatives, which are on the left all the way to the right across an infinite tower of d'Alembertians in the function $\mathcal{F}$. We can do this by utilizing the recursion property Eq. (B23) derived in Appendix B, which is appropriate since $\bar{\mathcal{D}}_{\mu\nu}$ is traceless. Accordingly, we observe

$$\bar{\nabla}_\nu \bar{\Box}^n \bar{\mathcal{D}}^{\mu\nu}\phi = \left(\bar{\Box} + \frac{5}{12}\bar{R}\right)^n \bar{\nabla}_\nu \bar{\mathcal{D}}^{\mu\nu}\phi = \left(\bar{\Box} + \frac{5}{12}\bar{R}\right)^n \left(\bar{\Box}\bar{\nabla}^\mu - \bar{\nabla}^\mu \frac{\bar{\Box}}{4}\right)\phi. \tag{3.16}$$

Next, we utilize Eq. (B6) along with Eq. (B23) to obtain

$$\bar{\nabla}_\mu \left(\bar{\Box} + \frac{5}{12}\bar{R}\right)^n \left(\bar{\Box}\bar{\nabla}^\mu - \bar{\nabla}^\mu \frac{\bar{\Box}}{4}\right)\phi = \left(\bar{\Box} + \frac{2}{3}\bar{R}\right)^n \bar{\nabla}_\mu \left(\bar{\Box}\bar{\nabla}^\mu - \bar{\nabla}^\mu \frac{\bar{\Box}}{4}\right)\phi = \left(\bar{\Box} + \frac{2}{3}\bar{R}\right)^n \left(\frac{3\bar{\Box} + \bar{R}}{4}\right)\bar{\Box}\phi. \tag{3.17}$$

Note that the number $2/3$ arises as $5/12 + 1/4$. Returning to Eq. (3.15), we can now write down the cumulative expression

$$\int dx^4 \sqrt{-\bar{g}}[\phi \bar{\mathcal{D}}_{\mu\nu}\mathcal{F}(\bar{\Box})\bar{\mathcal{D}}^{\mu\nu}\phi] = \int dx^4 \sqrt{-\bar{g}}\left[\phi \mathcal{F}\left(\bar{\Box} + \frac{2}{3}\bar{R}\right)\left(\frac{3\bar{\Box} + \bar{R}}{4}\right)\bar{\Box}\phi\right]. \tag{3.18}$$

Finally, adding all the terms, we have

$$s_2 \xrightarrow{\phi} \frac{\lambda}{32}\int dx^4 \sqrt{-\bar{g}}\phi\left[(3\bar{\Box} + \bar{R})\hat{\mathcal{F}}_2(\bar{\Box}) + \bar{\Box}\hat{\mathcal{F}}_2\left(\bar{\Box} + \frac{2}{3}\bar{R}\right)\right]\left(\frac{3\bar{\Box} + \bar{R}}{4}\right)\phi. \tag{3.19}$$

Finally, for the s_3 part involving $r^{\sigma\mu}_{\nu\rho}$'s, we must carefully count all the nonvanishing products and respective coefficients. We start by noticing that cross products of terms with and without $\bar{\mathcal{D}}_{\mu\nu}$ again vanish as the trace of $\bar{\mathcal{D}}_{\mu\nu}$ (which is 0) arises inevitably. Then the simplest contribution is the one free of $\bar{\mathcal{D}}_{\mu\nu}$, and it reads

$$\phi\left(\frac{3\bar{\Box} + \bar{R}}{48}\right)\mathcal{F}_3(\bar{\Box})\left(\frac{3\bar{\Box} + \bar{R}}{48}\right)\phi(\delta^\sigma_\nu\delta^\mu_\rho - \delta^\sigma_\rho\delta^\mu_\nu)(\delta^\nu_\sigma\delta^\rho_\mu - \delta^\rho_\sigma\delta^\nu_\mu) = \phi\left(\frac{3\bar{\Box} + \bar{R}}{2}\right)\mathcal{F}_3(\bar{\Box})\left(\frac{3\bar{\Box} + \bar{R}}{48}\right)\phi. \tag{3.20}$$

Terms with $\bar{\mathcal{D}}_{\mu\nu}$ produce the following expression:

$$\frac{1}{8}\phi(\bar{\mathcal{D}}^\sigma_\nu\delta^\mu_\rho + \bar{\mathcal{D}}^\mu_\rho\delta^\sigma_\nu - \bar{\mathcal{D}}^\mu_\nu\delta^\sigma_\rho - \bar{\mathcal{D}}^\sigma_\rho\delta^\mu_\nu)\mathcal{F}_3(\bar{\Box})\frac{1}{8}(\bar{\mathcal{D}}^\nu_\sigma\delta^\rho_\mu + \bar{\mathcal{D}}^\rho_\mu\delta^\nu_\sigma - \bar{\mathcal{D}}^\nu_\mu\delta^\rho_\sigma - \bar{\mathcal{D}}^\rho_\sigma\delta^\nu_\mu)\phi = \frac{1}{8}\phi\bar{\mathcal{D}}^\nu_\sigma\mathcal{F}_3(\bar{\Box})\bar{\mathcal{D}}^\sigma_\nu\phi. \tag{3.21}$$

This is a term of a type such as in Eq. (3.18), with the function $\hat{\mathcal{F}}_3$ inside, and with the coefficient $1/8$. Summing up the contributions, we get

$$s_3 \xrightarrow{\phi} \frac{\lambda}{32}\int dx^4 \sqrt{-\bar{g}}\phi\left[\left(\frac{3\bar{\Box} + \bar{R}}{6}\right)\hat{\mathcal{F}}_3(\bar{\Box}) + \frac{1}{2}\bar{\Box}\hat{\mathcal{F}}_3\left(\bar{\Box} + \frac{2}{3}\bar{R}\right)\right](3\bar{\Box} + \bar{R})\phi. \tag{3.22}$$

Putting all the four terms $s_{0,1,2,3}$ together we now have the complete action for the scalar mode,

$$S_0 = \frac{1}{32}\int dx^4 \sqrt{-\bar{g}}\phi(3\bar{\Box} + \bar{R})\left\{-\left(\frac{M_P^2}{2} + \lambda c_{1,0}\bar{R}\right) + \frac{\lambda}{2}\left[2\hat{\mathcal{F}}_1(\bar{\Box})(3\bar{\Box} + \bar{R}) + \frac{1}{2}\left(\hat{\mathcal{F}}_2\left(\bar{\Box} + \frac{2}{3}\bar{R}\right)\bar{\Box} + \hat{\mathcal{F}}_2(\bar{\Box})(3\bar{\Box} + \bar{R})\right) + \left(\bar{\Box}\hat{\mathcal{F}}_3\left(\bar{\Box} + \frac{2}{3}\bar{R}\right) + \hat{\mathcal{F}}_3(\bar{\Box})\left(\frac{3\bar{\Box} + \bar{R}}{3}\right)\right)\right]\right\}\phi. \tag{3.23}$$

Notice that even though the functions $\mathcal{F}$ carry hats, the stand-alone $c_{1,0}$ term does not have a hat and this is the value corresponding to the function $\mathcal{F}_1$. This latter action can actually be condensed slightly by reintroducing the functions $\mathcal{F}$'s without hats,

$$S_0 = \frac{1}{32}\int dx^4 \sqrt{-\bar{g}}\phi(3\bar{\Box} + \bar{R})\left\{-\left(\frac{M_P^2}{2} + \lambda c_{1,0}\bar{R}\right) + \frac{\lambda}{2}\left[2\mathcal{F}_1(\bar{\Box})(3\bar{\Box} + \bar{R}) + \frac{1}{2}\mathcal{F}_2\left(\bar{\Box} + \frac{2}{3}\bar{R}\right)\bar{\Box}\right]\right\}\phi. \tag{3.24}$$

We note here that absence of the $\mathcal{F}_3$ function in the last formula was to be expected. Indeed, if we restrict $h_{\mu\nu}$ to the ϕ part, the complete metric takes the form $g_{\mu\nu} = (1 - \frac{1}{4}\phi)\bar{g}_{\mu\nu}$. This is clearly a conformal scaling of the metric. The Weyl tensor of rank (1,3) is invariant under such a scaling and as a consequence a fully contracted square of the Weyl tensor is invariant as well. This implies that no contribution could arise from the Weyl tensor piece in the action; see Eq. (2.4).

C. Tensor modes

Let us now turn our attention to the tensorial terms. Contracting Eq. (3.7) with the Kronecker delta over the first and third indices and commuting the covariant derivatives, we get

$$r_{\mu\nu}(h^{\perp}_{\nu\mu}) = -\frac{1}{2}\left(\bar{\Box} - \frac{\bar{R}}{6}\right)h^{\perp}_{\nu\mu}, \tag{3.25}$$

and consequently,

$$r(h^{\perp}_{\nu\mu}) = 0. \tag{3.26}$$

We also have the well-known result

$$\delta_0(h^{\perp}_{\nu\mu}) = \frac{1}{4}h^{\perp}_{\nu\mu}\left(\bar{\Box} - \frac{\bar{R}}{6}\right)h^{\perp\nu\mu}, \tag{3.27}$$

as this is the only term that appears in pure GR.

We now have all the pieces to compute the action. The most challenging part was how to compute the term containing $r^{\mu\nu}_{\rho\sigma}$. The trick which eventually allowed us to accomplish the task was to roll back to $\delta C^{\mu\nu}_{\rho\sigma}$. To reduce the clutter we denote it as $c^{\mu\nu}_{\rho\sigma}$. It can be obtained as a linear combination of $r^{\mu\nu}_{\rho\sigma}$, r^{μ}_{ν} and r. It enjoys all the symmetry (i.e. nondifferential) properties of the Weyl tensor. Presently, we are interested in tensor modes only. This means that we have $r = 0$. Although, generically a variation of a traceless tensor does not have to be traceless, in our case the background is a conformally flat space-time. For such space-times the Weyl tensor is 0. As a consequence $c^{\mu\nu}_{\rho\sigma}$ is totally traceless similar to the Weyl tensor.

To see why the setup discussed here is so important, let us write down a generic term originating from the part of the action Eq. (2.16) with the function $\hat{\mathcal{F}}_3$. It is of the form

$$\int dx^4\sqrt{-\bar{g}}[h^{\perp}_{\bullet\bullet}\mathcal{O}_{L\bullet\bullet}\mathcal{F}(\bar{\Box})\mathcal{O}_R^{\bullet\bullet}h^{\perp\bullet\bullet}]. \tag{3.28}$$

Here $\mathcal{O}_{L,R}$ can be either a metric (delta symbol) or two covariant derivatives. Of course, the result is a scalar; all indices must be contracted. Notice, however, that indices are always contracted across the function $\mathcal{F}$, i.e. they are never contracted for tensor modes on one side of the $\mathcal{F}$-factor (because this generates either a trace or transverse combination for $h^{\perp}_{\mu\nu}$, and both are 0). The most tedious possibility is

$$h^{\perp\bullet}{}_{\bullet}\bar{\nabla}_{\bullet}\bar{\nabla}^{\bullet}\mathcal{F}\bar{\nabla}_{\bullet}\bar{\nabla}^{\bullet}h^{\perp\bullet}{}_{\bullet},$$

where indices can come in a large number of variations.

The biggest difficulty is to find a way of moving derivatives from the left of $\mathcal{F}$ to the right of it. After rigorous computations, we realized that we needed to utilize recursion relations analogous to Eq. (3.16), which can only be obtained if tensors on the right have special symmetric properties. $c^{\mu\nu}_{\rho\sigma}$ became an appropriate choice, and indeed we obtained the following recursion relation [see (B36)]:

$$\int dx^4\sqrt{-\bar{g}}[h^{\perp}_{\bullet\bullet}\bar{\nabla}_{\bullet}\bar{\nabla}_{\bullet}\mathcal{F}(\bar{\Box})c^{\bullet\bullet\bullet\bullet}] = \int dx^4\sqrt{-\bar{g}}\left[h^{\perp}_{\bullet\bullet}\bar{\nabla}_{\bullet}\mathcal{F}\left(\bar{\Box} + \frac{\bar{R}}{4}\right)\bar{\nabla}_{\bullet}c^{\bullet\bullet\bullet\bullet}\right]. \tag{3.29}$$

Here only two derivatives and not a metric can be on the left side of $\mathcal{F}$, as everything else in this expression is totally traceless.

So, we first rewrite the action Eq. (2.16) as

$$\delta^2 S = \int dx^4\sqrt{-\bar{g}}\left[\left(\frac{M_P^2}{2} + \lambda c_{1,0}\bar{R}\right)\delta_0 + \frac{\lambda}{2}(r\tilde{\mathcal{F}}_1(\bar{\Box})r + r^{\mu}_{\nu}\tilde{\mathcal{F}}_2(\bar{\Box})r^{\nu}_{\mu} + c^{\mu\alpha}{}_{\nu\beta}\tilde{\mathcal{F}}_3(\bar{\Box})c_{\mu\alpha}{}^{\nu\beta})\right] = \tilde{s}_0 + \tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3, \tag{3.30}$$

where the identification is obvious and the following shorthand notations are introduced:

$$\tilde{\mathcal{F}}_1(\Box) = \mathcal{F}_1(\bar{\Box}) - \frac{1}{4}\mathcal{F}_2(\bar{\Box}), \qquad \tilde{\mathcal{F}}_2(\bar{\Box}) = \mathcal{F}_2(\bar{\Box}), \qquad \tilde{\mathcal{F}}_3(\bar{\Box}) = \mathcal{F}_3(\bar{\Box}). \tag{3.31}$$

The tensorial part from $\tilde{s}_3$ can now be computed as follows:

$$\begin{aligned}
\tilde{s}_3 &\xrightarrow{h^{\perp}} \frac{\lambda}{2}\int dx^4\sqrt{-\bar{g}}[c^{\mu\alpha}{}_{\nu\beta}(h^{\perp}_{\mu\nu})\mathcal{F}_3(\bar{\Box})c_{\mu\alpha}{}^{\nu\beta}(h^{\perp}_{\mu\nu})],\\
&= \frac{\lambda}{2}\int dx^4\sqrt{-\bar{g}}\left[\frac{1}{2}(\bar{\nabla}_{\nu}\bar{\nabla}^{\alpha}h^{\perp\mu}_{\beta} - \bar{\nabla}_{\nu}\bar{\nabla}^{\mu}h^{\perp\alpha}_{\beta} - \bar{\nabla}_{\beta}\bar{\nabla}^{\alpha}h^{\perp\mu}_{\nu} + \bar{\nabla}_{\beta}\bar{\nabla}^{\mu}h^{\perp\alpha}_{\nu})\mathcal{F}_3(\bar{\Box})c_{\mu\alpha}{}^{\nu\beta}(h^{\perp}_{\mu\nu})\right],\\
&= \frac{\lambda}{2}\int dx^4\sqrt{-\bar{g}}[2h^{\perp\mu}_{\beta}\bar{\nabla}^{\alpha}\bar{\nabla}_{\nu}\mathcal{F}_3(\bar{\Box})c^{\nu\beta}{}_{\mu\alpha}(h^{\perp}_{\mu\nu})],\\
&= \frac{\lambda}{2}\int dx^4\sqrt{-\bar{g}}\left[2h^{\perp\mu}_{\beta}\bar{\nabla}^{\alpha}\mathcal{F}_3\left(\bar{\Box} + \frac{\bar{R}}{4}\right)\bar{\nabla}_{\nu}c^{\nu\beta}{}_{\mu\alpha}(h^{\perp}_{\mu\nu})\right].
\end{aligned}$$

Passing from the first to second line all other terms in the first ${c_{\mu\alpha}}^{\nu\beta}$ are dropped as they have at least one δ-symbol. The second transformation is solely due to the symmetry properties of the tensor ${c^{\nu\beta}}_{\mu\alpha}$. Next we compute the rank-3 tensor $\nabla_\nu {c^{\nu\beta}}_{\mu\alpha}$, and the result of quite a lengthy computation is

$$\bar{\nabla}_\nu c^{\nu\beta}_{\mu\alpha}(h^\perp_{\mu\nu}) = \frac{1}{4}\left(\bar{\Box} - \frac{\bar{R}}{4}\right)(\bar{\nabla}_\alpha h^{\perp\beta}_{\mu} - \bar{\nabla}_\mu h^{\perp\beta}_{\alpha}). \quad (3.32)$$

One can check that the latter tensor satisfies all the properties required for the use of recursion relation Eq. (B32) apart from manifest symmetry with respect to permutation of the first two indices. This however is not necessary, as those indices are anyway contracted with a symmetric tensor $h^\perp_{\mu\nu}$ on the left. Moving forward we get

$$\begin{aligned}
\tilde{s}_3 &\overset{h^\perp}{\rightarrow} \int dx^4 \sqrt{-\bar{g}} \frac{\lambda}{2}\left[\frac{1}{2} h^{\perp\mu}_{\ \ \beta} \bar{\nabla}^\alpha \mathcal{F}_3\left(\bar{\Box} + \frac{\bar{R}}{4}\right)\left(\bar{\Box} - \frac{\bar{R}}{4}\right)(\bar{\nabla}_\alpha h^{\perp\beta}_{\mu} - \bar{\nabla}_\mu h^{\perp\beta}_{\alpha})\right], \\
&= \int dx^4 \sqrt{-\bar{g}} \frac{\lambda}{2}\left[\frac{1}{2} h^{\perp\mu}_{\ \ \beta} \mathcal{F}_3\left(\bar{\Box} + \frac{\bar{R}}{3}\right)\bar{\nabla}^\alpha\left(\bar{\Box} - \frac{\bar{R}}{4}\right)(\bar{\nabla}_\alpha h^{\perp\beta}_{\mu} - \bar{\nabla}_\mu h^{\perp\beta}_{\alpha})\right], \\
&= \int dx^4 \sqrt{-\bar{g}} \frac{\lambda}{2}\left[\frac{1}{2} h^{\perp\mu}_{\ \ \beta} \mathcal{F}_3\left(\bar{\Box} + \frac{\bar{R}}{3}\right)\left(\bar{\Box} - \frac{\bar{R}}{6}\right)(\bar{\Box} h^{\perp\beta}_{\mu} - \bar{\nabla}^\alpha\bar{\nabla}_\mu h^{\perp\beta}_{\alpha})\right], \\
&= \int dx^4 \sqrt{-\bar{g}} \frac{\lambda}{2}\left[\frac{1}{2} h^{\perp\mu}_{\ \ \beta} \mathcal{F}_3\left(\bar{\Box} + \frac{\bar{R}}{3}\right)\left(\bar{\Box} - \frac{\bar{R}}{6}\right)\left(\bar{\Box} - \frac{\bar{R}}{3}\right) h^{\perp\beta}_{\mu}\right].
\end{aligned} \quad (3.33)$$

Since r vanishes for the tensorial part we do not get any contribution from the $\tilde{s}_1$ part of the action, while the contribution from the $\tilde{s}_2$ part can be easily written down as $r_{\mu\nu}$ contains no covariant derivatives and therefore no commutations need to be performed. Accordingly, we have the final action for the tensor modes,

$$S_2 = \frac{1}{4}\int dx^4 \sqrt{-\bar{g}} h^\perp_{\nu\mu}\left(\bar{\Box} - \frac{\bar{R}}{6}\right)\left\{\frac{M_P^2}{2} + \lambda c_{1,0}\bar{R} + \frac{\lambda}{2}\left[\mathcal{F}_2(\bar{\Box})\left(\bar{\Box} - \frac{\bar{R}}{6}\right) + 2\mathcal{F}_3\left(\bar{\Box} + \frac{\bar{R}}{3}\right)\left(\bar{\Box} - \frac{\bar{R}}{3}\right)\right]\right\} h^{\perp\nu\mu}. \quad (3.34)$$

Please note that our final result contains functions $\mathcal{F}$ without tildes.

IV. PHYSICAL EXCITATIONS AND CONSISTENCY CONDITIONS

A. Canonical action

We are finally ready to discuss the physics of the fluctuations, the main goal of our study. At $\mathcal{O}(h^2)$ the gravitational action has been neatly decomposed into a scalar and tensor part,

$$S_q = S_0 + S_2, \quad (4.1)$$

with

$$\begin{aligned}
S_2 = \frac{1}{2}\int dx^4 \sqrt{-\bar{g}} \widetilde{h^\perp}^{\mu\nu}\left(\bar{\Box} - \frac{\bar{R}}{6}\right) &\times \left\{1 + \frac{2}{M_p^2}\lambda c_{1,0}\bar{R} + \frac{\lambda}{M_p^2}\left[\left(\bar{\Box} - \frac{\bar{R}}{6}\right)\mathcal{F}_2(\bar{\Box})\right.\right. \\
&\left.\left. + 2\left(\bar{\Box} - \frac{\bar{R}}{3}\right)\mathcal{F}_3\left(\bar{\Box} + \frac{\bar{R}}{3}\right)\right]\right\} \widetilde{h^\perp}_{\mu\nu},
\end{aligned} \quad (4.2)$$

and

$$\begin{aligned}
S_0 = -\frac{1}{2}\int dx^4 \sqrt{-\bar{g}} \tilde{\phi}\left(\bar{\Box} + \frac{\bar{R}}{3}\right)&\left\{1 + \frac{2}{M_p^2}\lambda c_{1,0}\bar{R}\right. \\
&\left. - \frac{\lambda}{M_p^2}\left[2(3\bar{\Box} + \bar{R})\mathcal{F}_1(\bar{\Box}) + \frac{1}{2}\bar{\Box}\mathcal{F}_2\left(\bar{\Box} + \frac{2}{3}\bar{R}\right)\right]\right\}\tilde{\phi},
\end{aligned} \quad (4.3)$$

where we have introduced canonical fields

$$\widetilde{h^\perp}_{\mu\nu} = \frac{1}{2} M_p h^\perp_{\mu\nu}, \qquad \tilde{\phi} = \sqrt{\frac{3}{32}} M_p \phi. \quad (4.4)$$

In the Minkowski limit this yields the following spin-two and spin-0 propagators:

$$\Pi_2 = \frac{i}{p^2\{1 - \frac{2p^2}{M_p^2}[\mathcal{F}_2(-p^2) + 2\mathcal{F}_3(-p^2)]\}}, \quad (4.5)$$

$$\Pi_0 = \frac{-i}{p^2\{1 + \frac{2p^2}{M_p^2}[6\mathcal{F}_1(-p^2) + \frac{1}{2}\mathcal{F}_2(-p^2)]\}}, \quad (4.6)$$

where we have also chosen $\lambda = 2$ to compare with the results obtained in [28], and henceforth we proceed with this identification. This agrees precisely[7] with the results in [62], once one realizes that the Ricci and Riemann tensors were used to define $\mathcal{F}$'s instead of the TR tensor and Weyl tensor that we use here. The translation is rather simple; the $\hat{\mathcal{F}}_1$ and $\hat{\mathcal{F}}_2$ defined in Eq. (2.18) are nothing but the $\mathcal{F}_1$ and $\mathcal{F}_2$ in [62], while the $\mathcal{F}_3$ is unchanged.

One more useful check comes from comparing our results with the Gauss-Bonnet term, provided the curvature squared modification comes with the form factors which must vanish, since the Gauss-Bonnet term is a topological invariant. This explicitly corresponds to fixing

$$\mathcal{F}_1 = c_{1,0} = \frac{1}{6} f_0, \quad \mathcal{F}_2 = c_{2,0} = -2 f_0, \quad \mathcal{F}_3 = c_{3,0} = f_0.$$

One can check that expressions in curly brackets in both S_0 and S_2 reduce to 1, and one restores the pure GR results.

B. Ghost-free condition

The condition for absence of ghosts in our theory is equivalent to the following:

(1) absence of new 0's in the spin-two quadratic form as compared to the pure GR limit, and

(2) presence of at most one more 0, say at $\Box = m^2$ with $m^2 > 0$, in the spin-0 quadratic form as compared to the pure GR limit. An additional 0, if present, corresponds to the Brans-Dicke scalar mode usual in pure $F(R)$ gravity modifications.

The above conditions mean that

$$\mathcal{T}(\bar{R}, \bar{\Box}) \equiv 1 + \frac{4\bar{R}}{M_p^2} c_{1,0} + \frac{2}{M_p^2}\left[\left(\bar{\Box} - \frac{\bar{R}}{6}\right)\mathcal{F}_2(\bar{\Box}) + 2\left(\bar{\Box} - \frac{\bar{R}}{3}\right)\mathcal{F}_3\left(\bar{\Box} + \frac{\bar{R}}{3}\right)\right], \tag{4.7}$$

should not have any 0's, and

$$\mathcal{S}(\bar{R}, \bar{\Box}) \equiv 1 + \frac{4\bar{R}}{M_p^2} c_{1,0} - \frac{2}{M_p^2}\left[2(3\bar{\Box} + \bar{R})\mathcal{F}_1(\bar{\Box}) + \frac{1}{2}\bar{\Box}\mathcal{F}_2\left(\bar{\Box} + \frac{2}{3}\bar{R}\right)\right] \tag{4.8}$$

can at least have a single 0. Note that since the 0 of the scalar mode at $\Box = 0$ has a wrong sign in the residue, the new 0, if present, is guaranteed to have the correct residue sign, and therefore is a ghost, while the constraint $m^2 > 0$ ensures that it is not tachyonic either. Mathematically, we could express the two functions as

$$\mathcal{T}(\bar{R}, \bar{\Box}) \equiv e^{\tau(\bar{\Box})}, \tag{4.9}$$

and

$$\mathcal{S}(\bar{R}, \bar{\Box}) \equiv \left(1 - \frac{\bar{\Box}}{m^2}\right)^{\epsilon} e^{\sigma(\bar{\Box})}, \tag{4.10}$$

with τ and σ being entire functions, $\epsilon = 0$, 1 and m being real. $\epsilon = 0$ corresponds to no extra scalar mode and $\epsilon = 1$ corresponds to the Brans-Dicke scalar.

C. Illustrative examples

In this subsection we provide a few simple examples of gravitational models which are consistent around dS or AdS backgrounds. In particular, we focus on the cases when only one of the three functions, $\mathcal{F}_{1,2,3}$'s, is nonzero. In this process, we also extend the IDG model with only quadratic curvature terms that has been shown previously [28] to consistently modify the graviton propagator, and ameliorate some of the UV problems of GR without introducing any new degrees of freedom, ghosts or otherwise. By explicit construction, we show how by including nonlinear (in curvature) terms in the action it is possible to have a gravitational theory that is not only consistent around the Minkowski background ($\Lambda = 0$), but also the curved dS ($\Lambda > 0$) or AdS ($\Lambda < 0$) backgrounds. It becomes evident that nonlinear terms are necessary in order to achieve this, and corroborates the idea that requiring the quantum theory of gravity to be consistent around all possible backgrounds may be a powerful way to constrain the modifications to GR.

1. $\mathcal{F}_1 \neq 0$, but $\mathcal{F}_2 = \mathcal{F}_3 = 0$

In this case, we have manifestly no extra poles in the spin-two propagator. However, we must ensure that

$$1 + \frac{4\bar{R}}{M_p^2} c_{1,0} = 1 + \frac{16\Lambda}{M_p^4} c_{1,0} > 0, \tag{4.11}$$

and also guarantee that

$$\mathcal{S}(\bar{R}, \bar{\Box}) = 1 + \frac{4\bar{R}}{M_p^2} c_{1,0} - \frac{4}{M_p^2}(3\bar{\Box} + \bar{R})\mathcal{F}_1(\bar{\Box}) = \left(1 - \frac{\bar{\Box}}{m^2}\right)^{\epsilon} e^{\sigma(\bar{\Box})}. \tag{4.12}$$

We note that for the constant terms to match on both sides we must have $\sigma(0) = 0$. Then, the function $\mathcal{F}_1$ has to be of the form

$$\mathcal{F}_1(\bar{\Box}) = \frac{1 + \frac{4\bar{R}}{M_p^2} c_{1,0} - (1 - \frac{\bar{\Box}}{m^2})^{\epsilon} e^{\sigma(\bar{\Box})}}{\frac{4\bar{R}}{M_p^2}(1 + \frac{3\bar{\Box}}{\bar{R}})} = \frac{1 + \frac{16\Lambda}{M_p^4} c_{1,0} - (1 - \frac{\bar{\Box}}{m^2})^{\epsilon} e^{\sigma(\bar{\Box})}}{\frac{16\Lambda}{M_p^4}(1 + \frac{3M_p^2\bar{\Box}}{4\Lambda})}. \tag{4.13}$$

[7]To get a precise agreement with [28], one also has to put $M_p = 1$ and also realize that the scalar projection operator and the canonical field used here differ by a factor half; this is just a matter of convention.

A couple of comments are now in order: First, as one looks at the last expression, it is clear that for any set of parameters, $\Lambda, c_{1,0}, m, \epsilon$, and analytic function $\sigma(\Box)$, we have a $\mathcal{F}_1$ that gives rise to consistent fluctuations around a specific dS or AdS background, as long as Eq. (4.11) is satisfied, and $\sigma(0) = 0$. However, by inspection it is also clear that the function, $\mathcal{F}_1$, depends on the value of Λ, and it is not possible therefore for a single $\mathcal{F}_1$ function to be simultaneously consistent for two different Λ's. In other words, if we are starting from a model with only quadratic curvature terms, it is impossible for the theory to be consistent for arbitrary values of the cosmological constant. On the other hand, one may imagine that a complete theory of gravity should be consistent in the presence of any arbitrary stress-energy tensor, and in particular, for any value of the cosmological constant. The form of $\mathcal{F}_1$ in terms of $\bar{R}$ actually suggests a simple way out of this problem. If we instead allow for nonlinear terms in our original gravitational action, such as an action of the form

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \Lambda + R\mathcal{F}_1(\Box, R)R\right] \quad \text{with}$$

$$\mathcal{F}_1(\Box, R) = \frac{1 + \frac{4R}{M_p^2}c_{1,0} - (1 - \frac{\Box}{m^2})^{\epsilon}e^{\sigma(\Box)}}{\frac{4R}{M_p^2}(1 + \frac{3\Box}{R})}, \tag{4.14}$$

then such an action will be consistent for any arbitrary value of Λ, and in fact, around a given background (Λ) the equivalent $\mathcal{F}_1$ will be precisely given by Eq. (4.13).

Secondly, let us point out that this is a theory where essentially we have GR coupled to a scalar field theory. If σ identically vanishes, then $\epsilon = 1$ reproduces Brans-Dicke theory. For instance, if $\sigma(\Box) = -\Box$, and $\epsilon = 0$, then we have a p-adic-type scalar field coupled to gravity. Such systems have been studied in detail in the context of cosmology [63]. On the other hand, if we have $\sigma(\Box) = -\Box$, and $\epsilon = 1$, this corresponds to a SFT-type tachyonic field coupled to gravity whose cosmological implications have also been studied in previous literature [45]. Furthermore, this also provides an extension of Starobinsky's original model of inflation [64] to seek an UV completion; see [65] and detailed perturbation analysis in [66,67].

2. $\mathcal{F}_2 \neq 0$, but $\mathcal{F}_1 = \mathcal{F}_3 = 0$

In this case we get two constraints on the same function $\mathcal{F}_2$,

$$\mathcal{T}(\bar{R}, \bar{\Box}) \equiv 1 + \frac{2}{M_p^2}\left(\bar{\Box} - \frac{\bar{R}}{6}\right)\mathcal{F}_2(\bar{\Box}) = e^{\tau(\bar{\Box})}, \tag{4.15}$$

$$\mathcal{S}(\bar{R}, \bar{\Box}) \equiv 1 - \frac{2}{M_p^2}\frac{1}{2}\bar{\Box}\mathcal{F}_2\left(\bar{\Box} + \frac{2}{3}\bar{R}\right) = (\bar{\Box} - m^2)^{\epsilon}e^{\sigma(\bar{\Box})}, \tag{4.16}$$

and the only solution is the trivial case $\mathcal{F}_2 = 0$. In other words, we get back to GR.

3. $\mathcal{F}_3 \neq 0$, but $\mathcal{F}_1 = \mathcal{F}_2 = 0$, the case of only gravitons

A particularly interesting and illuminating case, which has been previously discussed in the context of black hole and big bang singularity, is when no scalar degrees of freedom are present, and the presence of infinite covariant derivatives only modify the graviton propagator without introducing any new states. This is ensured by demanding that $\mathcal{S}$ is just a constant, which, in particular, can be achieved by setting $\mathcal{F}_1 = \mathcal{F}_2 = 0$. We are then left with a graviton quadratic form that reads

$$\widetilde{h^{\perp}}_{\mu\nu}\left(-\bar{\Box} + \frac{\bar{R}}{6}\right)\left[1 - \frac{4}{M_P^2}\left(-\bar{\Box} + \frac{\bar{R}}{3}\right)\mathcal{F}_3\left(\bar{\Box} + \frac{\bar{R}}{3}\right)\right]\widetilde{h^{\perp}}^{\mu\nu}. \tag{4.17}$$

In order to illustrate how one can obtain a graviton propagator involving an infinite set of higher derivatives, let us consider the simplest case where the modified quadratic form is as follows:

$$\widetilde{h^{\perp}}_{\mu\nu}\left(-\bar{\Box} + \frac{\bar{R}}{6}\right)e^{\alpha(-\bar{\Box} + \frac{\bar{R}}{6})}\widetilde{h^{\perp}}^{\mu\nu}, \tag{4.18}$$

where α is a positive constant. This provides an exponential suppression at high momentum, which has been found to ameliorate the black hole and big bang singularities, but does not alter the Newtonian limit as the residue at $\bar{\Box} = \bar{R}/6$ remains unaltered. We note that the exponential suppression must be interpreted with care as it cannot be consistently defined in a Lorentzian signature. One must perform quantum loop computations in Euclidean space and then analytically continue the resulting answer to Minkowski space-time. This idea was originally introduced in the first paper in [12] and has found more systematic realizations in [16,36,38,68,69]. However, one still needs to work out all the mathematical details in order to have this theory defined in a Lorentzian signature in a general context.

It is easy to obtain the form of $\mathcal{F}_3$ function which gives rise to an inverse propagator, such as Eq. (4.18),

$$\mathcal{F}_3(\Box) = \frac{M_P^2}{4}\left[\frac{e^{-\alpha(\Box - \frac{R}{2})} - 1}{\Box - \frac{2R}{3}}\right]. \tag{4.19}$$

Again, just as our construction in Eq. (4.14), such a nonlinear function ensures that the model remains consistent in the presence of any cosmological constant; the coefficients adjust appropriately when perturbed around any given dS/AdS/Minkowski background so that no new poles are introduced.

The above function does have a pole at $\Box = 2R/3$, which is perfectly acceptable as the propagators are still well defined. However, it is also possible to construct analytic functions. For instance,

$$\mathcal{F}_3(\Box) = \frac{M_p^2}{4}\left[\frac{e^{\alpha(\Box-\frac{R}{2})(\Box-\frac{2R}{3})}-1}{\Box-\frac{2R}{3}}\right] \tag{4.20}$$

is an analytic function yielding the spin-two quadratic form

$$\widetilde{h^{\perp}}_{\mu\nu}\left(-\bar{\Box}+\frac{\bar{R}}{6}\right)e^{\alpha(-\bar{\Box}+\frac{\bar{R}}{6})(-\bar{\Box}+\frac{\bar{R}}{3})}\widetilde{h^{\perp}}_{\mu\nu} \tag{4.21}$$

that again only contains the graviton pole and an exponential suppression at high momenta.

V. CONCLUSIONS

In this paper we have provided an algorithm to construct the most general parity invariant and torsion-free covariant action of gravity that is consistent around constant curvature backgrounds, as long as the action has a well-defined Minkowski limit. This action is given by formula (2.4). In particular, we have studied dS and AdS backgrounds. Our analysis smoothly reduces to the Minkowski space-time limit which was studied before by BGKM in [28]. Our prescription is generic, and is equally applicable for both UV and IR higher derivative modifications that have found various cosmological and stringy applications. We also checked our analysis against some well-known cases; for instance, when the scale of nonlocality $M\to\infty$, the class of consistent gravitational actions reduces to the popular local models of 4 derivative gravity, the $F(R)$ and $F(G)$ theories. We paid special attention to the Gauss-Bonnet action as it provided us with some nontrivial checks on our derivations.

We found that the most general action can indeed contain *infinite* covariant derivatives, acting on the Ricci scalar, Ricci tensor or Weyl tensor. The infinite derivatives can be expressed in terms of form factors, whose forms can be constrained by demanding that the action is perturbatively ghost and tachyon free around constant curvature backgrounds. In order to verify this, it was sufficient to perturb the gravitational action up to order $\mathcal{O}(h^2)$ around dS and AdS backgrounds. We computed explicitly the second order variation of the Einstein-Hilbert term as well as the higher order terms involving the form factors. For pedagogical reasons, we have provided details of our conventions in Appendix A, properties of maximally symmetric space-times in A 2, and perturbations in A 3. Some very useful and powerful identities around constant curvature backgrounds were obtained in Appendix B.

In four dimensions, in order to obtain the true propagating degrees of freedom in space-time, we had to decompose the metric tensor into its degrees of freedom corresponding to transverse-traceless $h^{\perp}_{\mu\nu}$, a transverse vector field, $A^{\perp}_{\mu}$, and two scalars, B, h. After performing a systematic decomposition we were able to show that the only viable propagating degrees of freedom are the transverse-traceless spin-two field, $h^{\perp}_{\mu\nu}$, and a scalar combination which gives rise to the spin-0 component, $\phi=\bar{\Box}B-h$. The transverse vector component, $A^{\perp}_{\mu}$, and $\nabla_\mu\nabla_\nu B$ vanish identically, leaving the massless graviton and possibly a Brans-Dicke scalar to propagate around dS, AdS or Minkowski backgrounds. The details of the latter computation can be found in Appendixes C 1 and C 2.

In order to make the graviton propagator perturbatively ghost free in dS and AdS backgrounds, one has to make sure that the second variation of the spin-two component does not introduce any new pole in the propagator, and must recover the pure GR limit at low energies. Similarly, the spin-0 component can allow at most one extra pole, say at mass, $m^2>0$, in its propagator. The latter condition is necessary to ensure that an additional pole representing the Brans-Dicke scalar, if present as in $F(R)$ gravity modifications, is not a tachyon. If both spin-two and spin-0 propagators do not have any extra poles, then both can be expressed in terms of an exponential of an entire function, which does not introduce any pole except the essential singularity at the boundary corresponding to the UV limit, $\Box\to\infty$. In the IR limit, when $\Box\to 0$, the second variation of both spin-two and spin-0 components recovers the GR limit in dS and AdS backgrounds.

We have illustrated that we can recover various limits from the most generic IDG action. In the limit when $\mathcal{F}_2=0, \mathcal{F}_3=0, \mathcal{F}_1\neq 0$, the theory effectively reduces to scalar-tensor theory around dS and AdS backgrounds. When $\mathcal{F}_1=0, \mathcal{F}_3=0$, the ghost-free condition enforces a simple solution where $\mathcal{F}_2=0$, too, thereby reducing to pure GR. In the last scenario, when $\mathcal{F}_1=0, \mathcal{F}_2=0, \mathcal{F}_3\neq 0$, the theory space reduces to pure spin-two excitations, and no physical spin-0 mode.

Our analysis indeed opens up new avenues for higher derivative theories of gravity, including IDG, which can be made consistent at both classical and quantum level around dS and AdS backgrounds. This should have important implications for both cosmology and for AdS/CFT correspondence. In the dS case, it provides the possibility to realize a big bounce [16], avoiding the big bang singularity problem. This scenario also presents the first viable UV generalization of Starobinsky inflation [64], and also provides an interesting connection between the graviton degrees of freedom propagating in space-time with the avoidance of focusing the null congruences in a time dependent background [50].

Furthermore, our analysis also provides conditions for stable ghost and tachyon-free modifications of GR in the IR, in the context of dark energy problems. In this context it is rather useful if one can extend the class of actions we have analyzed to include terms that do not necessarily have a well-defined Minkowski limit as such actions have been discussed considerably in the literature to address the dark energy problem.

In the case of AdS, a consistent IDG provides an ideal platform to study connection between gravity in the UV and the corresponding CFT in the boundary. After all, presence of IDG is inevitable in closed string theory, in terms of α' corrections. At present, computing all order α' corrections in AdS background in the closed string sector is indeed challenging, and although our analysis does not involve supersymmetry, we believe that constructing a stable and consistent theory of gravity around AdS will help us in ascertaining how such an action can be derived from closed string field theory.

Finally, we emphasize that several of the computations and strategies that we developed in analyzing perturbations around dS/AdS should carry over to more nontrivial backgrounds such as FLRW or black hole space-times. As mentioned before, most of the applications and tests of GR involve encoding the physics around certain highly symmetric background space-times and small perturbations around them. Thus, we believe that our analysis could go a long way in making progress towards analyzing such important physical space-times for very general covariant theories of gravity.

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APPENDIX A: NOTATIONS

1. General backgrounds

Here we introduce the notations used at the background level.

The metric is

$$g_{\mu\nu} = (-,+,+,+,\cdots), \qquad g_{\mu\nu}g^{\mu\nu} = D = 4.$$

The dimension is always 4 and four-dimensional indices are small greek letters. The metric connection (Cristoffel symbol) is

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}).$$

The covariant derivative is ∇_μ and acts as

$$\nabla_\mu F^{\cdot\alpha\cdot}_{\cdot\beta\cdot} = \partial_\mu F^{\cdot\alpha\cdot}_{\cdot\beta\cdot} + \Gamma^{\alpha}_{\mu\chi}F^{\cdot\chi\cdot}_{\cdot\beta\cdot} - \Gamma^{\chi}_{\mu\beta}F^{\cdot\alpha\cdot}_{\cdot\chi\cdot}.$$

It follows that $\nabla_\rho g^{\rho\nu} \equiv 0$. The Riemann tensor, curvatures, and Einstein tensor are defined as

$$R^{\sigma}_{\mu\nu\rho} = \partial_\nu\Gamma^{\sigma}_{\mu\rho} - \partial_\rho\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\sigma}_{\chi\nu}\Gamma^{\chi}_{\mu\rho} - \Gamma^{\sigma}_{\chi\rho}\Gamma^{\chi}_{\mu\nu},$$
$$R_{\mu\rho} = R^{\sigma}_{\mu\sigma\rho}, R = R^{\mu}_{\mu}, \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}.$$

The symmetry properties are

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = -R_{\nu\mu\rho\sigma} = R_{\rho\sigma\mu\nu},$$
$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0, \qquad R_{\mu\nu} = R_{\nu\mu}.$$

The commutator of covariant derivatives is

$$[\nabla_\mu, \nabla_\nu]A_\rho = R^{\chi}_{\rho\nu\mu}A_\chi.$$

The Bianchi identity is given by

$$\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\sigma R_{\mu\nu\lambda\rho} + \nabla_\rho R_{\mu\nu\sigma\lambda} = 0.$$

It implies

$$\nabla_\mu R^{\mu}_{\nu} = \frac{1}{2}\partial_\nu R, \qquad \nabla_\nu\nabla_\mu R^{\mu\nu} = \frac{1}{2}\Box R,$$
$$\nabla_\lambda R_{\nu\sigma} - \nabla_\sigma R_{\nu\lambda} + \nabla^\mu R_{\mu\nu\sigma\lambda} = 0, \qquad \nabla_\mu G^{\mu}_{\nu} = 0.$$

The d'Alambertian (the box) is defined as $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, and acts in a fully covariant way. Another useful operator is

$$\mathcal{D}_{\mu\nu} = \nabla_\mu\nabla_\nu - \frac{1}{4}g_{\mu\nu}\Box.$$

It is traceless and this simplifies certain computations. The traceless analog of the Einstein tensor is given by

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{D}Rg_{\mu\nu}, \qquad S^{\mu}_{\mu} = 0.$$

The Weyl tensor follows from the Ricci decomposition, and this is given by (in D space-time dimensions)

$$C^{\mu}_{\alpha\nu\beta} = R^{\mu}_{\alpha\nu\beta} - \frac{1}{D-2}(\delta^{\mu}_{\nu}R_{\alpha\beta} - \delta^{\mu}_{\beta}R_{\alpha\nu} + R^{\mu}_{\nu}g_{\alpha\beta} - R^{\mu}_{\beta}g_{\alpha\nu})$$
$$+ \frac{R}{(D-2)(D-1)}(\delta^{\mu}_{\nu}g_{\alpha\beta} - \delta^{\mu}_{\beta}g_{\alpha\nu}).$$

The Weyl tensor has all the symmetry properties of the Riemann tensor and also it is absolutely traceless, i.e. $C^{\mu}_{\alpha\mu\beta} = 0$. Moreover this rank (1, 3) tensor is invariant under the conformal scaling of the metric. The latter implies that the Weyl tensor is 0 on conformally flat manifolds, i.e. on the manifolds where the metric can be brought to the form $ds^2 = a(x)^2\eta_{\mu\nu}dx^\mu dx^\nu$, with $\eta_{\mu\nu}$ being the Minkowski metric with the same signature as the original one. The following quadratic relations always hold:

$$S^2_{\mu\nu} = R^2_{\mu\nu} - \frac{1}{4}R^2, \qquad C^2_{\mu\nu\alpha\beta} = R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + \frac{1}{3}R^2.$$

The Gauss-Bonnet term can be written as

$$G = R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\alpha\beta}^2 = \frac{1}{6}R^2 - 2S_{\mu\nu}^2 + C_{\mu\nu\alpha\beta}^2.$$

2. Maximally symmetric space-times

By definition maximally symmetric space-times have $\frac{1}{2}D(D+1)$ linearly independent Killing vectors. This translates into the fact that

$$R^\sigma_{\mu\nu\rho} = \frac{R}{D(D-1)}(\delta^\sigma_\nu g_{\mu\rho} - \delta^\sigma_\rho g_{\mu\nu}). \tag{A1}$$

In general R does not have to be constant. One however can prove using the Bianchi identities that for $D > 2$ this form of the Riemann tensor implies $R = \text{const}$. Consequently, in such space-times

$$\nabla_\lambda R^\sigma_{\mu\nu\rho} = 0.$$

Also, one readily sees

$$S_{\mu\nu} = 0, \qquad C^\sigma_{\mu\nu\rho} = 0. \tag{A2}$$

Both AdS and dS space-times are maximally symmetric ones [i.e. they satisfies Eq. (A1)]. $R = \text{const} > 0$ for dS, and $R = \text{const} < 0$ AdS. Minkowski has $R^\sigma_{\mu\nu\rho} = 0$, and can be seen as the $R \to 0$ limit of the (A)dS space-time.

3. Perturbations

Here we introduce notations and quantities relevant for computing the second variation of the action around an (A) dS background. The metric variation is

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}.$$

Bars are used to designate the background quantities. The following relations are relevant for perturbation analysis in this paper:

$$\begin{aligned}
&g^{\mu\nu} \to \bar{g}^{\mu\nu} - h^{\mu\nu}, \qquad \sqrt{-g} \to \sqrt{-\bar{g}}\left(1 + \frac{h}{2} + \frac{h^2}{8} - \frac{h^\mu_\nu h^\nu_\mu}{4}\right), \qquad h = \bar{g}^{\mu\nu}h_{\mu\nu},\\
&\Gamma^\mu_{\nu\rho} \to \bar{\Gamma}^\mu_{\nu\rho} + \gamma^\mu_{\nu\rho}, \qquad \gamma^\mu_{\nu\rho} = \frac{1}{2}(\bar{\nabla}_\nu h^\mu_\rho + \bar{\nabla}_\rho h^\mu_\nu - \bar{\nabla}^\mu h_{\nu\rho}), \qquad \gamma^\rho_{\mu\rho} = \frac{1}{2}\partial_\mu h,\\
&R^\sigma_{\mu\nu\rho} \to \bar{R}^\sigma_{\mu\nu\rho} + \tilde{r}^\sigma_{\mu\nu\rho}, \qquad \tilde{r}^\sigma_{\mu\nu\rho} = \bar{\nabla}_\nu \gamma^\sigma_{\mu\rho} - \bar{\nabla}_\rho \gamma^\sigma_{\mu\nu},\\
&R^{\sigma\mu}{}_{\nu\rho} \to \bar{R}^{\sigma\mu}{}_{\nu\rho} + r^{\sigma\mu}{}_{\nu\rho}, \qquad r^{\sigma\mu}{}_{\nu\rho} = -h^{\mu\tau}\bar{R}^\sigma_{\tau\nu\rho} + \bar{g}^{\mu\tau}\tilde{r}^\sigma_{\tau\nu\rho},\\
&R_{\mu\rho} \to \bar{R}_{\mu\rho} + \tilde{r}_{\mu\rho} + \gamma^\sigma_{\chi\sigma}\gamma^\chi_{\mu\rho} - \gamma^\sigma_{\chi\rho}\gamma^\chi_{\sigma\mu}, \qquad \tilde{r}_{\mu\rho} = \bar{\nabla}_\nu \gamma^\nu_{\mu\rho} - \bar{\nabla}_\rho \gamma^\nu_{\mu\nu} = \frac{1}{2}(\bar{\nabla}_\nu\bar{\nabla}_\mu h^\nu_\rho + \bar{\nabla}_\nu\bar{\nabla}_\rho h^\nu_\mu - \bar{\Box}h_{\mu\rho} - \bar{\nabla}_\rho\partial_\mu h),\\
&R^\mu_\rho \to \bar{R}^\mu_\rho + r^\mu_\rho, \qquad r^\mu_\rho = -h^{\mu\sigma}\bar{R}_{\sigma\rho} + \bar{g}^{\mu\sigma}\tilde{r}_{\sigma\rho},\\
&R \to \bar{R} + r, \qquad r = -h^{\mu\rho}\bar{R}_{\mu\rho} + \bar{g}^{\mu\rho}(\bar{\nabla}_\nu\gamma^\nu_{\mu\rho} - \bar{\nabla}_\rho\gamma^\nu_{\mu\nu}) = (-\bar{R}_{\mu\nu} + \bar{\nabla}_\mu\bar{\nabla}_\nu - \bar{g}_{\mu\nu}\bar{\Box})h^{\mu\nu}.
\end{aligned} \tag{A3}$$

Here the arrow means that the rhs is equal to the lhs up to higher order corrections. The order of expansion is either linear or quadratic in the above expressions. One can show that other (higher order in h) terms do not contribute to the quadratic variation of the action. They either generate more than quadratic corrections to the quadratic action variation or become total derivatives. Note that

$$\delta F^\mu = \delta(g^{\mu\nu}F_\nu) = -h^{\mu\nu}F_\nu + \bar{g}^{\mu\nu}\delta F_\nu \neq \bar{g}^{\mu\nu}\delta F_\nu.$$

The two quantities which are often actually used are

$$\begin{aligned}
r^\mu_\rho &= -\frac{\bar{R}}{4}h^\mu_\rho + \frac{1}{2}(\bar{\nabla}_\nu\bar{\nabla}^\mu h^\nu_\rho + \bar{\nabla}_\nu\bar{\nabla}_\rho h^{\nu\mu} - \bar{\Box}h^\mu_\rho - \bar{\nabla}_\rho\partial^\mu h),\\
r^{\sigma\mu}{}_{\nu\rho} &= \frac{\bar{R}}{24}(\delta^\mu_\nu h^\sigma_\rho - \delta^\mu_\rho h^\sigma_\nu - \delta^\sigma_\nu h^\mu_\rho + \delta^\sigma_\rho h^\mu_\nu)\\
&\quad + \frac{1}{2}(\bar{\nabla}_\nu\bar{\nabla}^\mu h^\sigma_\rho - \bar{\nabla}_\nu\bar{\nabla}^\sigma h^\mu_\rho - \bar{\nabla}_\rho\bar{\nabla}^\mu h^\sigma_\nu + \bar{\nabla}_\rho\bar{\nabla}^\sigma h^\mu_\nu).
\end{aligned} \tag{A4}$$

Note that

$$r = \delta^\rho_\sigma r^\sigma_\rho, \qquad r^\sigma_\rho = \delta^\nu_\mu r^{\mu\sigma}{}_{\nu\rho}.$$

Upon acting on scalars, the variation of the $\Box$ operator is

$$\delta(\Box)\varphi = (-h^{\mu\nu}\bar{\nabla}_\mu\partial_\nu - \bar{g}^{\mu\nu}\gamma^\rho_{\mu\nu}\partial_\rho)\varphi. \tag{A5}$$

As a warm-up exercise, given pure Einstein-Hilbert action with a cosmological term, i.e.

$$S_0 = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \Lambda\right],$$

one obtains a second variation around any arbitrary background, as

$$\begin{aligned}
\delta^2 S_0 = \int dx^4\sqrt{-\bar{g}}\frac{M_P^2}{2}\Bigg[&\left(\frac{1}{4}h_{\mu\nu}\bar{\Box}h^{\mu\nu} - \frac{1}{4}h\bar{\Box}h + \frac{1}{2}h\bar{\nabla}_\mu\bar{\nabla}_\rho h^{\mu\rho} + \frac{1}{2}\bar{\nabla}_\mu h^{\mu\rho}\bar{\nabla}_\nu h^\nu_\rho\right)\\
&+ (hh^{\mu\nu} - 2h^\mu_\sigma h^{\sigma\nu})\left(\frac{1}{8}\bar{g}_{\mu\nu}\bar{R} - \frac{1}{4}\bar{g}_{\mu\nu}\frac{\Lambda}{M_P^2} - \frac{1}{2}\bar{R}_{\mu\nu}\right)\\
&- \left(\frac{1}{2}\bar{R}_{\sigma\nu}h^\sigma_\rho h^{\nu\rho} + \frac{1}{2}\bar{R}^\sigma_{\rho\nu\mu}h^\mu_\sigma h^{\nu\rho}\right)\Bigg].
\end{aligned} \tag{A6}$$

This result can be checked against, e.g., [70]. Note that the indices of perturbed quantities are raised and lowered by the background metric.

APPENDIX B: COMMUTATION RELATIONS

Here we collect important commutation relations which are used in transforming the expressions. All the expressions in this section are written with maximally symmetric space-times in mind. Also, all the derivative operators, metrics, and curvatures take their background values, and we simply omit bars for clarity.

Given an arbitrary scalar φ, we have

$$\nabla_\mu\nabla_\alpha\varphi = \nabla_\alpha\nabla_\mu\varphi, \tag{B1}$$

$$\nabla_\mu\nabla_\alpha\nabla_\beta\varphi = \nabla_\alpha\nabla_\beta\nabla_\mu\varphi + \frac{R}{12}(g_{\beta\mu}\nabla_\alpha - g_{\alpha\beta}\nabla_\mu)\varphi, \tag{B2}$$

$$\nabla_\mu\Box\varphi = \left(\Box - \frac{R}{4}\right)\nabla_\mu\varphi. \tag{B3}$$

Given an arbitrary vector t_μ, we have

$$\nabla_\mu\nabla_\alpha t^\mu = \nabla_\alpha\nabla_\mu t^\mu + \frac{R}{4}t_\alpha, \tag{B4}$$

$$\begin{aligned}\nabla_\mu\nabla_\alpha\nabla_\beta t^\mu = {}& \nabla_\alpha\nabla_\beta\nabla_\mu t^\mu + \frac{R}{4}(\nabla_\alpha t_\beta + \nabla_\beta t_\alpha) \\ &+ \frac{R}{12}(\nabla_\alpha t_\beta - g_{\alpha\beta}\nabla^\mu t_\mu),\end{aligned} \tag{B5}$$

$$\nabla_\mu\Box t^\mu = \left(\Box + \frac{R}{4}\right)\nabla_\mu t^\mu, \tag{B6}$$

$$\begin{aligned}\nabla_\nu\nabla^\mu\nabla_\rho t^\sigma = {}& \nabla_\rho\nabla^\mu\nabla_\nu t^\sigma + (\delta^\sigma_\nu(\nabla_\rho t^\mu + \nabla^\mu t_\rho) \\ &- g_{\mu\sigma}(\nabla_\rho t_\nu - \nabla_\nu t_\rho) - \delta^\sigma_\rho(\nabla_\nu t^\mu + \nabla^\mu t_\nu) \\ &+ \delta^\mu_\nu\nabla_\rho t^\sigma - \delta^\mu_\rho\nabla_\nu t^\sigma).\end{aligned} \tag{B7}$$

Therefore, for a transverse vector $\nabla_\mu A^\mu = 0$, one obtains

$$\nabla_\mu\nabla_\alpha A^\mu = \frac{R}{4}A_\alpha, \tag{B8}$$

$$\nabla_\mu\nabla_\alpha\nabla_\beta A^\mu = \frac{R}{4}(\nabla_\alpha A_\beta + \nabla_\beta A_\alpha) + \frac{R}{12}\nabla_\alpha A_\beta, \tag{B9}$$

$$\nabla_\mu\Box A^\mu = 0. \tag{B10}$$

The very last formula tells us that $\Box A_\mu$ is also a transverse vector.

Given an arbitrary (symmetric) tensor $t_{\mu\nu}$ we have

$$\nabla_\mu\nabla_\alpha t^{\mu\nu} = \nabla_\alpha\nabla_\mu t^{\mu\nu} + \frac{R}{3}t^\nu_\alpha - \frac{R}{12}t^\mu_\mu\delta^\nu_\alpha, \tag{B11}$$

$$\begin{aligned}\nabla_\mu\nabla_\alpha\nabla_\beta t^{\mu\nu} = {}& \nabla_\alpha\nabla_\beta\nabla_\mu t^{\mu\nu} + \frac{5R}{12}\nabla_\alpha t^\nu_\beta + \frac{R}{3}\nabla_\beta t^\nu_\alpha \\ &- \frac{R}{12}(\delta^\nu_\beta\nabla_\alpha + \delta^\nu_\alpha\nabla_\beta)t^\mu_\mu - \frac{R}{12}g_{\alpha\beta}\nabla_\mu t^{\mu\nu},\end{aligned} \tag{B12}$$

$$\nabla_\mu\Box t^{\mu\nu} = \left(\Box + \frac{5R}{12}\right)\nabla_\mu t^{\mu\nu} - \frac{R}{6}\nabla^\nu t^\mu_\mu, \tag{B13}$$

$$\begin{aligned}\nabla^\sigma\nabla_\rho\nabla_\sigma t_{\mu\nu} = {}& \Box\nabla_\rho t_{\mu\nu} + \frac{R}{12}(g_{\mu\rho}\nabla_\sigma t^\sigma_\nu + g_{\nu\rho}\nabla_\sigma t^\sigma_\mu) \\ &- \frac{R}{12}(\nabla_\mu t_{\rho\nu} + \nabla_\nu t_{\rho\mu}),\end{aligned} \tag{B14}$$

$$\begin{aligned}= {}& \nabla_\rho\Box t_{\mu\nu} - \frac{R}{12}(g_{\mu\rho}\nabla_\sigma t^\sigma_\nu + g_{\nu\rho}\nabla_\sigma t^\sigma_\mu) \\ &+ \frac{R}{12}(\nabla_\mu t_{\rho\nu} + \nabla_\nu t_{\rho\mu}) + \frac{R}{4}\nabla_\rho h_{\mu\nu}.\end{aligned} \tag{B15}$$

Therefore, for a transverse (symmetric) tensor $\nabla_\mu T^{\mu\nu} = 0$, one similarly obtains

$$\nabla_\mu\nabla_\alpha T^{\mu\nu} = \frac{R}{3}T^\nu_\alpha - \frac{R}{12}T^\mu_\mu\delta^\nu_\alpha, \tag{B16}$$

$$\nabla_\mu\nabla_\alpha\nabla_\beta T^{\mu\nu} = \frac{5R}{12}\nabla_\alpha T^\nu_\beta + \frac{R}{3}\nabla_\beta T^\nu_\alpha - \frac{R}{12}(\delta^\nu_\beta\nabla_\alpha + \delta^\nu_\alpha\nabla_\beta)T^\mu_\mu, \tag{B17}$$

$$\nabla_\mu\Box T^{\mu\nu} = -\frac{R}{6}\nabla^\nu T^\mu_\mu, \tag{B18}$$

$$\nabla^\sigma\nabla_\rho\nabla_\sigma T_{\mu\nu} = \Box\nabla_\rho T_{\mu\nu} - \frac{R}{12}(\nabla_\mu T_{\rho\nu} + \nabla_\nu T_{\rho\mu}), \tag{B19}$$

$$= \nabla_\rho\Box T_{\mu\nu} + \frac{R}{12}(\nabla_\mu T_{\rho\nu} + \nabla_\nu T_{\rho\mu}) + \frac{R}{4}\nabla_\rho T_{\mu\nu}. \tag{B20}$$

Next, for a traceless (symmetric) tensor $H^\mu_\mu = 0$ one gets

$$\nabla_\mu\nabla_\alpha H^{\mu\nu} = \nabla_\alpha\nabla_\mu H^{\mu\nu} + \frac{R}{3}H^\nu_\alpha, \tag{B21}$$

$$\begin{aligned}\nabla_\mu\nabla_\alpha\nabla_\beta H^{\mu\nu} = {}& \nabla_\alpha\nabla_\beta\nabla_\mu H^{\mu\nu} + \frac{5R}{12}\nabla_\alpha H^\nu_\beta \\ &+ \frac{R}{3}\nabla_\beta H^\nu_\alpha - \frac{R}{12}g_{\alpha\beta}\nabla_\mu H^{\mu\nu},\end{aligned} \tag{B22}$$

$$\nabla_\mu\Box H^{\mu\nu} = \left(\Box + \frac{5R}{12}\right)\nabla_\mu H^{\mu\nu}. \tag{B23}$$

Moreover, for a transverse and traceless (symmetric) tensor $\nabla^\mu h^\perp_{\mu\nu} = h^{\perp\mu}{}_\mu = 0$, one obtains

$$\nabla_\mu\nabla_\alpha h^{\perp\mu\nu} = \frac{R}{3}h^{\perp\nu}{}_\alpha, \tag{B24}$$

$$\nabla_\mu\nabla_\alpha\nabla_\beta h^{\perp\mu\nu} = \frac{5R}{12}\nabla_\alpha h^{\perp\nu}{}_\beta + \frac{R}{3}\nabla_\beta h^{\perp\nu}{}_\alpha, \tag{B25}$$

$$\nabla_\mu\Box h^{\perp\mu\nu} = 0. \tag{B26}$$

The very last formula tells us that $\Box h^\perp_{\mu\nu}$ is also a transverse and traceless tensor. For completeness, we also note

$$\nabla_\rho \Box h^{\perp}{}_{\mu\nu} = \Box \nabla_\rho h^{\perp}{}_{\mu\nu} - \frac{R}{6}(\nabla_\mu h^{\perp}{}_{\rho\nu} + \nabla_\nu h^{\perp}{}_{\rho\mu}) - \frac{R}{4}\nabla_\rho h^{\perp}{}_{\mu\nu}. \tag{B27}$$

From all the above three recursion relations, we can deduce a simple relation. For a scalar [from Eq. (B3)]

$$\nabla_\mu \Box \varphi = \left(\Box - \frac{R}{4}\right)\nabla_\mu \varphi \Rightarrow \nabla_\mu \Box^n \varphi = \left(\Box - \frac{R}{4}\right)^n \nabla_\mu \varphi. \tag{B28}$$

For any arbitrary vector, no transversality is required [from (B6)],

$$\nabla_\mu \Box t^\mu = \left(\Box + \frac{R}{4}\right)\nabla_\mu t^\mu \Rightarrow \nabla_\mu \Box^n t^\mu = \left(\Box + \frac{R}{4}\right)^n \nabla_\mu t^\mu. \tag{B29}$$

For a traceless (symmetric) and not necessarily transverse tensor [from (B23)]

$$\nabla_\mu \Box H^{\mu\nu} = \left(\Box + \frac{5R}{12}\right)\nabla_\mu H^{\mu\nu} \Rightarrow \nabla_\mu \Box^n H^{\mu\nu} = \left(\Box + \frac{5R}{12}\right)^n \nabla_\mu H^{\mu\nu}. \tag{B30}$$

Two more extremely essential commutators are needed. The first is for a rank-3 tensor $t^{\beta\mu\alpha}$. One can compute the following relation:

$$\nabla_\alpha \Box t^{\beta\mu\alpha} = \left(\Box + \frac{R}{4}\right)\nabla_\alpha t^{\beta\mu\alpha} + \frac{R}{6}\nabla_\rho(t^{\rho\mu\beta} + t^{\beta\rho\mu}) - \frac{R}{6}(\nabla^\beta t_\alpha{}^{\mu\alpha} + \nabla^\mu t_{\beta\alpha}{}^{\alpha}). \tag{B31}$$

We note that on the rhs, the second term is a specific linear combination of the initial tensor with some index permutations, while the last piece is a combination of various traces. Given a tensor $V^{\beta\mu\alpha}$, which enjoys the following properties,

$$V^{\beta\mu\alpha} + V^{\mu\alpha\beta} + V^{\alpha\beta\mu} = 0, \qquad V^{\beta\mu\alpha} = V^{\mu\beta\alpha}, \qquad V^{\alpha}_{\ \mu\alpha} = V^{\mu\alpha}_{\ \ \alpha} = 0,$$

one comes to a simple relation

$$\nabla_\alpha \Box V^{\beta\mu\alpha} = \left(\Box + \frac{R}{12}\right)\nabla_\alpha V^{\beta\mu\alpha}. \tag{B32}$$

Notice that outlined symmetry properties are very much similar (not identical though) to those of the so-called *Cotton tensor*. A recursion relation following from the latter formula reads

$$\nabla_\alpha \Box^n V^{\beta\mu\alpha} = \left(\Box + \frac{R}{12}\right)^n \nabla_\alpha V^{\beta\mu\alpha}. \tag{B33}$$

The last relation we need is for a rank-4 tensor $t^{\mu\alpha\nu\beta}$. One can compute the following relation:

$$\nabla_\mu \Box t^{\mu\alpha\nu\beta} = \left(\Box + \frac{R}{4}\right)\nabla_\mu t^{\mu\alpha\nu\beta} + \frac{R}{6}\nabla_\rho(t^{\alpha\rho\nu\beta} + t^{\nu\alpha\rho\beta} + t^{\beta\alpha\nu\rho}) - \frac{R}{6}(\nabla^\alpha t^\mu{}_\mu{}^{\nu\beta} + \nabla^\nu t^{\mu\alpha}{}_\mu{}^\beta + \nabla^\beta t^{\mu\alpha\nu}{}_v). \tag{B34}$$

One immediately sees that on the rhs, the second term is a specific linear combination of the initial tensor with some index permutations while the last piece is a combination of various traces. Given a tensor $W^{\mu\alpha\nu\beta}$, which enjoys all the symmetric properties of the Weyl tensor, and is also totally traceless, one comes to a simple relation

$$\nabla_\mu \Box W^{\mu\alpha\nu\beta} = \left(\Box + \frac{R}{4}\right)\nabla_\mu W^{\mu\alpha\nu\beta}. \tag{B35}$$

A recursion relation following from the latter formula reads

$$\nabla_\mu \Box^n W^{\mu\alpha\nu\beta} = \left(\Box + \frac{R}{4}\right)^n \nabla_\mu W^{\mu\alpha\nu\beta}. \tag{B36}$$

APPENDIX C: CANCELLATION OF MODES

1. Vector mode

In this section all the derivative operators, metrics, and curvatures take their background values, and we omit the bars. For the corresponding piece of $r^{\mu\sigma}_{\nu\rho}$, we have

$$\begin{aligned} r^{\sigma\mu}_{\nu\rho}(A^\perp_\mu) = {} & \frac{R}{24}(\delta^\mu_\nu(\nabla^\sigma A^\perp_\rho + \nabla_\rho A^{\perp\sigma}) - \delta^\mu_\rho(\nabla^\sigma A^\perp_\nu + \nabla_\nu A^{\perp\sigma}) - \delta^\sigma_\nu(\nabla^\mu A^\perp_\rho + \nabla_\rho A^{\perp\mu}) + \delta^\sigma_\rho(\nabla^\mu A^\perp_\nu + \nabla_\nu A^{\perp\mu})) \\ & + \frac{1}{2}(\nabla_\nu\nabla^\mu(\nabla^\sigma A^\perp_\rho + \nabla_\rho A^{\perp\sigma}) - \nabla_\nu\nabla^\sigma(\nabla^\mu A^\perp_\rho + \nabla_\rho A^{\perp\mu}) - \nabla_\rho\nabla^\mu(\nabla^\sigma A^\perp_\nu + \nabla_\nu A^{\perp\sigma}) + \nabla_\rho\nabla^\sigma(\nabla^\mu A^\perp_\nu + \nabla_\nu A^{\perp\mu})). \end{aligned}$$

Now we do the following commutations in the two last lines. In the first term with $A^\perp{}_\rho$ and the first term with $A^\perp{}_\nu$, we commute σ and μ derivatives. In the second line for the terms with $A^{\perp\sigma}$ and $A^{\perp\mu}$, we exchange ρ and ν derivatives. After some algebra together with the Riemann tensor substitution, we get

$$\begin{aligned}\frac{24}{R}r^{\sigma\mu}_{\nu\rho}(A^{\perp}_{\mu}) &= \delta^{\mu}_{\nu}(\nabla^{\sigma}A^{\perp}_{\rho}+\nabla_{\rho}A^{\perp\sigma})-\delta^{\mu}_{\rho}(\nabla^{\sigma}A^{\perp}_{\nu}+\nabla_{\nu}A^{\perp\sigma})-\delta^{\sigma}_{\nu}(\nabla^{\mu}A^{\perp}_{\rho}+\nabla_{\rho}A^{\perp\mu})+\delta^{\sigma}_{\rho}(\nabla^{\mu}A^{\perp}_{\nu}+\nabla_{\nu}A^{\perp\mu})\\ &+(\delta^{\mu}_{\rho}\nabla_{\nu}A^{\perp\sigma}-\delta^{\sigma}_{\rho}\nabla_{\nu}A^{\perp\mu})+(\delta^{\sigma}_{\nu}(\nabla_{\rho}A^{\perp\mu}+\nabla^{\mu}A^{\perp}{}_{\rho})-g_{\mu\sigma}(\nabla_{\rho}A^{\perp}{}_{\nu}-\nabla_{\nu}A^{\perp}{}_{\rho})-\delta^{\sigma}_{\rho}(\nabla_{\nu}A^{\perp\mu}+\nabla^{\mu}A^{\perp}{}_{\nu})\\ &+\delta^{\mu}_{\nu}\nabla_{\rho}A^{\perp\sigma}-\delta^{\mu}_{\rho}\nabla_{\nu}A^{\perp\sigma})-(\delta^{\mu}_{\nu}(\nabla_{\rho}A^{\perp\sigma}+\nabla^{\sigma}A^{\perp}{}_{\rho})-g_{\mu\sigma}(\nabla_{\rho}A^{\perp}{}_{\nu}-\nabla_{\nu}A^{\perp}{}_{\rho})-\delta^{\mu}_{\rho}(\nabla_{\nu}A^{\perp\sigma}+\nabla^{\sigma}A^{\perp}{}_{\nu})\\ &+\delta^{\sigma}_{\nu}\nabla_{\rho}A^{\perp\mu}-\delta^{\sigma}_{\rho}\nabla_{\nu}A^{\perp\mu})-(\delta^{\mu}_{\nu}\nabla_{\rho}A^{\perp\sigma}-\delta^{\sigma}_{\nu}\nabla_{\rho}A^{\perp\mu})=0.\end{aligned}$$

Note that all the terms cancel explicitly.

Since r^{σ}_{ρ} and r are obtained by a simple contraction of $r^{\mu\sigma}_{\nu\rho}$ with the Kronecker delta our result implies that the piece $A^{\perp}{}_{\nu}$ is also absent in r and r^{σ}_{ρ}. Similarly,

$$\begin{aligned}\delta_0(A^{\perp}{}_{\mu}) &= \frac{1}{4}(\nabla_{\mu}A^{\perp}_{\nu}+\nabla_{\nu}A^{\perp}{}_{\mu})\Box(\nabla^{\mu}A^{\perp\nu}+\nabla^{\nu}A^{\perp\mu})+\frac{1}{2}\nabla_{\mu}(\nabla^{\mu}A^{\perp\rho}+\nabla^{\rho}A^{\perp\mu})\nabla_{\nu}(\nabla^{\nu}A^{\perp}{}_{\rho}+\nabla_{\rho}A^{\perp\nu})\\ &-\frac{R}{24}(\nabla_{\mu}A^{\perp}_{\nu}+\nabla_{\nu}A^{\perp}{}_{\mu})(\nabla^{\mu}A^{\perp\nu}+\nabla^{\nu}A^{\perp\mu}).\end{aligned} \tag{C1}$$

Note that δ_0 is an integrand, and we can integrate it by parts. Doing so in the first and last lines, and utilizing several commutation relations, we get

$$\begin{aligned}\delta_0(A^{\perp}{}_{\mu}) &= -\frac{1}{2}A^{\perp}_{\nu}\left(\Box+\frac{5R}{12}\right)\nabla_{\mu}(\nabla^{\mu}A^{\perp\nu}+\nabla^{\nu}A^{\perp\mu})+\frac{1}{2}A^{\perp\rho}\left(\Box+\frac{R}{4}\right)^2A^{\perp}{}_{\rho}+\frac{R}{12}A^{\perp\rho}\left(\Box+\frac{R}{4}\right)A^{\perp}{}_{\rho},\\ &= A^{\perp\rho}\left[-\frac{1}{2}\left(\Box+\frac{5R}{12}\right)+\frac{1}{2}\left(\Box+\frac{R}{4}\right)+\frac{R}{12}\right]\left(\Box+\frac{R}{4}\right)A^{\perp}{}_{\rho}=0.\end{aligned} \tag{C2}$$

2. Scalar mode, $\nabla_{\mu}\nabla_{\nu}B$

As in the previous section, all the derivative operators, metrics, and curvatures take their background values, and we omit the bars. For the corresponding piece of $r^{\mu\sigma}_{\nu\rho}$, we have

$$\begin{aligned}2r^{\mu\sigma}_{\nu\rho}(\nabla_{\mu}\nabla_{\nu}B) &= (\nabla_{\nu}\nabla^{\mu}\nabla^{\sigma}\nabla_{\rho}-\nabla_{\nu}\nabla^{\sigma}\nabla^{\mu}\nabla_{\rho}-\nabla_{\rho}\nabla^{\mu}\nabla^{\sigma}\nabla_{\nu}+\nabla_{\rho}\nabla^{\sigma}\nabla^{\mu}\nabla_{\nu})B\\ &+2\frac{R}{24}(\delta^{\mu}_{\nu}\nabla^{\sigma}\nabla_{\rho}-\delta^{\mu}_{\rho}\nabla^{\sigma}\nabla_{\nu}-\delta^{\sigma}_{\nu}\nabla^{\mu}\nabla_{\rho}+\delta^{\sigma}_{\rho}\nabla^{\mu}\nabla_{\nu})B.\end{aligned} \tag{C3}$$

Since B is a scalar, the two most right derivatives can always be commuted. Also, we can commute others in order to cancel explicit 4 derivative terms. Explicitly, we can commute σ and μ derivatives in the first and last terms in the first line. Doing so, together with the Riemann tensor substitution, we gain four 2 derivative terms as follows:

$$2\frac{12}{R}r^{\mu\sigma}_{\nu\rho}(\nabla_{\mu}\nabla_{\nu}B) = (\nabla_{\nu}(\delta^{\mu}_{\rho}\nabla^{\sigma}-\delta^{\sigma}_{\rho}\nabla^{\mu})+\nabla_{\rho}(\delta^{\sigma}_{\nu}\nabla^{\mu}-\delta^{\mu}_{\nu}\nabla^{\sigma}))B+(\delta^{\mu}_{\nu}\nabla^{\sigma}\nabla_{\rho}-\delta^{\mu}_{\rho}\nabla^{\sigma}\nabla_{\nu}-\delta^{\sigma}_{\nu}\nabla^{\mu}\nabla_{\rho}+\delta^{\sigma}_{\rho}\nabla^{\mu}\nabla_{\nu})B=0. \tag{C4}$$

Now we have only two derivatives everywhere acting on a scalar. We therefore can forget ordering those derivatives. An explicit cancellation of all terms is transparent.

Since r^{σ}_{ρ} and r are obtained by a simple contraction of $r^{\mu\sigma}_{\nu\rho}$ with the Kronecker delta, our result implies that the piece $\nabla_{\mu}\nabla_{\nu}B$ is also absent in r and r^{σ}_{ρ}.

Now,

$$\delta_0(\nabla_{\mu}\nabla_{\nu}B) = B\left(\frac{1}{4}\nabla_{\mu}\nabla_{\nu}\Box\nabla^{\mu}\nabla^{\nu}-\frac{1}{4}\Box^3+\frac{1}{2}\Box\nabla_{\nu}\nabla_{\rho}\nabla^{\nu}\nabla^{\rho}-\frac{1}{2}\nabla_{\mu}\nabla^{\rho}\nabla^{\mu}\nabla_{\nu}\nabla_{\rho}\nabla^{\nu}\right)B-\frac{R}{48}B(\Box^2+2\nabla^{\nu}\nabla_{\mu}\nabla_{\nu}\nabla^{\mu})B, \tag{C5}$$

where we implicitly used the fact that the actual computation goes under the integral. As a result we can employ integration by parts. Performing the first iteration of commutations, one yields

$$\delta_0(\nabla_\mu\nabla_\nu B) = B\left(\frac{1}{4}\nabla_\mu\left(\left(\Box+\frac{5}{12}R\right)\Box\nabla^\mu - \frac{R}{6}\nabla^\mu\Box\right) - \frac{1}{4}\Box^3 + \frac{1}{2}\Box^2\left(\Box+\frac{R}{4}\right)\right.$$
$$\left. - \frac{1}{2}\nabla_\mu\nabla^\rho\nabla^\mu\nabla_\rho\left(\Box+\frac{R}{4}\right)\right)B - \frac{R}{48}B\left(\Box^2+2\Box\left(\Box+\frac{R}{4}\right)\right)B. \tag{C6}$$

Performing the remaining possible commutations, one finally gets

$$\delta_0(\nabla_\mu\nabla_\nu B) = B\left(\frac{1}{4}\left(\left(\Box+\frac{2}{3}R\right)\left(\Box+\frac{R}{4}\right)\Box - \frac{R}{6}\Box^2\right) - \frac{1}{4}\Box^3 + \frac{1}{2}\Box^2\left(\Box+\frac{R}{4}\right) - \frac{1}{2}\Box\left(\Box+\frac{R}{4}\right)^2\right)B$$
$$- \frac{R}{48}B\left(\Box^2+2\Box\left(\Box+\frac{R}{4}\right)\right)B = 0. \tag{C7}$$

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