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Weyl current, scale-invariant inflation, and Planck scale generation

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Scalar fields, ϕ_i, can be coupled nonminimally to curvature and satisfy the general criteria: (i) the theory has no mass input parameters, including $M_P = 0$; (ii) the ϕ_i have arbitrary values and gradients, but undergo a general expansion and relaxation to constant values that satisfy a nontrivial constraint, $K(\phi_i) = \text{constant}$; (iii) this constraint breaks scale symmetry spontaneously, and the Planck mass is dynamically generated; (iv) there can be adequate inflation associated with slow roll in a scale-invariant potential subject to the constraint; (v) the final vacuum can have a small to vanishing cosmological constant; (vi) large hierarchies in vacuum expectation values can naturally form; (vii) there is a harmless dilaton which naturally eludes the usual constraints on massless scalars. These models are governed by a global Weyl scale symmetry and its conserved current, K_μ. At the quantum level the Weyl scale symmetry can be maintained by an invariant specification of renormalized quantities.

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I. INTRODUCTION

There has recently been considerable interest in scale symmetric general relativity, in conjunction with inflation and dynamically generated mass scales [1–14]. This is a theory containing fundamental scalar fields together with general covariance and nonminimal coupling of the scalars to curvature, but no Planck mass. Remarkably, starting with a scale-invariant action, it is possible to spontaneously generate the Planck mass scale itself and naturally produce significant inflation. The inflation can, moreover, lead to large hierarchies of scalar vacuum expectation values (VEVs). All of this occurs as one unified phenomenon.

The key ingredient of this mechanism is a *global* Weyl scale symmetry and its current, K_μ. Gravity drives the scale current density, K_0, to zero, much as any conserved current charge density dilutes to zero by general expansion. However, the particular structure of the K_μ current is such that it has a "kernel," i.e., $K_\mu = \partial_\mu K$. Hence, as the scale charge density is diluted away, $K_0 \to 0$, the kernel evolves as $K \to \text{constant}$. K is the order parameter that defines a spontaneous scale symmetry breaking and the Planck scale, $K = O(M_P^2)$. The breaking of scale symmetry here is "inertial," and is determined by the random initial values of the field VEVs that settle down to yield a random fixed value of K.

In the multifield case the role of the potential is to determine the relative VEVs of the scalar fields contributing to K. In this case the nonzero constant value of K defines a constraint on the scalar field VEVs, requiring that the VEVs lie on an ellipse in multiscalar-field space. The inflationary slow-roll conditions are consistent with constant K and an inflationary era readily occurs in which the field VEVs migrate along the ellipse, and ultimately flow to an infrared (IR) fixed point. For the special case that the potential has a flat direction the fixed point corresponds to the potential minimum, the field VEVs flow to it, and the final cosmological constant vanishes.

In the present paper we discuss how this "current algebra" works in detail, and how inflation and Planck scale generation emerge from it. We will first illustrate this phenomenon in Sec. II, in a simplified theory with a single scalar field, ϕ, and a nonminimal coupling to gravity $\sim -(1/12)\alpha\phi^2 R$. For us $\alpha < 0$, and a nonzero VEV of ϕ induces a positive Planck $(\text{mass})^2$. We allow scale-invariant potentials, such as $\lambda\phi^4$. This theory thus has a global Weyl scale symmetry, and a conserved scale current:

$$K_\mu = (1-\alpha)\phi\partial_\mu\phi. \tag{1}$$

The prefactor is relevant and nontrivial when we consider N-scalar fields (this current vanishes in the $\alpha = 1$ limit when the Weyl symmetry becomes local [7]).

The Weyl scale current kernel is $K = (1-\alpha)\phi^2/2$. The kernel, K, is driven to a constant during an initial period of expansion of the Universe, as K_0 is diluted to zero. There is no ellipse in the single field case, and the field comes to rest with a fixed, eternal VEV, $\phi = \sqrt{2K/(1-\alpha)}$. The theory acquires the Planck mass as $M_P^2 = -\alpha K/6(1-\alpha)$. and the resulting inflation is eternal.

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The Nambu-Goldstone theorem applies with the dynamical spontaneous scale symmetry breaking by nonzero K, and there is a dilaton. We will mention some of the properties of the dilaton, with a more detailed discussion in [15]. If the underlying Weyl scale symmetry is maintained throughout the full theory (including quantum corrections), then the massless dilaton has at most derivative coupling to matter and the Brans-Dicke constraints go away.

We discuss in Sec. III a model with two scalars, ϕ and χ. The generalization of the Weyl current is straightforward. After the initial expansionary phase establishing constant K, the fields readily generate a period of slow-roll inflation as their VEVs migrate along an ellipse defined by constant K. If the potential $V(\phi_i)$ is scale invariant and has a nontrivial minimum with nonvanishing VEVs, it follows that $V(\phi_i)$ vanishes at its minimum and that it has a flat direction corresponding to a definite ratio of the scalar field VEVs. The slow-roll inflationary period is terminated by a period of "reheating" in which the fields acquire large kinetic energy which is rapidly damped by expansion. Subsequently the fields flow toward an IR fixed point that determines the ratio of their VEVs in terms of the couplings appearing in the scalar potential (this was studied in a two field example in Ref. [13]). The fixed point is the intersection of the potential flat direction with the ellipsoid. If the potential does not have a nontrivial minimum, gravitational effects prevent the roll to the scale-invariant minimum and the inflation is eternal, i.e., there is then a relic cosmological constant.

In Sec. IV we discuss the N-scalar scheme and the analytic solution for the inflationary phase in the two scalar scheme. We consider generalized inflationary fixed point of the N-scalar schemes, and the $N = 3$ model is examined in detail.

If scale symmetry is broken through quantum loops, the resulting trace anomaly would imply that K_μ is no longer conserved. Then the field VEVs, hence K, would relax to zero, and with it would go the Planck mass. To avoid this it is necessary to maintain the Weyl symmetry throughout. One of our main theses is that this is possible, i.e., the Weyl symmetry can be maintained at the quantum level if no external mass scales are introduced into the theory during the process of renormalization.

In Sec. V we turn to the quantum effects. We first describe how the Einstein and Klein-Gordon equations are conventionally modified by scale anomalies, leading to the modifield K_μ current and the kernel K. Our main goal here is to describe and construct effective Coleman-Weinberg-Jackiw [16,17] actions where the couplings run with fields.

In Weyl invariant theories there can be no absolute meaning to mass; only Weyl invariant dimensionless ratios of mass scales will occur. It is therefore crucial that no "external mass scales" are introduced at the quantum level in renormalizing the theory. This implies that counterterms must be field dependent and are ultimately specified by the overall constraint that the renormalized action remains Weyl invariant. In the effective action the running couplings must therefore depend exclusively upon Weyl invariant ratios of values of field VEVs, e.g., $\lambda(\phi_c/\chi_c)$, rather than ratios involving some external mass scale, e.g., $\lambda(\phi_c/M)$. This approach makes no specific reference to any particular regularization method (see [18,19]). The renormalization group (RG) with nontrivial β functions remains, however the running of parameters, is now given in terms of Weyl invariants.

We give general formal arguments in Sec. V and more details will be given elsewhere [20]. In Sec. V we explore a simple two scalar model of quantum effects with a particular choice of the running renormalized couplings which are expected to emerge in detailed calculations. Since the renormalization group running occurs in Weyl invariants such as ϕ_c/χ_c rather than ϕ_c/M, we find that the ellipse can be significantly distorted by these effects. K becomes constant, and a nontrivial ratio of VEVs ϕ_c/χ_c develops which is suggestive that a hierarchical relationship between M_P, M_{GUT} and m_{Higgs} might emerge from this dynamics in more detailed models. We follow with conclusions.

II. SINGLE NONMINIMAL SCALAR

A. The action

We begin by establishing some notation. A standard Einstein gravitation in our sign conventions with a minimally coupled massless scalar field, σ, and metric tensor g and cosmological constant, Λ, is an action of the form[1]:

$$S = \int \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \Lambda + \frac{1}{2}M_P^2 R\right) \qquad (2)$$

where the Einstein-Hilbert term contains the scalar curvature, R, and the Planck mass: $M_P^2 = (8\pi G)^{-1}$. For small σ this action describes a de Sitter universe with Hubble parameter:

$$H^2 = \frac{\Lambda}{3M_P^2}. \qquad (3)$$

Presently, we consider a theory of a real scalar field, ϕ, in which the Einstein-Hilbert term has been replaced with the nonminimal scalar coupling $-(1/12)\alpha\phi^2 R$, and we choose a scale-invariant potential $V(\phi) = \lambda\phi^4/4$:

$$S = \int \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\lambda}{4}\phi^4 - \frac{1}{12}\alpha\phi^2 R\right). \qquad (4)$$

[1]Our metric signature convention is $(1,-1,-1,-1)$, and our sign convention for the Riemann tensor is that of Weinberg [21]; our conventions are identically those of Ref. [22].

Assuming ϕ acquires a VEV, we would generate a Planck mass from Eq. (4) of the form $M_P^2 = -\alpha\phi^2/6$. We thus require $\alpha < 0$, to obtain the correct sign for the Einstein-Hilbert term, as in Eq. (2).

The theory of Eq. (4) is globally scale invariant. The invariant scale transformation corresponds to the global limit of the "Weyl transformation":

$$g_{\mu\nu} \to e^{-2\varepsilon(x)} g_{\mu\nu} \qquad \phi \to e^{\varepsilon(x)}\phi(x) \tag{5}$$

with $\varepsilon(x) = \varepsilon$ being constant in spacetime. If we perform an infinitesimal local transformation as in Eq. (5) on the action, we obtain, using Eq. (20) below, the Noether current:

$$K_\mu = \frac{\delta S}{\delta \partial^\mu \varepsilon} = (1-\alpha)\phi\partial_\mu\phi. \tag{6}$$

For the $N = 1$ single scalar case the prefactor of $(1-\alpha)$ appears spurious, but it is an essential normalization for $N > 1$ scalars where the α_i can take on different values, and this factor is generated when the Noether variation is performed, and it describes the vanishing of K_μ in the limit $\alpha \to 1$ which corresponds to a particular locally Weyl invariant theory [7].

The existence and conservation of K_μ follows by use of the equations of motion. From Eq. (4) we obtain the Einstein equation:

$$\frac{1}{6}\alpha\phi^2 G_{\alpha\beta} = \left(\frac{3-\alpha}{3}\right)\partial_\alpha\phi\partial_\beta\phi - g_{\alpha\beta}\left(\frac{3-2\alpha}{6}\right)\partial^\mu\phi\partial_\mu\phi + \frac{1}{3}\alpha(g_{\alpha\beta}\phi D^2\phi - \phi D_\beta D_\alpha\phi) + g_{\alpha\beta}V(\phi). \tag{7}$$

The trace of the Einstein equation becomes

$$-\frac{1}{6}\alpha\phi^2 R = (\alpha-1)\partial^\mu\phi\partial_\mu\phi + \alpha\phi D^2\phi + 4V(\phi). \tag{8}$$

We also have the Klein-Gordon (KG) equation for ϕ:

$$0 = \phi D^2\phi + \phi\frac{\delta}{\delta\phi}V(\phi) + \frac{1}{6}\alpha\phi^2 R. \tag{9}$$

We can combine the KG equation, Eq. (9), and trace equation, Eq. (8), to eliminate the $\alpha\phi^2 R$ term, and obtain

$$0 = (1-\alpha)\phi D^2\phi + (1-\alpha)\partial^\mu\phi\partial_\mu\phi + \phi\frac{\partial}{\partial\phi}V(\phi) - 4V(\phi) \tag{10}$$

This can be written as a current divergence equation:

$$D^\mu K_\mu = 4V(\phi) - \phi\frac{\partial}{\partial\phi}V(\phi) \tag{11}$$

where K_μ is given in Eq. (6). For the scale-invariant potential, $V(\phi) \propto \phi^4$, the rhs of Eq. (11) vanishes and the K_μ current is then covariantly conserved:

$$D^\mu K_\mu = 0. \tag{12}$$

We emphasize that this is an "on-shell" conservation law, i.e., it assumes that the gravity satisfies Eq. (7).

B. The kernel

It is clear that the scale current can be written as $K_\mu = \partial_\mu K$ where the kernel $K = (1-\alpha)\phi^2/2$. This has immediate implications for the dynamics of this theory. Consider a Friedman-Robertson-Walker (FRW) metric:

$$g_{\mu\nu} = [1, -a^2(t), -a^2(t), -a^2(t)] \qquad H = \frac{\dot{a}}{a}$$
$$G_{00} = -3\frac{\dot{a}^2}{a^2} \qquad R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \tag{13}$$

Starting with an arbitrary classical ϕ, after a period of general expansion, in some regions of space ϕ becomes approximately spatially constant, but time dependent. The conservation law of Eq. (12) becomes

$$\ddot{K} + 3H\dot{K} = 0. \tag{14}$$

If we take ϕ to be a function of time t only, we have by Eq. (14)

$$K(t) = c_1 + c_2\int_{t_0}^{t}\frac{dt'}{a^3(t')}, \tag{15}$$

where c_1 and c_2 are constants. Therefore we find that, under general initial conditions, $K(t)$ will evolve to a constant value, $K = K(t \to \infty)$. The (00) Einstein equation, with $G_{00} = -3H^2$, gives

$$H^2 = -\frac{\lambda\phi_0^2}{2\alpha}. \tag{16}$$

Thus, with $\alpha < 0$ we have a self-consistent, exponential relaxation to constant $\phi = \phi_0 = \sqrt{2K/(1-\alpha)}$, and eternal inflation.

Note that this situation contrasts what happens in conventional Einstein gravity with a fixed M_P and a $\lambda\phi^4/4$ potential. Inflation is possible for super-Plankian values of ϕ which slow roll to $\phi = 0$. Hence, while normal Einstein gravity causes ϕ to relax to zero, the scale-invariant gravity theory leads to constant nonzero $\phi = \phi_0$ which generates M_P and eternal inflation.

Anticipating our discussion in Sec. V, we can ask how the trace anomaly, arising through quantum effects, would affect these conclusions. The Weyl current is not conserved if there are trace anomalies, and Eq. (11) becomes

$$D^\mu K_\mu = 4V(\phi) - \phi\frac{\partial}{\partial\phi}V(\phi) = -\frac{\beta_\lambda(\phi)}{4}\phi^4 \tag{17}$$

where $\beta_\lambda(\phi) = d\lambda/d\ln\phi$ is the β function associated with the radiative corrections of the quartic coupling λ in Eq. (4).[2] Indeed, this anomaly enters the rhs of Eq. (12), and it would lead to slow-roll relaxation of ϕ to zero, $K \to 0$, and thus the Planck mass goes to zero as well. With nonzero trace anomaly, the enterprise of generating inflation and the Planck mass as a unified phenomenon would then fail. One of our main arguments here is that we can maintain the Weyl symmetry in any regularization scheme by renormalizing the theory with counterterms that maintain Weyl invariance. β functions then describe the running of couplings in terms of Weyl invariants, such as $\beta_\lambda(\phi) = d\lambda/\ln(\phi/\sqrt{R})$, but the trace anomaly is then zero, as discussed in Sec. V. This maintains the vanishing of the rhs of Eq. (12), and the Planck mass is then stabilized.

C. Weyl transformation and the dilaton

We can identify the spatially constant field ϕ with a new field, σ/f where f is a "decay constant" (analogue of f_π), and ϕ_0 is constant:

$$\phi = \phi_0\exp(\sigma/f), \tag{18}$$

and perform the metric transformation:

$$g_{\mu\nu} = \exp(-2\sigma/f)\tilde{g}_{\mu\nu}. \tag{19}$$

Using $g_{\mu\nu} = \exp(-2\varepsilon)\tilde{g}_{\mu\nu}$:

$$R \to \exp(2\varepsilon)\tilde{R} + 6\exp(2\varepsilon)(\partial^\mu\varepsilon\partial_\mu\varepsilon - \tilde{D}^\mu\partial_\mu\varepsilon) \tag{20}$$

where $\tilde{R}, (\tilde{D}^\mu)$ is the curvature (covariant derivative) expressed in terms of $\tilde{g}_{\mu\nu}$, we then have

$$S = \int\sqrt{-\tilde{g}}\Big[\frac{\phi_0^2}{2f^2}\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{\lambda}{4}\phi_0^4 - \frac{1}{2}\alpha\phi_0^2\Big(\frac{1}{6}\tilde{R} + \frac{1}{f^2}\partial^\mu\sigma\partial_\mu\sigma - \frac{1}{f}\tilde{D}^\mu\partial_\mu\sigma\Big)\Big]. \tag{21}$$

The canonical normalization of the σ field thus requires the decay constant $f = \sqrt{2K_0}$ where $K_0 = (1-\alpha)\phi_0^2/2$. Dropping a total divergence, and defining

$$\Lambda = \frac{\lambda}{4}\phi_0^4; \qquad M_P^2 = -\frac{1}{6}\alpha\phi_0^2 \tag{22}$$

we have

$$S = \int\sqrt{-\tilde{g}}\Big(\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \Lambda + \frac{1}{2}M_P^2\tilde{R}\Big). \tag{23}$$

Therefore, we see that the scale-invariant theory, Eq. (4), can be viewed as the "Jordan frame," equivalent to the "Einstein frame" action Eq. (23), as we originally wrote down in Eq. (2). The massless field σ is the dilaton, but this feature is virtually hidden in the Einstein frame, since there σ couples to gravity only through its stress tensor. Note the identical correspondence of Eq. (16) with Eq. (3).

Remarkably Eq. (3) contains a hidden Weyl symmetry. We see that Λ and M_P^2 are related to the ϕ_0^2, and can be written in terms of the dilaton decay constant as

$$\Lambda = \frac{\lambda}{4(1-\alpha)^2}f^4; \qquad M_P^2 = -\frac{1}{6(1-\alpha)}\alpha f^2. \tag{24}$$

These relations are the analogue, in a chiral Lagrangian, of the Goldberger-Treiman relation, $m_N = g_{NN\pi}f_\pi$ relating the mass of the nucleon, m_N, to f_π and the strong coupling constant $g_{NN\pi}$. The variation of the action of Eq. (23) with respect to σ/f yields the current, $K_\mu = f\partial_\mu\sigma$ which is the representation K_μ in the Einstein frame, and the analogue of the axial current, $f_\pi\partial_\mu\pi$, of the pion.

The dilaton reflects the fact that the exact scale symmetry remains, though hidden in the Einstein frame. We can rescale both the VEV $\phi_0 \to e^\varepsilon\phi_0$ and the Hubble constant $H_0 \to e^\varepsilon H_0$ while their ratio remains fixed:

$$\frac{H_0^2}{\phi_0^2} = \frac{\lambda}{2|\alpha|}. \tag{25}$$

It is straightforward to extend this effective Lagrangian to matter fields. If the dilaton develops a "hard coupling" to, e.g., the nucleon, then stars would develop dilatonic halo fields. This would then be subject to strict limits from Brans-Dicke theories, and the models would fail to give acceptable inflation. However, if all ordinary matter fields have masses that are ultimately associated with the spontaneous breaking of the Weyl scale symmetry, then the dilaton only couples derivatively. There are then no Brans-Dicke-like constraints, as no star or black hole, etc., will generate a σ field halo. In Ref. [15] we discuss the dilaton halo phenomenology in greater detail.

III. TWO SCALAR THEORY

A. Classical two scalar action

Consider an $N = 2$ model, with scalars (ϕ,χ), and the potential:

$$W(\phi,\chi) = \frac{\lambda}{4}\phi^4 + \frac{\xi}{4}\chi^4 + \frac{\delta}{2}\phi^2\chi^2. \tag{26}$$

[2]There is also an anomaly associated with the running of α.

The action takes the form:

$$S = \int \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - W(\phi,\chi) - \frac{1}{12}\alpha_1\phi^2 R - \frac{1}{12}\alpha_2\chi^2 R\right). \quad (27)$$

This has been studied in [1–4,13]. For example, [2] studied this theory in the context of a unimodular gravity and performed a Weyl transformation taking Eq. (27) from a Jordan frame to an Einstein frame. We follow the approach of [13] and work directly in the defining frame of Eq. (27), and then just follow the dynamics. The result is an effective, emergent Einstein gravity where the Planck mass is induced by the VEVs of ϕ and χ. We will see in Sec. IV that, due to the conserved K current, the slow-roll inflation of the classical system is amenable to an analytic treatment. We will also extend this to include quantum corrections that have a significant effect in the next section.

The sequence of steps follows those of the previous single scalar case. The Einstein equation is

$$\begin{aligned} M_P^2 G_{\alpha\beta} = {} & \left(1-\frac{1}{3}\alpha_1\right)\partial_\alpha\phi\partial_\beta\phi + \left(1-\frac{1}{3}\alpha_2\right)\partial_\alpha\chi\partial_\beta\chi \\ & - g_{\alpha\beta}\left(\frac{1}{2}-\frac{1}{3}\alpha_1\right)\partial^\mu\phi\partial_\mu\phi - g_{\alpha\beta}\left(\frac{1}{2}-\frac{1}{3}\alpha_2\right)\partial^\mu\chi\partial_\mu\chi \\ & + \frac{1}{3}\alpha_1(g_{\alpha\beta}\phi D^2\phi - \phi D_\beta D_\alpha\phi) \\ & + \frac{1}{3}\alpha_2(g_{\alpha\beta}\chi D^2\chi - \chi D_\beta D_\alpha\chi) + g_{\alpha\beta}W(\phi,\chi) \end{aligned} \quad (28)$$

where

$$M_P^2 = -\frac{1}{6}(\alpha_1\phi^2 + \alpha_2\chi^2). \quad (29)$$

The trace of the Einstein equation becomes

$$R = \frac{1}{M_P^2}((\alpha_1 - 1)\partial^\mu\phi\partial_\mu\phi + (\alpha_2 - 1)\partial^\mu\chi\partial_\mu\chi + \alpha_1\phi D^2\phi + \alpha_2\chi D^2\chi + 4W(\phi,\chi)). \quad (30)$$

The Klein-Gordon equations for the scalars are

$$\begin{aligned} 0 &= D^2\phi + \delta\phi^2\chi + \lambda\phi^3 + \frac{1}{6}\alpha_1\phi R \\ 0 &= D^2\chi + \delta\phi\chi^2 + \xi\chi^3 + \frac{1}{6}\alpha_2\chi R \end{aligned} \quad (31)$$

and we again use the trace equation to eliminate R:

$$\begin{aligned} 0 = {} & \phi D^2\phi - \frac{\alpha_1\phi^2}{6M_P^2}((1-\alpha_1)\partial^\mu\phi\partial_\mu\phi + (1-\alpha_2)\partial^\mu\chi\partial_\mu\chi \\ & - \alpha_1\phi D^2\phi - \alpha_2\chi D^2\chi - 4W) + \delta\phi^2\chi^2 + \lambda\phi^4 \\ 0 = {} & \chi D^2\chi - \frac{\alpha_2\chi^2}{6M_P^2}((1-\alpha_1)\partial^\mu\phi\partial_\mu\phi + (1-\alpha_2)\partial^\mu\chi\partial_\mu\chi \\ & - \alpha_1\phi D^2\phi - \alpha_2\chi D^2\chi - 4W) + \delta\phi^2\chi^2 + \xi\chi^4. \end{aligned} \quad (32)$$

We again see that the sum of the Klein-Gordon equations implies the conserved current, where the potential terms cancel owing to scale invariance:

$$0 = D_\mu[(1-\alpha_1)\phi\partial^\mu\phi + (1-\alpha_2)\chi\partial^\mu\chi] \quad (33)$$

so

$$K_\mu = (1-\alpha_1)\phi\partial^\mu\phi + (1-\alpha_2)\chi\partial^\mu\chi \quad (34)$$

is conserved $D_\mu K_\mu = 0$. The kernel is now given by

$$K = \frac{1}{2}[(1-\alpha_1)\phi^2 + (1-\alpha_2)\chi^2]. \quad (35)$$

B. Synopsis of two scalar dynamics

The two scalar theory has a number of interesting features, which we will summarize presently. We first discuss the classical case and, after the discussion of the scale-invariant renormalization procedure, we consider the modifications that can occur when including radiative corrections in Sec. V.

The potential of Eq. (26) has the general form:

$$W(\phi,\chi) = \frac{\xi}{4}(\chi^2 - \varsigma^2\phi^2)^2 + \frac{\lambda'}{4}\phi^4 \quad (36)$$

with $\lambda' = \lambda - \xi\varsigma^4$. For the case $\lambda' = 0$ the potential has a flat direction with $\chi = \varsigma\phi$, and the vacuum energy vanishes for nonzero VEVs of the fields.

The theory can lead to a realistic cosmological evolution as illustrated in Fig. 1 for a representative choice of parameters and initial conditions. In an initial "transient phase," the theory will redshift from arbitrary initial field values and velocities, $(\phi, \dot{\phi}; \chi, \dot{\chi})$. Owing to the conserved K current, the redshifting will cause $(\dot{\phi}, \dot{\chi}) \to 0$ and the K_0 charge density to dilute away as $\sim a(t)^{-3}$ leading to a state with constant kernel K. The arbitrary, nonzero value of K determines the scale of the Planck mass, $K \sim M_P^2$, and spontaneously breaks scale symmetry. The fields (ϕ, χ) are now approximately constant in space VEVs and are constrained to lie on the ellipse defined by Eq. (35). This initial location of the VEVs on the ellipse, $(\phi(0), \chi(0))$, is random.

As K settles down to its constant value, Einstein gravity has emerged with a fixed Planck mass. This can be seen

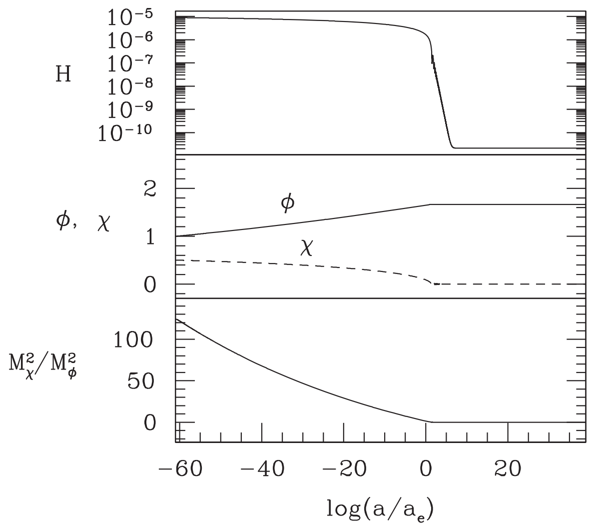

FIG. 1. Plot of the Hubble parameter, H, ϕ, χ and the ratio of the two components of the effective Planck mass, M_ϕ^2 and M_χ^2, as a function of a; we have normalized the x axis to the scale factor at the end of inflation, a_e. The chosen parameters are $\alpha_1 = -1.25 \times 10^{-2}$; $\alpha_2 = -6.19$; $\delta = 0$; $\lambda = 10^{-24}$; $\xi = 10^{-9}$; $\phi(-60) = 1$ and $\chi(-60) = 0.5$ in Planck units; initial velocities are set to zero.

analytically for the classical case as in Sec. III C below. The initial values of (ϕ_0, χ_0) are random and would not be expected to lie on the flat direction.

For a significant region of initial values the fields then slow roll along the ellipse, migrating toward a minimum of the potential and generating a period of inflation. The flat direction is a ray in the (ϕ, χ) plane that intersects the ellipse defined by the kernel, Eq. (35). If we assume $\varsigma \ll 1$ this intersection occurs near the rightmost end of the ellipse where $\chi \ll \phi$ in quadrant I $(\phi, \chi) > 0$ in Fig. 2. Note that $\varsigma \ll 1$ is a particular choice of the dynamics, since for $\varsigma \sim 1$ the flat direction can be arbitrary in the (ϕ, χ) plane, and the inflation can still be significant, but we will not then generate a large hierarchy in the VEVs of ϕ and χ.

The inflationary period ends when the slow-roll conditions are violated and the system enters a period of "reheating" when the potential energy is converted to kinetic energy which rapidly redshifts. Although this period cannot be solved analytically a numerical simulation shows that the kernel remains constant and that the fields ultimately resume slow roll with expectation values that are in the domain of attraction of an infrared fixed point [13].[3] The fixed point is determined by the parameters of the potential and the α_i and, if the potential has a nontrivial minimum corresponding to $\lambda' = 0$ in Eq. (36), the fixed point corresponds to the minimum of the potential with vanishing cosmological constant (otherwise the fixed point corresponds to nonvanishing VEVs, with nonvanishing potential energy, leading to eternal inflation).

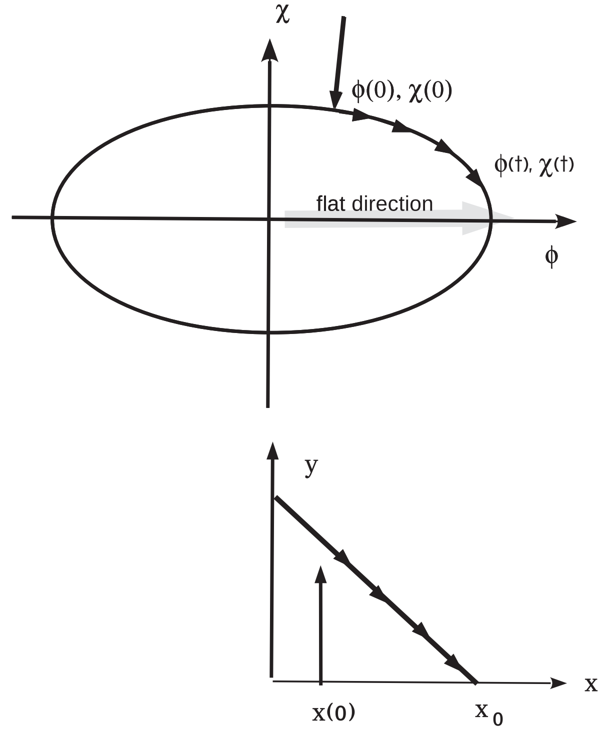

FIG. 2. The K ellipse in (ϕ, χ). The potential flat direction lies along the ϕ axis for the potential $\zeta\chi^4$. Initial values of fields and their velocities rapidly redshift to constant K and then slow roll on the ellipse toward the fixed point. The ellipse is mapped into $x + y = 1$; initial value of $x = x(0)$ slow rolls to the final fixed point x_0 (presently $x_0 \approx 1$).

C. Inflation in the two scalar scheme

In this section we give a detailed analysis of the inflationary era in the two scalar theory and determine the full analytic solution in the slow-roll regime.

In what follows we will be interested in a large hierarchy between the scalar VEVs that can develop after an initial period of inflation. In this case the large field VEV (we will choose parameters such that this is the ϕ VEV) sets the magnitude of the Planck scale while the small field VEV sets the scale in the "matter" sector characterized by the χ field.[4]

As before we assume the potential of Eq. (26), and that there is a Hubble size volume in which the fields are time dependent but spatially constant. Then following the argument in Sec. II B we see that the kernel K becomes a constant, which we take to be an arbitrary mass scale (related ultimately to the Planck mass $K \sim M_P^2$). The residual motion of the scalars during slow roll is

[3] By coupling χ to standard model fields one has that energy will be transferred—the Universe will "reheat"—during the oscillatory phase; the oscillations will be damped driving the dynamics to the fixed point (which remains unchanged).

[4] In [23,24] this field models the Higgs of the Standard Model.

constrained to lie on the $K = \text{constant}$ ellipse and is then described by a difference of the KG equations. We thus form the convenient combination:

$$\alpha_2 \frac{D^2\phi}{\phi} - \alpha_1 \frac{D^2\chi}{\chi} = -(\alpha_2\lambda - \alpha_1\delta)\phi^2 + (\alpha_1\xi - \alpha_2\delta)\chi^2. \quad (37)$$

We take the slow-roll limit of Eq. (37) and we pass to the "inflation derivative" $D^2\phi \to 3H\dot{\phi} = 3H^2\partial_N\phi$ where $N = \ln(a(t))$ hence $\partial_t\phi = H\partial_N\phi$:

$$\frac{3}{2}H^2\left(\alpha_2\frac{\partial_N\phi^2}{\phi^2} - \alpha_1\frac{\partial_N\chi^2}{\chi^2}\right) = -(\alpha_2\lambda - \alpha_1\delta)\phi^2 + (\alpha_1\xi - \alpha_2\delta)\chi^2. \quad (38)$$

We eliminate H^2 using the (00) Einstein equation in the slow-roll limit:

$$M_P^2 G_{00} = -\frac{1}{2}H^2(\alpha_1\phi^2 + \alpha_2\chi^2) \approx g_{00}W. \quad (39)$$

Without loss of generality we can choose the ellipse $K = 1$ and we can map quadrant I of the ellipse into the variables:

$$x = (1 - \alpha_1)\phi^2 \quad \text{and} \quad y = (1 - \alpha_2)\chi^2. \quad (40)$$

The ellipse then becomes the line segment $1 = x + y$ in quadrant I. With the fields constrained to be on the ellipse, we see that (x, y) are each constrained to range from 0 to 1.

The slow-roll differential equation on the ellipse, Eq. (38), can then be written as

$$\partial_N x = \frac{S(x)}{W(x)}x(1 - x)(x - x_0), \quad (41)$$

where

$$S(x) = \frac{2}{3}\frac{((1-\alpha_2)A + (1-\alpha_1)B)}{(1-\alpha_1)^2(1-\alpha_2)^2} \times \frac{((\alpha_1 - \alpha_2)x + \alpha_2(1-\alpha_1))}{(\alpha_2(1-x) + \alpha_1 x)} \quad (42)$$

$$A = (\alpha_2\lambda - \alpha_1\delta)$$
$$B = (\alpha_1\xi - \alpha_2\delta) \quad (43)$$

and

$$W(x) = \frac{\lambda x^2}{4(1-\alpha_1)^2} + \frac{\xi(1-x)^2}{4(1-\alpha_2)^2} + \frac{\delta x(1-x)}{2(1-\alpha_1)(1-\alpha_2)} \quad (44)$$

x_0 is the "fixed point" in x, as defined in [13], and takes the form:

$$x_0 = \frac{B(1-\alpha_1)}{A(1-\alpha_2) + B(1-\alpha_1)}. \quad (45)$$

The solutions to Eq. (41) depend critically on the behavior of $S(x)/W(x)$. To demonstrate there is a region of parameter space that does undergo slow-roll inflation we consider the case studied in Ref. [13], in which $\xi \gg \delta \gg \lambda$, such that $B \gg A$, $x_0 \approx 1$ and, during the initial inflationary era, $W \approx \xi(1-x)^2/4(1-\alpha_2)^2$. In this case

$$\partial_N x = -\frac{4}{3}x\frac{\alpha_1}{(1-\alpha_1)}\frac{((\alpha_1-\alpha_2)x + \alpha_2(1-\alpha_1))}{(\alpha_2 + (\alpha_1-\alpha_2)x)}. \quad (46)$$

The above result is an exact solution for slow roll in the model of [13]. The slow-roll conditions are readily satisfied for small, negative α_1 in which case:

$$\partial_N x \approx -\frac{4}{3}\alpha_1 x \quad (47)$$

and $x(t)$ will roll from an initial $x(0)$ toward $x(t_E) = x_0 \approx 1$ where t_E is the time at the end of inflation.

Equation (46) can readily be integrated:

$$\ln\frac{x(t)}{x(0)} - \alpha_1\ln\left(\frac{\alpha_2\alpha_1 - \alpha_1 x(t) - \alpha_2(1-x(t))}{\alpha_2\alpha_1 - \alpha_1 x(0) - \alpha_2(1-x(0))}\right) = -\frac{4}{3}\alpha_1(N(t) - N(0)). \quad (48)$$

In this limit of small α_1 Eq. (48) implies the number of e-folds of inflation, N, is given by

$$N = N(t_E) - N(0) = \frac{3}{4\alpha_1}\ln\left(\frac{x(0)}{x(t_E)}\right). \quad (49)$$

Inflation ends when slow roll ceases corresponding to the inflation parameter, ε, approaching unity: $\varepsilon = -(1/2)(d\ln H^2/dN) \approx 1$. This implies

$$\frac{2}{3}\left(\frac{2\alpha_1}{1-x(t_E)} - \frac{\alpha_1}{1-x(t_E) + \alpha_1/\alpha_2}\right)x(t_E) \approx 1 \quad (50)$$

hence, when $x(t_E) = 1 - O(\alpha_1)$. The number of e-folds of inflation is weakly governed by the initial value on the ellipse, $x(0)$. This is any value of order, but less than, unity, e.g., $x(0) \sim 0.5$, so to get large $(N(t_E) - N(0))$ we require $|\alpha_1| \ll 1$.

The resulting values for the spectral index, n_s, and the tensor to scalar fluctuation ration, r, are presented in [13]. An acceptable value for n_s is possible for $|\alpha_1| < 0.1$. The value of r is sensitive to α_2 and is between 1 and 2 orders of magnitude less than the current observational bound for $|\alpha_2| > 1$.

D. The "reheat" phase

Once $\varepsilon \approx 1$, the slow-roll conditions are violated and there is a period of rapid field oscillation—the "reheat" phase in which the scalar fields acquire large kinetic energy. We have not been able to find an analytic solution in this phase but a numerical study confirms this is the case.

An example is shown in Fig. (1) where it may be seen that after about 150 e-folds of inflation the Hubble parameter drops very rapidly before rolling to the infrared fixed point value. As the Hubble parameter drops the fields undergo very rapid oscillations (too rapid to show up in the Figure) after which they re-enter the slow-roll regime with values in the domain of attraction of the IR stable fixed point. During the "reheat" phase, and all subsequent evolution, the kernel, K, remains constant.

E. Infrared fixed point

After the "reheat" phase the fields enter a second slow-roll phase that is again described by Eq. (41). One may see that this equation has an IR stable fixed point given by

$$x(t \to \infty) = x_0. \tag{51}$$

This corresponds to the final ratio of the field VEVs given by

$$\frac{\langle \chi_f \rangle^2}{\langle \phi_f \rangle^2} = \frac{\alpha_2 \lambda - \alpha_1 \delta}{\alpha_1 \xi - \alpha_2 \delta}. \tag{52}$$

A large hierarchy between the "matter" sector scale and the Planck scale requires that the χ mass be hierarchically small compared to the Planck scale and this in turn requires $\delta \le \langle \chi_f \rangle^2 / \langle \phi_f \rangle^2$. In addition it is desirable that the cosmological constant after inflation be small or zero and this in turn requires a fine-tuning of the parameters in the potential so that it is (or is close to) a perfect square. For this to happen we need $\lambda \le \langle \chi_f \rangle^4 / \langle \phi_f \rangle^4$. Note that these choices are consistent with our assumption that $W \sim \xi \chi^4$ and $B \gg A$ during inflation when ϕ and χ are both large.

What happens to the scale factor in the IR? For static scalar fields the FRW equation is

$$3M^2 \left(\frac{\dot{a}}{a}\right)^2 = W = \left(\frac{\lambda}{4} + \frac{\xi \mu^4}{4} + \frac{\delta \mu^2}{2}\right) \phi_0^4 \tag{53}$$

(where $\mu^2 \equiv \langle \chi_f \rangle^2 / \langle \phi_f \rangle^2$) and we can define an effective cosmological constant $\Lambda_{\mathrm{eff}} = (\lambda/4 + \xi \mu^4/4 + \delta \mu^2/2) \phi_0^2 / (\alpha_1 + \alpha_2 \mu^2)$. With the ordering of the couplings discussed above $\Lambda_{\mathrm{eff}} \le \xi \chi_f^4 / 4M_P^2$. If this is nonzero there will be a late stage of eternal inflation. To obtain zero cosmological constant requires fine-tuning of the couplings corresponding to the potential having the form of a perfect square.

F. The dilaton

The dilaton effective action can be derived in analogy to the single scalar case in Sec. II C (see IV B below). Once the ratio of fields is fixed, the dilaton can readily be identified in the two scalar case from the fact that the scale current has the form $K_\mu \propto \partial_\mu \sigma$ and under a scale transformation $\sigma \to \sigma + \varepsilon$. Since the scale current has the form $K_\mu = \partial_\mu K$ with K given by Eq. (35) we know that σ must be some function of K. In order for scale symmetry to act as a shift symmetry implying

$$K = \frac{1}{2} f^2 e^{2\sigma/f} \tag{54}$$

with

$$f = \sqrt{2K_0} = \sqrt{(1-\alpha_1)\phi_0^2 + (1-\alpha_2)\chi_0^2}. \tag{55}$$

Upon passing to the "Einstein frame," the dilaton σ appears in the action only in its kinetic term as for the single scalar case, Eq. (23). The dilaton decoupling is due to the exact underlying global Weyl invariance that is broken only spontaneously via the VEV of K. This will be discussed in detail elsewhere [15].

IV. N-SCALAR CASE

The analysis generalizes readily to the case of N scalars. Here the scale current and its associated kernel are derived and the dilaton identified. It is also shown that the IR fixed point structure determines the ratios of all the scalar field VEVs in terms of the couplings entering the potential, so a hierarchical structure can emerge if the couplings are themselves hierarchical.

However, the existence of an initial inflationary era needs to be justified if there are large couplings between the fields as this can prevent a period of slow roll from occurring. This is of particular relevance if we treat the additional scalars as a model for the low-energy "matter" sector, for then there is no reason why the couplings should be anomalously small. To illustrate this we consider below the case of 3-scalar fields, ϕ_i, with large self and cross couplings between the two matter fields.

A. N-scalar action

The mathematical generalization to N scalars is straightforward. Consider a set of N-scalar quantum fields ϕ_i, $i = (1, 2, \ldots N)$ and action:

$$S = \int \sqrt{-g} \left(\sum_i \frac{1}{2} g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i - W(\phi_i) - \sum_i \frac{\alpha_i \phi_i^2}{12} R \right). \tag{56}$$

The Einstein equation is

$$\frac{1}{6}\sum_i \alpha_i \phi_i^2 G_{\alpha\beta} = g_{\alpha\beta} W(\phi_i) + \sum_i \left[\left(1 - \frac{\alpha_i}{3}\right) \partial_\alpha \phi_i \partial_\beta \phi_i - \left(\frac{1}{2} - \frac{\alpha_i}{3}\right) g_{\alpha\beta} \partial^\mu \phi_i \partial_\mu \phi_i + \left(\frac{\alpha_i}{3}\right) (g_{\alpha\beta} \phi_i D^2 \phi_i - \phi_i D_\beta D_\alpha \phi_i) \right]. \quad (57)$$

The trace of the Einstein equation becomes

$$-\frac{1}{6}\left(\sum_i \alpha_i \phi_i^2\right) R = 4W(\phi) + \sum_i [(\alpha_i - 1)\partial^\mu \phi_i \partial_\mu \phi_i + \alpha_i \phi_i D^2 \phi_i]. \quad (58)$$

The N Klein-Gordon equations are

$$0 = D^2 \phi_i + \frac{\delta}{\delta \phi_i} W(\phi) + \frac{1}{6} \alpha_i \phi_i R \quad (59)$$

and we can write the sum of the Klein-Gordon equations:

$$-\frac{1}{6}\left(\sum_i \alpha_i \phi_i^2\right) R = \sum_i \phi_i D^2 \phi_i + \phi_i \frac{\delta}{\delta \phi_i} W(\phi). \quad (60)$$

Combine Eqs. (58), (60) to eliminate R:

$$0 = \sum_i [(\alpha_i - 1)\partial^\mu \phi_i \partial_\mu \phi_i + (\alpha_i - 1)\phi_i D^2 \phi_i] + 4W(\phi) - \phi_i \frac{\delta}{\delta \phi_i} W(\phi). \quad (61)$$

If we assume a scale-invariant potential we have:

$$0 = 4W(\phi) - \sum_i \phi_i \frac{\delta}{\delta \phi_i} W(\phi). \quad (62)$$

We thus see that Eqs. (61), (62) implies a covariantly conserved current:

$$K_\mu = \sum_i (1 - \alpha_i)(\phi_i \partial_\mu \phi_i) \quad (63)$$

where $D_\mu K^\mu = 0$. The current K_μ arises from a "Weyl gauge transformation" and the K_μ current has a "kernel," i.e., it can be written as a gradient, $K_\mu = \partial_\mu K$ where

$$K = \frac{1}{2}\sum_i \phi_i^2 (1 - \alpha_i). \quad (64)$$

B. N-scalar dilaton

The scale symmetry is spontaneously broken by the constraint of Eq. (64). The fixed value of K has been generated inertially by the dynamical dilution of the bibliography charge density, K_0. The value of K is arbitrary and it can be shifted at no cost in energy due to overall Weyl invariance. This implies a dilaton. We can define the dilaton as

$$\sigma = \frac{f}{2} \log\left(\frac{2K}{f^2}\right). \quad (65)$$

To obtain the dilaton action we perform a local Weyl transformation using the dilaton field itself:

$$g_{\mu\nu}(x) \to \exp(-2\sigma(x)/f) g_{\mu\nu}(x)$$
$$\phi_i(x) \to \exp(\sigma(x)/f) \phi_i(x). \quad (66)$$

Hence, the action S of Eq. (56) becomes $S + \delta S$ with

$$\delta S = \int \sqrt{-g}\left[\frac{1}{f}\sum_i (1 - \alpha_i)\phi_i \partial_\mu \phi_i (\partial^\mu \sigma(x)) + \frac{1}{2f^2}\sum_i (1 - \alpha_i)\phi_i^2 (\partial_\rho \sigma(x) \partial^\rho \sigma(x))\right] = \int \sqrt{-g}\left[\frac{1}{f} K_\mu (\partial^\mu \sigma(x)) + \frac{K}{f^2}(\partial_\rho \sigma(x) \partial^\rho \sigma(x))\right]. \quad (67)$$

This implies

$$f = \sqrt{2K} \quad (68)$$

is the dilaton decay constant, (for constant K). We can integrate the first term by parts and use the covariant K_μ current divergence, $D_\mu K^\mu = 0$, leaving a decoupled dilaton in the Einstein frame. Technically, we should include a Lagrange multiplier to enforce the constraint of Eq. (64) on the ϕ_i.

C. Slow roll

The evolution equations take the form:

$$\begin{pmatrix} 1 + \frac{\alpha_1^2 \phi_1^2}{6M^2} & \frac{\alpha_1 \alpha_2 \phi_1 \phi_2}{6M^2} & \cdots & \frac{\alpha_1 \alpha_N \phi_1 \phi_N}{6M^2} \\ \frac{\alpha_1 \alpha_2 \phi_1 \phi_2}{6M^2} & 1 + \frac{\alpha_2^2 \phi_2^2}{6M^2} & \cdots & \frac{\alpha_2 \alpha_N \phi_2 \phi_N}{6M^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\alpha_1 \alpha_N \phi_1 \phi_N}{6M^2} & \frac{\alpha_2 \alpha_N \phi_2 \phi_N}{6M^2} & \cdots & 1 + \frac{\alpha_N^2 \phi_N^2}{6M^2} \end{pmatrix} \begin{pmatrix} 3H\dot{\phi}_1 \\ 3H\dot{\phi}_2 \\ \cdots \\ 3H\dot{\phi}_N \end{pmatrix} = -\begin{pmatrix} \frac{4\alpha_1 \phi_1}{6M^2} W + W_{\phi_1} \\ \frac{4\alpha_2 \phi_2}{6M^2} W + W_{\phi_2} \\ \cdots \\ \frac{4\alpha_N \phi_N}{6M^2} W + W_{\phi_N} \end{pmatrix}. \quad (69)$$

As before we assume that $U \equiv \lambda_N \phi_N^4$ dominates. We then have

$$\begin{pmatrix} \frac{4\alpha_1\phi_1}{6M^2}W + W_{\phi_1} \\ \frac{4\alpha_2\phi_2}{6M^2}W + W_{\phi_2} \\ \cdots \\ \frac{4\alpha_N\phi_N}{6M^2}W + W_{\phi_N} \end{pmatrix} = \frac{4U}{6M^2} \begin{pmatrix} \alpha_1\phi_1 \\ \alpha_2\phi_2 \\ \cdots \\ -\frac{\sum_i^{N-1}\alpha_i\phi_i^2}{\phi_N} \end{pmatrix}. \tag{70}$$

We can now solve this system to get

$$-3H \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \cdots \\ \dot{\phi}_N \end{pmatrix} = \frac{4U}{\sum_i^N \alpha_i(1-\alpha_i)\phi_i^2} \begin{pmatrix} -\alpha_1(1-\alpha_N)\phi_1 \\ -\alpha_2(1-\alpha_N)\phi_2 \\ \cdots \\ \frac{\sum_i^{N-1}\alpha_i(1-\alpha_i)\phi_i^2}{\phi_N} \end{pmatrix} \tag{71}$$

We now define $X_i = \alpha_i\phi_i^2$ to get

$$-\frac{3}{2}H \begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \cdots \\ \dot{X}_N \end{pmatrix} = \frac{4U}{\sum_i^N (1-\alpha_i)X_i} \times \begin{pmatrix} -\alpha_1(1-\alpha_N)X_1 \\ -\alpha_2(1-\alpha_N)X_2 \\ \cdots \\ \alpha_N\sum_i^{N-1}(1-\alpha_i)X_i \end{pmatrix}. \tag{72}$$

If we now change variables to $\ln a$ and use the FRW equation

$$3H^2 = \frac{U}{M^2} \tag{73}$$

we get

$$\begin{pmatrix} X_1' \\ X_2' \\ \cdots \\ X_N' \end{pmatrix} = \frac{4}{3}\frac{\sum_i^N X_i}{\sum_i^N(1-\alpha_i)X_i} \begin{pmatrix} -\alpha_1(1-\alpha_N)X_1 \\ -\alpha_2(1-\alpha_N)X_2 \\ \cdots \\ \alpha_N\sum_i^{N-1}(1-\alpha_i)X_i \end{pmatrix}. \tag{74}$$

If we now take the $X_N \gg X_i$ (with $i = 1, \ldots, N-1$) we get

$$\begin{pmatrix} X_1' \\ X_2' \\ \cdots \\ X_N' \end{pmatrix} = \frac{4}{3} \begin{pmatrix} -\alpha_1 X_1 \\ -\alpha_2 X_2 \\ \cdots \\ \frac{\alpha_N}{1-\alpha_N}\sum_i^{N-1}(1-\alpha_i)X_i \end{pmatrix}. \tag{75}$$

We can solve with $\nu_i = -\frac{4}{3}\alpha_i$ and $\gamma_i = \frac{\alpha_N(1-\alpha_i}{\alpha_i(1-\alpha_N)}$:

$$X_i = X_i^{(0)}e^{\nu_i \ln a} \qquad i = 1, \ldots, N-1$$
$$X_N = C + \sum_{i=1}^N \gamma_i X_i^{(0)} e^{\nu_i \ln a}. \tag{76}$$

D. Fixed point structure

The fixed points are found solving the N equations:

$$\frac{4\alpha_i\phi}{6M^2}W + W_{\phi_i} = 0. \tag{77}$$

We can rewrite this:

$$4\alpha_i\phi_i\sum_{jk}\phi_j^2 W_{jk}\phi_k^2 - 4\sum_j\alpha_j\phi_j^2\sum_k\phi_i W_{ik}\phi_k^2 = 0. \tag{78}$$

We divide out $\alpha_i\phi_i$ and define a set of N matrices (labeled by i):

$$\mathcal{A}_{jk}^{(i)} = W_{jk} - \frac{\alpha_j}{\alpha_i}W_{ik}. \tag{79}$$

We then have that the N quadratic forms satisfy

$$\sum_{jk}\phi_j^2\mathcal{A}_{jk}^{(i)}\phi_k^2 = 0. \tag{80}$$

If this is to be possible then we must have $\mathrm{Det}[\mathcal{A}] = 0$. But this is trivially so. If we pick the ith matrix, it will have that its ith line will be

$$\mathcal{A}_{ik}^{(i)} = W_{ik} - \frac{\alpha_i}{\alpha_i}W_{ik} = 0 \tag{81}$$

which means that its rank is less than or equal than $N-1$. If all the α_i are different, and if we assume W_{ik} is nonsingular, we have that the rank is $N-1$ and the solution will be a line in ϕ_i^2 space with one free parameter, the overall scale. Interestingly, if some of the α_i are degenerate, then the subspace will have a higher dimensionality.

E. Slow roll in a 3-scalar scheme

The fixed point structure proves to be important in the slow-roll regime for the case that more than one coupling is significant in the scalar potential during slow roll. We illustrate this presently in a particular 3-scalar example. Consider the case that the significant couplings during slow roll involve only two matter fields, ϕ_2 and ϕ_3. In this case the potential is dominantly of the form $W = U + Y + T$ where

$$U = a\phi_2^4, \qquad T = b\phi_3^4, \qquad V = c\phi_2^2\phi_3^2. \tag{82}$$

In writing the slow-roll equations it is convenient to define new fields:

$$X = -\alpha_1\phi_1^2, \qquad Y = -\alpha_2\phi_2^2, \qquad Z = -\alpha_3\phi_3^2. \tag{83}$$

Here ϕ_2 and ϕ_3 are the matter fields and we allow a, b and c to be $O(1)$. Then the evolution equations in the slow-roll region have the form:

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = -\frac{4}{3}\left(\frac{X+Y+Z}{\beta_1 X + \beta_2 Y + \beta_3 Z}\right)\frac{1}{(U+T+V)} \times \begin{pmatrix} -\alpha_1 X\{\beta_2(U+V) + \beta_3(T+V)\} \\ \alpha_2\left\{\beta_1 X\left(U+\frac{V}{2}\right) + \beta_3\left[Z\left(U+\frac{V}{2}\right) - Y\left(T+\frac{V}{2}\right)\right]\right\} \\ \alpha_3\left\{\beta_1 X\left(T+\frac{V}{2}\right) + \beta_2\left[Y\left(T+\frac{V}{2}\right) - Z\left(U+\frac{V}{2}\right)\right]\right\} \end{pmatrix} \quad (84)$$

where $\beta_i = 1 - \alpha_i$.

The problem is that, even if α_i are very small, the large couplings a, b and c cause the fields Y and Z to roll quickly and violate the slow-roll conditions used to derive the evolution equations. In the small α_1 regime we see from Eq. (84) that the dominant terms are proportional to $\pm(Z(U+V/2) - Y(T+V/2))$, respectively, with positive coefficients. These terms have an IR stable fixed point with

$$Z(U+V/2) = Y(T+V/2), \quad i.e. \quad \frac{\phi_2^2}{\phi_3^2} = \frac{2b\alpha_2 - c\alpha_3}{2a\alpha_3 - c\alpha_2}. \quad (85)$$

At this fixed point the evolution equation becomes

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = -\frac{4}{3}\left(\frac{X+Y+Z}{\beta_1 X + \beta_2 Y + \beta_3 Z}\right)\frac{1}{(U+T+V)} \times \begin{pmatrix} -\alpha_1 X\{\beta_2(U+V) + \beta_3(T+V)\} \\ \alpha_2\beta_1 X\left(U+\frac{V}{2}\right) \\ \alpha_3\beta_1 X\left(T+\frac{V}{2}\right) \end{pmatrix}. \quad (86)$$

As in the two scalar case all derivatives are proportional to X. Since X' is proportional to α_1, if α_1 is small the slow-roll constraints can indeed be satisfied. Also the evolution of Y and Z is much faster than X in the $\alpha_1 \ll \alpha_{2,3}$ regime, so the inflationary era in the three scalar case will be similar to that in the two scalar case.

V. QUANTUM EFFECTS AND THE K_μ CURRENT

We now consider the quantum effects. We first give a formal derivation of the conventional anomalies of the K_μ current, and show how this is realized in a Coleman-Weinberg-Jackiw effective action. We then discuss how Weyl invariance can be maintained in the renormalized theory. This implies that renormalized quantities satisfy renormalization group equations in which they run in Weyl invariant combinations of fields, such as the ratios of scalar fields. The trace anomaly is then absent and the K_μ current is identically conserved.

A. Weyl invariance and effective action

Scale symmetry of a theory is normally considered to be broken by quantum loops. However, this happens because at some stage in the renormalization procedure, we introduce explicit "external" mass scales into the theory by hand. These are mass scales that are not part of the defining action of the theory, and they lead to nonconservation of the scale current.

The renormalization procedure, however, can be made scale invariant if we specify these quantities, not by introducing external mass scales, but rather by using the VEVs of scalar fields that spontaneously break the scale symmetry but are part of the action itself. In this case, all logarithmic corrections arising in loops will have as their arguments scale-invariant ratios of the internal field VEVs. At the formal level, which we develop presently, the choice of dependencies of renormalized quantities appears arbitrary. However, calculations can be performed in which this arbitrariness is removed, and we will discuss this elsewhere [20].

We can see the usual "external mass parameter" renormalization in the famous paper of Coleman and Weinberg [16]. Starting with the classical $\lambda\phi^4/4$ theory, in their Eq. (3.7) to renormalize λ at one-loop level, they introduce a mass scale M. Once one injects M into the theory, one has broken scale symmetry. The one-loop effective potential then takes the form:

$$V(\phi) = \frac{\beta_\lambda}{4}\phi^4 \ln(\phi/M), \quad (87)$$

where β_λ is the one-loop approximation, ($\mathcal{O}(\hbar)$), to the β function, $\beta_\lambda = d\lambda(\mu)/d\ln\mu$.

The heart of our proposal is to replace M by the VEV of another dynamical field, e.g., χ, that is part of the action of our theory:

$$V = \frac{\beta_\lambda}{4}\phi^4 \ln(\phi/\chi). \quad (88)$$

We see that the Weyl symmetry, $\phi \to e^{\varepsilon}\phi$, $\chi \to e^{\varepsilon}\chi$, is now intact.

The manifestation of this can be seen in the trace of the improved stress tensor [22].[5] In a single scalar theory, the trace anomaly is the divergence of the scale current S_μ and, using Eq. (87), is given by [25]

$$\partial_\mu S^\mu = T^\mu_\mu = 4V(\phi) - \phi\frac{\partial}{\partial\phi}V(\phi) = -\frac{\beta_\lambda}{4}\phi^4. \tag{89}$$

We see, as usual, that the trace anomaly is directly associated with the β function of the coupling constant λ, and it exists on the rhs of Eq. (89) because we have introduced the explicit scale breaking into the theory by hand via M. On the other hand, with two scalars we have

$$\partial_\mu S^\mu = T^\mu_\mu = 4V - \phi\frac{\partial}{\partial\phi}V - \chi\frac{\partial}{\partial\chi}V = 0 \tag{90}$$

and this is vanishing with Eq. (88). In effect, the trace anomaly has been transferred onto the lhs of the divergence equation, and the overall scale current conservation is maintained. We will see that this applies to the Weyl current K_μ as well.

Of course, there is nothing wrong with the Coleman-Weinberg procedure, if one is only treating the effective potential as a subsector of the larger theory. That is, we are simply deferring the question of what is the true origin of M in the larger theory. If, however, scale symmetry is to be maintained as an exact invariance of the world, then M must be replaced by an internal mass scale that is part of action, i.e., M must then be the VEV a field appearing in the extended action, such as χ. If M is replaced by a dynamical field in our theory, we will still have renormalization group evolution, but the resulting physics can now depend only upon ratios of dynamical VEVs, and the running of couplings is given in terms of these ratios.

In fact, this is something we do in practice. All mass scales we measure in the laboratory are referred to other mass scales. Even derived scales, such as Λ_{QCD} can be viewed as arising from a specification of α_{QCD} at some higher energy scale, such as a grand-unification scale, or the Planck mass, M_{Planck}. With the boundary condition, specifying $\alpha_{\mathrm{QCD}}(M_{\mathrm{Planck}})$ then Λ_{QCD} is computed from the solution to the renormalization group equation. We obtain $\Lambda_{\mathrm{QCD}} = cM_{\mathrm{Planck}}$, where c is an exponentially small coefficient (at one loop $c \sim \exp(-2\pi/|b_0|\alpha_{\mathrm{QCD}}(M_{\mathrm{Planck}})))$. The question is then whether the fundamental reference scale, usually taken to be M_{Planck}, is an external input scale (such as the string constant), or the dynamical VEV of a field, such as χ. In the latter case, we can in principle maintain an overall Weyl symmetry, and derived mass scales, such as Λ_{QCD} become Weyl covariant: $\chi \to e^{\varepsilon}\chi$, $\Lambda_{\mathrm{QCD}} \to e^{\varepsilon}\Lambda_{\mathrm{QCD}}$.

[5] Technically, the improved stress tensor is defined only for $\alpha = 1$, and in the flat space limit, but its anomaly parallels that of the K_μ current; the K_μ current is the more relevant scale current for $\alpha \neq 1$ theories.

B. Conventional anomalies of the K_μ current

Let us first formulate the anomalies of the K_μ current in the conventional renormalization framework that introduces an external mass scale M, in a theory with fields $\phi, g_{\mu\nu}, \ldots$. The Weyl transformation is

$$\phi \to e^{\varepsilon}\phi, \qquad g_{\mu\nu} \to e^{-2\varepsilon}g_{\mu\nu}, \ldots \tag{91}$$

The contravariant metric must then transform as $g^{\mu\nu} \to e^{2\varepsilon}g^{\mu\nu}$. Here, if $\varepsilon(x)$ is a function of spacetime the transformation is local; if ε is a constant in spacetime the transformation is global.

It is useful to define a differential operator that acts upon fields:

$$\delta_W\phi = \phi\delta\varepsilon, \qquad \delta_W g_{\mu\nu} = -2g_{\mu\nu}\delta\varepsilon. \tag{92}$$

δ_W acts distributively, and, $\delta_W g^{\mu\nu} = +2g^{\mu\nu}\delta\varepsilon$, $\delta_W(\phi^{-1}) = -\phi^{-1}\delta\varepsilon$, and $\delta_W(\ln\phi) = \phi^{-1}\delta_W(\phi) = \delta\varepsilon$. In general, a field Φ of "mass dimension D" transforms covariantly as $\Phi \to e^{D\varepsilon}\Phi$ or $\delta_W\Phi = D\Phi\delta\varepsilon$.

Any locally Weyl invariant functional of fields $Q(\phi, g_{\mu\nu}\ldots)$ satisfies

$$\delta_W Q = 0. \tag{93}$$

We typically seek an effective Coleman-Weinberg-Jackiw action as a functional of classical fields for the study of inflation and spontaneous scale generation.

For the single scalar field ϕ, consider the effective action, constructed by adding sources to the fields, performing a Legendre transformation to the classical background fields, and integrating out quantum fluctuations [16,17]. The result for a single scalar field theory is a functional of local classical background fields $\phi(x)$ and $g_{\mu\nu}(x)$:

$$S = \int\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\lambda(\phi, g)}{4}\phi^4 - \frac{\alpha(\phi, g)}{12}\phi^2 R\right). \tag{94}$$

It is important to maintain locality in the Lagrangian, since general covariance is a local symmetry, and therefore requires that effective coupling constants be local functions of the fields.

Computing $\delta_W S$ we obtain the difference between the Einstein trace equation and the Klein-Gordon equations that yields the conservation law for K_μ. This calculation is simplified by noting the local Weyl invariants satisfy:

$$\delta_W \int \sqrt{-g}\left(\frac{1}{2}g^{\mu\upsilon}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}\phi^2 R\right) = 0$$
$$\delta_W \int \sqrt{-g}\phi^4 = 0. \qquad (95)$$

Hence

$$\delta_W S = -\int \sqrt{-g}\delta\varepsilon\left(D^\mu K_\mu + \frac{1}{4}(\delta_W\lambda)\phi^4 + \frac{1}{12}(\delta_W\alpha)\phi^2 R\right) = 0 \qquad (96)$$

where we integrate terms with $\partial_\mu(\delta\varepsilon)$, by parts and discard surface terms. K_μ is given by the usual expression, but now contains the field dependent $\alpha(\phi, g_{\alpha\beta})$:

$$K_\mu = \frac{1}{2}\partial_\mu(1-\alpha(\phi, g_{\alpha\beta}))\phi^2. \qquad (97)$$

Equation (96) defines the anomaly of the current:

$$D^\mu K_\mu = -\frac{1}{4}(\delta_W\lambda)\phi^4 - \frac{1}{12}(\delta_W\alpha)\phi^2 R. \qquad (98)$$

Consider the theory in a limit where we ignore all but internal ϕ loops. If we renormalize the effective action, introducing an external mass scale, M, then the β functions are

$$\phi\frac{\partial\lambda}{\partial\phi} = \beta_\lambda\left(=\frac{9\lambda^2}{8\pi^2}\right)$$
$$\phi\frac{\partial\alpha}{\partial\phi} = \beta_\alpha = (\alpha-1)\gamma_\alpha\ \left(\gamma_\alpha = \frac{3\lambda}{8\pi^2}\right) \qquad (99)$$

where in brackets we quote the 1-loop computed values that follow from the ϕ loops in this theory.

Renormalizing with an external mass scale M implies the constraint:

$$0 = \phi\frac{\partial\lambda}{\partial\phi} + M\frac{\partial\lambda}{\partial M}; \qquad 0 = \phi\frac{\partial\alpha}{\partial\phi} + M\frac{\partial\alpha}{\partial M}. \qquad (100)$$

The $\partial/\partial M$ terms in the above equations are not due to the loop calculations, but rather, are external conditions we impose upon the couplings. That is, Eq. (100) defines the functional dependence of the counterterms in the theory upon the external mass parameter M.

Note that the RG equation for α is $\propto(\alpha-1)$, which is why we introduce the factor γ_α into its β-function definition. We can write $\phi\partial\alpha'/\partial\phi = \alpha'\gamma_\alpha$ where, $\alpha' = \alpha - 1$, and this leads, for approximately constant γ_α, to the solution Eq. (101) below. The solutions to the RG equations in the approximation of a fairly constant or small λ, i.e., small β_λ, are

$$\lambda(\phi) = \beta_\lambda \ln\left(\frac{c\phi}{M}\right)\alpha(\phi) = 1 + (\alpha_0 - 1)\left(\frac{\phi}{M}\right)^{\gamma_\alpha} \qquad (101)$$

where the constants c and α_0 define the RG trajectories of the running couplings, λ and α.

Equation (98) with Eq. (101) then implies the form of the K_μ anomalies:

$$D^\mu K_\mu = -\frac{1}{4}\beta_\lambda\phi^4 - \frac{1}{12}\beta_\alpha\phi^2 R. \qquad (102)$$

The nonconservation of the K_μ current arises because the external mass parameter, M, breaks the Weyl scale symmetry.

Armed with the solutions of Eq. (101) we then have the effective action, where Weyl symmetry is broken by the effect of M:

$$S = \int \sqrt{-g}\left(\frac{1}{2}g^{\mu\upsilon}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}\beta_\lambda \ln\left(\frac{c\phi}{M}\right)\phi^4 - \frac{1}{12}\left(1 + (\alpha_0-1)\left(\frac{\phi}{M}\right)^{\gamma_\alpha}\right)\phi^2 R\right). \qquad (103)$$

We cannot have our program of a stable, dynamically generated Planck mass without maintaining the Weyl symmetry, and we must therefore eliminate the explicit M dependence and, hence, the anomalies in the K_μ current.

C. Maintaining exact Weyl scale symmetry in renormalized quantum theory

1. The single scalar theory

To preserve the Weyl invariance, we need to eliminate the anomaly, which requires replacing the constraint Eq. (100) that introduces the external mass scale M. From Eq. (98) we see that we can maintain the Weyl invariance of Eq. (92) in the renormalized theory provided the running coupling constants are Weyl invariant:

$$\delta_W\lambda = 0; \qquad \delta_W\alpha = 0. \qquad (104)$$

Equation (104) is thus a new constraint that replaces Eq. (100). Hence, together with Eq. (99), imposing Eq. (104) we see from Eq. (96) that $D^\mu K_\mu = 0$.

This is an almost obvious result: the coupling constants must be local functions of Weyl invariants in order to maintain the Weyl symmetry. However, just as the $\partial/\partial M$ terms in Eq. (100) are not due to the loop calculations, and are really part of the UV completion of the theory, neither do the dependencies upon various compensating fields implicit in Eq. (104) necessarily arise from the loops alone. These are external conditions that presumably come from the UV completion.

Logically, this procedure is analogous to having a theory in which we have a chiral anomaly that violates a given axial current which we may want to gauge. This is usually done explicitly by judicious choice of fermion representations in the theory. However, it can also be done by constructing a Wess-Zumino-Witten term that generates the anomaly through bosonic fields and can be used to cancel the fermionic chiral anomaly. For example, the Wess-Zumino-Witten term for the original Weinberg model of a single lepton pair (ν, e), can be written in terms of the 0^- and 1^- mesons of QCD, and the W, Z and γ. Including this term into the original Weinberg model gives the an anomaly free description for first generation lepton (ν, e) and the visible states of low-energy QCD (and correctly describes $B+L$ violation, see [26]). Of course, this represents the effects of the underlying confined (u, d) quarks. In our present situation we do not know what the underlying Weyl invariant UV complete theory of gravity and scalars is, but we can imitate the Wess-Zumino-Witten term by demanding an overall Weyl invariant constraint that maintains the renormalization group (the ϕ loops).

The solutions to the constraint Eq. (104) are coupling constants that are functions of Weyl invariants. These clearly must be Lorentz scalars, and also invariant under general coordinate transformations (diffeomorphisms). In the single scalar theory, we only have at our disposal the Weyl invariant objects, $\phi^2 g_{\mu\nu}$, and $\phi^{-2} g^{\mu\nu}$, which are obviously not scalars. The quantity $\sqrt{g}\phi^4$ is Weyl invariant, but is a scalar density and not diffeomorphism invariant. This leaves the Ricci scalar, $R(\phi^2 g_{\mu\nu})$, expressed as a function of the invariant combination $\tilde{g}_{\mu\nu}$ where $\tilde{g}_{\mu\nu} = \phi^2 g_{\mu\nu}$ (and $\tilde{g}^{\mu\nu} = \phi^{-2} g^{\mu\nu}$):

$$R(\phi^2 g) = \phi^{-2} R(g) + 6\phi^{-3} g^{\mu\nu} D_\mu \partial_\nu \phi. \tag{105}$$

Therefore, we can consider the arguments of the logs to be general functions $F_i[R(\phi^2 g)]$. The coupling constants become

$$\lambda(\phi) = \frac{1}{2}\beta_\lambda \ln\left(F_\lambda[R(\phi^2 g)]\right)$$
$$\alpha = 1 + (\alpha_0 - 1)(F_\alpha[R(\phi^2 g)])^{\gamma_\alpha/2}. \tag{106}$$

For example, we might choose

$$F_i = \frac{c_i \phi^2}{R(g) + \frac{6}{\phi} g^{\mu\nu} D_\mu \partial_\nu \phi + c_i' \phi^2}. \tag{107}$$

With the solutions of Eq. (101) we have the Weyl invariant Coleman-Weinberg effective action:

$$S = \int \sqrt{-g}\bigg(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4}\beta_\lambda \ln\left(F_\lambda[R(\phi^2 g), \phi^2]\right)\phi^4$$
$$- \frac{1}{12}(1 + (\alpha_0 - 1)(F_\alpha[R(\phi^2 g), \phi^2])^{\gamma_\alpha/2})\phi^2 R\bigg). \tag{108}$$

The renormalization group equations Eq. (99) are now modified:

$$F_\lambda \frac{\partial \lambda}{\partial F_\lambda} = \beta_\lambda$$
$$F_\alpha \frac{\partial \alpha}{\partial F_\alpha} = (\alpha - 1)\gamma_\alpha. \tag{109}$$

In writing Eq. (109) we have solved the constraint of Eq. (104). Since this is a constraint, it only dictates that the functional form of the F_i, be Weyl invariant. In lieu of an exact calculation is at this stage, the F_i arbitrary. However, such a calculation of the Coleman-Weinberg potential can be done (in a simple locally Weyl invariant two scalar theory) and it yields a specific functional form, as will be presented elsewhere [20].

We can specify F_i if we match onto the calculated β functions from ϕ loops For example, our choice in Eq. (107) will be consistent with the computed β functions of Eq. (99) from ϕ loops, (but not necessarily with calculated functions associated with graviton loops). It is interesting to note that while β_λ of Eq. (99) produces a Landau pole in the running of λ with large ϕ, the choice of nonzero c_λ' implies that asymptotically $\lambda(\phi)$ approaches a constant, $\lambda(c/c')$.[6]

2. The two scalar theory

In the case of the two scalar scheme, defined by Eqs. (26), (27), we have the five couplings, $(\lambda, \xi, \delta, \alpha_1, \alpha_2)$ and will have RG equations for running in ϕ or χ. For the sake of discussion we will presently assume that the field VEVs ϕ and χ are large compared to curvature R. If we consider a typical coupling constant λ we therefore have the scale-invariant constraint:

$$\frac{\delta_W \lambda}{\delta \varepsilon} = \phi \frac{\partial \lambda}{\partial \phi} + \chi \frac{\partial \lambda}{\partial \chi} = 0. \tag{110}$$

We reinterpret the usual RG equations in terms of $\lambda(F)$ with running in a Weyl invariant function of ϕ and χ, such as an arbitrary function of the ratio, $F_\lambda = F(\phi/\chi)$, for example, $F = \phi/\chi$. The renormalization group β function is now:

$$\beta_\lambda = F \frac{\partial \lambda}{\partial F}. \tag{111}$$

Hence, we can maintain the Weyl symmetry while having β functions that now describe the running of couplings in Weyl invariants. Elsewhere we will demonstrate how to

[6]There is a characteristic difference between RG running in field VEVs and running in momentum space. E.g., the top quark, etc., never decouples if the Higgs VEV runs into the IR. RG running for deep scattering processes in momentum will be standard and remains sensitive to the Landau pole as usual.

obtain this result by a direct calculation of the Coleman-Weinberg effective potential while maintaining a local Weyl symmetry [20].

3. Relation to other scale-invariant schemes

There have been several proposals for maintaining Weyl invariance that focus on the regularization schemes e.g., see [18,27–32].

a.Dimensional regularization
Extensively studied is the case of dimensional regularization in which the external mass scale, μ is replaced by a combination of fields, $\mu(\phi,\chi)$. In this approach the Coleman-Weinberg formula for the 1-loop correction scalar potential:

$$-i\int d^4p\,\mathrm{Tr}\ln[p^2 - V(\phi,\chi) + i\varepsilon] \tag{112}$$

is continued to d-dimensions. This gives

$$V(\phi,\chi) = \mu(\phi,\chi)^{4-d}V_0(\phi,\chi) \tag{113}$$

where $V_0(\phi,\chi)$ is the potential in 4D. The first factor gives additional corrections to V that, due to the divergent structure of the integral in 4D, give finite contributions to the scalar potential (see [31,32]). Weyl invariance is maintained by choosing μ to be a function of ϕ and χ of scaling dimension 1.

For the very simple choice $\mu \propto \phi$ the resulting corrections are of the form $\chi^6/\phi^2 + \cdots$ and the theory must be viewed as an effective field theory valid for $\chi^2/\phi^2 \ll 1$. Arbitrariness obviously enters here in the choice of $\mu(\phi,\chi)$, and will affect the β functions as we have discussed above.

b."Renormalized" perturbation theory
In the case of renormalized perturbation theory the Feynman rules are derived from the Lagrangian computed in terms of the physical parameters of the theory. In this case the potential will have a dependence on the scale M at which the couplings are determined. Writing M as a function of ϕ and χ of scaling dimension 1, Weyl invariance can be maintained. However the field dependence of $M = M(\phi,\chi)$ will, as in the case of dimensional regularization, give additional contributions to M^2 that give rise to non-renormalizable and arbitrary corrections of the form found in dimensional regularization.

c."Bare" perturbation theory
An alternative possibility is bare perturbation theory in which the Feynman rules are based on the bare Lagrangian. In this case the bare potential has no dependence on the scale M and so there are no new contributions to the potential of the form discussed above. Weyl invariance can be maintained by identifying the cutoff scale, M, in the loop calculations with a function of the fields of scaling dimension 1 and is equivalent to the procedure proposed in Sec. V C.

D. An ansatz for a quantum corrected theory

What might be the physical effects that arise from Weyl invariant renormalization? In the following we initially consider a general form, $F(x)$ for the argument of the log and then specialize the case where $F = x$. We shall see that this will lead to modifications during inflation to elliptic path in $(\phi,\, \chi)$ that we described above.

The one-loop Coleman-Weinberg (CW) action (neglecting terms in δ) can then take the form of Eq. (27) with the potential:

$$\begin{aligned} W(\phi,\chi) &\simeq \frac{\lambda\phi^4}{4} + \frac{\beta_\xi}{4}\chi^4\ln(cF(\phi/\chi)) \\ &= \frac{\lambda\phi^4}{4}\left[1 + \frac{\beta_\xi}{\lambda x^4}\ln(cF(x))\right] \end{aligned} \tag{114}$$

where $x = \phi/\chi$ and c is a constant. A nontrivial minimum exists for the field values (ϕ_0,χ_0) if

$$\begin{aligned} &\frac{\partial W}{\partial x} = 0 \to cF(x_0) = \exp(x_0F'(x_0)/4F(x_0)) \\ &\frac{\partial W}{\partial \phi} = 0 \to 1 + \frac{\beta_\xi}{\lambda x_0^4}\ln(cF(x_0)) = 0. \end{aligned} \tag{115}$$

Combining gives us one combination of the equations:

$$\frac{1}{x_0^3}\frac{F'(x_0)}{F(x_0)} = -\frac{4\lambda}{\beta_\xi}. \tag{116}$$

An independent combination of the equations gives us a fine-tuning constraint on c.

We can consider the simple case, $F \equiv F_\xi = 1/x = \chi_0/\phi_0$ and we thus find $x_0 = \phi_0/\chi_0 = (\beta_\xi/4\lambda)^{1/4}$. Note however the consistency condition $\ln(cF(x_0)) = -\lambda x_0^4/\beta = -1/4$ requires that c is fine-tuned as $c = x_0\exp(-1/4)$.

Once tuned, this not only corresponds to a minimum but also to a zero of the potential, i.e. a locus in field evolution of fixed $x_0 = \phi_0/\chi_0$ with no cosmological constant. It is straightforward to consider the more general case with a fixed point and late time accelerated expansion, generalizing the results we found in the previous sections.

Including a running α_1 term and $\alpha_2 \approx$ constant, we have the action:

$$\begin{aligned} S = \int \sqrt{-g}\Big\{&\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - W(\phi,\chi) \\ &-\frac{1}{12}[1 + (\alpha_0 - 1)F(x)^{\gamma_1}]\phi^2R - \frac{1}{12}\alpha_2\phi^2R\Big\}. \end{aligned} \tag{117}$$

The quantum corrections deform the ellipse shown in Fig. 2, arising from the running of the α_i (mainly α_1

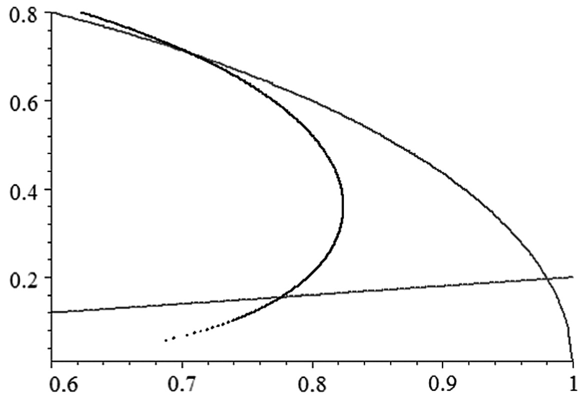

FIG. 3. We expand the right-handed quadrant, $(\phi,\chi) > 0$ where the classical ellipse has rightmost endpoint at $\chi = 1$. The "quantum ellipse" turns back toward the origin due to the quantum running of α_1 where ϕ tracks χ as $\phi \propto \chi^{\gamma_a/(2+\gamma_1)}$. The potential flat direction is indicated as the nearly horizontal line.

presently). γ_1 is a parameter appearing in the β function for α_1, and $\alpha_{1_0} < 0$ is an initial value of α_1 at the "scale" $\phi/\chi = 1$. We reiterate that, in the Weyl invariant framework, one must get used to the notion that there are no fundamental mass scales anymore, and only invariant ratios of field VEVs can arise in scale-invariant physical quantities such as dimensionless couplings like the α_i.

Hence, given the fixed value of K, we have in the classical and quantum cases:

$$\text{classical: } 2K = (1-\alpha_1)\phi^2 + (1-\alpha_2)\chi^2$$
$$\text{quantum: } 2K = (1-\alpha_{1_0})\phi^2[F(x)]^{\gamma_1} + (1-\alpha_2)\chi^2. \quad (118)$$

If we now specialize to $F(x) \equiv F_\alpha = x$, we find the differences illustrate in Fig. 3. In this case, the Planck mass is now given by

$$\text{classical: } 6M_P^2 = -\alpha_1\phi^2 - \alpha_2\chi^2$$
$$\text{quantum: } 6M_P^2 = -\left(1 - (1-\alpha_{1_0})\left(\frac{\phi}{\chi}\right)^{\gamma_1}\right)\phi^2 - \alpha_2\chi^2. \quad (119)$$

At the rightmost end of the ellipse, where $\chi \to 0$ we thus have, approximately

$$\text{classical: } 2K \approx (1-\alpha_1)\phi^2$$
$$\text{quantum: } 2K \approx (1-\alpha_{1_0})\phi^2\left(\frac{\phi}{\chi}\right)^{\gamma_1}. \quad (120)$$

The fields ultimately reach the fixed point, the intersection of the ellipse and the flat direction, and then satisfy the potential minimum constraint $\chi = \varepsilon\phi$.

In the classical case the results are simple. We see that ϕ is determined by Eq. (120), and likewise, $6M_P^2 \approx -\alpha_1\phi^2$ follows in $\chi \to 0$ limit from Eq. (119). We also have that $\chi = \varepsilon\phi$ is determined from the flat direction of the potential. Hence, $M_P^2 = -\alpha_1 K/3(1-\alpha_1)$. This defines the vacuum of the theory, and the slow-roll migration along the ellipse to the fixed point can generate many e-foldings of inflation, $N \propto -1/\alpha$ (see Sec. V).

The quantum case is somewhat different. As $\chi \to 0$, we see that the constraint of fixed K and the running of α_1 cause ϕ to track χ:

$$\phi \propto \chi^{\gamma_1/(2+\gamma_1)}. \quad (121)$$

Hence χ approaches zero quickly, while ϕ also tends to zero but does so more slowly. The Planck mass, however, approaches a constant:

$$\text{quantum: } 6M_P^2 \approx (1-\alpha_{1_0})\phi^2\left(\frac{\phi}{\chi}\right)^{\gamma_1} \quad (122)$$

thus we see that at the end of the ellipse with $\chi \to 0$:

$$M_P^2 = \frac{1}{6}\left((1-\alpha_{10})\left(\frac{\phi}{\chi}\right)^{\gamma_1}\right)\phi^2 = \frac{1}{3}K. \quad (123)$$

While ϕ and χ are becoming smaller, they do not become zero. The fixed point, the terminus of inflation, corresponds to the minimum of the potential in the flat direction, hence

$$\chi = \varepsilon\phi. \quad (124)$$

Combining, this yields the final VEVs of the fields:

$$\phi = M_P\left(\frac{6\varepsilon^{\gamma_1}}{1-\alpha_{1_0}}\right)^{1/2} \qquad \chi = \varsigma\phi. \quad (125)$$

It is interesting to speculate about the implications of this result in realistic models. The present model supposes only the potential interactions amongst ϕ and χ and the nonminimal gravitational interactions. The quantities γ_i in the present scheme are determined by the quartic couplings λ, δ, ξ and involves mixing induced by δ. If the only relevant term was λ, as in the single scalar model, we compute $\gamma_1 = 3\lambda/8\pi^2$. However, with the flat direction we have $\lambda = -\varsigma^2\delta$, and mixing effects in γ_i are dominant. In any case, if the potential coupling contributions to γ_i are small and if they are the only effects, we would have the classical result, $\phi = c_0 M_P$ with $c_0 = \sqrt{6/(1-\alpha_{1_0})}$ of order unity.

However, other schemes would likely have additional interactions, including gauge interactions. For example, ϕ and χ could have separate $U(1)_i$ gauge groups and gauge couplings (e_1, e_2), hence $\gamma_i \sim ke_i^2/16\pi^2$. Moreover, what is relevant is the "UV" behavior of these couplings i.e., the large ϕ/χ limit, and they could become large. Hence, is possible that in such schemes γ_1 can become large, perturbatively ranging, perhaps, from ~0.1 to 1, and nonperturbatively even larger. We thus would have $\phi = c_0 M_P \epsilon^{\gamma_1/2}$ and

$$\phi = c_0 M_P^{2/(2+\gamma_1)} \chi^{\gamma_1/(2+\gamma_1)} \qquad \chi = \varsigma\phi. \tag{126}$$

If we then identify χ with the Higgs VEV, $v_H = 175$ GeV, then we determine $\phi = c_0 M$ where $M = 2.6 \times 10^{13}$ GeV with $\gamma = 1$ and $M = 1.6 \times 10^{18}$ GeV with $\gamma = 0.1$. So, it is possible that the quantum running of α_{1_0} plays a role in establishing the grand-unification scale, identified with the VEV of ϕ. Even more extreme, if we identify χ with the QCD scale, 0.1 GeV and allow a nonperturbative at large ϕ/χ, $\gamma_1 \approx 10$, then we find $\chi \approx v_H \approx 175$ GeV. Perhaps χ could then be identified with the Higgs boson itself (this would be a "Higgs inflation model" with a dynamically generated Planck mass), where $M_P \sim m_H(m_H/\Lambda_{\mathrm{QCD}})_1^{\gamma}$.

The quantum effects are clearly of great interest. A detailed study of the renormalization of this theory and various models is beyond the scope of the present paper (see [20]). In particular the worked example of the ellipse we have presented involves a particular choice of an "ansatz" of $F(x)$, that might be anticipated from full calculation. Full details will be presented elsewhere [20].

VI. CONCLUSIONS

In the present paper we have discussed how inflation and Planck scale generation emerge from a dynamics associated with global Weyl symmetry and its current, K_μ. In the preinflationary universe, the scale current density, K_0, is driven to zero by general expansion. However, K_μ has a kernel structure, i.e., $K_\mu = \partial_\mu K$, and as $K_0 \to 0$, the kernel evolves as $K \to$ constant. This resulting constant K defines the scale symmetry breaking, indeed, defines M_P^2. The breaking of scale symmetry is thus determined by random initial values of the field VEVs. In addition, a scale-invariant potential of the theory ultimately determines the relative VEVs of the scalar fields contributing to K.

This mechanism entails a new form of dynamical scale symmetry breaking driven by the formation of a nonzero kernel, K, as the order parameter of scale symmetry breaking. The scale breaking has nothing to do with the potential in the theory, but is dynamically generated by gravity. The potential ultimately sculpts the structure of the vacuum (together with any quantum effects that may distort the K ellipse). There is a harmless dilaton associated with the dynamical symmetry breaking.

We illustrated this phenomenon in a single scalar field theory, ϕ, with nonminimal coupling to gravity $\sim -(1/12)\alpha\phi^2 R$, and a $\lambda\phi^4$ potential. The theory has a conserved current, $K_\mu = (1-\alpha)\phi\partial_\mu\phi$. The scale current charge density dilutes to zero in the preinflationary phase $K_0 \sim (a(t))^{-3}$. Hence, the kernel, $K = (1-\alpha)\phi^2/2$, and the VEV of ϕ are driven to a constant. With $\alpha < 0$, this induces a positive Planck $(\text{mass})^2$. The resulting inflation is eternal. However, if we allowed for breaking of scale symmetry through quantum loops, by conventional scale breaking renormalization, the resulting trace anomaly would imply that K_μ is no longer conserved. Then ϕ would relax to zero, and so too the Planck mass.

In multiscalar-field theories we see that the generalized $K = \sum_i (1-\alpha_i)\phi_i^2/2$. As this is driven to a constant by gravity, it defines an ellipsoidal constraint on the scalar field VEVs, and the Planck scale is again generated $\propto K$. An inflationary slow -roll is then associated with the field VEVs migrating along the ellipse, ultimately flowing to an infrared fixed point. This is shown to be amenable to analytic treatment, again owing to the Weyl symmetry. If the potential has a flat direction, which is a ray in field space that intersects the ellipse, then the fixed point corresponds to the potential minimum, and the field VEVs flow to it. This is associated with a period of rapid reheating and relaxation to the vacuum. This terminal phase of inflation is similar to standard ϕ^4 inflation, since the effective theory is now essentially Einstein gravity with a fixed M_P^2. The vacuum is determined by the intersection of the flat direction and the ellipse. The final cosmological constant vanishes by the scale symmetry.

These classical models illustrate the essential requirement of maintaining the Weyl symmetry, including quantum effects throughout. Any Weyl breaking effect will show up as a nonzero divergence in the K_μ current. Quantum anomalies will occur with conventional running couplings constants (β functions). We show that a Weyl invariant condition can be imposed on renormalized coupling constants to enforce the symmetry in the renormalized action. The coupling "constants" are then functions of Weyl invariant quantities. For example, λ, which previously ran with ϕ/M, now runs with the Weyl invariant function of the fields, $F_\lambda(\phi, \chi, g_{\mu\nu})$. This preserves all of the features of the classical global Weyl invariant model, but enforces a constraint on the original β functions that can only be satisfied by introducing field dependent counterterms. This is similar to adding the Wess-Zumino-Witten term to a theory as a counterterm to cancel (or provide) unwanted (or desired) chiral anomalies. We will explore detailed calculations that explicitly exhibit these results elsewhere [20].

We have experimented with the anticipated effects of quantum corrections in a simple ansatz model of the quantum effects. Here we see that the ellipse may be significantly distorted near the intersection with a potential flat direction. The final phase of inflation can involve a trajectory in which both scalar field VEVs shrink, but subject to a constraint that maintains constant K, and thus constant M_P^2. If the quantum effects are large, we may generate multiple hierarchies with possible intriguing relationships, such as $M_P = M_{GUT}(M_{GUT}/m_{\mathrm{Higgs}})^{\gamma}$.

The Nambu-Goldstone theorem applies in these models, with the dynamical scale symmetry breaking by nonzero K, and there is a dilaton. We touch upon some of the properties of the dilaton, with a more detailed discussion of its phenomenology in a subsequent work [20]. If the underlying exact Weyl scale symmetry (though spontaneously broken via K) is maintained throughout the theory, then the massless dilaton has at most derivative coupling to matter, becomes harmlessly decoupled, and any putative Brans-Dicke constraints go away [33]. Again, here it is essential that quantum breaking of global Weyl scale symmetry be suppressed to maintain the decoupling of the dilaton.

An unsolved problem in these schemes is that the flat direction generally can exist only for the special case of a fine-tuned parameter. This has been argued to be enforced in certain cases by a symmetry, such as in an $SO(1,1)$ invariant potential, $\sim\lambda(\phi^2-\chi^2)^2$ [5]. However, there is no such symmetry in the full theory as, e.g., the ϕ and χ kinetic terms are $O(2)$ invariant, and these symmetries will clash in loop order, and the flat direction will be lifted. If c is not fine-tuned, then we get either a trivial minimum at $\phi_0=\chi_0=0$, or a saddle point. Hence, a fundamental problem for us is how to naturally maintain flat directions.

Though we have not discussed it in detail presently, we expect there are implications here for novel UV completions of gravity. There is an inherent UV "softening" of quantum general relativity in these schemes since, essentially, we have no graviton propagator in this theory until the Planck scale forms. The low-energy Einstein gravity is then emergent. The UV completion of gravity would have to be scale-free and it might be viewed as a theory that contains only a metric, matter fields with nonminimal couplings, general covariance, but no stand-alone curvature terms. The construction of such a theory is beyond the scope of the present paper.

Global Weyl invariance may be a veritable and profound constraint on nature. It hints at intriguing consequences, dramatically including a dynamical origin of inflation and M_P as a unified phenomenon, dynamically generated mass hierarchies, including new effects that involve the running to the nonminimal coupling parameters, and leading ultimately to a vacuum with (near) zero cosmological constant.

After completing this paper we received a related work by Kannike, *et al.*, [14], who discuss the effect of explicit sources of scale invariance breaking on the stability of the Planck scale with nonminimally coupled scalars, including Coleman-Weinberg potentials. The authors find it challenging to construct viable models, lending support to the result here that Weyl symmetry must be maintained and its breaking can only be spontaneous.

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[1] M. Shaposhnikov and D. Zenhausern, Quantum scale invariance, cosmological constant and hierarchy problem, Phys. Lett. B **671**, 162 (2009).
[2] M. Shaposhnikov and D. Zenhausern, Scale invariance, unimodular gravity and dark energy, Phys. Lett. B **671**, 187 (2009).
[3] D. Blas, M. Shaposhnikov, and D. Zenhausern, Scale-invariant alternatives to general relativity, Phys. Rev. D **84**, 044001 (2011).
[4] J. Garcia-Bellido, J. Rubio, M. Shaposhnikov, and D. Zenhausern, Higgs-dilaton cosmology: From the early to the late universe, Phys. Rev. D **84**, 123504 (2011).
[5] R. Kallosh and A. Linde, Non-minimal inflationary attractors, J. Cosmol. Astropart. Phys. 10 (2013) 033; J. J. M. Carrasco, R. Kallosh, and A. Linde, Cosmological attractors and initial conditions for inflation, Phys. Rev. D **92**, 063519 (2015).
[6] K. Allison, C. T. Hill, and G. G. Ross, An ultra-weak sector, the strong *CP* problem and the pseudo-Goldstone dilaton, Nucl. Phys. **B891**, 613 (2015); Ultra-weak sector, Higgs boson mass, and the dilaton, Phys. Lett. B **738**, 191 (2014).
[7] R. Jackiw and S. Y. Pi, Fake Conformal symmetry in conformal cosmological models, Phys. Rev. D **91**, 067501 (2015).
[8] K. Kannike, G. Htsi, L. Pizza, A. Racioppi, M. Raidal, A. Salvio, and A. Strumia, Dynamically induced Planck scale and inflation, J. High Energy Phys. 05 (2015) 065.
[9] G. K. Karananas and J. Rubio, On the geometrical interpretation of scale-invariant models of inflation, Phys. Lett. B **761**, 223 (2016).
[10] G. K. Karananas and M. Shaposhnikov, Scale invariant alternatives to general relativity. II. Dilaton properties, Phys. Rev. D **93**, 084052 (2016).
[11] I. Quiros, Scale invariance: Fake appearances, arXiv:1405.6668; Scale invariant theory of gravity and the standard model of particles, arXiv:1401.2643.
[12] N. K. Singh and S. Panda, Local scale invariance and inflation, arXiv:1605.02543; P. Jain, P. Karmakar, S. Mitra, S. Panda, and N. K. Singh, Cosmological perturbation analysis in a scale invariant model of gravity, Class. Quant. Grav. **28**, 215010 (2011).
[13] P. G. Ferreira, C. T. Hill, and G. G. Ross, Scale-independent inflation and hierarchy generation, Phys. Lett. B **763**, 174 (2016).

[14] K. Kannike, M. Raidal, C. Spethmann, and H. Veerme, Evolving Planck mass in classically scale-invariant theories, arXiv:1610.06571.

[15] P. G. Ferreira, C. T. Hill, and G. G. Ross, No fifth force in a scale invariant universe, arXiv:1612.03157.

[16] S. R. Coleman and E. J. Weinberg, Radiative Corrections as the Origin of Spontaneous Symmetry Breaking, Phys. Rev. D **7**, 1888 (1973).

[17] R. Jackiw, Functional evaluation of the effective potential, Phys. Rev. D **9**, 1686 (1974).

[18] R. Percacci, Renormalization group flow of Weyl invariant dilaton gravity, New J. Phys. **13**, 125013 (2011).

[19] W. A. Bardeen, Report No. Fermilab-Conf-95-391-T; (private communication); Report No. Fermilab-Conf-08-118-T.

[20] P. G. Ferreira, C. T. Hill, and G. G. Ross, (to be published).

[21] S. Weinberg, *Gravitation and Cosmology* (Wiley and Sons, New York, 1972); The Riemann-Christoffel tensor is defined in Weinberg's Eq. (6.1.5); this sign is opposite to that in common use, e.g., as given in R. Wald, *General Relativity* (University Of Chicago Press, Chicago, 1984). Both Weinberg and Wald use the metric signature convention $(-1, 1, 1, 1)$. Reference [21] uses the Weinberg definition of curvature, but adopts the metric signature $(1, -1, -1, -1)$. In the present paper we follow the conventions of Ref. [21].

[22] C. G. Callan, Jr., S. R. Coleman, and R. Jackiw, A new improved energy-momentum tensor, Ann. Phys. (N.Y.) **59**, 42 (1970).

[23] F. L. Bezrukov and M. Shaposhnikov, The standard model Higgs boson as the inflaton, Phys. Lett. B **659**, 703 (2008).

[24] F. L. Bezrukov, A. Magnin, and M. Shaposhnikov, Standard model Higgs boson mass from inflation, Phys. Lett. B **675**, 88 (2009).

[25] C. T. Hill, Is the Higgs boson associated with Coleman-Weinberg dynamical symmetry breaking?, Phys. Rev. D **89**, 073003 (2014).

[26] J. A. Harvey, C. T. Hill, and R. J. Hill, Standard model gauging of the Wess-Zumino-Witten term: Anomalies, global currents, and pseudo-Chern-Simons interactions, Phys. Rev. D **77**, 085017 (2008); Anomaly Mediated Neutrino-Photon Interactions at Finite Baryon Density, Phys. Rev. Lett. **99**, 261601 (2007); O. Kaymakcalan, S. Rajeev, and J. Schechter, Non-Abelian anomaly and vector meson decays, Phys. Rev. D **30**, 594 (1984).

[27] F. Englert, C. Truffin, and R. Gastmans, Conformal invariance in quantum gravity, Nucl. Phys. **B117**, 407 (1976).

[28] S. Deser, Scale invariance and gravitational coupling, Ann. Phys. (N.Y.) **59**, 248 (1970).

[29] M. Shaposhnikov and D. Zenhausern, Quantum scale invariance, cosmological constant and hierarchy problem, Phys. Lett. B **671**, 162 (2009).

[30] R. Armillis, A. Monin, and M. Shaposhnikov, Spontaneously broken conformal symmetry: Dealing with the trace anomaly, J. High Energy Phys. 10 (2013) 030.

[31] D. M. Ghilencea, Manifestly scale-invariant regularization and quantum effective operators, Phys. Rev. D **93**, 105006 (2016); C. Tamarit, Running couplings with a vanishing scale anomaly, J. High Energy Phys. 12 (2013) 098.

[32] D. M. Ghilencea, Z. Lalak, and P. Olszewski, Two-loop scale-invariant scalar potential and quantum effective operators, Eur. Phys. J. C **76**, 656 (2016); D. M. Ghilencea, One-loop potential with scale invariance and effective operators, *Proc. Sci.*, CORFU2015 (2016) 040 [arXiv:1605.05632].

[33] Y. Fujii, Scalar-tensor theory of gravitation and spontaneous breakdown of scale invariance, Phys. Rev. D **9**, 874 (1974).