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$f_2(1810)$ as a triangle singularity

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We perform calculations showing that a source producing $K^*\bar{K}^*$ in $J=2$ and $L=0$ gives rise to a triangle singularity at 1810 MeV with a width of about 200 MeV from the mechanism $K^*\to\pi K$ and then $K\bar{K}^*$ merging into the $a_1(1260)$ resonance. We suggest that this is the origin of the present $f_2(1810)$ resonance and propose to look at the $\pi a_1(1260)$ mode in several reactions to clarify the issue.

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I. INTRODUCTION

Triangle singularities, discussed long ago by Landau [1], are catching the attention of hadron physicists recently. The large amount of phenomenology gathered in different facilities studying intermediate energy reactions and hadron spectra has widened the range of possible cases where triangle singularities are relevant. In essence the singularities appear from Feynmann diagrams with three particles in a loop when the particles are placed on shell and the momenta are collinear. Yet, some condition is also necessary for the singularity to appear and it is that the mechanisms reflect a classical problem in which the external particle A decays into 1 and 2, particle 1 decays into 3 and an external particle B and then 2 and 3 fuse to give an external particle C. This is the content of the Coleman-Norton theorem [2]. A very simple analytical way to see when the triangle singularity appears can be seen in Ref. [3], where a critical discussion of the suggestion made in Refs. [4–6] associating the narrow peak of the pentaquark $P_c(4450)$ observed in the LHCb collaboration [7] to a triangle singularity is made.

Recent examples of relevant triangle singularities can be seen in the $\eta(1405)\to\pi a_0(980)$ and $\eta(1405)\to\pi f_0(980)$ decays [8], the latter one violating isospin, which is abnormally enhanced due to the triangle singularities [9–11]. Another example is the case of the "$a_1(1420)$" claimed as a new resonance by the COMPASS collaboration [12], which, as suggested in Ref. [13] and shown explicitly in Refs. [14,15], represents the decay mode of the $a_1(1260)$ into $\pi f_0(980)$ due to a triangle singularity coming from the $a_1(1260)\to K^*\bar{K}$, $K^*\to K\pi$ and $K\bar{K}$ combining to give the $f_0(980)$.

A more recent example is given in Ref. [16], where the enhancement in the cross section of the $\gamma p\to K^+\Lambda(1405)$ reaction around the γp center of mass energy $W=2110$ MeV is associated to a triangle singularity stemming from the formation of a resonance called $N^*(2030)$ that is dynamically generated from the vector-baryon interaction [17]. This resonance finds support in the $\gamma p\to K^0\Sigma^+$ reaction close to the $K^*\Lambda$ and $K^*\Sigma$ thresholds [18] (see Ref. [19] for the theoretical analysis). For the present problem the resonance decays into $K^*\Sigma$, then $K^*\to\pi K$ and the $\pi\Sigma$ merge to give the $\Lambda(1405)$.

In this work we wish to show that a peak appears precisely at 1810 MeV due to a process induced by the nearby $f_2(1640)$ going to $K^*\bar{K}^*$, $K^*\to\pi K$ and $\bar{K}^*K$ merging into the $a_1(1260)$ resonance. The strength of the peak will have the same quantum numbers as the resonance from which it comes from, but the singularity appears at 1810 MeV, producing a peak that has given rise to the claim of the $f_2(1810)$ resonance [20].

The information on the $f_2(1810)$ in the particle data group book (PDG) [20] is scarce. It has been seen in a few experiments and the mass and width are quoted as 1815 ± 12 MeV and 197 ± 22 MeV, respectively. The decay modes reported are $\pi\pi$, $K\bar{K}$, $\eta\eta$, $4\pi^0$, and $\gamma\gamma$, but one finds there $\Gamma_{\eta\eta}/\Gamma_{\rm total}\sim 0.008^{+0.028}_{-0.003}$ [21], $\Gamma_{\pi\pi}/\Gamma_{4\pi^0}<0.75$ [22], $\Gamma_{4\pi^0}/\Gamma_{\eta\eta}\sim 0.8\pm0.3$ [22], $\Gamma_{K^+K^-}/\Gamma_{\rm total}\sim 0.003^{+0.019}_{-0.002}$ [21], from which one concludes that the sum of branching fractions of all decay channels, where the resonance is observed, is only a small fraction of the total. The main decay mode is still not identified. From the study presented here we would conclude that the main decay mode of the peak should be the $\pi a_1(1260)$, which can be seen in the $\pi\pi\rho$ channel, a decay mode not investigated so far.

We should note that in the present edition of the PDG [23] the $f_2(1810)$ resonance appears with the cautionary labels, "omitted from the summary table" and "needs confirmation." The present work will contribute to clarify the situation.

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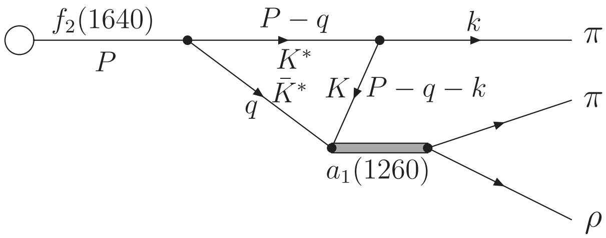

FIG. 1. Triangle singularity for the production of the $\pi a_1(1260)$. The circle indicates an external source that produces the $f_2(1640)$ albeit with an energy different than 1640 MeV. The momenta of the particles are also shown.

II. FORMALISM AND INGREDIENTS

The tensor mesons have been for a long time an example of a successful classification in $SU(3)$ multiplets with a $q\bar{q}$ structure [24–26]. However, the low lying tensor mesons, $f_2(1270)$, $f_2'(1525)$, and $K_2^*(1430)$ qualify well as dynamically generated states from the vector-vector interaction [27,28] (see Ref. [29] for a list of reactions supporting this picture). In this sense, the $f_2'(1525)$ is mostly made from $K^*\bar{K}^*$ [28]. If we allow this resonance to decay into $K^*\bar{K}^*$, then $K^* \to \pi K$ and $\bar{K}^* K \to a_1(1260)$ (with a mass slightly above the mass threshold of $\bar{K}^* K$), we find a triangle singularity at 1810 MeV.[1] Since the width of the $f_2'(1525)$ is only 73 MeV, the strength of this resonance at 1810 MeV is very small and the chances of the singularity at 1810 MeV to show up due to the decay of the $f_2'(1525)$ in that particular channel are very dim. Yet, an inspection of the PDG shows that in between the $f_2'(1525)$ and $f_2(1810)$ there are two f_2 resonances, the $f_2(1565)$ and $f_2(1640)$. Due to the proximity of the mass of the $f_2(1640)$ to the 1810 MeV, this latter resonance has more chances to influence the 1810 MeV region. The $f_2(1640)$ is also not a very well studied resonance, but the decay modes $\omega\omega$, 4π and $K\bar{K}$ have been observed. If the $f_2(1640)$ couples to $\omega\omega$, then it should also couple to $K^*\bar{K}^*$, which guarantees that it also decays into $K\bar{K}$ if $K^*\bar{K}^* \to K\bar{K}$ via π exchange. The mechanism that we study is depicted in Fig. 1.

The coupling of the $f_2(1640)$ has the structure of a tensor [27,30], where the polarizations of $K^*\bar{K}^*$ couple to $J=2$ but in $L=0$. We have

$$-iV_{f_2,K^*\bar{K}^*} \equiv g_{f_2,K^*\bar{K}^*}\left(\frac{1}{2}[\epsilon_i(1)\epsilon_j(2)+\epsilon_j(1)\epsilon_i(2)] - \frac{1}{3}\epsilon_m(1)\epsilon_m(2)\delta_{ij}\right), \tag{1}$$

where ϵ_i are the polarization vectors, 1 for K^* and 2 for $\bar{K}^*$, and only spatial components are considered, neglecting the three momentum of the vector mesons versus their masses, as done in Refs. [27,28]. In Eq. (1) $g_{f_2,K^*\bar{K}^*}$ is the coupling constant of the f_2 meson to the $K^*\bar{K}^*$ channel.

We also need the coupling of $K^* \to \pi K$ which is easily obtained from the Lagrangian,

$$\mathcal{L}_{VPP} = -ig\langle V^\mu[P,\partial_\mu P]\rangle, \tag{2}$$

where V and P are the $SU(3)$ vector meson matrix and pseudoscalar meson matrix [27,28], respectively. The coupling g is,

$$g = \frac{M_V}{2f_\pi}, \qquad M_V = 780\ \text{MeV}, \qquad f_\pi = 93\ \text{MeV}. \tag{3}$$

Similarly we need the coupling of the $a_1(1260)$ to $\bar{K}^* K$ and $\pi\rho$, which are a constant times $\vec{\epsilon}(A)\cdot\vec{\epsilon}(V)$ [31], where $\vec{\epsilon}(A)$ stands for the polarization vector of the axial vector meson $a_1(1260)$ and $\vec{\epsilon}(V)$ for the one of the vector meson $\bar{K}^*$ or ρ. We are only concerned about the shape of the amplitude compared to a pure $f_2(1640)$ propagator, hence the couplings do not play a role at this step. Then we have for the diagram of Fig. 1

$$t = \frac{1}{P^2 - M_{f_2}^2 + iM_{f_2}\Gamma_{f_2}} t_T, \tag{4}$$

where M_{f_2} and Γ_{f_2} are the mass and width of the $f_2(1640)$ meson. In this work we take $M_{f_2} = 1639$ MeV and $\Gamma_{f_2} = 150$ MeV as in Refs. [20,32]. Meanwhile, t_T is the triangle amplitude

$$\begin{aligned} t_T = i\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2 - m_{K^*}^2 + i\epsilon}\frac{1}{(P-q)^2 - m_{K^*}^2 + i\epsilon} \\ \times \frac{1}{(P-q-k)^2 - m_K^2 + i\epsilon}\left(\frac{1}{2}[\epsilon_i(1)\epsilon_j(2)+\epsilon_j(1)\epsilon_i(2)]\right. \\ \left. -\frac{1}{3}\epsilon_m(1)\epsilon_m(2)\delta_{ij}\right)\vec{\epsilon}(1)\cdot(2\vec{k}-\vec{P}+\vec{q})\vec{\epsilon}(2)\cdot\vec{\epsilon}(A) \\ \times \frac{1}{(P-k)^2 - M_A^2 + iM_A\Gamma_A}\vec{\epsilon}(A)\cdot\vec{\epsilon}(\rho), \end{aligned} \tag{5}$$

where the momenta are shown in Fig. 1 and M_A and Γ_A are the mass and width of the $a_1(1260)$ resonance. We take $M_A = 1230$ MeV and $\Gamma_A = 425$ MeV as in Ref. [20]. Besides, $\vec{\epsilon}(\rho)$ stands for the polarization vector of the ρ meson.

Using the following property

$$\sum_{\text{pol}} \vec{\epsilon}_i\vec{\epsilon}_j = \delta_{ij} \tag{6}$$

and taking $\vec{P}=0$ for the $f_2(1640)$ at rest, we find the vertex combination

[1]It can be easily obtained with the formalism shown in Eq. (8) of Ref. [15]. The position of the singularity in case of zero width for the loop particles can be easily obtained by means of Eq. (18) of Ref. [3].

$$\frac{1}{2}\left[(2k+q)_i\epsilon_j(\rho)+(2k+q)_j\epsilon_i(\rho)-\frac{1}{3}(2k+q)_l\epsilon_l(\rho)\delta_{ij}\right]. \tag{7}$$

Since when integrating over $\vec{q}$, the only remaining vector is $\vec{k}$, we can write

$$\int d^3\vec{q}q_iF(\vec{q},\vec{k})\equiv Ak_i, \tag{8}$$

with $F(\vec{q},\vec{k})$ the rest of the integrand. From this we can get

$$A=\int d^3\vec{q}\frac{\vec{q}\cdot\vec{k}}{|\vec{k}|^2}F(\vec{q},\vec{k}). \tag{9}$$

We also perform analytically the q^0 integration as done in Ref. [33] and finally we find

$$t_T=I_2V_{ij}, \tag{10}$$

with

$$V_{ij}=k_i\epsilon_j(\rho)+k_j\epsilon_i(\rho)-\frac{2}{3}k_l\epsilon_l(\rho)\delta_{ij}, \tag{11}$$

and

$$I_2=\frac{1}{M_{\rm inv}^2-M_A^2+iM_A\Gamma_A}I_2', \tag{12}$$

$$\begin{aligned} I_2' = \int\frac{d^3\vec{q}}{(2\pi)^3}\left(2+\frac{\vec{q}\cdot\vec{k}}{|\vec{k}|^2}\right)&\frac{1}{8\omega_{\bar{K}^*}(|\vec{q}|)\omega_{K^*}(|\vec{q}|)\omega_K(|\vec{q}+\vec{k}|)}\frac{1}{k^0-\omega_K(|\vec{q}+\vec{k}|)-\omega_{K^*}(|\vec{q}|)}\\ &\times\frac{1}{P^0+\omega_{\bar{K}^*}(|\vec{q}|)+\omega_K(|\vec{q}+\vec{k}|)-k^0}\frac{1}{P^0-\omega_{\bar{K}^*}(|\vec{q}|)-\omega_K(|\vec{q}+\vec{k}|)-k^0+i\Gamma_{K^*}/2}\\ &\times\frac{1}{P^0-\omega_{K^*}(|\vec{q}|)-\omega_{\bar{K}^*}(|\vec{q}|)+i\Gamma_{K^*}}\\ &\times\{P^0\omega_{\bar{K}^*}(|\vec{q}|)+k^0\omega_K(|\vec{q}+\vec{k}|)-[\omega_{\bar{K}^*}(|\vec{q}|)+\omega_K(|\vec{q}+\vec{k}|)]\times[\omega_{\bar{K}^*}(|\vec{q}|)+\omega_K(|\vec{q}+\vec{k}|)+\omega_{K^*}(|\vec{q}|)]\}, \end{aligned} \tag{13}$$

where $\omega_{\bar{K}^*}(|\vec{q}|)=\sqrt{m_{K^*}^2+|\vec{q}|^2}$, $\omega_{K^*}(|\vec{q}|)=\sqrt{m_{K^*}^2+|\vec{q}|^2}$, and $\omega_K(|\vec{q}+\vec{k}|)=\sqrt{m_K^2+|\vec{q}+\vec{k}|^2}$ are the energies of $\bar{K}^*$, K^*, and K in the triangle loop, respectively. We take the mass of K^* meson $m_{K^*}=893.1$ MeV and width $\Gamma_{K^*}=49.1$ MeV. In addition, $M_{\rm inv}$ is the invariant mass of the $\pi\rho$ system[2] decaying from the $a_1(1260)$ resonance, and $|\vec{k}|$ is the π momentum in the $f_2(1640)$ rest frame,

$$|\vec{k}|=\frac{\sqrt{[s-(m_\pi+M_{\rm inv})^2][s-(m_\pi-M_{\rm inv})^2]}}{2\sqrt{s}}, \tag{14}$$

with $s=P^2$ the invariant mass squared of the initial $f_2(1640)$ meson.

In order to evaluate $|t|^2$ we must sum over polarizations

$$\sum_{ij}\sum_{\rm pol}V_{ij}V_{ij}=\frac{20}{3}|\vec{k}|^2, \tag{15}$$

and then

$$\bar{\sum}\sum|t|^2=\frac{20|\vec{k}|^2}{3}\left|\frac{1}{s-M_{f_2}^2+iM_{f_2}\Gamma_{f_2}}\right|^2|I_2|^2. \tag{16}$$

[2]In this work, when we talk about the $\pi\rho$ system, the π is always from the decay of the $a_1(1260)$. Otherwise, the π is from the decay of the K^* in the triangle loop.

The differential mass distribution for the $\pi\rho$ decaying from the $a_1(1260)$ is given by

$$\frac{d\Gamma}{dM_{\rm inv}}=\frac{1}{(2\pi)^3}\frac{|\vec{k}|\tilde{p}_\rho}{4s}\bar{\sum}\sum|t|^2, \tag{17}$$

where $\tilde{p}_\rho$ is the ρ momentum in the rest frame of the $a_1(1260)$ meson,

$$\tilde{p}_\rho=\frac{\sqrt{[M_{\rm inv}^2-(m_\rho+m_\pi)^2][M_{\rm inv}^2-(m_\rho-m_\pi)^2]}}{2M_{\rm inv}}, \tag{18}$$

with $m_\rho=775.26$ MeV and $m_\pi=138.04$ MeV.

Finally, we obtain Γ by integrating Eq. (17) in $M_{\rm inv}$,

$$\Gamma=\int_{m_\rho+m_\pi}^{\sqrt{s}-m_\pi}\frac{d\Gamma}{dM_{\rm inv}}dM_{\rm inv}. \tag{19}$$

This step allows the contribution of a range of masses for the $a_1(1260)$ weighted by its spectral function, which is relevant since the mass of $\bar{K}^*K$ is 1383 MeV that is above the nominal mass of the $a_1(1260)$.

III. NUMERICAL RESULTS

In Fig. 2 we show the results for Γ of Eq. (19) as a function of $\sqrt{s}$, removing the propagator of the $f_2(1640)$ in

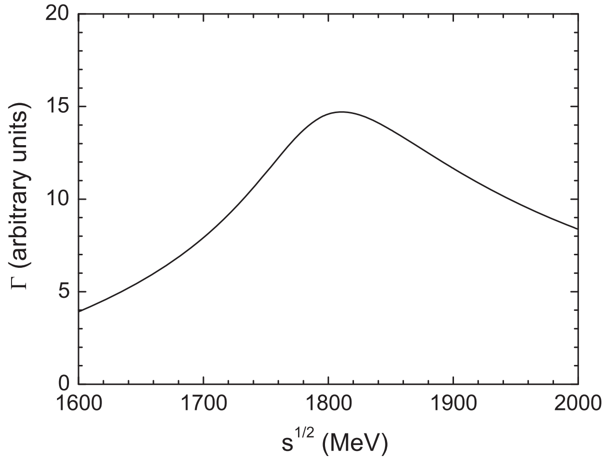

FIG. 2. Γ as a function of $\sqrt{s}$ for Model A without the $f_2(1640)$ propagator.

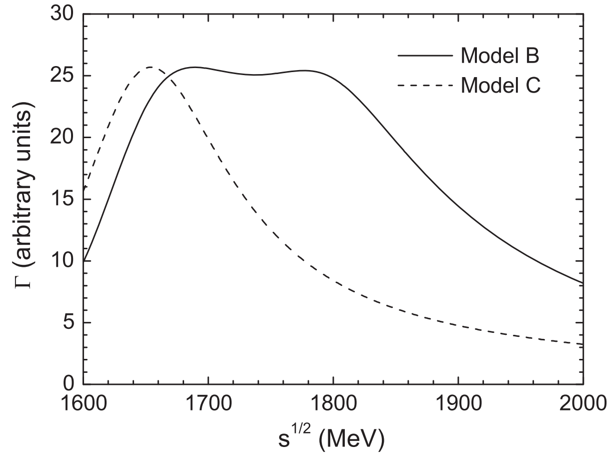

FIG. 3. Γ as a function of $\sqrt{s}$ for Model B (solid curve) and Model C (dashed curve).

Eq. (16) and the $|\vec{k}|^3$ factor[3] (Model A). This is done to show the strength of the triangle singularity alone. We see a broad peak around 1810 MeV.

Although a triangle singularity gives indeed rise to an infinite amplitude when the particles in the triangle diagram have zero width, in practice some of them have width and the amplitude becomes finite. The explicit consideration of the width of the intermediate K^*, $\bar{K}^*$ and the mass distribution of the $a_1(1260)$ renders the results finite and the singular peak becomes the broad peak that we observe in Fig. 2.

In Fig. 3 we plot the full width of Eq. (19), taking into account the $f_2(1640)$ propagator in Eq. (16) and the $|\vec{k}|^3$ factor (Model B). For comparison we also show the shape of the modulus squared of the propagator (Model C), removing t_T and normalizing the two curves to the peak. We can see clear differences between the two curves, with a large enhancement of the results in the region around 1800 MeV where the triangle singularity appears. The result with the solid line gives the shape that we predict if one looks at the decay mode of the $\pi a_1(1260)$ in the region of 1600–1900 MeV. A resonancelike bump shows clearly around 1800 MeV as a combination of the $f_2(1640)$ propagator and the structure of the singularity shown in Fig. 2.

In order to further clarify the issue we take the amplitude corresponding to the diagram of Fig. 1, removing the spin operator V_{ij} of Eq. (11). Hence, we consider the amplitude

$$\tilde{t} = \frac{1}{s - M_{f_2}^2 + iM_{f_2}\Gamma_{f_2}} I_2', \tag{20}$$

in analogy to Eq. (4), which corresponds to the production of $\pi a_1(1260)$ from an external source with the $a_1(1260)$ treated as a stable particle.

We plot the results for $|T_1|^2$ ($T_1 = -i\tilde{t}$) in Fig. 4 with a red-solid curve. In addition, the modulus squared of the $f_2(1640)$ propagator, $|T_2|^2 = |\frac{1}{s-M_{f_2}^2+iM_{f_2}\Gamma_{f_2}}|^2$, is shown in Fig. 4 with a black-solid curve, which is normalized to the peak of $|T_1|^2$. Like in Fig. 3 we see the bump corresponding to the "$f_2(1810)$" resonance. In the figure we also show the real and imaginary parts of T_1 with green-dashed and blue-dashed curves, respectively. We observe two structures. Around $\sqrt{s} = 1640$ MeV, looking at $\tilde{t}$ instead of T_1, we see the typical Breit-Wigner (BW) structure that we have introduced by hand in the propagator of $f_2(1640)$ in Eq. (20), where $\mathrm{Im}(\tilde{t})$ [see $\mathrm{Re}(T_1)$ in Fig. 4] has a minimum and $\mathrm{Re}(\tilde{t})$ [see $\mathrm{Im}(T_1)$ in Fig. 4] changes sign at the $f_2(1640)$ resonance mass. Then, when multiplying $\tilde{t}$ by the phase $-i$ we get a second structure in T_1 that also looks like a BW with a minimum for $\mathrm{Im}(T_1)$, and $\mathrm{Re}(T_1)$ changing sign around $\sqrt{s} = 1780$ MeV. We adopt now an experimental attitude and try to fit this second structure of T_1 by means of a BW amplitude

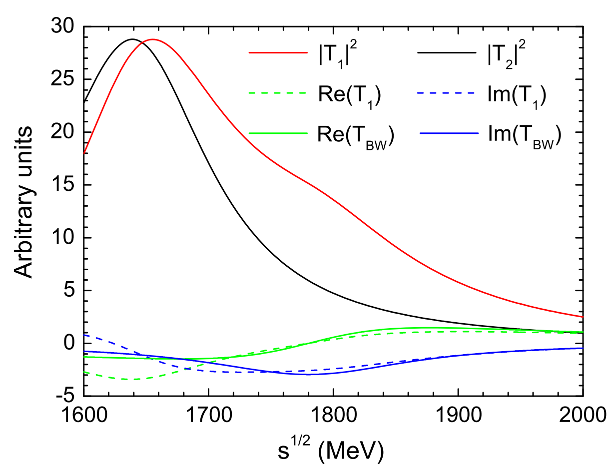

FIG. 4. Results of $|T_1|^2$, $|T_2|^2$, $\mathrm{Re}(T_1)$, $\mathrm{Im}(T_1)$, $\mathrm{Re}(T_{\mathrm{BW}})$, and $\mathrm{Im}(T_{\mathrm{BW}})$ as a function of $\sqrt{s}$.

[3]This factor is from the p-wave decay of $f_2(1640) \to \pi a_1(1260)$ and the phase space.

$$T_{\rm BW} = \frac{\beta}{s - M_R^2 + iM_R\Gamma_R}, \tag{21}$$

where $M_R = 1780$ MeV can be easily obtained at the zero point of $\mathrm{Re}(T_1)$. We then fit β to the value of $\mathrm{Im}(T_1)$ at $s = M_R^2$ and also get Γ_R from the results shown in Fig. 4, $\Gamma_R \simeq 200$ MeV. Then

$$\beta = -M_R\Gamma_R \times \mathrm{Im}(T_1)|_{s=M_R^2}. \tag{22}$$

The results of $\mathrm{Re}(T_{\rm BW})$ and $\mathrm{Im}(T_{\rm BW})$ are shown in Fig. 4 with green-solid and blue-solid curves. We find an excellent agreement with T_1 from $\sqrt{s} = 1750$ MeV on. From the experimental point of view, this would qualify as a resonance, and the exercise we have done, shows that it is in the $\pi a_1(1260)$ channel [we have taken the nominal mass of the $a_1(1260)$ to make Fig. 4] where this resonant structure shows up. This allows us to trace the behavior to the triangle singularity. Apart from the fit of the amplitude one wishes to see the meaning of the structure observed experimentally. One wishes to associate resonances to states stemming from the interaction of quarks and gluons [34–36] or from the interaction of hadrons making molecules [27,28,37–40]. However, in the present case the structure observed comes from neither of these. It comes from a singularity in a triangle diagram that contains $K^*\bar{K}^*K$ in the intermediate states and has nothing to do with the interaction of quarks or the interaction of $\pi a_1(1260)$ which is not taken into account in our work.

Since one important aim in hadron physics is to know the origin of the resonances and their nature, it is important to single out those cases where a resonant structure can be attributed to a triangle singularity or a threshold effect. Identifying them is an important task, and it would be better to exclude them as "genuine" resonances to prevent the misleading work of trying to get them from theories of quarks or hadron interactions.

IV. CONCLUSIONS

We have studied a triangle singularity driven by a source giving rise to $K^*\bar{K}^*$ in $J = 2$, with $K^* \to \pi K$ and $K\bar{K}^*$ merging to give the axial vector resonance $a_1(1260)$. We have shown that this triangle singularity gives rise to a resonancelike structure with a peak at 1810 MeV and a width of about 200 MeV, consistent with the basic properties of the $f_2(1810)$ listed in the PDG [20]. We have taken an arbitrary external source and have chosen the nearby $f_2(1640)$ resonance to be the driving element giving rise to $K^*\bar{K}^*$ with the $f_2(1640)$ quantum numbers, the same as the catalogued $f_2(1810)$ "resonance." We find a natural explanation in the singularity to explain the observed peak, and predict that the main decay mode should be the $\pi a_1(1260)$. This does not contradict the information on the decay modes of the $f_2(1810)$ tabulated in the PDG [23] since the few decay modes where it was observed account for only a small fraction of the total width.

It would be interesting to look at the $\pi a_1(1260)$ channel in the region of 1600–1900 MeV in some decay processes to eventually find the clear structure that we predict in our calculations. Possible reactions to see this decay mode would be $J/\psi \to \phi\pi a_1(1260)$, $J/\psi \to \omega\pi a_1(1260)$, or $J/\psi \to \gamma\pi a_1(1260)$. Actually, the mode $J/\psi \to \gamma f_2(1810)$ with $f_2(1810) \to \eta\eta$ is measured with a branching ratio $5.4^{+3.5}_{-2.4} \times 10^{-5}$ [41]. According to our discussion, the mode $J/\psi \to \gamma f_2(1810) \to \gamma\pi a_1(1260)$ should have a much bigger rate. Actually, the signal for the $f_2(1810)$ in Ref. [41] is weak and only extracted through partial wave analysis with some ambiguities. The detection of the $\pi a_1(1260)$ mode should show a much clearer signal around 1810 MeV, according to the results obtained here. We have shown that in the $\pi a_1(1260)$ mode the amplitude shows indeed a resonant structure that can be cast into a Breit-Wigner form. Yet, this structure does not stem from the interaction of quarks or hadrons but from the triangle diagram that we have discussed, which produce a kinematical singularity. It is important to know that to distinguish structures of this type from other ones corresponding to genuine states that have a dynamical origin from the interaction of more elementary components. We can only encourage the performance of experiments like those quoted above that can shed light on the $f_2(1810)$ and pave the way to investigate other peaks that might have a similar origin as the one discussed here.

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[1] L. D. Landau, On analytic properties of vertex parts in quantum field theory, Nucl. Phys. **13**, 181 (1959).

[2] S. Coleman and R. E. Norton, Singularities in the physical region, Nuovo Cimento **38**, 438 (1965).

[3] M. Bayar, F. Aceti, F. K. Guo, and E. Oset, A discussion on triangle singularities in the $\Lambda_b \to J/\psi K^- p$ reaction, Phys. Rev. D **94**, 074039 (2016).

[4] F. K. Guo, U. G. Meißner, W. Wang, and Z. Yang, How to reveal the exotic nature of the $P_c(4450)$, Phys. Rev. D **92**, 071502 (2015).

[5] X. H. Liu, Q. Wang, and Q. Zhao, Understanding the newly observed heavy pentaquark candidates, Phys. Lett. B **757**, 231 (2016).

[6] F. K. Guo, U. G. Meißner, J. Nieves, and Z. Yang, Remarks on the P_c structures and triangle singularities, Eur. Phys. J. A **52**, 318 (2016).

[7] R. Aaij *et al.* (LHCb Collaboration), Observation of $J/\psi p$ Resonances Consistent with Pentaquark States in $\Lambda_b^0 \to J/\psi K^- p$ Decays, Phys. Rev. Lett. **115**, 072001 (2015).

[8] M. Ablikim *et al.* (BESIII Collaboration), First Observation of $\eta(1405)$ Decays into $f_0(980)\pi^0$, Phys. Rev. Lett. **108**, 182001 (2012).

[9] J. J. Wu, X. H. Liu, Q. Zhao, and B. S. Zou, The Puzzle of Anomalously Large Isospin Violations in $\eta(1405/1475) \to 3\pi$, Phys. Rev. Lett. **108**, 081803 (2012).

[10] X. G. Wu, J. J. Wu, Q. Zhao, and B. S. Zou, Understanding the property of $\eta(1405/1475)$ in the J/ψ radiative decay, Phys. Rev. D **87**, 014023 (2013).

[11] F. Aceti, W. H. Liang, E. Oset, J. J. Wu, and B. S. Zou, Isospin breaking and $f_0(980)$-$a_0(980)$ mixing in the $\eta(1405) \to \pi^0 f_0(980)$ reaction, Phys. Rev. D **86**, 114007 (2012).

[12] C. Adolph *et al.* (COMPASS Collaboration), Observation of a New Narrow Axial-Vector Meson $a_1(1420)$, Phys. Rev. Lett. **115**, 082001 (2015).

[13] X. H. Liu, M. Oka, and Q. Zhao, Searching for observable effects induced by anomalous triangle singularities, Phys. Lett. B **753**, 297 (2016).

[14] M. Mikhasenko, B. Ketzer, and A. Sarantsev, Nature of the $a_1(1420)$, Phys. Rev. D **91**, 094015 (2015).

[15] F. Aceti, L. R. Dai, and E. Oset, $a_1(1420)$ peak as the $\pi f_0(980)$ decay mode of the $a_1(1260)$, Phys. Rev. D **94**, 096015 (2016).

[16] E. Wang, J. J. Xie, W. H. Liang, F. K. Guo, and E. Oset, The role of a triangle singularity in the $\gamma p \to K^+ \Lambda(1405)$ reaction, Phys. Rev. C **95**, 015205 (2017).

[17] E. Oset and A. Ramos, Dynamically generated resonances from the vector octet-baryon octet interaction, Eur. Phys. J. A **44**, 445 (2010).

[18] R. Ewald *et al.* (CBELSA/TAPS Collaboration), Anomaly in the $K_S^0 \Sigma^+$ photoproduction cross section off the proton at the K^* threshold, Phys. Lett. B **713**, 180 (2012).

[19] A. Ramos and E. Oset, The role of vector-baryon channels and resonances in the $\gamma p \to K^0 \Sigma^+$ and $\gamma n \to K^0 \Sigma^0$ reactions near the $K^* \Lambda$ threshold, Phys. Lett. B **727**, 287 (2013).

[20] K. A. Olive *et al.* (Particle Data Group Collaboration), Review of particle physics, Chin. Phys. C **38**, 090001 (2014).

[21] R. S. Longacre *et al.*, A measurement of $\pi^- p \to K^0(s) K^0(s) n$ at 22-GeV/c and a systematic study of the 2^{++} meson spectrum, Phys. Lett. B **177**, 223 (1986).

[22] D. Alde *et al.* (Serpukhov-Brussels-Los Alamos-Annecy (LAPP) Collaboration), Neutral mesons which decay into 4 pi0, Phys. Lett. B **198**, 286 (1987).

[23] C. Patrignani *et al.* (Particle Data Group Collaboration), Review of particle physics, Chin. Phys. C **40**, 100001 (2016).

[24] F. Giacosa, T. Gutsche, V. E. Lyubovitskij, and A. Faessler, Decays of tensor mesons and the tensor glueball in an effective field approach, Phys. Rev. D **72**, 114021 (2005).

[25] E. Klempt and A. Zaitsev, Glueballs, hybrids, multiquarks. Experimental facts versus QCD inspired concepts, Phys. Rep. **454**, 1 (2007).

[26] V. Crede and C. A. Meyer, The experimental status of glueballs, Prog. Part. Nucl. Phys. **63**, 74 (2009).

[27] R. Molina, D. Nicmorus, and E. Oset, The rho rho interaction in the hidden gauge formalism and the $f_0(1370)$ and $f_2(1270)$ resonances, Phys. Rev. D **78**, 114018 (2008).

[28] L. S. Geng and E. Oset, Vector meson-vector meson interaction in a hidden gauge unitary approach, Phys. Rev. D **79**, 074009 (2009).

[29] J. J. Xie, E. Oset, and L. S. Geng, Photoproduction of the $f_2'(1525)$, $a_2(1320)$, and $K_2^*(1430)$, Phys. Rev. C **93**, 025202 (2016).

[30] J. J. Xie and E. Oset, Photoproduction of the $f_2(1270)$ resonance, Eur. Phys. J. A **51**, 111 (2015).

[31] L. Roca, E. Oset, and J. Singh, Low lying axial-vector mesons as dynamically generated resonances, Phys. Rev. D **72**, 014002 (2005).

[32] D. V. Bugg, I. Scott, B. S. Zou, V. V. Anisovich, A. V. Sarantsev, T. H. Burnett, and S. Sutlief, Further amplitude analysis of $J/\psi \to \gamma(\pi^+\pi^-\pi^+\pi^-)$, Phys. Lett. B **353**, 378 (1995).

[33] F. Aceti, J. M. Dias, and E. Oset, $f_1(1285)$ decays into $a_0(980)\pi_0$, $f_0(980)\pi_0$ and isospin breaking, Eur. Phys. J. A **51**, 48 (2015).

[34] S. Godfrey and N. Isgur, Mesons in a relativized quark model with chromodynamics, Phys. Rev. D **32**, 189 (1985).

[35] D. R. Entem, F. Fernandez, and A. Valcarce, Chiral quark model of the N N system within a Lippmann-Schwinger resonating group method, Phys. Rev. C **62**, 034002 (2000).

[36] M. M. Giannini, E. Santopinto, and A. Vassallo, Hypercentral constituent quark model and isospin dependence, Eur. Phys. J. A **12**, 447 (2001).

[37] J. A. Oller and E. Oset, Chiral symmetry amplitudes in the S wave isoscalar and isovector channels and the σ, $f_0(980)$, $a_0(980)$ scalar mesons, Nucl. Phys. **A620**, 438 (1997); Erratum, Nucl. Phys. **652**, 407(E) (1999).

[38] N. Kaiser, $\pi\pi$ S wave phase shifts and nonperturbative chiral approach, Eur. Phys. J. A **3**, 307 (1998).

[39] J. Nieves and E. Ruiz Arriola, Bethe-Salpeter approach for unitarized chiral perturbation theory, Nucl. Phys. **A679**, 57 (2000).

[40] A. H. Fariborz, N. W. Park, J. Schechter, and M. Naeem Shahid, Gauged linear sigma model and pion-pion scattering, Phys. Rev. D **80**, 113001 (2009).

[41] M. Ablikim *et al.* (BESIII Collaboration), Partial wave analysis of $J/\psi \to \gamma\eta\eta$, Phys. Rev. D **87**, 092009 (2013); Erratum, Phys. Rev. D **87**, 119901(E) (2013).