$\eta-\eta'$ mixing and the derivative of the topological susceptibility at zero momentum transfer

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The couplings of the isosinglet axial-vector currents to the η and η' mesons are evaluated in a stable, model-independent way by use of polynomial kernels in dispersion integrals. The corrections to the Gell-Mann–Oakes–Renner relation in the isoscalar channel are deduced. The derivative of the topological susceptibility at the origin is calculated taking into account instantons and instanton screening.

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I. INTRODUCTION

The subject of $\eta-\eta'$ mixing has been a topic of discussion since $SU(3)$ flavor symmetry was proposed [1–6]. The gluon axial anomaly and the corresponding topological charges of the isoscalar mesons imply that the $SU(3)$ singlet axial-vector current is not conserved in the chiral limit. Initially, the octet-singlet mixing was described by an angle θ which was thought to be small and later given larger values [1]. It was later realized that the couplings of the isoscalar axial currents to the pseudoscalar mesons need not be dependent and that the single-angle description is inadequate. A number of theoretical approaches have been used to compute these couplings. Apart from chiral perturbation theory [7] and QCD sum rules [8], Shore [9] has used the generalized Gell-Mann–Oakes–Renner [10] relation to evaluate the couplings.

A related topic is the calculation of the topological susceptibility and its derivative at zero momentum transfer. The results obtained show a wide dispersion [11–13]. Such a dispersion in the results and instabilities in the parameters which enter the calculations is inherent in the Borel (Laplace) sum rules [14] used by the authors.

This method starts from a dispersion integral,

$$\text{Residue} = \frac{1}{\pi}\int_{\text{th}}^{\infty} dt e^{-t/M^2}\text{Im}P(t). \qquad (1.1)$$

The residue contains the physical quantity of interest and the integral runs from the physical threshold to infinity. The integral is then split into two parts,

$$\int_{\text{th}}^{\infty} dt e^{-t/M^2}\text{Im}P(t) = \int_{\text{th}}^{t_0} dt e^{-t/M^2}\text{Im}P(t) + \int_{t_0}^{\infty} dt e^{-t/M^2}\text{Im}P(t), \qquad (1.2)$$

where t_0 signals the onset of perturbative QCD. In the first integral on the rhs of the equation above $\text{Im}P(t)$ describes the unknown contribution of the resonances. The second integral takes into account the contribution of the QCD part of the amplitude when $P(t)$ is replaced by its QCD expression. M^2, the square of the Borel mass, is a parameter introduced in order to suppress the unknowns of the problem. If M^2 is small, the damping of the first unknown integral is good but the contribution of the unknown higher-order nonperturbative condensates increases rapidly. If M^2 increases, the contribution of the unknown condensates decreases but the damping in the resonances region worsens. An intermediate value of M^2 has to be chosen. Because M^2 is a nonphysical parameter the results should be independent of it in a relatively broad window; this is not the case in the problems at hand. The choice of the parameter t_0 which signals the onset of perturbative QCD is another source of uncertainty. In this work we shall use low-order polynomial kernels in order to suppress the contribution of the unknown continuum. The coefficients of these polynomials are determined by the masses of the isoscalar resonances and the method avoids the instabilities and arbitrariness which accompany the use of exponential kernels. Having determined the couplings of the isoscalar currents to the η and η' mesons, we shall turn to the study of the corrections to the Gell-Mann–Oakes–Renner relation [10] in the isoscalar channel and recover m_η. Finally, we shall evaluate $\chi'(0)$—the derivative of the topological susceptibility at zero momentum transfer—taking into account the effect of instantons and their possible screening which can be important, as has been emphasized by Forkel [15].

II. AXIAL CURRENTS AND THEIR COUPLING TO THE $\eta-\eta'$ MESONS

The isoscalar components of the octet of axial-vector currents couple to the physical η and η' mesons:

$$\begin{aligned}\langle 0|A_\mu^{(8)}|\eta(p)\rangle &= 2if_{8\eta}p_\mu,\\ \langle 0|A_\mu^{(0)}|\eta(p)\rangle &= 2if_{0\eta}p_\mu,\\ \langle 0|A_\mu^{(8)}|\eta'(p)\rangle &= 2if_{8\eta'}p_\mu,\\ \langle 0|A_\mu^{(0)}|\eta'(p)\rangle &= 2if_{0\eta'}p_\mu.\end{aligned} \qquad (2.1)$$

In the $SU(3)$ limit $f_{8\eta} = f_\pi = 92.4$ MeV, and in the two-mixing-angle description adopted here the coupling constants above are independent quantities. The axial-vector currents are written in terms of the quark fields:

$$A_\mu^{(8)} = \frac{1}{\sqrt{3}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d - 2\bar{s}\gamma_\mu\gamma_5 s),$$
$$A_\mu^0 = \sqrt{\frac{2}{3}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s). \quad (2.2)$$

In the limit $m_u = m_d = 0$, the divergences of these currents are

$$\partial_\mu A_\mu^8 = \frac{2}{\sqrt{3}}(-2im_s\bar{s}\gamma_5 s),$$
$$\partial_\mu A_\mu^0 = -\sqrt{\frac{2}{3}}(-2im_s\bar{s}\gamma_5 s) + \sqrt{\frac{2}{3}}Q, \quad (2.3)$$

where $Q = \frac{3\alpha_s}{4\pi}G\tilde{G}$ is the anomaly with $G\tilde{G} = G_{\mu\nu}\tilde{G}^{r\gamma}$, with $G_{\mu\nu}$ being the gluon field strength tensor and $\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$ its dual. Consider now the correlator

$$\Pi_{\mu\nu}^{ij} = \int dx e^{iqx}\langle 0|TA_\mu^{(i)}(x)A_\nu^{(j)}(0)|0\rangle, \quad (2.4)$$

where $i,\ j = 0,\ 8$

It can be decomposed,

$$\Pi_{\mu\nu}(q^2) = (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi(q^2)^{(1)} + q_\mu q_\nu \Pi(q^2)^{(0)}, \quad (2.5)$$

and we let

$$\Pi(t = q^2) = \Pi^{(1)}(t) + \Pi^{(0)}(t). \quad (2.6)$$

Start with $\Pi^{88}(t)$. At low energies it has two poles,

$$\Pi^{88}(t) = \frac{-4f_{8\eta}^2}{t - m_{\eta^2}} - \frac{4f_{8\eta'}^2}{t - m_{\eta'^2}} + \ldots, \quad (2.7)$$

and a cut on the real positive t axis running from the continuum threshold to ∞.

The amplitude also possesses a QCD expansion, valid in the complex t plane for $|t|$ large and not too close to the physical cut. The aim of the calculation is to relate the residues of the poles to the QCD parameters,

$$\Pi_{\rm QCD}^{88}(t) = \Pi_{\rm pert}^{88} + \frac{C_1^{88}}{t} + \frac{C_2^{88}}{t^2} + \ldots \quad (2.8)$$

The perturbative part is known to five loops in the chiral limit [16],

$$\frac{1}{\pi}{\rm Im}\Pi_{\rm pert}^{88} = 2\frac{1}{4\pi^2}\Bigg\{1 + a_s + a_s^2(F_3 + \beta_{\frac{1}{2}}L_\mu) + a_s^3\left[F_4 + \left(\beta_1 F_3 + \frac{\beta_2}{2}\right)L_\mu + \frac{\beta_1^2}{4}L_\mu^2\right] + a_s^4\Bigg[k_3 - \frac{\pi^2}{4}\beta_1^2 F_3 - \frac{5}{24}\pi^2\beta_1\beta_2 + \left(\frac{3}{2}\beta_1 F_4 + \beta_2 F_3 + \frac{\beta_3}{2}\right)L_\mu + \frac{\beta_1}{2}\left(\frac{3}{2}\beta_1\right)F_3 + \frac{5}{4}\beta_2)L_\mu^3 + \frac{\beta_1^3}{8}L_\mu^3\Bigg]\Bigg\}, \quad (2.9)$$

where $a_s = \frac{\alpha_s(\mu^2)}{\pi}$, $L_\mu = \ln(\frac{-t}{\mu^2})$, $\beta_1 = -\frac{1}{2}(11 - \frac{2}{3}n_f)$, $\beta_2 = -\frac{1}{8}(102 - \frac{38}{3}n_f)$, $\beta_3 = -\frac{1}{32}(\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2)$, $F_3 = 1.9857 - .1153n_f$, $F_4 = 18.2427 - \frac{\pi^2}{3}(\frac{\beta_1}{2})^2 - 4.2158n_f + .0862n_f^2$, and $k_3 = 49.076$.

The strong coupling constant is likewise known to five-loop order [17] in terms of $\frac{\alpha_s^{(1)}}{\pi} \equiv \frac{-2}{\beta_1 L}$ with $L = \ln(\frac{-t}{\Lambda^2})$, where Λ^2 defines the standard $\overline{\rm MS}$ scale to be used here.

$$C_1^{88} = \frac{2}{\pi^2}(1 + 2a_s)m_s^2 \quad (2.10)$$

is a correction to the perturbative part proportional to m_s^2 [18] and

$$C_2^{88} = \frac{1}{6}\left(1 - \frac{11}{18}a_s\right)\langle a_s G\tilde{G}\rangle + \frac{8}{3}\left(1 - \frac{7}{3}a_s - \frac{75}{6}a_s^2\right)\langle m_s\bar{s}s\rangle,$$
$$C_3^{88} = \frac{-448}{\pi^2}a_s\langle\bar{u}u\rangle^2. \quad (2.11)$$

Consider next the contour C shown in Fig. 1 consisting of two straight lines parallel to the real axis, located just above and just below the cut, and running from the

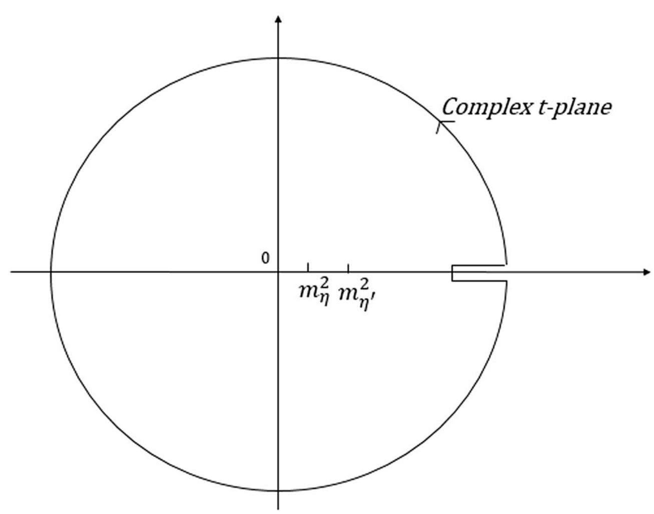

FIG. 1. The contour of integration C.

continuum threshold to a large value R and the circle of radius R.

And consider the integral

$$\int_c dt f(t)\Pi(t),$$

where $f(t)$ is an entire function. On the circle, $\Pi(t)$ can be replaced by $\Pi_{\rm QCD}(t)$ to a good approximation.

Applying Cauchy's theorem leads to

$$\begin{aligned}4f_{8\eta}^2 f(m_\eta^2)+4f_{8\eta'}^2 f(m_{\eta'}^2) &= -\frac{1}{\pi}\int_{\rm th}^R dt f(t){\rm Im}\Pi(t)\\ &\quad -\frac{1}{2Mi}\oint dt f(t)\Pi_{\rm pert}(t)\\ &\quad -\frac{1}{2Mi}\oint dt f(t)\Pi_{\rm rep}(t).\end{aligned} \tag{2.12}$$

The first term on the rhs of the equation above—which represents the contribution of the physical continuum—constitutes the main uncertainty of the calculation. The choice of the so-far arbitrary function $f(t)$ aims at reducing this term as much as possible so that it may be neglected. All that is known about the continuum is that it is dominated by the pseudoscalar excitations $\eta(1295)$ and $\eta(1440)$, as well as the axial-vector isoscalars $f_1(1285)$ and $f_1(1420)$ with practically the same masses.

We shall choose for $f(t)$ a simple polynomial,

$$f(t)=p(t)=1-a_1t-a_2t^2,$$

where the coefficients a_1 and a_2 annihilate $p(t)$ at the masses of the resonances, i.e.,

$$p(t)=1\text{–}1.090\ {\rm GeV}^{-2}t+.294\ {\rm GeV}^{-4}t^2. \tag{2.13}$$

With this choice the integrand is reduced to only a few percent of its initial value on the interval $1.5\ {\rm GeV}^2\leq t\leq 2.5\ {\rm GeV}^2$ and the contribution of the continuum is thus practically annihilated.

$\Pi_{\rm pert}(t)$ has a different analytical structure than the physical amplitude: it has a cut on the real t axis which starts at the origin so that $\frac{1}{2\pi i}\oint_{C'} dt f(t)\Pi_{\rm pert}(t)=0$, where C' is the contour shown in Fig. 2.

It then follows that

$$\frac{1}{2Mi}\oint dt f(t)\Pi_{\rm pert}(t)=-\frac{1}{\pi}\int_0^R dt f(t){\rm Im}\Pi_{\rm pert}(t). \tag{2.14}$$

Also,

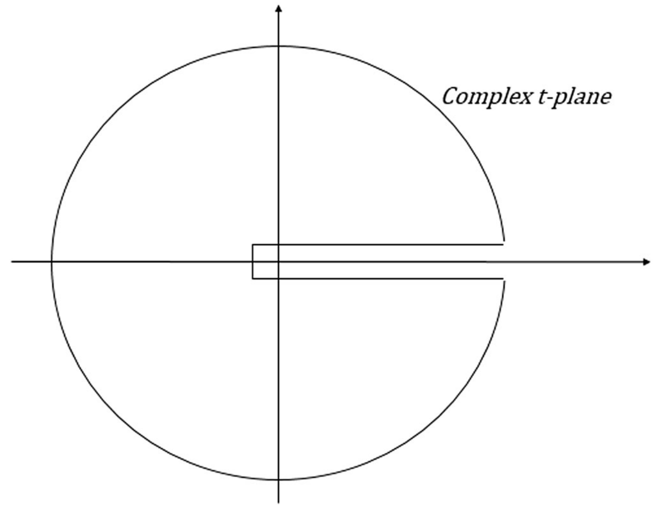

FIG. 2. The contour of integration C' used to transform the integral $\Pi_{\rm pert}(t)$ over the circle into an integral over the real axis.

$$\begin{aligned}&\frac{1}{2Mi}\oint dt f(t)\Pi_{\rm rep}(t)\\ &\quad=-\frac{1}{2Mi}\oint dt(1-a_1t-a_2t^2)\left(\frac{C_1^{88}}{t}+\frac{C_2^{88}}{t^2}+\frac{C_3^{88}}{t^3}+\cdots\right)\\ &\quad=C_1^{88}-a_1C_2^{88}-a_2C_3^{88}.\end{aligned} \tag{2.15}$$

The second term on the rhs of Eq. (2.12) equals the contribution of the integral over the circle of $\Pi_{\rm pert}(t)$ and provides the main contribution. The last two terms are contributed by the corresponding ones in Eq. (2.8). Thus,

$$\begin{aligned}&4f_{8\eta}^2p(m_\eta^2)+4f_{8\eta'}^2p(m_{\eta'}^2)\\ &\quad=\frac{1}{\pi}\int_0^R dt\,p(t){\rm Im}\Pi_{\rm pert}(t)-C_1^{88}+a_1C_2^{88}+a_2C_3^{88}.\end{aligned} \tag{2.16}$$

The choice of R is determined by stability considerations. It should not be too small as this would invalidate the operator product expansion on the circle, nor should it be too large because $p(t)$ would start enhancing the contribution of the continuum instead of suppressing it. We seek an intermediate range of R for which the integral in Eq. (2.16) is stable. This turns out to be the case for $1.5\ {\rm GeV}^2\leq R\leq 2.5\ {\rm GeV}^2$. The integral provides the main contribution to the rhs of Eq. (2.16).

A similar treatment of the amplitude $\Pi^{00}(t)$ leads to

$$\begin{aligned}&4f_{0\eta}^2p(m_\eta^2)+4f_{0\eta'}^2p(m_{\eta'}^2)\\ &\quad=\frac{1}{\pi}\int_0^R dt p(t){\rm Im}\Pi_{\rm pert}^{00}(t)-C_1^{00}+a_1C_2^{00}\\ &\qquad+a_2C_3^{00},\end{aligned} \tag{2.17}$$

where $\Pi_{\rm pert}^{00}=\Pi_{\rm pert}^{88}$, and C_1^{00} and C_2^{00} are the nonperturbative coefficients of the QCD expansion,

$$\Pi^{00}_{\rm QCD}(t) = \Pi^{00}_{\rm pert}(t) + \frac{C^{00}_1}{t} + \frac{C^{00}_2}{t^2} + \cdots,$$
$$C^{00}_1 = \frac{1}{\pi^2}(1+2a_s)m_s^2,$$
$$C^{00}_2 = \frac{1}{6}\left(1 - \frac{11}{18}a_s\right)\langle a_s GG\rangle + \frac{4}{3}\left(1 - \frac{7}{3}a_s - \frac{75}{6}a_s^2\right)\langle m_s \bar{s}s\rangle,$$
$$C^{00}_3 = -\frac{448}{81}\pi^2 a_s \langle \bar{u}u\rangle^2. \tag{2.18}$$

Finally, we turn to the mixed amplitude $\Pi^{08}(t)$, with the result

$$4f_{8\eta}f_{0\eta}p(m_\eta^2) + 4f_{8\eta'}f_{0\eta'}p(m_{\eta'}^2) = -C^{08}_1 + a_1 C^{08}_2 + a_2 C^{08}_3, \tag{2.19}$$

with

$$C^{08}_1 = \frac{-\sqrt{2}}{\pi^2}(1+2a_s)m_s^2,$$
$$C^{08}_2 = \frac{-8\sqrt{2}}{3}\left(1 - \frac{7}{3}a_s - \frac{75}{6}a_s^2\right)\langle m_s \bar{s}s\rangle,$$
$$C^{08}_3 \simeq 0. \tag{2.20}$$

Equation (2.20) is distinguished from Eqs. (2.16) and (2.17) in that the dominant perturbative contribution is now absent and the smallness of its rhs will result in the smallness of the $\eta - \eta'$ mixing, i.e., of the couplings $f_{0\eta}$ and $f_{8\eta'}$.

Equations (2.16), (2.17), and (2.19) are however insufficient to determine all four couplings. An additional equation is obtained by considering the integral $\frac{1}{2\pi i}\int_c dt\, t p(t)\Pi^{08}(t)$.

The fast convergence of the amplitude—due to the absence of the perturbative part in the asymptotic expansion—guarantees the reliability of the result. This yields

$$4f_{8\eta}f_{0\eta}p(m_\eta^2)m_\eta^2 + 4f_{8\eta'}f_{0\eta'}p(m_{\eta'}^2)m_{\eta'}^2 = -C^{08}_2 + a_1 C^{08}_3. \tag{2.21}$$

The numbers used for the condensates are
$m_s = (.10 \pm .01)$ GeV,
$-\langle \bar{s}s\rangle = (.012 \pm .002)$ GeV^3,
$\langle a_s G\tilde{G}\rangle = .013$ GeV^4,
and the value of the integral in Eqs. (2.16) and (2.17) at the stability values of R is $\frac{1}{\pi}\int_0^R dt p(t) {\rm Im}\Pi_{\rm pert}(t) = .034$ GeV^2, as shown in Fig. 3.

These finally yield for the couplings

$$f_{8\eta} = .104 \text{ GeV}, \quad f_{8\eta'} = -.046 \text{ GeV},$$
$$f_{0\eta} = .042 \text{ GeV}, \quad f_{0\eta'} = .160 \text{ GeV}, \tag{2.22}$$

which correspond to the mixing angles

$$\theta_8 = \tan^{-1}\left(\frac{f_{8\eta'}}{f_{8\eta}}\right) = -24° \quad \text{and}$$
$$\theta_0 = \tan^{-1}\left(\frac{-f_{0\eta}}{f_{0\eta'}}\right) = -14.7°. \tag{2.23}$$

The values obtained above can be used in the calculation of the corrections to the Gell-Mann–Oakes–Renner relation [10] in the isoscalar channel. A Ward identity introduces a subtraction which improves the convergence of the dispersion relation and therefore its reliability.

Start with the correlator

$$T^{88}(t) = \int idx e^{iqx}\langle 0|TD^{(8)}(x)D^{(8)}(0)|0\rangle, \tag{2.24}$$

where $D^{(8)} = \partial_\mu A^{(8)}_\mu$,

$$T^{88}(t) = \frac{-4f_{8\eta}^2 m_\eta^4}{t - m_\eta^4} - \frac{4f_{8\eta'}^2 m_{\eta'}^4}{t - m_{\eta'}^4} + \ldots, \tag{2.25}$$

which satisfies the Ward identity

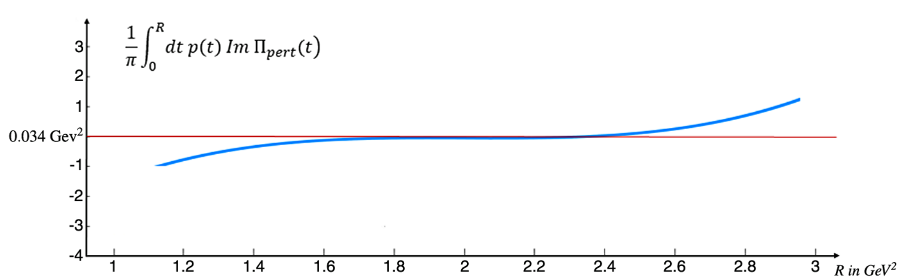

FIG. 3. The variation of $\frac{1}{\pi}\int_0^R dt f(t){\rm Im}\Pi(t)$ as a function of R.

$$T^{88}(0) = -\frac{16}{3}\langle m_s\bar{s}s\rangle.$$

Introducing a subtraction consists in considering the integral $\frac{1}{2\pi}\int_c \frac{dt}{t} p(t)\Pi^{88}(t)$.

This gives

$$f_{8\eta}^2 m_\eta^2 p(m_\eta^2) + f_{8\eta'}^2 m_{\eta'}^2 p(m_{\eta'}^2) = -\frac{4}{3}\langle m_s\bar{s}s\rangle + m_s^2\left\{\frac{1}{2\pi^2}\left(1+\frac{17}{3}a_s\right)\int_0^R dt p(t) + \frac{4}{3}a_1\left(2\langle m_s\bar{s}s\rangle - \frac{1}{4}\langle a_s G\tilde{G}\rangle\right)\right\}. \quad (2.26)$$

Numerically,

$$f_{8\eta}^2 m_\eta^2 p(m_\eta^2) = .002 \text{ GeV}^4$$

which results in recovering m_η,

$$m_\eta = (500 \pm 30) \text{ MeV}. \quad (2.27)$$

The uncertainly is estimated from the one in the parameters.

III. THE TOPOLOGICAL SUSCEPTIBILITY AND ITS DERIVATIVE AT ZERO MOMENTUM TRANSFER

The topological susceptibility

$$\chi(t) = i\int dx e^{iqx}\langle 0|TQ(x)Q(0)|0\rangle \quad (3.1)$$

has poles at the pseudoscalar mesons

$$\chi(t) = -\frac{\langle 0|Q|\pi\rangle^2}{t-m_\pi^2} - \frac{\langle 0|Q|\eta\rangle^2}{t-m_\eta^2} - \frac{\langle 0|Q|\eta'\rangle^2}{t-m_\eta^2} + \ldots \quad (3.2)$$

Consider again the integral $\frac{1}{2Mi}\int_c \frac{dt}{t} p(t)\chi(t)$ with the same polynomial $p(t)$ introduced in order to suppress the contribution of the physical continuum. It gives

$$\chi(0) = \frac{\langle 0|Q|\pi^0\rangle^2}{m_\pi^2} + \frac{\langle 0|Q|\eta\rangle^2}{m_\eta^2} p(m_\eta^2) + \frac{\langle 0|Q|\eta'\rangle^2}{m_{\eta'}^2} p(m_{\eta'}^2) + \frac{1}{2\pi i}\oint \frac{dt}{t} p(t)\chi^{\mathrm{QCD}}(t), \quad (3.3)$$

and the derivative is

$$\chi'(0) - a_1\chi(0) = \frac{\langle 0|Q|\pi^0\rangle^2}{m_\pi^4} + \frac{\langle 0|Q|\eta\rangle^2}{m_\eta^4} p(m_\eta^2) + \frac{\langle 0|Q|\eta'\rangle^2}{m_{\eta'}^4} + \frac{1}{2\pi i}\oint \frac{dt}{t^2} p(t)\chi^{\mathrm{QCD}}(t). \quad (3.4)$$

The coupling $\langle 0|Q|\pi^0\rangle$ was given in Ref. [19],

$$\langle 0|Q|\pi^0\rangle = \frac{i}{4} f_\pi m_\pi^2\left(\frac{m_d - m_u}{m_d + m_u}\right), \quad (3.5)$$

and the couplings $\langle 0|Q|\eta\rangle$ and $\langle 0|Q|\eta'\rangle$ are obtained by sandwiching Eq. (2.3) between the vacuum and the η, η' states,

$$\langle 0|Q|\eta\rangle = \sqrt{\frac{1}{12}}(f_{8\eta} - \sqrt{2}f_{0\eta})m_\eta^2,$$
$$\langle 0|Q|\eta'\rangle = \sqrt{\frac{1}{12}}(f_{8\eta'} - \sqrt{2}f_{0\eta'})m_{\eta'}^2. \quad (3.6)$$

The QCD expression is [11–13]

$$\chi^{\mathrm{QCD}}(t) = C_{21}t^2\ln -t + C_{22}t^2(\ln -t)^2 + C_{01}\ln -t + C_{00} + \frac{C_{-1}}{t} + \frac{C_{-2}}{t^2} + I(t), \quad (3.7)$$

where $I(t)$ stands for the instanton contribution and

$$C_{21} = -\left(\frac{\alpha_s}{8\pi}\right)\frac{2}{\pi^2}\left(1+\frac{83}{4}\frac{\alpha_s}{\pi}\right),$$
$$C_{22} = \frac{9}{4}\left(\frac{\alpha_s}{\pi}\right)C_{21},$$
$$C_{01} = \frac{9}{64}\left(\frac{\alpha_s}{\pi}\right)^2\left\langle\frac{\alpha_s}{\pi}G\tilde{G}\right\rangle,$$
$$C_{-1} = \frac{1}{8}\left(\frac{\alpha_s}{\pi}\right)\left\langle 9_s\frac{\alpha_s}{\pi}G\tilde{G}\right\rangle,$$
$$C_{-2} = -\frac{15}{128}\pi^2\left(\frac{\alpha_s}{\pi}\right)\left\langle\frac{\alpha_s}{\pi}G\tilde{G}\right\rangle^2,$$
$$C_{00} = -\frac{1}{16}\left(\frac{\alpha_s}{\pi}\right)\left\langle\frac{\alpha_s}{\pi}G\tilde{G}\right\rangle. \quad (3.8)$$

When calculations are carried out and numbers are inserted, Eq. (3.3) yields

$$\chi(0) = .94\times 10^{-3}\text{ GeV}^4 + \delta_1, \quad (3.9)$$

where

$$\delta_1 = \frac{1}{2\pi i}\oint \frac{dt}{t} p(t)I(t) \quad (3.10)$$

denotes the instanton contribution. The derivative is

$$\chi'(0) = a_1\chi(0) + 1.30\times 10^{-3}\text{ GeV}^2 + \delta_2, \quad (3.11)$$

where

$$\delta_2 = \frac{1}{2\pi i}\oint \frac{dt}{t^2} p(t)I(t). \qquad (3.12)$$

The instanton term $I(t)$ is model dependent; the form used by Ioffe and Samsonov [11] is

$$I(t) = t^2 \int dp n(\rho)\rho^4 K_2^2(Q\rho),$$

where

$$n(\rho) = n_0\delta(\rho - \rho_c), \qquad \rho_c = 1.5\ \text{GeV}^{-1}, \qquad (3.13)$$

and $K_2(Q\rho)$ is the MacDonald function. It should be noted however that important screening corrections (as has been emphasized by Forkel [15]) can considerably modify Eq. (3.13).

I shall take the screening corrections into account simply by considering the overall factor as a free parameter to be determined by the calculation. Thus, let

$$I(t) = ct^2 K_2^2(\rho_c\sqrt{-t}). \qquad (3.14)$$

Asymptotic forms of $K_2(x)$ were given by Dwight [20]: these are used to compute integrals of the form $I_n = \frac{1}{2\pi i}\int dt t^n K_2^2(t)$, which give the ratio

$$\frac{\delta_2}{\delta_1} = \frac{I_0 - a_1 I_1 - a_2 I_2}{I_1 - a_1 I_2 - a_2 I_3} = -.53\ \text{GeV}^{-2} \qquad (3.15)$$

and yield

$$\chi'(0) = 1.75\times 10^{-3}\ \text{GeV}^{+2} + .55\ \text{GeV}^{-2}\chi(0). \qquad (3.16)$$

$\chi(0)$ has been computed on the lattice [21] with the result $\chi(0) = -1.33\times 10^{-3}\ \text{GeV}^4$. Another value is the one given by the Witten-Veneziano [22,23] formula obtained in the large-N_c limit,

$$\chi(0) = -\frac{f_\pi^2}{2n_f}(m_{\eta'}^2 + m_\eta^2 - 2m_k^2)$$
$$= -1.05\times 10^{-3}\ \text{GeV}^4, \qquad (3.17)$$

yielding finally

$$\chi'(0) = (1.1 \pm .1)\times 10^{-3}\ \text{GeV}^2. \qquad (3.18)$$

IV. RESULTS AND CONCLUSION

The subject of octet-singlet mixing of the pseudoscalar mesons has been studied and the couplings of the η and η' mesons to the axial currents A_μ^0 and A_μ^8 were evaluated, yielding the mixing angles $\theta_8 = -24°$ and $\theta_0 = -14.7°$. The corrected Gell-Mann–Oakes–Renner relation reproduces the value of m_η. The topological susceptibility at the origin has also been computed with the effects of instantons and instanton screening taken into account, resulting in $\chi'(0) = (1.1 \pm .1)\times 10^{-3}\ \text{GeV}^2$.

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