PHYSICAL REVIEW D **95,** 031901(R) (2017)

Quantum phase transition in many-flavor supersymmetric QED$_3$

Jorge G. Russo[1,2] and Miguel Tierz[3]

[1]*Institució Catalana de Recerca i Estudis Avançats (ICREA), Pg. Lluis Compayns 23–08010 Barcelona, Spain*

[2]*Departament de Física Cuántica i Astrofísica and ICC, Universitat de Barcelona—Martí Franquès, 1, 08028 Barcelona, Spain*

[3]*Departamento de Matemática, Grupo de Física Matemática, Faculdade de Ciências, Universidade de Lisboa, Campo Grande, Edifício C6, 1749-016 Lisboa, Portugal*

(Received 1 November 2016; published 13 February 2017)

We study $\mathcal{N}=4$ supersymmetric QED in three dimensions, on a 3-sphere, with $2N$ massive hypermultiplets and a Fayet-Iliopoulos parameter. We identify the exact partition function of the theory with a conical (Mehler) function. This implies a number of analytical formulas, including a recurrence relation and a second-order differential equation, associated with an integrable system. In the large N limit, the theory undergoes a second-order phase transition on a critical line in the parameter space. We discuss the critical behavior and compute the two-point correlation function of a gauge invariant mass operator, which is shown to diverge as one approaches criticality from the subcritical phase. Finally, we comment on the asymptotic $1/N$ expansion and on mirror symmetry.

DOI: 10.1103/PhysRevD.95.031901

The study of quantum electrodynamics in three dimensions (QED$_3$) has been a subject of interest since the early 1980s, due to its connection to finite-temperature QCD via dimensional reduction and the fact that, as QCD in four dimensions, QED$_3$ also exhibits spontaneous chiral symmetry breaking and confinement [1].

The theory has experienced a remarkably renewed interest in recent years due to, in great part, the relevance of relativistic field theories of particles moving in two dimensions in the description of the pseudogap phase of cuprates [2], the spin-liquid phase of quantum antiferromagnets [3] and the low-energy electronic excitations of graphene [4]. This reinvigorated relevance of the dynamics of a $U(1)$ gauge field and N_f fermions in three dimensions extends to the case with supersymmetry, where the theory appears for example in descriptions of the physics of half-filled Landau levels in terms of Dirac fermions [5], in 3D Bosonization [6] and nonperturbative descriptions of renormalization group flows [7]. Supersymmetric $U(1)$ theories in three dimensions can be related to the study of quantum phase transitions in quantum antiferromagnets and provide examples of quantum phase transitions beyond the Landau-Ginzburg paradigm [8]. In these discussions, the existence of mirror symmetry in the supersymmetric gauge theory is important [6,8] and it is precisely the recent, more detailed, analysis of dualities that has bolstered great interest in QED$_3$. In particular for example, it has been recently shown that the fermionic vortex of QED$_3$ is a free Dirac fermion [9,10]. Around this result lurks a number of connections between topological insulators, spin liquids and quantum Hall physics, making QED$_3$ a subject of considerable physical interest.

At the same time, the development of tools for studying supersymmetric gauge theories on curved manifolds, in particular localization [11,12], has increased the means at our disposal to obtain exact analytical results. Examples of works that use localization and the F-theorem in the study of QED$_3$ (with or without supersymmetry) includes Refs. [13–15]. In this paper, we shall study supersymmetric $U(1)$ gauge theory on $\mathbb{S}^3$, but, instead of massless matter as in Ref. [14], or the cases studied in Refs. [7,15], we include massive $\mathcal{N}=4$ hypermultiplets and a Fayet-Iliopoulos (FI) term. As we will see, this leads to a dramatic change in the dynamics of the theory. For the sake of simplicity, we consider the case of N hypermultiplets with mass m and N hypermultiplets with mass $-m$. More general mass configurations will be discussed at the end. The total number of flavors N_f is therefore $2N$. In addition, as mentioned, there is a FI term, and it is actually the interplay between the difference of masses and the FI parameter η, the one which is responsible for producing novel behavior. In particular, it is responsible for the emergence of a second-order quantum phase transition in the large N limit.

We thus consider an $\mathcal{N}=4$ supersymmetric $U(1)$ theory consisting of $2N$ massive $\mathcal{N}=4$ (flavor) hypermultiplets (N of mass m and N of mass $-m$), coupled to an $\mathcal{N}=4$ vector multiplet. Localization readily leads to an integral representation for the partition function [16]

$$
\begin{aligned}
Z_{\mathrm{QED}_3} &= \int_{-\infty}^{\infty} dx \frac{e^{i\eta x}}{[2\cosh(\frac{x+m}{2})2\cosh(\frac{x-m}{2})]^N} \\
&= 2^{-N}\int_{-\infty}^{\infty} dx \frac{e^{i\eta x}}{[\cosh x+\cosh m]^N}. \qquad (1)
\end{aligned}
$$

In what follows, we drop the 2^N factor, which is inessential to our discussion, and we have set the radius of $\mathbb{S}^3$ to $r=1/(2\pi)$. In the case when the hypermultiplets have masses m_1 and m_2, the parameter m is $m = (m_1 - m_2)/2$. Thus, in the discussion below, increasing m corresponds to separating the two mass scales. It is important to note that the parameters m, while they correspond to mass deformations of the Lagrangian, represent curved-space analogs of the more familiar flat space mass parameters, as here they are measured in units of the radius of the $\mathbb{S}^3$ (the partition function only depends on the combination mr).

Notice that, by writing the integral representation (1) in the latter form, one can immediately identify it as the integral representation of a conical (Mehler) function [17], which is an associated Legendre function with a complex index. We find

$$Z_{\mathrm{QED}_3} = \sqrt{2\pi}\frac{\Gamma(N+i\eta)\Gamma(N-i\eta)}{\Gamma(N)(\sinh(m))^{N-\frac{1}{2}}}P^{\frac{1}{2}-N}_{-\frac{1}{2}+i\eta}(\cosh(m)). \quad (2)$$

It can also be conveniently represented in terms of a hypergeometric function,

$$Z_{\mathrm{QED}_3} = \frac{\sqrt{2\pi}\Gamma(N+i\eta)\Gamma(N-i\eta)}{\Gamma(N)\Gamma(N+\frac{1}{2})(1+z)^{N-\frac{1}{2}}} \times {}_2F_1\left(\frac{1}{2}-i\eta,\frac{1}{2}+i\eta,N+\frac{1}{2};\frac{1}{2}(1-z)\right), \quad (3)$$

with $z \equiv \cosh(m)$. In specific cases, the expression simplifies. In particular, for two and four flavors, we find

$$Z^{N=1}_{\mathrm{QED}_3} = \frac{2\pi\sin(m\eta)}{\sinh(m)\sinh(\pi\eta)}, \quad (4)$$

$$Z^{N=2}_{\mathrm{QED}_3} = \frac{2\pi(\cosh m\sin(m\eta) - \eta\sinh m\cos(m\eta))}{\sinh^3(m)\sinh(\pi\eta)}. \quad (5)$$

These can also be obtained from residue integration [18]. The expression (4) already exhibits some of the general properties of the conical function and, hence, of the partition function, such as the oscillatory behavior, which depends on both m and η. It is well known that supersymmetric QED_3 with two flavors is self-dual [19]. Notice that indeed we find that (4) is invariant under the exchange $m \leftrightarrow \pi\eta$ implied by the duality transformation.

The oscillatory behavior is related to the fact that the function has an infinite number of zeros, all of them real, precisely in the physical region $m \geq 0$ [17] and the function is monotonic until the appearance of the first zero of the function, after which the behavior is oscillatory. This transition between a monotonic and an oscillatory region when the first zero appears, in the large N limit, becomes a phase transition, which we characterize below by computing the saddle points of (1).

The identification with a conical function and the ensuing hypergeometric representation has interesting consequences. To begin with, from a standard recurrence relation for the Legendre functions, we obtain that the partition function also satisfies a recurrence relation:

$$(2N-1)\cosh(m)Z_N = \frac{(N-1)^2+\eta^2}{N-1}Z_{N-1} + N\sinh^2(m)Z_{N+1}. \quad (6)$$

For short, here we defined $Z_N \equiv Z_{\mathrm{QED}_3}(m,\eta,N)$. By this formula, we can easily generate any Z_N from the above expressions (4) and (5) for $N=1$ and $N=2$. In addition, the representation (3) can be used to study a small mass expansion, since the radius of convergence of a Gauss hypergeometric function is $|x| < 1$ in the variable. In the massless limit, the hypergeometric becomes 1, and Z_{QED_3} is given by the first line in (3). In order to find the large mass behavior, we use an Euler hypergeometric transformation and write the partition in the form

$$Z_{\mathrm{QED}_3} = \frac{\Gamma(-i\eta)\Gamma(N+i\eta)}{2^{i\eta}\Gamma(N)(\cosh(m)+1)^{N+i\eta}} \times {}_2F_1\left(\frac{1}{2}+i\eta,N+i\eta,1+2i\eta;\mathrm{sech}^2\frac{m}{2}\right) + \mathrm{c.c.} \quad (7)$$

Using (7), we then obtain

$$Z^{mr\gg 1}_{\mathrm{QED}_3} = \frac{2^N\Gamma(-i\eta)\Gamma(N+i\eta)}{\Gamma(N)}e^{-mr(N+i\eta)} + \mathrm{c.c.} \quad (8)$$

Here, we have restored the $\mathbb{S}^3$ radius dependence to exhibit the fact that this regime can also be interpreted as a decompactification limit. Another nontrivial consequence of the relation of the partition function to a hypergeometric function is the fact that then Z_N satisfies a second-order differential equation:

$$\frac{d^2Z_N}{dm^2} + 2N\coth(m)\frac{dZ_N}{dm} + (\eta^2+N^2)Z_N = 0. \quad (9)$$

By defining $\tilde{Z}_{\mathrm{QED}_3} = (\sinh(m))^N Z_{\mathrm{QED}_3}$, this equation can be written as a Schrödinger equation with a hyperbolic Pöschl-Teller potential [20], which is a well-known solvable one-dimensional quantum mechanical problem,

$$\frac{d^2\tilde{Z}_{\mathrm{QED}_3}}{dm^2} + \left(\eta^2 + \frac{N(1-N)}{\sinh^2 m}\right)\tilde{Z}_{\mathrm{QED}_3} = 0.$$

The present theory has a large N limit, with fixed $\lambda \equiv \eta/N$. In this limit, the partition function (1) can be computed by the saddle-point method. The integrand in (1) can be written as $e^{-NS(\lambda)}$ where the *action* S is

$$S(\lambda,x,z) = -i\lambda x + \log(\cosh x + \cosh m).$$

The saddle-point equation is then

$$-i\lambda + \frac{\sinh x}{\cosh x + \cosh m} = 0,$$

which has as solutions

$$x_{1,2} = \log\left(\frac{-\lambda\cosh m \pm i\Delta}{i+\lambda}\right) + 2\pi i n, \qquad (10)$$

where $n \in \mathbb{Z}$ and $\Delta \equiv \sqrt{1-\lambda^2\sinh^2 m}$. In what follows, we show that the theory undergoes a large N phase transition at $\lambda_c \equiv 1/\sinh m$, or, more generally, at the critical line $\lambda\sinh(m) = 1$ in the (λ, m) space, where $\Delta = 0$.

Subcritical phase ($\lambda\sinh(m) < 1$).—In this case, all saddle points lie on the imaginary axis. We find that the saddle point x_1 with $n = 0$ is the relevant one, and, to leading order for large N, the partition function becomes

$$Z_{\mathrm{QED}_3} \approx \frac{\sqrt{2\pi}}{\sqrt{NS''(x_1)}}\exp(-NS(x_1)), \qquad (11)$$

where $S''(z,x) = (z\cosh(x)+1)/(\cosh(x)+z)^2$. We numerically checked that, for large N, this formula reproduces the analytic expression (3) with great accuracy. In the subcritical phase, the large N free energy $F = -\log Z_{\mathrm{QED}_3}$ is given by $NS(x_1)$. We obtain

$$F_{\mathrm{sub}} = -\frac{i\lambda N}{2}\log\left(\frac{(-z\lambda+i\Delta)(i-\lambda)}{(z\lambda+i\Delta)(i+\lambda)}\right) + N\log\left(\frac{z+\Delta}{1+\lambda^2}\right).$$

Supercritical phase ($\lambda\sinh(m) > 1$).—The two saddle points move to the complex plane, with $x_2 = -x_1^*$. The action is complex, with $\mathrm{Re}(S(x_1)) = \mathrm{Re}(S(x_2))$, $\mathrm{Im}(S(x_1)) = -\mathrm{Im}(S(x_2))$. Therefore, both saddle points contribute with equal weights and need to be taken into account. The partition function $Z = Z_{\mathrm{QED}_3}$ is now

$$Z \approx \sqrt{\frac{2\pi}{N}}e^{-N\mathrm{Re}(S(x_1))}\left(\frac{e^{-iN\mathrm{Im}(S(x_1))}}{\sqrt{S''(x_1)}} + \mathrm{c.c.}\right).$$

For $N \gg 1$, this expression agrees with the exact analytic expression (3). For large mass, it is also in precise agreement with the large mass formula (8). Taking the log to get the free energy, we see that at large N, the leading contribution proportional to N is given by the real part of the action $F \approx N\mathrm{Re}(S(x_1))$, giving

$$F_{\mathrm{super}} = \frac{N}{4}\left(2\log\frac{z^2-1}{1+\lambda^2} - 2i\lambda\log\frac{\lambda-i}{\lambda+i}\right).$$

The free energy and its first derivative are continuous at the critical point, while the second derivative gives

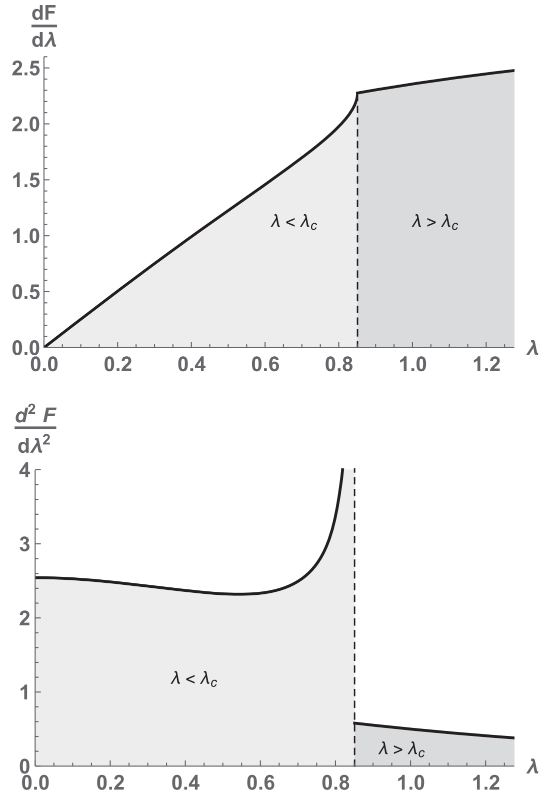

FIG. 1. (a) Behavior of $dF/d\lambda$. (b) Discontinuity of $d^2F/d\lambda^2$ at the transition point ($m = 1$).

$$\frac{d^2F}{d\lambda^2} = \frac{N}{1+\lambda^2}\left(1 + \frac{\cosh(m)}{\sqrt{1-\lambda^2\sinh^2(m)}}\right), \qquad \lambda < \lambda_c,$$

$$\frac{d^2F}{d\lambda^2} = \frac{N}{1+\lambda^2}, \qquad \lambda \geq \lambda_c.$$

Thus, $d^2F/d\lambda^2$ is discontinuous, implying a second-order phase transition. In addition, in the subcritical regime, the susceptibility $\chi = -\frac{d^2F}{d\lambda^2}$ diverges as the critical line is approached, $\chi \sim (\lambda_c - \lambda)^{-\gamma}$, $\gamma = 1/2$, which is a recurrent behavior in second-order phase transitions. The critical behavior is shown in Figs. 1(a) and 1(b) for fixed $m = 1$ and in Fig. 2 for the whole phase diagram.

Next, consider the analytic properties of the free energy in crossing the critical line by varying the mass parameter at fixed coupling λ. By differentiating the free energy with respect to the mass m, one generates correlators of the gauge invariant mass operator [15]

$$J_3 = \frac{1}{N}(\tilde{Q}_{1,i}Q_1^i - \tilde{Q}_{2,i}Q_2^i),$$

where Q_1 are the hypermultiplets of mass m and Q_2 are the hypermultiplets of mass $-m$. Because of supersymmetry, these correlators are independent of the position [15]. For example, for the simple $N = 1$ case, we have that

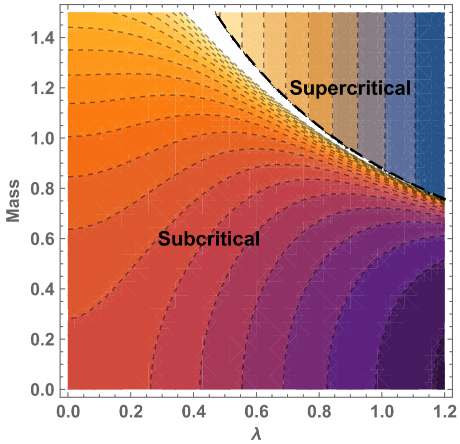

FIG. 2. Phase diagram. The critical line (dashed) $\lambda \sinh(m) = 1$ separates the two phases. The plot also shows the contour lines of $d^2F/d\lambda^2$ (increasing from dark to light).

$$\langle J_3 \rangle \propto \frac{dF}{dm} = \eta \cot(m\eta) - \coth(m),$$
$$\langle J_3 J_3 \rangle - \langle J_3 \rangle \langle J_3 \rangle \propto \frac{d^2F}{dm^2} = -\frac{\eta^2}{\sin^2(m\eta)} + \frac{1}{\sinh^2(m)}.$$

This extends a result of Ref. [15] to the case $\eta \neq 0$. Returning to the large N free energy, we find that $\langle J_3 \rangle$ is continuous, whereas

$$\left(\frac{d^2F}{dm^2}\right)_{\lambda<\lambda_c} = \frac{1}{N \sinh^2 m}\left(1 - \frac{\cosh m}{\sqrt{1-\lambda^2 \sinh^2 m}}\right)$$
$$\left(\frac{d^2F}{dm^2}\right)_{\lambda>\lambda_c} = \frac{1}{N \sinh^2 m}.$$

Thus, d^2F/dm^2 is discontinuous, implying a discontinuity in the two-point function of the operator J_3. Moreover, the two-point correlation function diverges as the critical line is approached from the subcritical phase.

The theory has an asymptotic $1/N$ expansion, which we now briefly outline. For concreteness, we consider the subcritical phase. The first $1/N$ correction arises from the term $(x - x_1)^4$ in the expansion of the integrand of (1) around x_1. A closely related expansion of the conical functions in inverse powers of $(N - 1/2)$ was discussed in Refs. [17,21]. An interesting approach is described in Ref. [7].

An elegant treatment, which exhibits the asymptotic character of the $1/N$ series, is as follows. We introduce a new integration variable by the transformation

$$S(x) - S_0 = \beta t. \tag{12}$$

This leads to

$$Z_{\mathrm{QED}_3} = \beta e^{-NS_0} \int_{\mathcal{C}} dt e^{-N\beta t} \mathcal{B}(t), \qquad \mathcal{B}(t) \equiv \frac{1}{S'(x(t))}, \tag{13}$$

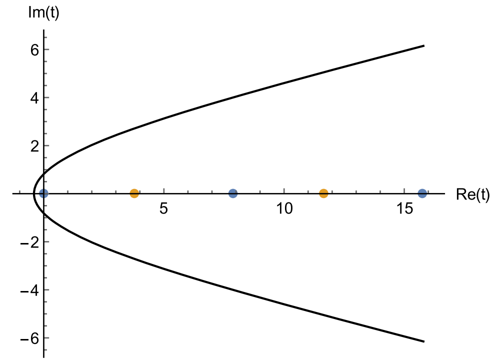

FIG. 3. Integration contour $\mathcal{C}$ in (13) and location of singularities of the integrand ($m = 1$, $\lambda = 0.3\lambda_c$).

with $S_0 \equiv S(x_1)$, $\beta \equiv S''(x_1)/2$. The contour $\mathcal{C}$ in the complex t-plane is determined by the transformation (12) in varying x from $-\infty$ to ∞ (see Fig. 3). The contour surrounds singularities at $t_1^{(n)}$ and $t_2^{(n)}$ lying on the positive real axis, which are associated with the saddles at x_1, with $n = 0, 1, 2, \ldots$, x_2, with $n = 1, 2, \ldots$. All singularities in $\mathcal{B}(t)$ are branch points of the form $(t - t_{1,2}^{(n)})^{-1/2}$. The $1/N$ expansion is generated upon Taylor expanding $\mathcal{B}(t)$ in powers of t,

$$\mathcal{B}(t) = \frac{1}{2\beta\sqrt{t}} \sum_{k=0}^{\infty} b_k t^k, \qquad b_0 = 1.$$

This expansion has a finite radius of convergence, determined by the location of the singularity that is closest to the origin. We are left with the integral

$$\int_{\mathcal{C}} dt e^{-N\beta t} t^{k-\frac{1}{2}} = 2\int_0^{\infty} dt e^{-N\beta t} t^{k-\frac{1}{2}} = \frac{2\Gamma(k+1/2)}{(\beta N)^{k+\frac{1}{2}}},$$

where we have deformed the contour to the positive real axis (note that in this integral, there is no singularity on the positive real axis). Thus, we get the asymptotic series

$$Z_{\mathrm{QED}_3} = \frac{e^{-NS_0}}{(\beta N)^{\frac{1}{2}}} \sum_{k=0}^{\infty} b_k \frac{\Gamma(k+1/2)}{\beta^k N^k}. \tag{14}$$

The term $k = 0$ just reproduces the earlier formula (11).

Here, we have expanded $\mathcal{B}(t)$ around $t = 0$. By expanding $\mathcal{B}(t)$ around some $t_{1,2}^{(n)}$, $n = 1, 2, \ldots$ one finds an extra factor $e^{-2\pi n\lambda}$ coming from $e^{-NS(t)}$. The presence of an infinite number of saddle points suggests that the $1/N$ expansion can be more conveniently treated in terms of resurgent trans-series. In deforming the contour, one crosses Stokes discontinuities which may imply resurgent relations in the different trans-series coefficients (see Refs. [22–26] for examples). It would be extremely interesting to understand the origin of nonperturbative effects that render the large N expansion asymptotic, as well as the resurgent properties of the series and how the existence of a phase transition is encoded in the $1/N$ expansions below and above the phase transition.

Now, consider more general masses. The theory with N hypermultiplets of mass m_1 and N hypermultiplets of mass m_2 is equivalent to the one we discussed. By a shift in the integration variable, one gets the same partition function (1) with an extra phase $e^{-i\eta m_+}$ and m replaced by m_-, with $m_\pm = (m_1 \pm m_2)/2$. In general, for N_f flavors, there are $N_f - 1$ mass parameters associated with the Cartan generators of $SU(N_f)$ flavor symmetry, satisfying $\sum_i m_i = 0$. One can have independent parameters m_1 and m_2 by adding an extra hypermultiplet of mass $m_3 = -N(m_1 + m_2)$. At large N, this decouples (its one-loop partition function becomes a constant, $1/\cosh m_3$), and the large N physics is then the same as in (1). More general mass assignments with similar phase transition are possible. The reason is that the mechanism that triggers the phase transition is also at work in more general cases: on the imaginary axis, the one-loop partition function provides a periodic potential with infinite number of vacua; as the constant force, represented by the FI parameter, is increased, there is a critical point where this overcomes the maximum force from the periodic potential. Beyond this point, equilibrium is not possible, and the saddle points move to the complex plane.

For 3D $\mathcal{N} = 4$ theories, mirror symmetry involves two or more theories with a different UV description flowing to the same superconformal point in the IR. Mirror symmetry interchanges Coulomb and Higgs branches of the theory, where FI parameters are interchanged with some linear combination of mass parameters [19]. The present theory is known to be dual to a A_{N-1} quiver gauge theory [19,27]. Particularizing to our model, we see that the dual theory is a $U(1)^{2N-1}$ quiver gauge theory with a FI parameter $2m$ and a single mass for all hypermultiplets $-\eta/2N$. Our results show that the quiver gauge theory also has a novel type of phase transition in the limit when the number of *quiver nodes* goes to infinity.

To conclude, the partition function of supersymmetric QED_3 with the FI term is given in terms of the conical function (3). It is remarkable that this simple formula encapsulates very rich physical phenomena such as large N phase transitions, asymptotic $1/N$ expansion, the emergence of complex saddle points, nonperturbative effects and aspects of mirror symmetry.

ACKNOWLEDGMENTS

J. R. acknowledges financial support from Projects No. FPA2013-46570 (MINECO) and No. 2014-SGR-104 (Generalitat de Catalunya). M. T. is supported by the Fundãçao para a Ciência e Tecnologia (program Investigador FCT IF2014), under Contract No. IF/01767/2014.

[1] T. Appelquist and R. D. Pisarski, Phys. Rev. D **23**, 2305 (1981).
[2] M. Franz and Z. Tesanovic, Phys. Rev. Lett. **87**, 257003 (2001).
[3] W. Rantner and X. G. Wen, Phys. Rev. B **66**, 144501 (2002).
[4] V. N. Kotov, B. Uchoa, V. M. Pereira, A. H. C. Neto, and F. Guinea, Rev. Mod. Phys. **84**, 1067 (2012).
[5] S. Kachru, M. Mulligan, G. Torroba, and H. Wang, Phys. Rev. B **92**, 235105 (2015).
[6] S. Kachru, M. Mulligan, G. Torroba, and H. Wang, Phys. Rev. D **94**, 085009 (2016).
[7] S. Gukov, arXiv:1608.06638.
[8] S. Sachdev and X. Yin, Ann. Phys. (Amsterdam) **325**, 2 (2010).
[9] M. A. Metlitski and A. Vishwanath, Phys. Rev. B **93**, 245151 (2016).
[10] C. Wang and T. Senthil, Phys. Rev. X **5**, 041031 (2015).
[11] V. Pestun, Commun. Math. Phys. **313**, 71 (2012).
[12] V. Pestun *et al.*, arXiv:1608.02952.
[13] S. Giombi, I. R. Klebanov, and G. Tarnopolsky, J. Phys. A **49**, 135403 (2016).
[14] I. R. Klebanov, S. S. Pufu, S. Sachdev, and B. R. Safdi, J. High Energy Phys. 05 (2012) 036.
[15] M. Dedushenko, S. S. Pufu, and R. Yacoby, arXiv:1610.00740.
[16] A. Kapustin, B. Willett, and I. Yaakov, J. High Energy Phys. 03 (2010) 089.
[17] A. Gil, J. Segura, and N. M. Temme, SIAM J. Sci. Comput. **31**, 1716 (2009).
[18] S. Benvenuti and S. Pasquetti, J. High Energy Phys. 05 (2012) 099.
[19] K. A. Intriligator and N. Seiberg, Phys. Lett. B **387**, 513 (1996).
[20] S. Schindler, Trans. Am. Math. Soc. **155**, 257 (1971).
[21] T. M. Dunster, Proceedings Section A, Mathematical and physical sciences/The Royal Society of Edinburgh **119**, 311 (1991).
[22] S. Pasquetti and R. Schiappa, Ann. Inst. Henri Poincare, Sect. C **11**, 351 (2010).
[23] M. Mariño, *Instantons and Large N: An Introduction to Non-Perturbative Methods in Quantum Field Theory* (Cambridge University Press, Cambridge, England, 2015).
[24] G. V. Dunne and M. Unsal, J. High Energy Phys. 11 (2012) 170.
[25] I. Aniceto, J. G. Russo, and R. Schiappa, J. High Energy Phys. 03 (2015) 172.
[26] R. Couso-Santamaría, R. Schiappa, and R. Vaz, Ann. Phys. (Amsterdam) **356**, 1 (2015).
[27] A. Kapustin, B. Willett, and I. Yaakov, J. High Energy Phys. 10 (2010) 013.