

Action principle and the supersymmetrization of Chern-Simons terms in eleven-dimensional supergravity

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We develop computational tools for calculating supersymmetric higher-order derivative corrections to eleven-dimensional supergravity using the action principle approach. We show that, provided the superspace Bianchi identities admit a perturbative solution in the derivative expansion, there are at least two independent superinvariants at the eight-derivative order of eleven-dimensional supergravity. Assuming the twelve superforms associated to certain anomalous Chern-Simons terms are Weil trivial, there will be a third independent superinvariant at this order. Under certain conditions, at least two superinvariants will survive to all orders in the derivative expansion. However only one of them will be present in the quantum theory: the supersymmetrization of the Chern-Simons terms of eleven-dimensional supergravity required for the cancellation of the M5-brane gravitational anomaly by inflow. This superinvariant can be shown to be unique at the eight-derivative order, assuming it is quartic in the fields. On the other hand, a necessary condition for the superinvariant to be quartic is the exactness, in τ -cohomology, of $X_{0,8}$, the purely spinorial component of the eight superform related by descent to the M5-brane anomaly polynomial. In that case it can also be shown that the solution of the Weil-triviality condition of the corresponding twelve form, which is a prerequisite for the explicit construction of the superinvariant, is guaranteed to exist. We prove that certain highly nontrivial necessary conditions for the τ -exactness of $X_{0,8}$ are satisfied. Moreover any potential superinvariant associated to anomalous Chern-Simons terms at the eight-derivative order must necessarily contain terms cubic or lower in the fields.

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I. INTRODUCTION

Eleven-dimensional supergravity [1] is believed to be the low-energy limit of M-theory [2], the conjectured non-perturbative completion of string theory. As such it is expected to receive an infinite tower of higher-order corrections in an expansion in the Planck length or, equivalently, in the derivative expansion. At present such higher-order corrections cannot be systematically constructed within M-theory, so one must resort to indirect approaches.

One such approach is to calculate the higher-order corrections within perturbative string theory, in particular, type IIA in ten dimensions, which is related to eleven-dimensional supergravity by dimensional reduction. The effective action of string theory can be systematically constructed perturbatively in a loop expansion in the string coupling,

$$S_{\text{eff}} = \sum_{g=0}^{\infty} g_s^{2g-2} \int d^{10}x \sqrt{G} \mathcal{L}_g, \quad (1.1)$$

where g is the loop order (equivalently, the genus of the Riemann surface), g_s is the string coupling constant, G is

the spacetime metric, and \mathcal{L}_g is the effective action at order g . Each \mathcal{L}_g admits a perturbative expansion in an infinite series of higher-order derivative terms. Moreover it is expected that each \mathcal{L}_g should correspond to an independent superinvariant in ten dimensions; see, e.g., [3].

The bosonic part of the tree-level effective action takes schematically the following form,

$$\mathcal{L}_0 = \mathcal{L}_{\text{IIA}} + \alpha^3 \left(I_0(R) - \frac{1}{8} I_1(R) + \dots \right) + \mathcal{O}(\alpha^4), \quad (1.2)$$

where \mathcal{L}_{IIA} is the (two-derivative) Lagrangian of ten-dimensional IIA supergravity, and the ellipses stand for terms that have not been completely determined yet. Unlike the case of $\mathcal{N} = 1$ superstrings, the first higher-derivative correction starts at order α^3 (eight derivatives). The I_0, I_1 in (1.2) are defined as follows,

$$\begin{aligned} I_0(R) &= t_8 t_8 R^4 + \frac{1}{2} \varepsilon_{10} t_8 B R^4, \\ I_1(R) &= -\varepsilon_{10} \varepsilon_{10} R^4 + 4 \varepsilon_{10} t_8 B R^4. \end{aligned} \quad (1.3)$$

These were constructed in [4], to which we refer for further details, by directly checking invariance under part of the supersymmetry transformations. The terms in (1.3) linear in

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B are, up to a numerical coefficient, Hodge dual to the Chern-Simons term $B \wedge X_8$ [5,6]. The eight form X_8 , see (4.2) below, is related by descent to the M5-brane anomaly polynomial and is a linear combination of $(\text{tr}R^2)^2$ and $\text{tr}R^4$. Note that the Chern-Simons (CS) term drops out of (1.2).

The R^4 part of the tree-level effective action was determined in [7,8] via four-graviton scattering amplitudes and in [9–11] from the vanishing of the world sheet beta-function at four loops. The NSNS sector of the four-field part of the effective action (common to all superstring theories in ten dimensions) was determined in [12]: it is captured by the simple replacement $R \rightarrow \hat{R}$, where \hat{R} is a modified Riemann tensor with torsion that includes the NSNS three form and the dilaton.¹ The $\varepsilon_{10}\varepsilon_{10}R^4$ term does not contribute to tree-level four-point scattering amplitudes, but gives a nonvanishing contribution to the five-graviton scattering amplitude. The complete tree-level four-point effective action for type-II superstrings was first determined in [13] and, in addition to the NSNS sector, consists of terms of the form $(\partial F)^2 \hat{R}^2$ and $\partial^4 F^4$, where F stands for all RR flux.

The superinvariant I_0 can be further decomposed into two separate $\mathcal{N} = 1$ superinvariants in ten dimensions [4], $I_0 = -6I_{0a} + 24I_{0b}$, where

$$\begin{aligned} I_{0a} &= \left(t_8 + \frac{1}{2} \varepsilon_{10} B \right) (\text{tr}R^2)^2 + \dots, \\ I_{0b} &= \left(t_8 + \frac{1}{2} \varepsilon_{10} B \right) \text{tr}R^4 + \dots \end{aligned} \quad (1.4)$$

correspond to the supersymmetrization of the $B \wedge (\text{tr}R^2)^2$ and $B \wedge \text{tr}R^4$ Chern-Simons terms, respectively. As we show in the following, if the uplift of I_{0a} , I_{0b} gives rise to two separate superinvariants in eleven dimensions, they will necessarily have to be cubic or lower in the fields.

The one-loop effective action takes the following form [7,14]:

$$\mathcal{L}_1 = \alpha^3 \left(I_0(R) + \frac{1}{8} I_1(R) + \dots \right) + \mathcal{O}(\alpha^4). \quad (1.5)$$

In particular, we see that in this case the Chern-Simons term does not drop out, cf., (1.3). The ellipses above indicate terms that are not completely known, although partial results exist thanks to five- and six-point amplitude

¹Note that [12] contains an error that has unfortunately caused some confusion in the literature: the expansion of the $t_8 t_8 R^4$ term in Eq. (2.11) in that reference indeed has the form of the term in the square brackets on the right-hand side of Eq. (2.13) therein. However, if one replaces R by the modified Riemann tensor \hat{R} , given in Eq. (2.12) therein, Eq. (2.13) no longer gives the correct expansion of $t_8 t_8 \hat{R}^4$.

computations [15–18]. Contrary to the tree-level superinvariant \mathcal{L}_0 that is suppressed at strong coupling, the uplift of the one-loop superinvariant \mathcal{L}_1 is expected to survive in eleven dimensions, and thus to be promoted to an eleven-dimensional superinvariant. We refer to the latter as the supersymmetrization of the Chern-Simons term $C \wedge X_8$, the uplift of the ten-dimensional Chern-Simons term, where C is the three-form potential of eleven-dimensional supergravity.

An argument of [19], which we review in the following, guarantees that if the supersymmetrization of the Chern-Simons term is quartic or higher in the fields, then it is unique at the eight-derivative order.² The uniqueness of this superinvariant is also supported by the results of [20–22], which use the Noether procedure to implement part of the supersymmetry transformations of eleven-dimensional supergravity. The results of these references constrain the supersymmetrization of the Chern-Simons term to be of the form

$$\begin{aligned} \Delta \mathcal{L} &= l^6 \left(t_8 t_8 R^4 - \frac{1}{4!} \varepsilon_{11} \varepsilon_{11} R^4 - \frac{1}{6} \varepsilon_{11} t_8 C R^4 + R^3 G^2 + \dots \right) \\ &\quad + \mathcal{O}(l^7), \end{aligned} \quad (1.6)$$

where l is the Planck length. The ellipses indicate terms that were not determined by the analysis of [20–22], while the $R^3 G^2$ terms were only partially determined. The reduction of the above to ten dimensions is consistent, as expected, with the one-loop IIA superinvariant (1.5). In addition the quartic interactions $R^2 (\partial G)^2$ and $(\partial G)^4$ were determined in [23] by eleven-dimensional superparticle one-loop computations in the light cone, and in [24–26] by a different method that uses tree amplitudes instead.³ The $t_8 t_8 R^4$ terms have also been obtained by four-graviton one-loop amplitudes in eleven dimensions [27,28], while it can be shown [29] that higher loops do not contribute to the superinvariant (1.6).

In the present paper we reexamine the problem of calculating supersymmetric higher-order derivative corrections to eleven-dimensional supergravity from the point of view of the action principle approach. This method relies on the superspace formulation of the theory and is particularly well suited to the supersymmetrization of Chern-Simons terms. Given an eleven-dimensional Chern-Simons term there is an associated gauge-invariant twelve superform obtained by exterior differentiation. The action principle approach can be carried out provided the twelve form is Weil trivial, i.e., exact on the space of on-shell superfields. Computing the superinvariant then boils

²The existence of independent superinvariants starting at an order higher than eight in the derivative expansion course spoils the uniqueness of the superinvariant at higher orders.

³There is disagreement between [23,25] concerning part of the $(\partial G)^4$ terms.

down to explicitly solving the Weil-triviality condition for the twelve form.

We show that, provided the superspace Bianchi identities admit a perturbative solution in the derivative expansion, there are at least two independent superinvariants at the eight-derivative order. If we also assume that the twelve superforms associated to the anomalous (in the presence of an M5-brane) Chern-Simons terms, $C \wedge (\text{Tr}R^2)^2$ and $C \wedge \text{Tr}R^4$, are separately Weil trivial, there is a third independent superinvariant at this order. Moreover we argue that, under certain conditions, at least two of the superinvariants should be expected to survive to all orders in the derivative expansion. However only one of those would correspond to the supersymmetrization of $C \wedge X_8$, cf., (1.6).

As already noted this superinvariant can be shown to be unique, assuming it is quartic in the fields. On the other hand, a necessary condition for the superinvariant to be quartic is the exactness, in the so-called τ -cohomology, of $X_{0,8}$, the purely spinorial component of X_8 . In that case we also show that the solution of the Weil-triviality condition of the corresponding twelve form is guaranteed to exist. Proving the τ -exactness of $X_{0,8}$ is the first and arguably most difficult step in obtaining the explicit solution to the Weil-triviality condition of the twelve form, and therefore constructing the superinvariant using the action principle.

To tackle this computationally intensive problem we have built on the computer program [30], to supplement it, among other things, with functionalities related to Young tableaux [31]. By a combination of calculational techniques involving the implementation of Fierz identities and Young tableaux projections we prove that certain highly nontrivial necessary conditions for the τ -exactness of $X_{0,8}$ are satisfied. As a corollary of our work, it follows that any potential superinvariant associated to the anomalous Chern-Simons terms, $C \wedge (\text{Tr}R^2)^2$ and $C \wedge \text{Tr}R^4$, must necessarily contain terms cubic or lower in the fields.

The plan of the rest of the paper is as follows. In Sec. II we review the different superspace cohomologies that are useful in the following. In Sec. III we introduce the action principle approach and in Sec. III A we show how to obtain the eleven-dimensional supergravity of [1] in this framework. In Sec. III B we apply the action principle to derive the five-derivative correction. Section IV considers the eight-derivative correction. In Sec. IV A we examine the number of independent superinvariants at the eight-derivative order. Section IV B addresses the problem of the τ -exactness of $X_{0,8}$. In Sec. IV C we discuss the conditions for the existence of the superinvariants to all orders in the perturbative expansion. We conclude in Sec. V. Further technical details are included in the appendixes.

II. COHOMOLOGY IN SUPERSPACE

In this section we review the various superspace cohomology groups that are useful in the following.

This is not new material, but we are including it here to make the paper self-contained and for the benefit of the readers who may not be familiar with the relevant literature.

Let us start by explaining our conventions: Eleven-dimensional superspace [32,33] consists of eleven even (bosonic) and thirty-two odd (fermionic) dimensions, with the structure group being the eleven-dimensional spin group. Let $A = (a, \alpha)$ be flat tangent superindices, where $a = 0, \dots, 10$ is a Lorentz vector index and $\alpha = 1, \dots, 32$ is a Majorana spinor index. Curved superindices are denoted by $M = (m, \mu)$, with the corresponding supercoordinates denoted by $Z^M = (x^m, \theta^\mu)$. The supercoframe is denoted by $E^A = (E^a, E^\alpha)$ while the dual superframe is denoted by $E_A = (E_a, E_\alpha)$. We can pass from the coframe to the coordinate basis using the supervielbein, $E^A = dz^M E_M^A$.

We assume the existence of a connection one form Ω_A^B with values in the Lie algebra of the Lorentz group. In particular, this implies that

$$\Omega_{(a^c \eta_b)_c} = 0, \quad \Omega_\alpha^\beta = \frac{1}{4} (\gamma^a_b)_\alpha^\beta \Omega_a^b, \quad \Omega_a^\beta = 0 = \Omega_\alpha^b. \quad (2.1)$$

The associated supertorsion and supercurvature tensors are then given by

$$T^A = DE^A := dE^A + E^B \wedge \Omega_B^A = \frac{1}{2} E^C \wedge E^B T_{BC}^A, \\ R_A^B = d\Omega_A^B + \Omega_A^C \wedge \Omega_C^B = \frac{1}{2} E^D \wedge E^C R_{CDA}^B, \quad (2.2)$$

where the exterior derivative is given by $d = dz^M \partial_M$. The assumption of a Lorentzian structure group implies that the components of the curvature two form obey a set of equations analogous to (2.1). The super-Bianchi identities (BI) for the torsion and the curvature,

$$DT^A = E^B \wedge R_B^A, \\ DR_A^B = 0, \quad (2.3)$$

follow from the definitions (2.2). Moreover, a theorem due to Dragon [34] ensures that for a Lorentz structure group the second BI above follows from the first and need not be considered separately. Once constraints are imposed the BI cease to be automatically satisfied. As was shown in [33], by imposing the conventional constraint

$$T_{\alpha\beta}^c = i\gamma_{\alpha\beta}^c, \quad (2.4)$$

and solving the torsion BI, one recovers ordinary eleven-dimensional supergravity. In particular, one determines in this way all components of the torsion.

In addition one can construct a closed superfour form G_4 and a superseven form G_7 obeying [35,36]⁴

$$dG_4 = 0, \quad dG_7 + \frac{1}{2}G_4 \wedge G_4 = 0, \quad (2.5)$$

whose bosonic components correspond to the eleven-dimensional supergravity four form and its Hodge dual, respectively,

$$G_{m_1 \dots m_7} = (\star G)_{m_1 \dots m_7}. \quad (2.6)$$

The solution of the eleven-dimensional superspace BI is reviewed in Appendix E.

A. De Rham cohomology and Weil triviality

Let Ω^n be the space of n superforms. Thanks to the nilpotency of the exterior superderivative, one can define de Rham cohomology groups in superspace in the same way as in the case of bosonic space,

$$H^n = \{\omega \in \Omega^n | d\omega = 0\} / \{\omega \sim \omega + d\lambda, \lambda \in \Omega^{n-1}\}. \quad (2.7)$$

The fact that the topology of the odd directions is trivial means that the de Rham cohomology of a supermanifold coincides with the de Rham cohomology of its underlying bosonic manifold, also known as the *body* of the supermanifold. In the remainder of the paper we assume that the body has trivial topology. This is the simplest type of supermanifold, sometimes called a *graded manifold*. It implies, in particular, that every d-closed superform is d exact.

There is an important caveat to the previous statement: it is only valid when the cohomology is computed on the space of unconstrained superfields. Once constraints are imposed it ceases to be automatically satisfied. Adopting the terminology of [37], we call Weil trivial those d-closed

superforms that are also d exact on the space of constrained (also referred to as ‘‘on-shell,’’ or ‘‘physical’’) superfields. The cohomology groups computed on the space of constrained superfields are denoted by $H^n(\text{phys})$, as in [19]. As already emphasized, there is no *a priori* reason why $H^n(\text{phys})$ should coincide with the cohomology of the body of the supermanifold.

B. τ -cohomology

The space of superforms can be further graded according to the even, odd degrees of the forms. We denote the space of forms with p even and q odd components by $\Omega^{p,q}$ so that

$$\Omega^n = \bigoplus_{p+q=n} \Omega^{p,q}. \quad (2.8)$$

A (p, q) superform $\omega \in \Omega^{p,q}$ can be expanded as follows:

$$\omega = \frac{1}{p!q!} E^{\beta_q} \dots E^{\beta_1} E^{\alpha_p} \dots E^{\alpha_1} \omega_{a_1 \dots a_p \beta_1 \dots \beta_q}. \quad (2.9)$$

In the following we use the notation $\Phi_{(p,q)} \in \Omega^{p,q}$ for the projection of a superform $\Phi \in \Omega^n$ onto its (p, q) component.

The exterior superderivative, $d: \Omega^{p,q} \rightarrow \Omega^{p+1,q} + \Omega^{p,q+1} + \Omega^{p-1,q+2} + \Omega^{p+2,q-1}$, when written out in this basis gives rise to components of the torsion as it acts on the coframe. Following [38] we split d into its various components with respect to the bigrading,

$$d = d_b + d_f + \tau + t, \quad (2.10)$$

where d_b, d_f are even, odd derivatives respectively, such that $d_b: \Omega^{p,q} \rightarrow \Omega^{p+1,q}$, $d_f: \Omega^{p,q} \rightarrow \Omega^{p,q+1}$. The operators τ and t are purely algebraic and can be expressed in terms of the torsion. Explicitly, for any $\omega \in \Omega^{p,q}$ we have

$$\begin{aligned} (d_b \omega)_{a_1 \dots a_{p+1} \beta_1 \dots \beta_q} &= (p+1)(D_{[a_1} \omega_{a_2 \dots a_{p+1}] \beta_1 \dots \beta_q} + \frac{p}{2} T_{[a_1 a_2]}^c \omega_{c[a_3 \dots a_{p+1}] \beta_1 \dots \beta_q} \\ &\quad + q(-1)^p T_{[a_1 | (\beta_1]^\gamma} \omega_{a_2 \dots a_{p+1} \gamma | \beta_2 \dots \beta_q}), \\ (d_f \omega)_{a_1 \dots a_p \beta_1 \dots \beta_{q+1}} &= (q+1)((-1)^p D_{(\beta_1} \omega_{a_1 \dots a_p | \beta_2 \dots \beta_{q+1})} + \frac{q}{2} T_{(\beta_1 \beta_2]}^\gamma \omega_{a_1 \dots a_p \gamma | \beta_3 \dots \beta_{q+1}} \\ &\quad + p(-1)^p T_{(\beta_1 | [a_1]^\gamma} \omega_{c[a_2 \dots a_p] | \beta_2 \dots \beta_{q+1}}), \\ (\tau \omega)_{a_1 \dots a_{p-1} \beta_1 \dots \beta_{q+2}} &= \frac{1}{2}(q+1)(q+2) T_{(\beta_1 \beta_2]}^c \omega_{c a_1 \dots a_{p-1} | \beta_3 \dots \beta_{q+2}}, \\ (t \omega)_{a_1 \dots a_{p+2} \beta_1 \dots \beta_{q-1}} &= \frac{1}{2}(p+1)(p+2) T_{[a_1 a_2]}^\gamma \omega_{a_3 \dots a_{p+2} \gamma | \beta_1 \dots \beta_{q-1}}. \end{aligned} \quad (2.11)$$

The nilpotency of the exterior derivative, $d^2 = 0$, implies the following identities:

⁴The G_7 BI receives a correction at the eight-derivative order, cf., (4.3) below.

$$\begin{aligned}
\tau^2 &= 0, \\
d_f \tau + \tau d_f &= 0, \\
d_f^2 + d_b \tau + \tau d_b &= 0, \\
d_b d_f + d_f d_b + \tau t + t \tau &= 0, \\
d_b^2 + d_f t + t d_f &= 0, \\
d_b t + t d_b &= 0, \\
t^2 &= 0.
\end{aligned} \tag{2.12}$$

The first and the last of these equations are algebraic identities and are always satisfied. On the other hand, as a consequence of the splitting of the tangent bundle into even and odd directions, the remaining identities are only satisfied provided the torsion tensor obeys its Bianchi identity.

The first of the equations in (2.12), the nilpotency of the τ operator, implies that we can consider the cohomology of τ , as first noted in [38] (see also [35] for some related concepts). Explicitly we set

$$H_\tau^{p,q} = \{\omega \in \Omega^{p,q} | \tau\omega = 0\} / \{\omega \sim \omega + \tau\lambda, \lambda \in \Omega^{p+1,q-2}\}. \tag{2.13}$$

As in the case of de Rham cohomology, one could make a distinction between cohomology groups computed on the space of unconstrained superfields and those computed on the space of physical fields.

Suppose now that the conventional constraint (2.4) is imposed so that τ reduces to a gamma matrix. It was conjectured in [19], consistently with the principle of maximal propagation of [39], that in this case the only potentially nontrivial τ -cohomology appears as a result of the so-called M2-brane identity,

$$(\gamma^a)_{(\alpha_1\alpha_2} (\gamma_{ab})_{\alpha_3\alpha_4)} = 0. \tag{2.14}$$

Explicitly, for $p = 0, 1, 2$, one may form the following τ -closed (p, q) superforms,

$$\begin{aligned}
\omega_{\alpha_1 \dots \alpha_q} &= S_{\alpha_1 \dots \alpha_q}; & \omega_{a\alpha_1 \dots \alpha_q} &= (\gamma_{ab})_{(\alpha_1\alpha_2} P^b_{\alpha_3 \dots \alpha_q)}; \\
\omega_{ab\alpha_1 \dots \alpha_q} &= (\gamma_{ab})_{(\alpha_1\alpha_2} U_{\alpha_3 \dots \alpha_q)},
\end{aligned} \tag{2.15}$$

with S, P, U being arbitrary superfields. It can be seen using (2.14) that the forms ω above correspond to nontrivial elements of $H_\tau^{p,q}$ with $p = 0, 1, 2$. The conjecture of [19] means that all nontrivial cohomology is thus generated, and that all $H_\tau^{p,q}$ groups are trivial for $p \geq 3$. This was subsequently proven in [40–44].

C. Spinorial cohomology

Following [19], let us now define a spinorial derivative d_s that acts on elements of τ -cohomology,

$d_s: H_\tau^{p,q} \rightarrow H_\tau^{p,q+1}$. For any $\omega \in [\omega] \in H_\tau^{p,q}$ we set

$$d_s[\omega] := [d_f \omega]. \tag{2.16}$$

To check that this is well defined, one first shows that $d_f \omega$ is τ closed,

$$\tau d_f \omega = -d_f \tau \omega = 0, \tag{2.17}$$

where we used the second equation in (2.12). Moreover, $d_s[\omega]$ is independent of the choice of representative,

$$[d_f(\omega + \tau\lambda)] = [d_f \omega - \tau d_f \lambda] = [d_f \omega]. \tag{2.18}$$

Furthermore it is simple to check that $d_s^2 = 0$,

$$d_s^2[\omega] = d_s[d_f \omega] = [d_f^2 \omega] = -[(d_b \tau + \tau d_b)\omega] = 0, \tag{2.19}$$

where we took into account the third equation in (2.12). We can therefore define the corresponding spinorial cohomology groups $H_s^{p,q}$ as follows:

$$H_s^{p,q} = \{\omega \in H_\tau^{p,q} | d_s \omega = 0\} / \{\omega \sim \omega + d_s \lambda, \lambda \in H_\tau^{p,q-1}\}. \tag{2.20}$$

The notion of spinorial cohomology was originally introduced in [39,45] and applied in a series of papers with the aim of computing higher-order corrections to supersymmetric theories [46–50], and more recently in [51–53]. The spinorial cohomology as presented above was introduced in [19] and is independent of the value of the dimension-zero torsion. It reduces to the spinorial cohomology of [39,45] upon imposing the conventional constraint (2.4).

D. Pure-spinor cohomology

It was first pointed out by P. Howe [54] and subsequently elaborated in [19] that in the case where the dimension-zero torsion is flat, cf., (2.4), the cohomology groups $H_s^{0,q}$ are isomorphic to Berkovits's pure-spinor cohomology groups [55]. Therefore, in view of what was said in Sec. II C, the latter are also isomorphic to the spinorial cohomology groups that had been computed a few months earlier in [39]. In the following we briefly explain the equivalence between the two formulations.

The pure-spinor cohomology groups are defined as follows. Consider an eleven-dimensional pure spinor, λ^α , à la Berkovits, i.e., such that it obeys⁵

$$\lambda^\alpha \gamma_{\alpha\beta}^a \lambda^\beta = 0. \tag{2.21}$$

⁵This definition is different from an eleven-dimensional pure spinor à la Cartan, used in [56], which obeys $\lambda^\alpha \gamma_{\alpha\beta}^{ab} \lambda^\beta = 0$ in addition to (2.21).

The pure spinor λ^α is assigned ghost number 1. Furthermore we define a form of ghost number q as a multipure spinor,

$$\omega = \lambda^{\alpha_1} \dots \lambda^{\alpha_q} \omega_{\alpha_1 \dots \alpha_q}. \quad (2.22)$$

Note that the above definition implies that $\omega \in [\omega] \in H_\tau^{0,q}$: indeed any shift of $\omega_{\alpha_1 \dots \alpha_q}$ by a τ -exact term would drop out of the right-hand side above due to the contractions with the pure spinors; moreover, $\omega_{\alpha_1 \dots \alpha_q}$ is trivially τ closed.

The pure-spinor BRST operator is defined as follows,

$$Q := \lambda^\alpha D_\alpha, \quad (2.23)$$

where D_α is the spinor component of the covariant derivative defined in flat superspace. Therefore the action of Q on omega,

$$Q\omega = \lambda^{\alpha_1} \dots \lambda^{\alpha_q} \lambda^{\alpha_{q+1}} D_{\alpha_{q+1}} \omega_{\alpha_1 \dots \alpha_q}, \quad (2.24)$$

corresponds precisely to the action of d_s defined in (2.16). Indeed, for flat superspace the torsion terms drop out and d_f reduces to D_α , cf., the second line of Eq. (2.11). Moreover the contraction with the pure spinors on the right-hand side above implies that $Q\omega \in [Q\omega] \in H_\tau^{0,q+1}$, for the same reasons noted below (2.22). In other words, in the linearized limit the pure-spinor cohomology groups of ghost number q are isomorphic to the spinorial cohomology groups $H_s^{0,q}$. For an extended review of pure-spinor superfields, see [57].

III. THE ACTION PRINCIPLE

The action principle, also known as ectoplasmic integration, is a way of constructing superinvariants in D spacetime dimensions as integrals of closed D superforms [58,59]. Indeed if L is a closed D superform, the following action is invariant under supersymmetry,

$$S = \frac{1}{D!} \int d^D x \varepsilon^{m_1 \dots m_D} L_{m_1 \dots m_D} |, \quad (3.1)$$

where a vertical bar denotes the evaluation of a superfield at $\theta^\mu = 0$. This can be seen as follows. Consider an infinitesimal superdiffeomorphism generated by a supervector field ξ . The corresponding transformation of the action reads

$$\delta L = \mathcal{L}_\xi L = (di_\xi + i_\xi d)L = di_\xi L, \quad (3.2)$$

where we took into account that L is closed. On the other hand, local supersymmetry transformations and spacetime diffeomorphisms are generated by $\xi|$ and, in view of (3.2), the integrand in (3.1) transforms as a total derivative under

such transformations. The action is thus invariant assuming boundary terms can be neglected.

This method is particularly well suited to actions with CS terms and indeed has been used to construct all Green-Schwarz brane actions [60,61]; see [62,63] for more recent applications to other theories and [64] for applications to higher-order corrections. The idea is as follows: let Z_D be the CS term and $W_{D+1} = dZ_D$ be its exterior derivative. Obviously W_{D+1} is a closed form. On the other hand one might be led to conclude that the de Rham cohomology group of rank $D+1$ must be trivial on a supermanifold whose body is D dimensional; hence W_{D+1} must also be exact. This means that it can be written as $W_{D+1} = dK_D$ where now, contrary to Z_D , K_D is a globally defined (gauge-invariant) superform. It follows that $L_D := Z_D - K_D$ is a closed superform, and can therefore be used to construct a supersymmetric action as in (3.1).

Eleven-dimensional supergravity is another example of an action with Chern-Simons terms, and we turn to the application of the action principle to this case in the following sections. Unfortunately there is a caveat to the previous argument that W_{D+1} is exact. As already noted in Sec. II A, this argument can be applied only in the case where the cohomology is computed on the space of unconstrained superfields, but is not *a priori* true on the space of physical (on-shell) superfields. Interestingly it does turn out to be true in all known cases. As we see in the following this includes the case of ordinary eleven-dimensional supergravity as well as its supersymmetric corrections with five derivatives. In Sec. IV B we show that a sufficient condition for the Weil triviality of the eight-derivative correction is the τ -exactness of $X_{0,8}$.

We parametrize the derivative expansion in terms of the Planck length l , so that the Cremmer-Julia-Scherk two-derivative action (CJS) corresponds to zeroth order in l . In Sec. IV we show that, provided the four- and seven-form BI are satisfied at order $\mathcal{O}(l^6)$, cf., (4.3), there are at least two Weil-trivial twelve forms W_{12} and hence at least two independent supersymmetric actions with eight derivatives. Provided the twelve forms associated to certain anomalous CS terms are Weil trivial, cf., (4.16) below, there is a third independent superinvariant at this order. We argue that at least two of those superinvariants exist to all orders in the derivative expansion.

As we see in detail in the following, in practice one solves for the flat components of the closed superform L_D in a stepwise fashion in increasing engineering dimension. Once all flat components of L_D have been determined in this way, the explicit form of the action (3.1) can be extracted using the formula

$$\begin{aligned} L_{m_1 \dots m_D} | &= e_{m_D}^{\alpha_D} \dots e_{m_1}^{\alpha_1} L_{\alpha_1 \dots \alpha_D} | \\ &+ D e_{m_D}^{\alpha_D} \dots e_{m_2}^{\alpha_2} \psi_{m_1}^{\alpha_1} L_{\alpha_1 \alpha_2 \dots \alpha_D} | + \dots \\ &\dots + \psi_{m_D}^{\alpha_D} \dots \psi_{m_1}^{\alpha_1} L_{\alpha_1 \dots \alpha_D} |, \end{aligned} \quad (3.3)$$

where $\psi_m^\alpha := E_m^\alpha$ and $e_m^a := E_m^a$ are identified as the gravitino and the vielbein of (bosonic) spacetime, respectively. In particular, the bosonic terms of the Lagrangian can be read off immediately from $L_{a_1 \dots a_D}$.

A. CJS supergravity in the action principle formulation

The eleven-dimensional supergravity action reads [1]

$$S = \int \left(R \star 1 - \frac{1}{2} G_4 \wedge \star G_4 - \frac{1}{6} C_3 \wedge G_4 \wedge G_4 \right), \quad (3.4)$$

where $dC_3 = G_4$ is the three-form potential; it is understood that only the bosonic (11,0) components of the forms enter the formula above, as in (3.1).

This action can also be understood from the point of view of the action principle as follows. The twelve form corresponding to the CS term reads

$$\begin{aligned} W_{12} &= -\frac{1}{6} G_4 \wedge G_4 \wedge G_4 = dZ_{11}; \\ Z_{11} &= -\frac{1}{6} C_3 \wedge G_4 \wedge G_4. \end{aligned} \quad (3.5)$$

Using the BI (2.5) this can also be written in a manifestly Weil-trivial form,

$$W_{12} = dK_{11}; \quad K_{11} = \frac{1}{3} G_4 \wedge G_7. \quad (3.6)$$

Taking $L_{11} = Z_{11} - K_{11}$ we obtain that the following action is invariant under supersymmetry:

$$S = \int \left(-\frac{1}{3} G_4 \wedge G_7 - \frac{1}{6} C_3 \wedge G_4 \wedge G_4 \right). \quad (3.7)$$

This can then be put in the form (3.4) by using the on-shell conditions $\star G_4 = G_7$ and $G_4 \wedge \star G_4 = 6R \star 1$, cf., Appendix A. Therein we also give the details of the solution of the superspace equation $W_{12} = dK_{11}$ and we show, as a byproduct, that the solution for K_{11} given in (3.6) is unique up to exact terms.

B. The $\mathcal{O}(l^3)$ correction (five derivatives)

It was shown in [49], by directly computing the relevant spinorial cohomology group, that there is a unique superinvariant at the five-derivative level (order l^3 in the Planck length).⁶ The modified eleven-dimensional action to order l^3 reads

⁶As explained in [49], on a topologically trivial spacetime manifold this superinvariant can be removed by an appropriate field redefinition of the three-form superpotential. However on a spacetime with nonvanishing first Pontryagin class the superinvariant cannot be redefined away without changing the quantization condition of the four-form field strength.

$$S = \int \left(R \star 1 - \frac{1}{2} G_4 \wedge \star G_4 - \frac{1}{6} C_3 \wedge G_4 \wedge G_4 + l^3 (C_3 \wedge G_4 \wedge \text{tr} R^2 + 2 \text{tr} R^2 \wedge \star G_4) \right), \quad (3.8)$$

where an arbitrary numerical coefficient has been absorbed in the definition of l and $\text{tr} R^2 := R_a^b \wedge R_b^a$; it is understood that only the bosonic (11,0) components of the forms enter the formula above. This action can also be easily understood from the point of view of the action principle as follows. Consider the twelve form corresponding to the CS term at order l^3 ,

$$W_{12} = G_4 \wedge G_4 \wedge \text{tr} R^2 = dZ_{11}; \quad Z_{11} = C_3 \wedge G_4 \wedge \text{tr} R^2. \quad (3.9)$$

Using the BI (2.3) and (2.5) this can also be written in a manifestly Weil-trivial form,

$$W_{12} = dK_{11}; \quad K_{11} = -2G_7 \wedge \text{tr} R^2. \quad (3.10)$$

Taking $L_{11} = Z_{11} - K_{11}$ we obtain the following superinvariant at order l^3 :

$$\Delta S = \int \left(C_3 \wedge G_4 \wedge \text{tr} R^2 + 2G_7 \wedge \text{tr} R^2 \right). \quad (3.11)$$

This can be seen, using the Hodge duality relation $G_7 = \star G_4$, to precisely correspond to the order- l^3 terms in (3.8).⁷

In Appendix B we work out in detail the superspace equation $W_{12} = dK_{11}$ and confirm that the solution (3.10) for K_{11} is unique up to exact terms, in accordance with the spinorial cohomology result of [49].

IV. THE $\mathcal{O}(l^6)$ CORRECTION (EIGHT DERIVATIVES)

As was shown in [6,65], the requirement that the M5-brane gravitational anomaly is canceled by inflow from eleven dimensions implies the existence of certain CS terms Z_{11} at the eight-derivative order in the eleven-dimensional theory. The corresponding twelve form reads

$$W_{12} = l^6 G_4 \wedge X_8 = dZ_{11}; \quad Z_{11} = l^6 C_3 \wedge X_8, \quad (4.1)$$

where X_8 is related to the M5-brane anomaly polynomial by descent,

⁷The Hodge duality relation between G_7 and G_4 is expected to receive higher-order corrections [see below (4.11)]. These can be neglected here since ΔS is already a higher-order correction.

$$X_8 = \text{tr}R^4 - \frac{1}{4}(\text{tr}R^2)^2, \quad (4.2)$$

and we have set $(\text{tr}R^2)^2 := \text{tr}R^2 \wedge \text{tr}R^2$, $\text{tr}R^4 := R_a{}^b \wedge R_b{}^c \wedge R_c{}^d \wedge R_d{}^a$. At eight derivatives the modified four- and seven-form BI read

$$dG_4 = 0; \quad dG_7 + \frac{1}{2}G_4 \wedge G_4 = l^6 X_8, \quad (4.3)$$

where a numerical coefficient has been absorbed in the definition of l . We expand the forms perturbatively in l ,

$$G_4 = G_4^{(0)} + l^6 G_4^{(1)} + \dots; \quad G_7 = G_7^{(0)} + l^6 G_7^{(1)} + \dots, \quad (4.4)$$

and similarly for the supercurvature $R_A{}^B$. Note that in the expansion above the bosonic components of the lowest-order fields, $G_{m_1 \dots m_4}^{(0)}$ etc., are identified with the field strengths of the supergravity multiplet, while the higher-order fields $G_4^{(1)}$ etc. are composite higher-derivative fields that are polynomial in the field strengths of the supergravity fields.

Solving perturbatively the BI at each order in l , taking into account that the exterior superderivative $d = dz^M \partial_M$ is zeroth order in l , implies

$$\begin{aligned} dG_4^{(0)} = 0; \quad dG_4^{(1)} = 0; \\ dG_7^{(0)} + \frac{1}{2}G_4^{(0)} \wedge G_4^{(0)} = 0; \quad dG_7^{(1)} + G_4^{(1)} \wedge G_4^{(0)} = X_8^{(1)}, \end{aligned} \quad (4.5)$$

where we have set $l^6 X_8 = l^6 X_8^{(1)} + \dots$. Note that $X_8^{(1)}$ only involves the lowest-order curvature $R^{(0)}$. Let us expand the twelve-form W_{12} perturbatively in l , $W_{12} = l^6 W_{12}^{(1)} + \dots$, so that

$$W_{12}^{(1)} = X_8^{(1)} \wedge G_4^{(0)} = dZ_{11}; \quad Z_{11} = X_8^{(1)} \wedge C_3^{(0)}. \quad (4.6)$$

It then follows from (4.5) that this can also be written in a manifestly Weil-trivial form as follows:

$$W_{12}^{(1)} = dK_{11}; \quad K_{11} = G_7^{(1)} \wedge G_4^{(0)} - 2G_7^{(0)} \wedge G_4^{(1)}. \quad (4.7)$$

In particular, we see that it suffices to solve the four- and seven-form BI in order to determine the order- l^6 superinvariant corresponding to $L_{11} = Z_{11} - K_{11}$,

$$\boxed{\Delta S = l^6 \int (X_8^{(1)} \wedge C_3^{(0)} - G_7^{(1)} \wedge G_4^{(0)} + 2G_7^{(0)} \wedge G_4^{(1)})}, \quad (4.8)$$

where it is understood that only the bosonic (11,0) components of the forms enter. This is the superinvariant corresponding to the supersymmetrization of the CS term (4.1). The action would then read to this order,

$$\begin{aligned} S = \int \left(R^{(0)} \star 1 - \frac{1}{2} G_4^{(0)} \wedge \star G_4^{(0)} - \frac{1}{6} C_3^{(0)} \wedge G_4^{(0)} \wedge G_4^{(0)} \right. \\ \left. + l^6 (X_8^{(1)} \wedge C_3^{(0)} - G_7^{(1)} \wedge G_4^{(0)} + 2G_7^{(0)} \wedge G_4^{(1)}) \right), \end{aligned} \quad (4.9)$$

where $R^{(0)}$, $G^{(0)}$ are identified with the field strengths of the physical fields in the supergravity multiplet, while the first-order fields $R^{(1)}$, $G^{(1)}$ should be thought of as gauge-invariant functions of the physical fields. We see that the action above is in agreement with the expectation that the bosonic part of the derivative-corrected supergravity action should be of the form

$$\begin{aligned} S = \int \left(R \star 1 - \frac{1}{2} G_4 \wedge \star G_4 - \frac{1}{6} C_3 \wedge G_4 \wedge G_4 \right. \\ \left. + l^6 (X_8 \wedge C_3 + \Delta L \star 1) \right), \end{aligned} \quad (4.10)$$

with ΔL being a function of R , G and their derivatives. Since ΔL is gauge invariant, we see, in particular, that the CS terms do not receive higher-order corrections beyond eight derivatives.

Varying (4.10) with respect to C_3 implies

$$d \star G_4 + \frac{1}{2} G_4 \wedge G_4 = X_8 + \frac{\delta}{\delta C_3} (\Delta L \star 1). \quad (4.11)$$

It is straightforward to see that the second term on the right-hand side above is exact by virtue of the fact that ΔL is gauge invariant and thus only depends on C_3 through G_4 . Indeed the variation of the C_3 -dependent terms in the ΔL part of the action (4.10) can be written (possibly up to integration by parts) in the form $\int \Phi_7 \wedge d\delta C_3$, for some seven-form Φ_7 . Therefore by appropriately correcting the lowest-order duality relation by higher-derivative terms, $G_7 = \star G_4 + \mathcal{O}(l^6)$, one arrives at the modified BI (4.3).

A. How many superinvariants?

We have seen that provided the modified BI (4.3) are satisfied, there will be at least one superinvariant at eight derivatives, cf., (4.8). A second independent superinvariant can also be similarly constructed as follows. Consider the twelve form,

$$W'_{12} = \frac{1}{6} G_4 \wedge G_4 \wedge G_4. \quad (4.12)$$

Expanding perturbatively to order l^6 we obtain

$$\begin{aligned} W'_{12} &= \frac{1}{2} G_4^{(0)} \wedge G_4^{(0)} \wedge G_4^{(1)} = dZ_{11}; \\ Z_{11} &= \frac{1}{2} G_4^{(1)} \wedge G_4^{(0)} \wedge C_3^{(0)}. \end{aligned} \quad (4.13)$$

The above can also be written in a manifestly Weil-trivial form using (4.5),

$$W'_{12} = dK_{11}; \quad K_{11} = -G_7^{(0)} \wedge G_4^{(1)}. \quad (4.14)$$

The order- l^6 superinvariant corresponding to $Z_{11} - K_{11}$ then reads

$$\Delta S' = l^6 \int G_4^{(1)} \wedge \left(\frac{1}{2} G_4^{(0)} \wedge C_3^{(0)} + G_7^{(0)} \right), \quad (4.15)$$

where it is understood that only the bosonic (11,0) components of the forms enter. The above superinvariant does not contain the correct CS terms required by anomaly

$$\Delta S_U = l^6 \int (\text{tr} R^4 \wedge C_3^{(0)} - K_U); \quad \Delta S_V = l^6 \int ((\text{tr} R^2)^2 \wedge C_3^{(0)} - K_V). \quad (4.17)$$

By the argument at the end of the last section, $\Delta S_U, \Delta S_V$ should correspond to a modified BI obtained by replacing the right-hand side of the second equation in (4.3) by $\text{tr} R^4, (\text{tr} R^2)^2$ respectively. Then K_U, K_V would still be given by (4.7) but with $G_4^{(1)}, G_7^{(1)}$ solutions of the new modified BI.

Together with the superinvariant $\Delta S'$ of (4.15), we would then have a total of at least three independent superinvariants at the eight-derivative order, with only one linear combination thereof, ΔS of (4.8), corresponding to the quantum-mechanically consistent eight-derivative correction. As we see in Sec. IV B, if $\Delta S_U, \Delta S_V$ exist they must necessarily be cubic or lower in the fields.

B. τ -exactness of X_8

Based on what is known about superinvariants in $D < 11$ dimensions [66], it is plausible to assume that the superinvariant (4.8) corresponding to the supersymmetrization of the CS term (4.1) should be quartic or higher in the fields. As pointed out in [19], a necessary condition for the superinvariant to be quartic is that the order- l^6 seven form should be quartic or higher in the fields. Since $G_{0,7}^{(1)}$ cannot be quartic or higher in the fields, as can be seen by dimensional analysis, the order- l^6 seven-form BI (4.5) must be solved for $G_{0,7}^{(1)} = 0$. It then follows that the purely spinorial component of the

cancellation, cf., (4.10), and should therefore be excluded by the requirement of quantum consistency of the theory. However if one is only interested in counting superinvariants at order l^6 in the classical theory, the above superinvariant is perfectly acceptable and its existence is guaranteed provided the BI are obeyed to order l^6 .

Dropping the requirement of quantum consistency, relying on classical supersymmetry alone, one may also consider the following two twelve forms,

$$U_{12} = l^6 G_4 \wedge \text{tr} R^4; \quad V_{12} = l^6 G_4 \wedge (\text{tr} R^2)^2, \quad (4.16)$$

so that $U - \frac{1}{4}V$ is the Weil-trivial twelve form corresponding to the CS terms of eleven-dimensional supergravity required for anomaly cancellation, cf., (4.1). It follows that U and V are both Weil trivial, or neither U nor V is Weil trivial. If the former is true, there would exist gauge-invariant eleven forms K_U, K_V so that at order l^6 we have $U^{(1)} = dK_U, V^{(1)} = dK_V$. One can then construct two corresponding superinvariants using the action principle,

M5-brane anomaly eight form $X_{0,8}$ is τ -exact. Explicitly, the first nontrivial component (at dimension four) of the seven-form BI then reads

$$\gamma_{(\alpha_1 \alpha_2 |}^f G_{f | \alpha_3 \dots \alpha_8)}^{(1)} = X_{\alpha_1 \dots \alpha_8}^{(1)}. \quad (4.18)$$

As explained in detail in Appendix E, taking the form of $G^{(0)}$ into account, cf., (E1), it follows that the Weil-triviality condition,

$$W_{12} = dK_{11}, \quad (4.19)$$

is solved up to dimension 7/2 for $K_{0,11} = K_{1,10} = K_{2,9} = 0$. At dimension 4, condition (4.19) then takes the form

$$\gamma_{(\alpha_1 \alpha_2 |}^f K_{f ab | \alpha_3 \dots \alpha_{10})} = W_{aba_1 \dots \alpha_{10}}^{(1)}. \quad (4.20)$$

From this it follows that (4.19) is solved, up to τ -exact terms, for $K_{3,8}$ given in terms of $G_{1,6}^{(1)}$, cf., (4.18),

$$K_{abca_1 \dots \alpha_8} = 3(\gamma_{ab})_{\alpha_1 \alpha_2} G_{ca_3 \dots \alpha_8}^{(1)}, \quad (4.21)$$

where it is understood that all bosonic (spinor) indices are antisymmetrized (symmetrized). Note that the solution for $K_{3,8}$ above relies on the M2-brane identity (2.14).

Moreover, it can be shown that all higher components of K_{11} solving (4.19) are automatically guaranteed to exist. To see this, let us define the twelve form,

$$I_{12} := W_{12} - dK_{11}, \quad (4.22)$$

which is closed by construction,

$$\begin{aligned} 0 = (dI)_{p,13-p} &= \tau I_{p+1,11-p} + d_f I_{p,12-p} \\ &+ d_b I_{p-1,13-p} + t I_{p-2,14-p}. \end{aligned} \quad (4.23)$$

On the other hand, as we saw above, provided (4.18) holds, condition (4.19) is solved up to dimension 4, i.e., $I_{p,12-p} = 0$ for $p = 0, 1, 2$. Setting $p = 2$ in (4.23) then gives $\tau I_{3,9} = 0$, which implies $I_{3,9} = 0$ up to a τ -exact piece that can be absorbed in $K_{4,7}$, since all τ -cohomology groups $H_\tau^{p,12-p}$ are trivial for $p \geq 3$, cf., Sec. II B. By induction we easily see that $I_{p,12-p} = 0$, for all $p \geq 3$. In other words, provided (4.18) holds, the Weil-triviality condition (4.19) is guaranteed to admit a solution.

In the present paper we provide highly nontrivial evidence corroborating (4.18). We give the outline of the argument here, relegating the technical details to Appendix C. The component $X_{0,8}$ of the anomaly polynomial in (4.18) contains a large number of terms of the form G^4 , which can be organized in terms of irreducible representations of B_5 . Using certain Fierz identities, cf., Appendix D, we have been able to show that almost all of these terms are indeed τ exact. There are only three irreducible representations of B_5 corresponding to terms that can potentially be present in $X_{0,8}$ and are not τ exact. These are (04000), (03002), and (02004), where we use the Dynkin notation for B_5 ; see, e.g., Appendix C of [19].

On the other hand we show that, after Fierzing, $X_{0,8}$ can be put in the form

$$\begin{aligned} X_{0,8} &= (\gamma^{a_1 a_2})(\gamma^{a_3 a_4})(\gamma^{a_5 a_6})(\gamma^{a_7 a_8}) G_{a_1 a_2; a_3 a_4; a_5 a_6; a_7 a_8}^4 \\ &+ (\gamma^{a_1 a_2})(\gamma^{a_3 a_4})(\gamma^{a_5 a_6})(\gamma^{a_7 \dots a_{12}}) G_{a_1 a_2; a_3 a_4; a_5 a_6; a_7 \dots a_{12}}^4 \\ &+ (\gamma^{a_1 a_2})(\gamma^{a_3 a_4})(\gamma^{a_5 \dots a_9})(\gamma^{a_{10} \dots a_{14}}) \\ &\times G_{a_1 a_2; a_3 a_4; a_5 \dots a_9; a_{10} \dots a_{14}}^4, \end{aligned} \quad (4.24)$$

where $G_{a_1 a_2; \dots; a_7 a_8}^4$, $G_{a_1 a_2; \dots; a_7 \dots a_{12}}^4$, $G_{a_1 a_2; \dots; a_{10} \dots a_{14}}^4$ denote certain sums of G^4 terms with 8,4,2 indices contracted respectively, cf., (C6), and we have suppressed spinorial indices for simplicity of notation. Furthermore we show that (04000) can only be potentially present in the projection of $G_{a_1 a_2; \dots; a_7 a_8}^4$ onto the Young diagram associated to the partition [4, 4], while (02004) can only be potentially present in the projection of $G_{a_1 a_2; \dots; a_{10} \dots a_{14}}^4$ onto the Young diagram [4, 4, 2, 2, 2]. Therefore a necessary condition for $X_{0,8}$ to be τ exact is that the two aforementioned projections should vanish identically up to τ -exact terms,

$$\Pi_{\square\square\square\square} G_{a_1 a_2; \dots; a_7 a_8}^4 \approx 0; \quad \Pi_{\square\square\square\square} G_{a_1 a_2; \dots; a_{10} \dots a_{14}}^4 \approx 0. \quad (4.25)$$

In the above \approx denotes equality up to τ -exact terms. These two constraints are highly nontrivial, involving seemingly miraculous cancellations between hundreds of terms. We have shown that, remarkably, (4.25) are indeed identically satisfied.

Furthermore we show that the required cancellations for (4.25) to hold crucially rely on the relative coefficient between $\text{tr}R^4$ and $(\text{tr}R^2)^2$ in X_8 . In other words we show that the purely spinorial components of $\text{tr}R^4$, $(\text{tr}R^2)^2$ are *not* separately τ exact. Consequently, if the twelve forms U , V are Weil trivial, the corresponding modified order- l^6 BI is solved for some $G_7^{(1)}$ that are cubic or lower in the fields. [Indeed if $G_7^{(1)}$ were quartic or higher, $G_{0,7}^{(1)}$ would vanish and the purely spinorial components of $\text{tr}R^4$, $(\text{tr}R^2)^2$ would be τ exact.] It then follows from (4.7) that also K_U , K_V are cubic or lower, and similarly for ΔS_U , ΔS_V , cf., (4.17).

C. Integrability

The perturbative expansion of the curved components following from (4.4) reads

$$G_{M_1 \dots M_4} = G_{M_1 \dots M_4}^{(0)} + l^6 G_{M_1 \dots M_4}^{(1)} + \dots, \quad (4.26)$$

and similarly for G_7 and $R_A{}^B$. Note that in terms of flat components there is a mixing between zeroth order and order l^6 due to

$$\Phi = E^A \Phi_A = E^{(0)A} \Phi_A^{(0)} + l^6 (E^{(0)A} \Phi_A^{(1)} + E^{(1)A} \Phi_A^{(0)}) + \dots, \quad (4.27)$$

where we have expanded the coframe, $E^A = E^{(0)A} + l^6 E^{(1)A} + \dots$, and we have considered an arbitrary one-form Φ for simplicity. However, if one restricts to the top bosonic component of a superform at $\theta = 0$ as in (3.1), then there is no mixing,

$$\begin{aligned} \Phi_m^{(0)} &| = e_m{}^a \Phi_a^{(0)} | + \psi_m^\alpha \Phi_\alpha^{(0)} |; \\ \Phi_m^{(1)} &| = e_m{}^a \Phi_a^{(1)} | + \psi_m^\alpha \Phi_\alpha^{(1)} |, \end{aligned} \quad (4.28)$$

where $e_m{}^a$, ψ_m^α were defined below (3.3). Indeed the $\mathcal{O}(l^6)$ corrections to the coframe E^A only start at higher orders in the θ -expansion and could be systematically determined as in, e.g., [67] once the $\mathcal{O}(l^6)$ corrections to the torsion components have been determined.

In practice the BI are solved for the flat components of the superforms involved, $G_{A_1 \dots A_4}^{(0)}$, $G_{A_1 \dots A_4}^{(1)}$, etc., at each order in l . Consequently the corresponding BI, $dG_4 = 0$ etc., are only shown to be satisfied up to terms of the next

order in l . In principle, there may be an integrability obstruction to the solution of the BI at next-to-leading order in the derivative corrections, although that would most probably be prohibitively difficult to check in practice. In the following we see that the integrability of a certain superinvariant is guaranteed provided the BI admit solutions to all orders in l . Note however that all-order integrability need not be a consequence of the BI.

The phenomenon of inducing a higher-order correction at next-to-leading order is also well understood at the level of the component action, $S = S^{(0)} + l^6 S^{(1)} + \dots$. The condition of invariance of the action under supersymmetry transformations $\delta = \delta^{(0)} + l^6 \delta^{(1)} + \dots$ reads

$$\delta^{(0)} S^{(0)} = 0; \quad \delta^{(0)} S^{(1)} + \delta^{(1)} S^{(0)} = 0, \quad (4.29)$$

and similarly at higher orders. The term $\delta^{(1)} S^{(0)}$ in the second equation above is proportional to the lowest-order equations of motion. Therefore in constructing $S^{(1)}$ we only need to check its invariance with respect to the lowest-order supersymmetry transformations $\delta^{(0)}$ and only up to terms that vanish by virtue of the lowest-order equations of motion. This corresponds, in the superspace approach, to the fact that in solving the first-order BI one uses the zeroth-order equations for the various superfields. Once $S^{(1)}$ is thus constructed, the correction $\delta^{(1)}$ to the supersymmetry transformations can be read off. Since $\delta^{(1)} S^{(1)} \neq 0$ in general, this induces a correction $S^{(2)}$ to the action and a corresponding correction $\delta^{(2)}$ to the supersymmetry transformations, and so on.

The existence of an integrability obstruction can also be understood in the context of the Noether procedure. Indeed at next-to-leading order we have

$$\delta^{(2)} S^{(0)} + \delta^{(1)} S^{(1)} + \delta^{(0)} S^{(2)} = 0. \quad (4.30)$$

Therefore there must exist an action $S^{(2)}$ such that its variation with respect to lowest-order supersymmetry transformations is equal to $-\delta^{(1)} S^{(1)}$, up to terms that vanish by virtue of the lowest-order equations of motion. This condition is not automatically satisfied for every $S^{(1)}$.

In particular, one would like to know how many of the independent superinvariants at order l^6 presented in Sec. IV A survive to all orders in the derivative expansion. Assuming M-theory is a nonperturbatively consistent theory, we expect the superinvariant (4.8), corresponding to the supersymmetrization of the CS term required for anomaly cancellation, to be integrable to all orders. Moreover, assuming this superinvariant is at least quartic in the fields, a similar argument as the one detailed below (4.23) shows that it must be unique at order l^6 [19].

In addition, if one assumes that the BI admit a solution to all orders in a perturbative expansion in l , then there is one linear combination of the superinvariants presented in

Sec. IV A that is guaranteed to exist to all orders in l . Indeed in that case the twelve form,

$$W_{12} = \left(l^6 X_8 - \frac{1}{2} G_4 \wedge G_4 \right) \wedge G_4 = d(G_7 \wedge G_4), \quad (4.31)$$

is Weil trivial by virtue of (4.3), which should now be considered valid to all orders in l . However this is *not* the superinvariant that corresponds to the supersymmetrization of the anomaly term, cf., (4.8). Indeed by the usual action principle procedure the twelve form above would give rise to the superinvariant,

$$\Delta S = \int \left(l^6 X_8 \wedge C_3 - \frac{1}{2} G_4 \wedge G_4 \wedge C_3 - G_7 \wedge G_4 \right) \Big| . \quad (4.32)$$

Expanding to order l^6 and assuming G_4 receives a non-vanishing correction at this order, we see that (4.32) does not coincide with (4.8) and the corresponding l^6 -corrected action is different from (4.9).

In conclusion, under the aforementioned assumptions, we would then expect (at least) two independent superinvariants to exist to all orders in a perturbative expansion in l . Only one of these, the one corresponding to the supersymmetrization of the CS anomaly term, is quantum-mechanically consistent.

V. DISCUSSION

We have shown that the highly nontrivial constraints (4.25) are satisfied, corroborating the expectation that the purely spinorial component of X_8 is τ exact. Furthermore we have seen that the τ exactness of $X_{0,8}$ suffices for the existence of the superinvariant at order l^6 . Solving the τ exactness of $X_{0,8}$ is the first step, and arguably the most difficult, towards the explicit construction, via the action principle approach, of the supersymmetrization of the Chern-Simons term $C \wedge X_8$ of eleven-dimensional supergravity required for the quantum consistency of the theory.

Conclusively proving the τ exactness of $X_{0,8}$ would in addition require checking that the representation (03002) is absent from $X_{0,8}$. This representation is potentially present in two different Young diagrams. As a consequence, showing the cancellation would involve, after projecting onto the appropriate Young diagram, Fierzing hundreds of four- γ terms. This is equivalent to eight-spinor Fierzing, as opposed to the four-spinor Fierzing that is sufficient in order to show the absence of the (04000) and (02004) representations. At present, this seems prohibitively difficult even with the help of a computer.

As a corollary of this work, we have shown that if the anomalous Chern-Simons terms $C \wedge (\text{Tr} R^2)^2$ and

$C \wedge \text{Tr}R^4$ can be supersymmetrized independently, the corresponding superinvariants must necessarily contain terms cubic or lower in the fields. The existence of eleven-dimensional cubic superinvariants at the eight-derivative order has not been examined in the past. Their existence would presumably imply, by dimensional reduction, the presence of cubic terms in the ten-dimensional superinvariants I_{0a} , I_{0b} mentioned in the introduction. This would not be inconsistent with the results of [4] whose authors have excluded from the outset such terms in their analysis. This is an interesting open question to which we hope to return in the future.

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APPENDIX A: WEIL TRIVIALITY AT ℓ^0

In this section we give the details of the solution of the superspace equation $W_{12} = dK_{11}$ at lowest order in the Planck length. As a byproduct we see that the solution for K_{11} given in (3.6) is unique up to exact terms. We look for the solution to

$$dK_{11} = -\frac{1}{6}G_4 \wedge G_4 \wedge G_4, \quad (\text{A1})$$

with K_{11} gauge invariant, i.e., function of the field strengths of the physical fields. The explicit construction of K_{11} in

$$\begin{aligned} & \frac{7}{12}D_{a_1}K_{\alpha_2\dots\alpha_7a_1\dots a_5} - \frac{5}{12}D_{a_1}K_{\alpha_2\dots\alpha_5a_1\dots\alpha_7} \\ & + \frac{11}{2}\left(\frac{5}{33}T_{\alpha_1\alpha_2}{}^f K_{f\alpha_3\dots\alpha_7a_1\dots a_5} - \frac{7}{22}T_{a_1a_2}{}^\gamma K_{\gamma\alpha_3\dots\alpha_5a_1\dots\alpha_7} - \frac{35}{66}T_{a_1\alpha_1}{}^\gamma K_{\gamma\alpha_2\dots\alpha_7a_2\dots a_5}\right) = 0, \end{aligned}$$

and involves⁸ $K_{\alpha_1\dots\alpha_5a_1\dots a_6}^{(-1/2)}$, $K_{\alpha_1\dots\alpha_6a_1\dots a_5}^{(-1)}$, $K_{\alpha_1\dots\alpha_7a_1\dots a_4}^{(-3/2)}$, and $K_{\alpha_1\dots\alpha_8a_1\dots a_3}^{(-2)}$, which cannot be expressed in terms of the physical fields: the equation is thus trivially satisfied.

2. Dimension 0 ($A_1\dots A_6 \rightarrow \alpha_1\dots\alpha_6$, $A_7\dots A_{12} \rightarrow a_1\dots a_6$)

At dimension 0, Eq. (A2) reads

$$\begin{aligned} & \frac{1}{2}D_{a_1}\overbrace{K_{\alpha_2\dots\alpha_6a_1\dots a_6}}^0 + \frac{1}{2}D_{a_1}\overbrace{K_{\alpha_2\dots\alpha_6a_1\dots a_6}}^0 + \frac{11}{2}\left(\frac{5}{22}T_{\alpha_1\alpha_2}{}^f K_{f\alpha_3\dots\alpha_6a_1\dots a_6} + \frac{5}{22}T_{a_1a_2}{}^\gamma \underbrace{K_{\gamma\alpha_3\dots\alpha_6a_1\dots a_6}}_0 + \frac{12}{22}T_{a_1\alpha_1}{}^\gamma \underbrace{K_{\gamma\alpha_2\dots\alpha_6a_2\dots a_6}}_0\right) \\ & = -\frac{11!}{6(4!)^3}\frac{18}{77}G_{a_1a_2\alpha_1a_2}G_{a_3a_4\alpha_3\alpha_4}G_{a_5a_6\alpha_5\alpha_6}. \end{aligned}$$

⁸In the following we use superscripts to indicate the dimension. This should not be confused with the notation in the main text, e.g., (4.4) where the superscript denotes the order in the derivative expansion.

flat components proceeds by solving the BI at each engineering dimension in a stepwise fashion, from dimension -3 to 2 (i.e., from $K_{\alpha_1\dots\alpha_{11}}$ to $K_{a_1\dots a_{11}}$). In components the BI (A1) reads

$$\begin{aligned} & D_{[A_1}K_{A_2\dots A_{12}]} + \frac{11}{2}T_{[A_1A_2]}^F K_{F|A_3\dots A_{12})} \\ & = -\frac{11!}{6(4!)^3}G_{[A_1\dots A_4}G_{A_5\dots A_8}G_{A_9\dots A_{12}]}, \quad (\text{A2}) \end{aligned}$$

where the torsion term arises from the action of the exterior derivative on the supervielbein. The $[ABC)$ notation stands for symmetrization or antisymmetrization, depending on the bosonic or fermionic nature of the indices. In the following, antisymmetrization of the indices a_i and symmetrization of the indices α_i is always implied.

The engineering (mass) dimensions of the physical fields that are involved in the construction of K_{11} are

$$\begin{aligned} [D_{a_1}] &= 1 & [T_{a_1a_2}{}^\alpha] &= 3/2 & [G_{\alpha\beta\alpha\beta}] &= [T_{\alpha\beta}{}^\alpha] = 0 \\ [D_{\alpha_1}] &= 1/2 & [T_{\alpha\alpha}{}^\beta] &= 1 & [G_{abcd}] &= 1. \end{aligned}$$

1. From dimension -3 to $-1/2$

From dimension -3 (12 odd indices) to $-1/2$ (seven odd and five even indices), the right-hand side of (A1) always vanishes. Given the dimensions of the field strengths of the physical fields, the first nonvanishing component of K_{11} is $K_{\alpha_1\dots\alpha_4a_1\dots a_7}$, appearing for the first time in the 0-dimensional equation (six fermionic indices and six bosonic indices). For example, the equation (A2) at dimension $-1/2$ reads

Most terms vanish and the equation simplifies as follows:

$$(\gamma^f)_{\alpha_1\alpha_2} K_{fa_1\dots a_6\alpha_3\dots\alpha_6} = 90(\gamma_{a_1a_2})_{\alpha_1\alpha_2}(\gamma_{a_3a_4})_{\alpha_3\alpha_4}(\gamma_{a_5a_6})_{\alpha_5\alpha_6}.$$

Using the M2-brane identity as well as the so-called M5-brane identity,

$$(\gamma^e)_{\alpha_1\alpha_2}(\gamma_{ea_1\dots a_4})_{\alpha_3\alpha_4} = 3(\gamma_{a_1a_2})_{\alpha_1\alpha_2}(\gamma_{a_3a_4})_{\alpha_3\alpha_4}, \quad (\text{A3})$$

it is easy to check that the solution is given by

$$K_{a_1\dots a_7\alpha_1\dots\alpha_4} = 42(\gamma_{a_1\dots a_5})_{\alpha_1\alpha_2}(\gamma_{a_6a_7})_{\alpha_3\alpha_4}. \quad (\text{A4})$$

3. Dimension 1/2 ($A_1\dots A_5 \rightarrow \alpha_1\dots\alpha_5$, $A_6\dots A_{12} \rightarrow a_1\dots a_7$)

At dimension 1/2, Eq. (A2) reads

$$\begin{aligned} & \frac{5}{12} \overbrace{D_{\alpha_1} K_{\alpha_2\dots\alpha_5 a_1\dots a_7}}^{D_a(\gamma^{(5)}\gamma^{(2)})=0} - \frac{7}{12} D_{a_1} \overbrace{K_{a_2\dots a_7\alpha_1\dots\alpha_5}}^0 \\ & + \frac{11}{2} \left(\frac{5}{33} T_{\alpha_1\alpha_2}{}^f K_{f\alpha_3\dots\alpha_5 a_1\dots a_7} - \frac{35}{66} T_{a_1 a_2}{}^\gamma \underbrace{K_{\gamma a_3\dots a_7\alpha_1\dots\alpha_5}}_0 - \frac{7}{22} T_{a_1\alpha_1}{}^\gamma \underbrace{K_{\gamma a_2\dots\alpha_6 a_2\dots a_6}}_0 \right) = 0, \end{aligned}$$

which simplifies to

$$(\gamma^f)_{\alpha_1\alpha_2} K_{fa_1\dots a_7\alpha_3\dots\alpha_5} = 0.$$

Since $[K_{fa_1\dots a_7\alpha_3\dots\alpha_5}] = 1/2$ and there is no gauge-invariant field with that dimension, we conclude that

$$K_{a_1\dots a_8\alpha_1\dots\alpha_3} = 0. \quad (\text{A5})$$

4. Dimension 1 ($A_1\dots A_4 \rightarrow \alpha_1\dots\alpha_4$, $A_5\dots A_{12} \rightarrow a_1\dots a_8$)

At dimension 1, Eq. (A2) reads

$$\begin{aligned} & \frac{4}{12} D_{\alpha_1} \overbrace{K_{\alpha_2\dots\alpha_4 a_1\dots a_8}}^0 + \frac{8}{12} \overbrace{D_{a_1} K_{a_2\dots a_8\alpha_1\dots\alpha_4}}^{d_{a_1}(\gamma^{(5)}\gamma^{(2)})=0} \\ & + \frac{11}{2} \left(\frac{1}{11} T_{\alpha_1\alpha_2}{}^f K_{f\alpha_3\alpha_4 a_1\dots a_8} + \frac{14}{33} T_{a_1 a_2}{}^\gamma \underbrace{K_{\gamma a_3\dots a_8\alpha_1\dots\alpha_4}}_0 + \frac{16}{33} T_{a_1\alpha_1}{}^\gamma K_{\gamma a_2\dots\alpha_4 a_2\dots a_8} \right) \\ & = -\frac{11!}{6(4!)^3} \frac{12}{55} G_{a_1 a_2 a_3 a_4} G_{a_5 a_6 a_1 a_2} G_{a_7 a_8 a_3 a_4}, \end{aligned}$$

which becomes, using (E1),

$$\begin{aligned} (\gamma^f)_{\alpha_1\alpha_2} K_{fa_1\dots a_8\alpha_3\alpha_4} & = -\frac{56}{3} i G_{a_1 a_2 a_3 f} (\gamma^f)_{\alpha_1\alpha_2} (\gamma_{a_4\dots a_8})_{\alpha_3\alpha_4} \\ & + \frac{7}{18} i G_{fghi} (\gamma_{a_1\dots a_6}{}^{fghi})_{\alpha_1\alpha_2} (\gamma_{a_7 a_8})_{\alpha_3\alpha_4} \\ & + 70i (\gamma_{a_1 a_2})_{\alpha_1\alpha_2} (\gamma_{a_3 a_4})_{\alpha_3\alpha_4} G_{a_5\dots a_8}. \end{aligned} \quad (\text{A6})$$

The last term above can be expanded as

$$\begin{aligned} & 70i (\gamma_{a_1 a_2})_{\alpha_1\alpha_2} (\gamma_{a_3 a_4})_{\alpha_3\alpha_4} G_{a_5\dots a_8} \\ & = \frac{70}{3} i (\gamma^f)_{\alpha_1\alpha_2} (\gamma_{fa_1\dots a_4})_{\alpha_3\alpha_4} G_{a_5\dots a_8} \\ & = 42i (\gamma^f)_{\alpha_1\alpha_2} (\gamma_{[fa_1\dots a_4]})_{\alpha_3\alpha_4} G_{[a_5\dots a_8]} \\ & \quad - \frac{56}{3} i (\gamma^f)_{\alpha_1\alpha_2} (\gamma_{[a_1\dots a_5]})_{\alpha_3\alpha_4} G_{[a_6 a_7 a_8]f}. \end{aligned}$$

Similarly, the second term on the right-hand side of (A6) can be written in a manifestly τ -exact form,

$$\begin{aligned} & \frac{7}{18} (\gamma_{a_1\dots a_6}{}^{fghi})_{\alpha_1\alpha_2} (\gamma_{a_7 a_8})_{\alpha_3\alpha_4} \\ & = -\frac{7}{18} \epsilon_{ja_1\dots a_6}{}^{fghi} (\gamma^j)_{\alpha_1\alpha_2} (\gamma_{a_7 a_8})_{\alpha_3\alpha_4} \\ & = -\frac{1}{2} \epsilon_{[ja_1\dots a_6]}{}^{fghi} (\gamma^j)_{\alpha_1\alpha_2} (\gamma_{|a_7 a_8|})_{\alpha_3\alpha_4} \\ & \quad + \frac{1}{9} \epsilon_{a_1\dots a_7}{}^{fghi} \underbrace{(\gamma^j)_{\alpha_1\alpha_2} (\gamma_{a_8 j})_{\alpha_3\alpha_4}}_0. \end{aligned}$$

Then Eq. (A6) takes the following form:

$$\begin{aligned} (\gamma^j)_{\alpha_1\alpha_2} K_{ja_1\dots a_8\alpha_3\alpha_4} & = (\gamma^j)_{\alpha_1\alpha_2} (42i (\gamma_{[ja_1\dots a_4]})_{\alpha_3\alpha_4} G_{a_5\dots a_8}] \\ & \quad - \frac{1}{2} i \epsilon_{[ja_1\dots a_6]}{}^{i_1\dots i_4} (\gamma_{|a_7 a_8|})_{\alpha_3\alpha_4} G_{i_1\dots i_4}. \end{aligned}$$

Since the cohomology group $H_\tau^{9,2}$ is trivial, the solution to the above equation reads

$$K_{a_1\dots a_9\alpha_1\alpha_2} = 42i(\gamma_{a_1\dots a_5})_{\alpha_1\alpha_2} G_{a_6\dots a_9} - \frac{1}{2}i\epsilon_{a_1\dots a_7}{}^{i_1\dots i_4}(\gamma_{a_8a_9})_{\alpha_1\alpha_2} G_{i_1\dots i_4},$$

up to τ -exact terms.

5. Dimension 3/2 ($A_1\dots A_3 \rightarrow \alpha_1\dots\alpha_3$, $A_4\dots A_{12} \rightarrow a_1\dots a_9$)

At dimension 3/2, Eq. (A2) reads

$$\begin{aligned} & \frac{3}{12}D_{a_1}K_{\alpha_2\alpha_3a_1\dots a_9} - \frac{9}{12}D_{a_1}\overbrace{K_{a_2\dots a_9\alpha_1\dots\alpha_3}}^0 \\ & - \frac{11}{2}\left(\frac{1}{22}T_{\alpha_1\alpha_2}{}^f K_{fa_1\dots a_9\alpha_3} + \frac{6}{11}T_{a_1a_2}{}^\gamma K_{\gamma a_3\dots a_9\alpha_1\dots\alpha_3} + \frac{9}{22}T_{a_1\alpha_1}{}^\gamma \underbrace{K_{\gamma\alpha_2\alpha_3a_2\dots a_9}}_0\right) = 0, \end{aligned}$$

which becomes, using (E1),

$$\begin{aligned} (\gamma^f)_{\alpha_1\alpha_2}K_{fa_1\dots a_9\alpha_3} & = +252(\gamma_{a_1\dots a_5})_{\alpha_2\alpha_3}(\gamma_{a_6a_7})_{\alpha_1\gamma}T_{a_8a_9}{}^\gamma \\ & - 3\epsilon_{a_1\dots a_7}{}^{i_1\dots i_4}(\gamma_{a_8a_9})_{\alpha_1\alpha_2}(\gamma_{i_1i_2})_{\alpha_3\gamma}T_{i_3i_4}{}^\gamma \\ & + 504(\gamma_{a_1a_2})_{(\alpha_1\alpha_2)}(\gamma_{a_3\dots a_7})_{|\alpha_3\gamma)}T_{a_8a_9}{}^\gamma. \end{aligned} \quad (\text{A7})$$

The decomposition of $K_{fa_1\dots a_9\alpha_3}$ in irreducible components is given by

$$(10000) \otimes (00001) = (10001) \oplus (00001),$$

whereas $T_{ab}{}^\alpha$ is in the representation (01001). It follows that

$$K_{a_1\dots a_{10}\alpha_1} = 0, \quad (\text{A8})$$

and moreover the right-hand side of (A7) must vanish identically. This can be verified by, e.g., taking the Hodge dual of $(\gamma_{i_1i_2})_{\alpha_3\gamma}$ in the second term of (A7), and using the γ -tracelessness of $T_{ab}{}^\gamma$, cf., (E3).

6. Dimension 2 ($A_1A_2 \rightarrow \alpha_1\alpha_2$, $A_3\dots A_{12} \rightarrow a_1\dots a_{10}$)

At dimension 2, Eq. (A2) reads

$$\begin{aligned} & \frac{2}{12}D_{a_1}K_{\alpha_2a_1\dots a_{10}} + \frac{10}{12}D_{a_1}K_{a_2\dots a_{10}\alpha_1\alpha_2} \\ & + \frac{11}{2}\left(\frac{1}{66}T_{\alpha_1\alpha_2}{}^f K_{fa_1\dots a_{10}} + \frac{10}{33}T_{a_1a_2}{}^\gamma K_{\gamma a_3\dots a_{10}\alpha_1\alpha_2} + \frac{15}{22}T_{a_1\alpha_1}{}^\gamma K_{\gamma\alpha_2a_2\dots a_{10}}\right) \\ & = -\frac{11!}{6(4!)^3}\frac{18}{66}G_{a_1a_2a_3a_4}G_{a_5a_6a_7a_8}G_{a_9a_{10}\alpha_1\alpha_2}, \end{aligned}$$

which becomes, using (E1),

$$\begin{aligned} (\gamma^f)_{\alpha_1\alpha_2}K_{fa_1\dots a_{10}} & = -10i(42i(\gamma_{a_2\dots a_6})_{\alpha_1\alpha_2}d_{a_1}G_{a_7\dots a_{10}} - \frac{1}{2}i\epsilon_{a_2\dots a_8}{}^{i_1\dots i_4}(\gamma_{a_9a_{10}})_{\alpha_1\alpha_2}d_{a_1}G_{i_1\dots i_4}) \\ & - 20iT_{a_1\alpha_1}{}^\epsilon(42i(\gamma_{a_2\dots a_6})_{\epsilon\alpha_2}G_{a_7\dots a_{10}} - \frac{1}{2}i\epsilon_{a_2\dots a_8}{}^{i_1\dots i_4}(\gamma_{a_9a_{10}})_{\epsilon\alpha_2}G_{i_1\dots i_4}) \\ & - 1575G_{a_1\dots a_4}G_{a_5\dots a_8}(\gamma_{a_9a_{10}})_{\alpha_1\alpha_2}. \end{aligned} \quad (\text{A9})$$

Multiplying by $\gamma^{(1)}$ and taking the trace leads to

$$K_{a_1\dots a_{11}} = \frac{1}{72}\epsilon_{a_1\dots a_{11}}G_{d_1\dots d_4}G^{d_1\dots d_4}. \quad (\text{A10})$$

On the other hand contracting (A9) with $\gamma^{(2)}$ or $\gamma^{(5)}$ imposes that the contraction of the right-hand side must be identically 0. This can indeed be straightforwardly verified using (E3).

7. Dimension 5/2 ($A_1 \rightarrow \alpha_1, A_2 \dots A_{12} \rightarrow a_1 \dots a_{11}$)

The equation at dimension 5/2 does not contain any additional information, but serves as a consistency check for the expressions we found for $K_{a_1 \dots a_{11}}$. It reads

$$\frac{1}{12} D_{\alpha_1} K_{a_1 \dots a_{11}} - \frac{11}{12} D_{a_1} K_{a_2 \dots a_{11} \alpha_1} - \frac{11}{2} \left(\frac{2}{12} T_{a_1 a_2}{}^\gamma K_{\gamma a_3 \dots a_{11} \alpha_1} - \frac{10}{12} T_{a_1 \alpha_1}{}^\gamma K_{\gamma a_2 \dots a_{11}} \right) = 0,$$

which becomes, using (E1) and (A10),

$$\frac{1}{72} \epsilon_{a_1 \dots a_{11}} D_{\alpha_1} G_{abcd} G^{abcd} - 330i G_{a_1 a_2 gh} (\gamma^{gh}{}_{a_3 \dots a_9} T_{a_{10} a_{11}})_{\alpha_1} + 2310i G_{a_1 \dots a_4} (\gamma_{a_5 \dots a_9} T_{a_{10} a_{11}})_{\alpha_1}.$$

Using (E2) and (D1) we then obtain the constraint

$$0 = \epsilon_{a_1 \dots a_{11}} T_{d_1 d_2}{}^\delta (\gamma_{d_3 d_4}{}_\delta)_{\alpha_1} G^{d_1 \dots d_4} + \frac{77}{4} \epsilon_{a_1 \dots a_5}{}^{b_1 \dots b_6} (\gamma_{b_1 \dots b_6} T_{a_6 a_7})_{\alpha_1} G_{a_8 \dots a_{11}} - 990 \epsilon_{a_1 \dots a_7}{}^{b_1 b_2 gh} (\gamma_{b_1 b_2} T_{a_8 a_9})_{\alpha_1} G_{a_{10} a_{11} gh}, \quad (\text{A11})$$

which can be seen to be automatically satisfied by contracting (A11) with $\epsilon_{a_1 \dots a_{11}}$. The next equation (of dimension 3) is trivially satisfied, since the purely bosonic

component of a twelve form vanishes automatically in eleven dimensions.

8. Action at $\mathcal{O}(\ell^0)$

We have thus constructed the explicit expression of all components of K_{11} and have seen that it is unique up to exact terms. Its purely bosonic component in particular takes the following form,

$$K^{(2)} = \frac{1}{72 \cdot 11!} \epsilon_{a_1 \dots a_{11}} G_{d_1 \dots d_4} G^{d_1 \dots d_4} dx^{a_1} \wedge \dots \wedge dx^{a_{11}} = \frac{1}{3} G \wedge \star G \quad (\text{A12})$$

$$= -R \star 1 + \frac{1}{2} G \wedge \star G, \quad (\text{A13})$$

where in this subsection we have reverted to bosonic conventions for bosonic forms. Using the action principle then leads to the CJS action of Sec. III A.

The last two equalities in (A12) above can be seen as follows. The volume element is defined as

$$dV = \star 1 = \frac{1}{11!} \epsilon_{a_1 \dots a_{11}} dx^{a_1} \wedge \dots \wedge dx^{a_{11}},$$

from which it follows that

$$G \wedge \star G = \frac{1}{7!(4!)^2} G_{a_1 \dots a_4} \epsilon_{a_5 \dots a_{11}}{}^{b_1 \dots b_4} G_{b_1 \dots b_4} \overbrace{dx^{a_1} \wedge \dots \wedge dx^{a_{11}}}^{-\epsilon^{a_1 \dots a_{11}} dV} = \frac{1}{4!} G_{d_1 \dots d_4} G^{d_1 \dots d_4} dV.$$

Moreover, taking the trace of the third relation of (E3) gives

$$R \star 1 = \frac{1}{144} G_{d_1 \dots d_4} G^{d_1 \dots d_4} dV = \frac{1}{6} G \wedge \star G.$$

APPENDIX B: WEIL TRIVIALITY AT ℓ^3

In this section we are looking for the solution to the equation,

$$dK_{11} = G_4 \wedge G_4 \wedge R^{ab} \wedge R_{ba}. \quad (\text{B1})$$

We construct all components of K_{11} explicitly and confirm that the solution of Sec. III B is unique up to exact terms. In components the equation above takes the following form:

$$D_{[A_1} K_{A_2 \dots A_{12}]} + \frac{11}{2} T_{[A_1 A_2]}^F K_{F|A_3 \dots A_{12}]} = \frac{11!}{4(4!)^2} R_{[A_1 A_2] c_1 c_2} R_{[A_3 A_4] c_2 c_1} G_{|A_5 \dots A_8} G_{A_9 \dots A_{12}]} \cdot \quad (\text{B2})$$

The dimensions of the physical fields are the same as before, with the addition of $[R_{abcd}] = 2$. The dimensions of the various components of K range from $-1/2$ ($K_{\alpha_1 \dots \alpha_{11}}$) to 5 ($K_{a_1 \dots a_{11}}$).

1. Dimension 0 to 3/2

Since the dimension of $K_{\alpha_1 \dots \alpha_{11}}$ is $-1/2$, it must be set to 0 as it cannot be expressed in terms of the physical fields. The equation of dimension 0 then takes the form

$$D_{a_1} \underbrace{K_{a_2 \dots a_{12}}}_0 + \frac{11}{2} T_{a_1 a_2}{}^f K_{f a_3 \dots a_{12}} = \frac{11!}{4(4!)^2} R_{a_1 a_2 c_1 c_2} R_{a_3 a_4}{}^{c_2 c_1} \underbrace{G_{a_5 \dots a_8} G_{a_9 \dots a_{12}}}_0,$$

which simplifies to

$$(\gamma^f)_{a_1 a_2} K_{f a_3 \dots a_{12}} = 0. \quad (\text{B3})$$

Since $[K_{f\alpha_3\dots\alpha_{12}}] = 0$ and $H_\tau^{1,10}$ is nontrivial, a τ -nonexact solution involving only γ -matrices could exist. In that case $K_{f\alpha_3\dots\alpha_{12}}$ would necessary transform as a scalar, since the only available gauge-invariant superfield of zero dimension is a constant. On the other hand,

$$(10000) \otimes (00001)^{\otimes_3 10} = 1 \times (00000) + \dots,$$

i.e., the decomposition of $K_{f\alpha_3\dots\alpha_{12}}$ contains a unique scalar combination. It follows that

$$K_{f\alpha_3\dots\alpha_{12}} \propto (\gamma_f)_{\alpha_3\alpha_4} (\gamma^a)_{\alpha_5\alpha_6} (\gamma_a)_{\alpha_7\alpha_8} (\gamma^b)_{\alpha_9\alpha_{10}} (\gamma_b)_{\alpha_{11}\alpha_{12}}. \quad (\text{B4})$$

However it can be verified that this expression does not satisfy Eq. (B3), unless $K_{a_1\alpha_3\dots\alpha_{12}} = 0$.

The right-hand side of Eq. (B2) vanishes from dimension 0 to dimension 3/2, and the equations to solve are all similar to (B3): The component $K_{a_1a_2\alpha_1\dots\alpha_9}^{(1/2)}$ are set to 0 because there is no gauge-invariant field of dimension 1/2. The components $K_{a_1a_2a_3\alpha_1\dots\alpha_8}^{(1)}$, $K_{a_1a_2a_3a_4\alpha_1\dots\alpha_7}^{(3/2)}$ are set to 0, up to exact terms, as a consequence of the triviality of $H_\tau^{3,8}$, $H_\tau^{4,7}$.

2. Dimension 2 ($A_1\dots A_8 \rightarrow \alpha_1\dots\alpha_8$, $A_9\dots A_{12} \rightarrow a_1\dots a_4$)

This is the first equation with a nonzero right-hand side,

$$\begin{aligned} & \frac{8}{12} D_{\alpha_1} \overbrace{K_{a_2\dots a_8 a_1\dots a_4}}^0 + \frac{4}{12} D_{\alpha_1} \overbrace{K_{a_2\dots a_4 \alpha_1\dots \alpha_8}}^0 \\ & + \frac{11}{2} \left(\frac{14}{33} T_{\alpha_1\alpha_2}^f K_{fa_1\dots a_4 \alpha_3\dots \alpha_8} + \frac{1}{11} T_{\alpha_1\alpha_2}^\gamma \underbrace{K_{\gamma\alpha_1\dots \alpha_8 a_3 a_4}}_0 + \frac{16}{33} T_{\alpha_1\alpha_2}^\gamma \underbrace{K_{\gamma\alpha_3\dots \alpha_8 a_2\dots a_4}}_0 \right) \\ & = 3 \frac{4}{55} \frac{11!}{4(4!)^2} R_{\alpha_1\alpha_2 c_1 c_2} R_{\alpha_3\alpha_4}{}^{c_2 c_1} G_{a_1 a_2 \alpha_5 \alpha_6} G_{a_3 a_4 \alpha_7 \alpha_8}, \end{aligned}$$

which becomes, using (E1),

$$(\gamma^f)_{\alpha_1\alpha_2} K_{fa_1\dots a_4 \alpha_3\dots \alpha_8} = -180i (\gamma^f)_{\alpha_1\alpha_2} (\gamma_{fa_1\dots a_4})_{\alpha_3\alpha_4} R_{\alpha_5\alpha_6}{}^{c_1 c_2} R_{\alpha_7\alpha_8 c_2 c_1}.$$

Since $H_\tau^{5,6}$ is trivial, the solution reads

$$K_{a_1\dots a_5 \alpha_1\dots \alpha_6}^{(2)} = -180i (\gamma_{a_1\dots a_5})_{\alpha_1\alpha_2} R_{\alpha_3\alpha_4}{}^{c_1 c_2} R_{\alpha_5\alpha_6 c_2 c_1},$$

up to τ -exact terms.

3. Dimension 5/2 ($A_1\dots A_7 \rightarrow \alpha_1\dots\alpha_7$, $A_8\dots A_{12} \rightarrow a_1\dots a_5$)

At dimension 5/2, Eq. (B2) reads

$$\begin{aligned} & \frac{7}{12} D_{\alpha_1} K_{\alpha_2\dots \alpha_7 a_1\dots a_5} - \frac{5}{12} D_{\alpha_1} \overbrace{K_{a_2\dots a_5 \alpha_1\dots \alpha_7}}^0 \\ & - \frac{11}{2} \left(\frac{7}{22} T_{\alpha_1\alpha_2}^f K_{fa_1\dots a_5 \alpha_3\dots \alpha_7} + \frac{5}{33} T_{\alpha_1\alpha_2}^\gamma \underbrace{K_{\gamma\alpha_1\dots \alpha_7 a_3\dots a_5}}_0 + \frac{35}{66} T_{\alpha_1\alpha_2}^\gamma \underbrace{K_{\gamma\alpha_3\dots \alpha_7 a_2\dots a_5}}_0 \right) \\ & = \frac{11!}{4(4!)^2} \left(2 \frac{1}{11} R_{\alpha_1\alpha_2 c_1 c_2} R_{\alpha_3\alpha_4}{}^{c_2 c_1} G_{a_2 a_3 a_4 \alpha_5} G_{a_4 a_5 a_6 \alpha_7} \right), \end{aligned}$$

which becomes, using (E1),

$$\begin{aligned} (\gamma^f)_{\alpha_1\alpha_2} K_{fa_1\dots a_5 \alpha_3\dots \alpha_7} & = -120i (\gamma_{a_1\dots a_5})_{\alpha_1\alpha_2} R_{\alpha_3\alpha_4}{}^{c_1 c_2} \left((\gamma^{e_1 e_2})_{\alpha_5\alpha_6} (\gamma_{[c_1 c_2} T_{e_1 e_2]})_{\alpha_7} + \frac{1}{24} (\gamma_{c_1 c_2}{}^{e_1\dots e_4})_{\alpha_5\alpha_6} (\gamma_{e_1 e_2} T_{e_3 e_4})_{\alpha_7} \right) \\ & + 1800i (\gamma_{a_1 a_2})_{\alpha_1\alpha_2} (\gamma_{a_3 a_4})_{\alpha_3\alpha_4} R_{\alpha_5\alpha_6}{}^{c_1 c_2} R_{\alpha_7 a_5 c_2 c_1}. \end{aligned} \quad (\text{B5})$$

The second term in (B5) can be written as

$$\begin{aligned}
& 1800i(\gamma_{[a_1 a_2]})_{\alpha_1 \alpha_2} (\gamma_{|a_3 a_4|})_{\alpha_3 \alpha_4} R_{\alpha_5 \alpha_6}{}^{c_1 c_2} R_{\alpha_7 |a_5] c_2 c_1} \\
& = 600i(\gamma^g)_{\alpha_1 \alpha_2} (\gamma_{g[a_1 a_2 a_3 a_4]})_{\alpha_3 \alpha_4} R_{\alpha_5 \alpha_6}{}^{c_1 c_2} R_{\alpha_7 |a_5] c_2 c_1} \\
& = 600i(\gamma^g)_{\alpha_1 \alpha_2} \left(\frac{6}{5} (\gamma_{[g a_1 a_2 a_3 a_4]})_{\alpha_3 \alpha_4} R_{\alpha_5 \alpha_6}{}^{c_1 c_2} R_{\alpha_7 |a_5] c_2 c_1} + \frac{1}{5} (\gamma_{a_1 a_2 a_3 a_4 a_5})_{\alpha_3 \alpha_4} R_{\alpha_5 \alpha_6}{}^{c_1 c_2} R_{\alpha_7 g c_2 c_1} \right). \quad (B6)
\end{aligned}$$

One can then verify that the second term on the right-hand side of (B6) cancels with the first term on the right-hand side of (B5). Since the first term on the right-hand side of (B6) is in a τ -exact form and $H_\tau^{6,5}$ is trivial, the solution reads

$$K_{a_1 \dots a_6 \alpha_1 \dots \alpha_5}^{(5/2)} = 720i(\gamma_{a_1 \dots a_5})_{\alpha_1 \alpha_2} R_{\alpha_3 \alpha_4}{}^{c_1 c_2} R_{\alpha_5 a_6 c_2 c_1},$$

up to τ -exact terms.

4. Dimension 3 ($A_1 \dots A_6 \rightarrow \alpha_1 \dots \alpha_6$, $A_7 \dots A_{12} \rightarrow a_1 \dots a_6$)

At dimension 3, Eq. (B2) reads

$$\begin{aligned}
& \frac{1}{2} D_{\alpha_1} K_{a_2 \dots a_6 \alpha_1 \dots \alpha_6} + \frac{1}{2} D_{\alpha_1} \overbrace{K_{a_2 \dots a_6 \alpha_1 \dots \alpha_6}}^0 \\
& + \frac{11}{2} \left(\frac{5}{22} T_{\alpha_1 \alpha_2}{}^f K_{f a_1 \dots a_6 \alpha_3 \dots \alpha_6} + \frac{5}{22} T_{a_1 a_2}{}^\gamma \underbrace{K_{\gamma a_1 \dots a_6 \alpha_3 \dots \alpha_6}}_0 + \frac{6}{11} T_{a_1 \alpha_1}{}^\gamma \underbrace{K_{\gamma \alpha_3 \dots a_6 a_2 \dots \alpha_6}}_0 \right) \\
& = \frac{11!}{4(4!)^2} \left(-\frac{12}{77} R_{\alpha_1 \alpha_1}{}^{c_1 c_2} R_{\alpha_2 a_2 c_2 c_1} G_{a_3 a_4 \alpha_3 \alpha_4} G_{a_5 a_6 \alpha_5 \alpha_6} \right. \\
& + 2 \frac{1}{154} R_{\alpha_1 \alpha_2}{}^{c_1 c_2} R_{\alpha_3 \alpha_4 c_2 c_1} G_{a_1 a_2 a_5 a_6} G_{a_3 \dots a_6} \\
& \left. + 2 \frac{3}{77} R_{a_1 a_2}{}^{c_1 c_2} R_{\alpha_1 \alpha_2 c_2 c_1} G_{a_3 a_4 \alpha_3 \alpha_4} G_{a_5 a_6 \alpha_5 \alpha_6} \right),
\end{aligned}$$

which becomes, using (E1),

$$\begin{aligned}
& -\frac{5}{4} i(\gamma^f)_{\alpha_1 \alpha_2} K_{f a_1 \dots a_6 \alpha_3 \dots \alpha_6} \\
& = -\frac{1}{2} D_{\alpha_1} K_{a_2 \dots a_6 \alpha_1 \dots \alpha_6} \\
& -\frac{1}{2} d_{\alpha_1} K_{a_2 \dots a_6 \alpha_1 \dots \alpha_6} \\
& -3T_{a_1 \alpha_1}{}^\epsilon K_{\epsilon a_2 \dots a_6 a_2 \dots a_6} \\
& -2700 R_{\alpha_1 \alpha_1}{}^{c_1 c_2} R_{\alpha_2 a_2 c_2 c_1} G_{a_3 a_4 \alpha_3 \alpha_4} G_{a_5 a_6 \alpha_5 \alpha_6} \\
& + 225 R_{\alpha_1 \alpha_2}{}^{c_1 c_2} R_{\alpha_3 \alpha_4 c_2 c_1} G_{a_1 a_2 a_5 a_6} G_{a_3 \dots a_6} \\
& + 1350 R_{a_1 a_2}{}^{c_1 c_2} R_{\alpha_1 \alpha_2 c_2 c_1} G_{a_3 a_4 \alpha_3 \alpha_4} G_{a_5 a_6 \alpha_5 \alpha_6}.
\end{aligned}$$

Let us now examine separately each group of terms in the equation above with the same type of field content. There are four G^3 terms that read

$$\begin{aligned}
& -225iG_{a_1 \dots a_4} (\gamma_{a_5 a_6})_{\alpha_1 \alpha_2} R_{\alpha_3 \alpha_4}{}^{c_1 c_2} R_{\alpha_5 \alpha_6 c_1 c_2} \\
& -360(\gamma_{a_1 \dots a_5})_{\alpha_1 \alpha_2} R_{\alpha_3 \alpha_4 c_1 c_2} T_{c_2 a_5}{}^\epsilon T_{c_1}{}^\beta (\gamma_{a_6})_{\beta \alpha_6} \\
& + 720(\gamma_{a_1 \dots a_5})_{\alpha_1 \alpha_2} R_{\alpha_3 \alpha_4 c_1 c_2} T_{c_1 a_5}{}^\epsilon T_{a_6}{}^\beta (\gamma_{c_2})_{\beta \alpha_6} \\
& -540i(\gamma_{a_1 \dots a_5})_{\alpha_1 \epsilon} T_{a_6 a_2}{}^\epsilon R_{|\alpha_3 \alpha_4|}{}^{c_1 c_2} R_{|\alpha_5 \alpha_6 c_2 c_1}. \quad (B7)
\end{aligned}$$

The last term in (B7) can be split in two parts,

$$\begin{aligned}
& -540i \left(\frac{2}{6} (\gamma_{a_1 \dots a_5})_{\alpha_1 \epsilon} T_{a_6 a_2}{}^\epsilon R_{\alpha_3 \alpha_4}{}^{c_1 c_2} R_{\alpha_5 \alpha_6 c_2 c_1} \right. \\
& \left. + \frac{4}{6} (\gamma_{a_1 \dots a_5})_{\alpha_1 \alpha_2} T_{a_6 a_3}{}^\epsilon R_{\epsilon \alpha_4}{}^{c_1 c_2} R_{\alpha_5 \alpha_6 c_2 c_1} \right).
\end{aligned}$$

The first one leads to

$$\begin{aligned}
& (\gamma^g)_{\alpha_1 \alpha_2} \left(\frac{5}{8} i \epsilon_{g a_1 \dots a_6}{}^{b_1 \dots b_4} G_{b_1 \dots b_4} R_{\alpha_3 \alpha_4}{}^{c_1 c_2} R_{\alpha_5 \alpha_6 c_2 c_1} \right) \\
& + 225iG_{a_1 \dots a_4} (\gamma_{a_5 a_6})_{\alpha_1 \alpha_2} R_{\alpha_3 \alpha_4}{}^{c_1 c_2} R_{\alpha_5 \alpha_6 c_2 c_1},
\end{aligned}$$

where the first term is τ exact, and the second term cancels with the first one in (B7). It can then be verified that the three remaining G^3 terms cancel out.

Moreover there are three terms of the schematic form $G(DG)$,

$$\begin{aligned}
& -360i(\gamma^{a_1 \dots a_5})_{\alpha_1 \alpha_2} R_{\alpha_3 \alpha_4}{}^{c_1 c_2} d_{c_2} T_{c_1 a_5}{}^\beta (\gamma_{a_6})_{\beta \alpha_6} \\
& -720i(\gamma^{a_1 \dots a_5})_{\alpha_1 \alpha_2} R_{\alpha_3 \alpha_4}{}^{c_1 c_2} d_{c_1} T_{a_6 a_5}{}^\beta (\gamma_{c_2})_{\beta \alpha_6} \\
& -180i(\gamma^{a_1 \dots a_5})_{\alpha_1 \alpha_2} R_{\alpha_3 \alpha_4}{}^{c_1 c_2} d_{a_6} R_{\alpha_5 \alpha_6 c_2 c_1}, \quad (B8)
\end{aligned}$$

which cancel out.

There are two RG terms that read

$$- 1350(\gamma_{a_1 a_2})_{\alpha_1 \alpha_2} (\gamma_{a_3 a_4})_{\alpha_3 \alpha_4} R_{a_5 a_6}{}^{c_1 c_2} R_{a_5 a_6 c_2 c_1} \\ + 45(\gamma_{a_1 \dots a_5})_{\alpha_1 \alpha_2} R_{\alpha_3 \alpha_4}{}^{c_1 c_2} ((\gamma^{e_1 e_2} \gamma_{a_6})_{\alpha_5 \alpha_6} R_{c_2 c_1 e_1 e_2} - 2(\gamma^{e_1 e_2} \gamma_{c_2})_{\alpha_5 \alpha_6} R_{c_1 a_6 e_1 e_2}). \quad (\text{B9})$$

The first term of (B9) can be put in a τ -exact form,

$$- 1350(\gamma_{a_1 a_2})_{\alpha_1 \alpha_2} (\gamma_{a_3 a_4})_{\alpha_3 \alpha_4} R_{a_5 a_6}{}^{c_1 c_2} R_{a_5 a_6 c_2 c_1} \\ = (\gamma^g)_{\alpha_1 \alpha_2} (-630i(\gamma_{[g a_1 \dots a_4]})_{\alpha_3 \alpha_4} R_{|a_5 a_6]}{}^{c_1 c_2} R_{a_5 a_6 c_2 c_1} + 180(\gamma_{a_1 \dots a_5})_{\alpha_1 \alpha_2} R_{a_6 g}{}^{c_1 c_2} R_{a_5 a_6 c_2 c_1}),$$

while the remaining RG terms cancel out.

There are two T^2 terms that read

$$+ 2700(\gamma_{a_1 a_2})_{\alpha_1 \alpha_2} (\gamma_{a_3 a_4})_{\alpha_3 \alpha_4} R_{a_5 a_6}{}^{c_1 c_2} R_{a_6 a_6 c_2 c_1} \\ + 1080i(\gamma_{a_1 \dots a_5})_{\alpha_1 \alpha_2} ((\gamma^{e_1 e_2})_{\alpha_3 \alpha_4} (\gamma_{[c_1 c_2} T_{e_1 e_2]})_{\alpha_5}) + \frac{1}{24}(\gamma_{c_1 c_2}{}^{e_1 \dots e_4})_{\alpha_3 \alpha_4} (\gamma_{e_1 e_2} T_{e_3 e_4})_{\alpha_5} R_{a_6 a_6 c_2 c_1}. \quad (\text{B10})$$

The first term can be put in a τ -exact form,

$$2700(\gamma_{a_1 a_2})_{\alpha_1 \alpha_2} (\gamma_{a_3 a_4})_{\alpha_3 \alpha_4} R_{a_5 a_6}{}^{c_1 c_2} R_{a_6 a_6 c_2 c_1} \\ = \frac{1}{3} 2700(\gamma^g)_{\alpha_1 \alpha_2} \left(\frac{7}{5} (\gamma_{g a_1 \dots a_4})_{\alpha_3 \alpha_4} R_{a_5 a_6}{}^{c_1 c_2} R_{a_6 a_6 c_2 c_1} + \frac{2}{5} (\gamma_{a_1 \dots a_5})_{\alpha_3 \alpha_4} R_{a_5 a_6}{}^{c_1 c_2} R_{a_6 g c_2 c_1} \right),$$

while the remaining TT terms cancel out. Taking the triviality of $H_\tau^{7,4}$ into account, the nonvanishing terms extracted from the RG , T^2 , and G^3 terms lead to the solution,

$$K_{a_1 \dots a_7 \alpha_1 \dots \alpha_4}^{(3)} = 504i(\gamma_{a_1 \dots a_5})_{\alpha_1 \alpha_2} (-R_{a_6 a_7}{}^{c_1 c_2} R_{a_5 a_6 c_2 c_1} + 2R_{a_5 a_6}{}^{c_1 c_2} R_{a_6 a_7 c_2 c_1}) \\ - \frac{1}{2} i \epsilon_{a_1 \dots a_7}{}^{b_1 \dots b_4} G_{b_1 \dots b_4} R_{\alpha_1 \alpha_2}{}^{c_1 c_2} R_{\alpha_3 \alpha_4 c_2 c_1}, \quad (\text{B11})$$

up to τ -exact terms.

5. Dimensions 7/2 ($A_1 \dots A_5 \rightarrow \alpha_1 \dots \alpha_5$, $A_6 \dots A_{12} \rightarrow a_1 \dots a_7$)

At dimension 7/2, Eq. (B2) reads

$$\frac{5}{12} D_{a_1} K_{\alpha_2 \dots \alpha_5 a_1 \dots a_7} - \frac{7}{12} D_{a_1} K_{a_2 \dots a_7 \alpha_1 \dots \alpha_5} \\ - \frac{11}{2} \left(\frac{5}{33} T_{\alpha_1 \alpha_2}{}^f K_{f a_1 \dots a_7 \alpha_3 \dots \alpha_5} - \frac{7}{22} T_{a_1 a_2}{}^\gamma K_{\gamma \alpha_1 \dots \alpha_5 a_3 \dots a_7} + \frac{35}{66} T_{a_1 \alpha_1}{}^\gamma K_{\gamma \alpha_3 \dots \alpha_5 a_2 \dots a_7} \right) \\ = \frac{11!}{4(4!)^2} \left(2 \frac{1}{11} R_{\alpha_1 a_1}{}^{c_1 c_2} R_{a_2 a_3 c_2 c_1} G_{a_4 a_5 a_3 \alpha_4} G_{a_6 a_7 a_5 a_6} \right. \\ \left. + 4 \frac{1}{66} R_{\alpha_1 a_1}{}^{c_1 c_2} R_{a_2 a_3 c_2 c_1} G_{a_2 a_3 \alpha_4 a_5} G_{a_4 \dots a_7} \right).$$

The right-hand side of the equation above contains terms of the form $G(DT)$, $T(DG)$, TR , and TG^2 . The first two groups of terms simply vanish (without the use of any equations of motion or BI). Two τ -exact terms can be extracted from RT and TG^2 , and the remaining terms cancel out. This leads to the solution,

$$K_{a_1 \dots a_8 \alpha_1 \dots \alpha_3}^{(7/2)} = 2016i(\gamma_{a_1 \dots a_5})_{\alpha_1 \alpha_2} R_{a_6 a_7}{}^{c_1 c_2} R_{\alpha_3 a_8 c_2 c_1} \\ + 4\epsilon_{a_1 \dots a_7}{}^{b_1 \dots b_4} G_{b_1 \dots b_4} R_{\alpha_1 \alpha_2}{}^{c_1 c_2} R_{\alpha_3 a_8 c_2 c_1},$$

up to τ -exact terms.

6. Dimensions 4 ($A_1 \dots A_4 \rightarrow \alpha_1 \dots \alpha_4$, $A_5 \dots A_{12} \rightarrow a_1 \dots a_8$)

At dimension 4, Eq. (B2) reads

$$\begin{aligned} & \frac{4}{12} D_{\alpha_1} K_{a_2 \dots a_4 a_1 \dots a_8} + \frac{8}{12} D_{a_1} K_{a_2 \dots a_8 \alpha_1 \dots \alpha_4} \\ & + \frac{11}{2} \left(\frac{1}{11} T_{\alpha_1 \alpha_2}{}^f K_{f a_1 \dots a_8 \alpha_3 \alpha_4} + \frac{14}{33} T_{a_1 a_2}{}^\gamma K_{\gamma \alpha_1 \dots \alpha_4 a_3 \dots a_8} + \frac{16}{33} T_{a_1 \alpha_1}{}^\gamma K_{\gamma \alpha_2 \dots \alpha_4 a_2 \dots a_8} \right) \\ & = \frac{11!}{4(4!)^2} (1260 R_{a_1 a_2}{}^{c_1 c_2} R_{a_3 a_4 c_2 c_1} G_{a_5 a_6 \alpha_3 \alpha_4} G_{a_7 a_8 \alpha_5 \alpha_6} \\ & + 35 R_{\alpha_1 \alpha_2}{}^{c_1 c_2} R_{\alpha_3 \alpha_4 c_2 c_1} G_{a_1 a_2 a_3 a_4} G_{a_5 \dots a_8} \\ & + 4 \cdot 210 R_{\alpha_1 \alpha_2}{}^{c_1 c_2} R_{a_1 a_2 c_2 c_1} G_{a_3 a_4 a_5 a_6} G_{a_7 a_8 \alpha_3 \alpha_4} \\ & - 2 \cdot 840 R_{\alpha_1 \alpha_1}{}^{c_1 c_2} R_{\alpha_2 a_2 c_2 c_1} G_{a_3 a_4 \alpha_3 \alpha_4} G_{a_5 a_6 a_7 a_8}). \end{aligned}$$

The terms in the equation above can be cast in eight groups: R^2 , RG^2 , $R(DG)$, G^4 , $G^2(DG)$, GT^2 , $T(DT)$, and $G(DR)$. Parts of the terms of the form R^2 , G^2R , and GT^2 can be put in a τ -exact form, while the remaining terms cancel out. Taking into account the BI,

$$D_{a_1} R_{a_2 a_3 c_1 c_2} = -T_{a_1 a_2}{}^\gamma R_{\gamma a_3 c_1 c_2}, \quad (\text{B12})$$

we see that the term $G(DR)$ cancels against a term from GT^2 . Taking into account the equation of motion of G we see that a term from $G^2(DG)$ cancels against a term in G^4 ,

$$\epsilon_{a_1 \dots a_7}{}^{b_1 \dots b_4} D_{a_8} G_{b_1 \dots b_4} = \frac{1}{2} \epsilon_{a_1 \dots a_8}{}^{b_1 \dots b_3} D^c G_{c b_1 \dots b_3} = 105 G_{a_1 \dots a_4} G_{a_5 \dots a_8}. \quad (\text{B13})$$

We are thus led to the solution,

$$\begin{aligned} K_{a_1 \dots a_9 \alpha_1 \dots \alpha_2}^{(4)} & = -1512i (\gamma_{a_1 \dots a_5})_{\alpha_1 \alpha_2} R_{a_6 a_7}{}^{c_1 c_2} R_{a_8 a_9 c_2 c_1} \\ & - 6 \epsilon_{a_1 \dots a_7}{}^{b_1 \dots b_4} G_{b_1 \dots b_4} R_{\alpha_1 \alpha_2}{}^{c_1 c_2} R_{a_8 a_9 c_2 c_1} \\ & + 12 \epsilon_{a_1 \dots a_7}{}^{b_1 \dots b_4} G_{b_1 \dots b_4} R_{\alpha_1 a_8}{}^{c_1 c_2} R_{a_2 a_9 c_2 c_1}, \end{aligned}$$

up to τ -exact terms.

7. Dimensions 9/2 ($A_1 \dots A_3 \rightarrow \alpha_1 \dots \alpha_3$, $A_4 \dots A_{12} \rightarrow a_1 \dots a_9$)

At dimension 9/2, Eq. (B2) reads

$$\begin{aligned} & \frac{3}{12} D_{\alpha_1} K_{a_2 \alpha_3 a_1 \dots a_9} - \frac{9}{12} D_{a_1} K_{a_2 \dots a_9 \alpha_1 \dots \alpha_3} \\ & - \frac{11}{2} \left(\frac{1}{22} T_{\alpha_1 \alpha_2}{}^f K_{f a_1 \dots a_9 \alpha_3} + \frac{6}{11} T_{a_1 a_2}{}^\gamma K_{\gamma \alpha_1 \dots \alpha_3 a_3 \dots a_9} + \frac{9}{22} T_{a_1 \alpha_1}{}^\gamma K_{\gamma \alpha_2 a_3 a_2 \dots a_9} \right) \\ & = \frac{11!}{4(4!)^2} \left(2 \frac{1}{110} R_{\alpha_1 \alpha_2}{}^{c_1 c_2} R_{\alpha_3 a_1 c_2 c_1} G_{a_2 \dots a_5} G_{a_6 \dots a_9} \right. \\ & \left. + 4 \frac{3}{55} R_{a_1 a_2}{}^{c_1 c_2} R_{a_1 a_3 c_2 c_1} G_{a_2 \alpha_3 a_4 a_5} G_{a_6 \dots a_9} \right). \end{aligned}$$

The terms in the equation above can be cast in seven groups: $R(DT)$, RTG , $G^2(DT)$, G^3T , T^3 , $TG(DG)$, and $T(DR)$. One term of the form RTG is τ exact, while all the remaining terms can be seen to cancel out, using (B12) and (B13) to convert a term of the form $T(DR)$ to the form T^3 , and a term of the form $TG(DG)$ to the form G^3T . Up to τ -exact terms, the component of dimension 9/2 then reads

$$K_{a_1 \dots a_9 \alpha_1 \alpha_2}^{(9/2)} = 60i \epsilon_{a_1 \dots a_7}^{b_1 \dots b_4} G_{b_1 \dots b_4} R_{a_8 a_9}{}^{c_1 c_2} R_{\alpha_1 a_{10} c_2 c_1}.$$

8. Dimensions 5 ($A_1 A_2 \rightarrow \alpha_1 \alpha_2$, $A_3 \dots A_{12} \rightarrow a_1 \dots a_{10}$)

At dimension 5, Eq. (B2) reads

$$\begin{aligned} & \frac{2}{12} D_{a_1} K_{a_2 a_1 \dots a_9} + \frac{10}{12} D_{a_1} K_{a_2 \dots a_{10} \alpha_1 \alpha_2} \\ & + \frac{11}{2} \left(\frac{1}{66} T_{\alpha_1 \alpha_2}{}^f K_{f a_1 \dots a_{10}} + \frac{15}{22} T_{a_1 a_2}{}^\gamma K_{\gamma \alpha_1 \alpha_2 a_3 \dots a_{10}} + \frac{10}{33} T_{a_1 \alpha_1}{}^\gamma K_{\gamma \alpha_2 a_2 \dots a_{10}} \right) \\ & = \frac{11!}{4(4!)^2} \left(2 \frac{1}{66} R_{\alpha_1 \alpha_2}{}^{c_1 c_2} R_{a_1 a_2 c_2 c_1} G_{a_3 \dots a_6} G_{a_7 \dots a_{10}} \right. \\ & - 1 \frac{2}{33} R_{\alpha_1 a_1}{}^{c_1 c_2} R_{\alpha_2 a_2 c_2 c_1} G_{a_3 a_4 a_5 a_6} G_{a_7 \dots a_{10}} \\ & \left. + 2 \frac{1}{11} R_{a_1 a_2}{}^{c_1 c_2} R_{a_3 a_4 c_2 c_1} G_{a_5 a_6 \alpha_1 \alpha_2} G_{a_7 a_8 \alpha_3 \alpha_4} \right). \end{aligned}$$

The terms in the equation above can be cast in nine groups: RT^2 , $GT(DT)$, G^2T^2 , GR^2 , $GR(DG)$, RG^3 , $R(DR)$, $G^2(DR)$, and $T^2(DG)$. One term in GR^2 is τ exact, while all the remaining terms cancel out, as can be seen using Eqs. (B12) and (B13) to convert a term of the form $R(DR)$ to the form RT^2 , a term of the form $G^2(DR)$ to the form G^2T^2 , and a term of the form $T^2(DG)$ to the form G^2T^2 . Up to τ -exact terms, the component of dimension 5 then reads

$$K_{a_1 \dots a_{11}}^{(5)} = -165 \epsilon_{a_1 \dots a_7}^{b_1 \dots b_4} G_{b_1 \dots b_4} R_{a_8 a_9}{}^{c_1 c_2} R_{a_{10} a_{11} c_2 c_1}. \quad (\text{B14})$$

9. Dimensions 11/2 ($A_1 \rightarrow \alpha_1$, $A_2 \dots A_{12} \rightarrow a_1 \dots a_{11}$)

Since there is no new component of K appearing, this equation should be satisfied automatically,

$$\begin{aligned} & \frac{1}{12} D_{a_1} K_{a_1 \dots a_{11}} - \frac{11}{12} D_{a_1} K_{a_2 \dots a_{11} \alpha_1} \\ & - \frac{11}{2} \left(\frac{1}{6} T_{a_1 \alpha_2}{}^f K_{f a_2 \dots a_{11}} - \frac{5}{6} T_{a_1 a_2}{}^\gamma K_{\gamma \alpha_1 a_3 \dots a_{11}} \right) \\ & = \frac{11!}{4(4!)^2} \left(\frac{2}{6} R_{\alpha_1 a_1}{}^{c_1 c_2} R_{a_2 a_3 c_2 c_1} G_{a_4 \dots a_7} G_{a_8 \dots a_{11}} \right). \end{aligned}$$

The equation contains six types of terms: TR^2 , $GR(DT)$, G^2TR , $GT(DR)$, $RT(DG)$, and T^3G . As expected all the terms cancel out, as can be seen using (B12) and (B13) to convert a term of the form $GT(DR)$ to the form T^3G , and a term of the form $RT(DG)$ to the form G^2TR .

10. Action at $\mathcal{O}(l^3)$

We have constructed the explicit expression of each component of K_{11} and showed that it is unique up to exact

terms. In particular, the top component, given in Eq. (B14), precisely agrees with (3.10), leading to the superinvariant of Sec. IIIB.

APPENDIX C: WEIL TRIVIALITY AT l^6

The same method is used to generate the corrections at l^6 order, cf., Sec. IV. We look for the solution to the equation $dK_{11} = G_4 \wedge X_8^{(1)}$. In components this reads

$$\begin{aligned} & D_{[A_1} K_{A_2 \dots A_{12}]} + \frac{11}{2} T_{[A_1 A_2]}^F K_{F[A_3 \dots A_{12}]} \\ & = \frac{11!}{(4!)^4} \left(G_{[A_1 \dots A_4} R_{|A_5 A_6| c_1 c_2} R_{|A_7 A_8|}{}^{c_2 c_3} R_{|A_9 A_{10}| c_3 c_4} R_{|A_{11} A_{12}|}{}^{c_4 c_1} \right. \\ & \left. - \frac{1}{4} G_{[A_1 \dots A_4} R_{|A_5 A_6| c_1 c_2} R_{|A_7 A_8|}{}^{c_1 c_2} R_{|A_9 A_{10}| d_1 d_2} R_{|A_{11} A_{12}|}{}^{d_1 d_2} \right). \quad (\text{C1}) \end{aligned}$$

The dimensions of the various components of K_{11} now range from $[K_{\alpha_1 \dots \alpha_{12}}] = \frac{5}{2}$ to $[K_{a_1 \dots a_{11}}] = 8$.

1. Dimension 3 and 7/2

If we assume that the superinvariant at $\mathcal{O}(l^6)$ is quartic or higher in fields, the first potentially nonvanishing component of K_{11} appears at dimension 4 (it is of the form G^4). We thus obtain

$$K_{\alpha_1 \dots \alpha_{11}}^{(5/2)} = K_{a_1 \alpha_1 \dots \alpha_{10}}^{(3)} = K_{a_1 a_2 \alpha_1 \dots \alpha_9}^{(7/2)} = 0.$$

This is consistent with (C1), whose right-hand side vanishes for dimensions lower than 4.

2. Dimension 4 ($A_1 \dots A_{10} \rightarrow \alpha_1 \dots \alpha_{10}$, $A_{11} A_{12} \rightarrow a_1 a_2$)

Equation (C1) takes the following form,

$$\begin{aligned} & \frac{2}{12} D_{a_1} \overbrace{K_{a_2 a_1 \dots \alpha_{10}}^{(3)}}^0 + \frac{10}{12} D_{a_1} \overbrace{K_{a_2 \dots \alpha_{10} a_1 a_2}^{(7/2)}}^0 \\ & + \frac{11}{2} \left(\frac{15}{22} T_{a_1 a_2} f K_{f a_1 a_2 \alpha_3 \dots \alpha_{10}}^{(4)} + \frac{10}{33} T_{a_1 \alpha_1} \gamma \overbrace{K_{\gamma a_2 \dots \alpha_{10} a_2}^{(3)}}^0 + \frac{1}{66} T_{a_1 a_2} \gamma \overbrace{K_{\gamma a_1 \dots \alpha_{10}}^{(5/2)}}^0 \right) \\ & = \frac{11!}{4!4^2} G_{a_1 a_2 \alpha_1 a_2} \left(R_{\alpha_3 \alpha_4}{}^{c_1 c_2} R_{\alpha_5 \alpha_6 c_2 c_3} R_{\alpha_7 \alpha_8}{}^{c_3 c_4} R_{\alpha_9 \alpha_{10} c_4 c_1} - \frac{1}{4} R_{\alpha_3 \alpha_4}{}^{c_1 c_2} R_{\alpha_5 \alpha_6 c_1 c_2} R_{\alpha_7 \alpha_8}{}^{d_1 d_2} R_{\alpha_9 \alpha_{10} d_1 d_2} \right), \end{aligned}$$

which simplifies to

$$\begin{aligned} (\gamma^f)_{a_1 a_2} K_{f a_1 a_2 \alpha_3 \dots \alpha_{10}}^{(4)} & = 2520 (\gamma_{a_1 a_2})_{\alpha_1 \alpha_2} \left(R_{\alpha_3 \alpha_4}{}^{c_1 c_2} R_{\alpha_5 \alpha_6 c_2 c_3} R_{\alpha_7 \alpha_8}{}^{c_3 c_4} R_{\alpha_9 \alpha_{10} c_4 c_1} - \frac{1}{4} R_{\alpha_3 \alpha_4}{}^{c_1 c_2} R_{\alpha_5 \alpha_6 c_1 c_2} R_{\alpha_7 \alpha_8}{}^{d_1 d_2} R_{\alpha_9 \alpha_{10} d_1 d_2} \right) \\ & = (\gamma_{a_1 a_2})_{\alpha_1 \alpha_2} X_{\alpha_3 \dots \alpha_{10}}^{(8)}. \end{aligned} \quad (C2)$$

Explicitly, the term $(\text{tr}R^2)^2$ reads (omitting the factor $-1/4$)

$$\begin{aligned} & \frac{1}{6^4} (\gamma^{u_0 u_1}) (\gamma^{u_2 u_3}) (\gamma^{u_4 u_5}) (\gamma^{u_6 u_7}) G_{u_0 u_1}{}^{y_0 y_1} G_{u_2 u_3 y_0 y_1} G_{u_4 u_5 z_0 z_1} G_{u_6 u_7}{}^{z_0 z_1} \\ & \frac{4}{24 \cdot 6^4} (\gamma^{u_0 u_1}) (\gamma^{u_2 u_3}) (\gamma^{u_4 u_5}) (\gamma^{v_0 \dots v_5}) G_{u_0 u_1}{}^{y_0 y_1} G_{u_2 u_3 y_0 y_1} G_{u_4 u_5 v_0 v_1} G_{v_2 \dots v_5} \\ & \frac{2}{24^2 6^4} (\gamma^{u_0 u_1}) (\gamma^{u_2 u_3}) (\gamma^{v_0 \dots v_3 x_0 x_1}) (\gamma^{w_0 \dots w_3}{}_{x_0 x_1}) G_{u_0 u_1}{}^{y_1 y_2} G_{u_2 u_3 y_1 y_2} G_{v_0 \dots v_3} G_{w_0 \dots w_3} \\ & \frac{4}{24^2 6^4} (\gamma^{u_0 u_1}) (\gamma^{u_2 u_3}) (\gamma^{v_0 \dots v_5}) (\gamma^{w_0 \dots w_5}) G_{u_0 u_1 v_0 v_1} G_{u_2 u_3 w_0 w_1} G_{v_2 \dots v_5} G_{w_2 \dots w_5} \\ & \frac{4}{24^3 6^4} (\gamma^{u_0 u_1}) (\gamma^{v_0 \dots v_5}) (\gamma^{w_0 \dots w_3 y_0 y_1}) (\gamma^{x_0 \dots x_3}{}_{y_0 y_1}) G_{u_0 u_1 v_0 v_1} G_{v_2 \dots v_5} G_{w_0 \dots w_3} G_{x_0 \dots x_3} \\ & \frac{1}{24^4 6^4} (\gamma^{u_0 \dots u_3 y_0 y_1}) (\gamma^{v_0 \dots v_3}{}_{y_0 y_1}) (\gamma^{w_0 \dots w_3 z_0 z_1}) (\gamma_{x_0 \dots x_3}{}^{z_0 z_1}) G_{u_0 \dots u_3} G_{v_0 \dots v_3} G_{w_0 \dots w_3} G_{x_0 \dots x_3}, \end{aligned}$$

while the term $\text{tr}R^4$ reads

$$\begin{aligned} & \frac{1}{6^4} (\gamma^{u_0 u_1}) (\gamma^{u_2 u_3}) (\gamma^{u_4 u_5}) (\gamma^{u_6 u_7}) G_{u_0 u_1}{}^{y_0 y_1} G_{u_2 u_3 y_0 z_0} G_{u_4 u_5 y_1 z_1} G_{u_6 u_7}{}^{z_0 z_1} \\ & \frac{4}{24 \cdot 6^4} (\gamma^{u_0 u_1}) (\gamma^{u_2 u_3}) (\gamma^{u_4 u_5}) (\gamma^{v_0 \dots v_5}) G_{u_0 u_1}{}^{y_0 y_1} G_{u_2 u_3 v_0 y_0} G_{u_4 u_5 v_1 y_1} G_{v_2 \dots v_5} \\ & \frac{2}{24^2 6^4} (\gamma^{u_0 u_1}) (\gamma^{u_2 u_3}) (\gamma^{v_0 \dots v_5}) (\gamma^{w_0 \dots w_5}) G_{u_0 u_1 v_0 w_0} G_{u_2 u_3 v_1 w_1} G_{v_2 \dots v_5} G_{w_2 \dots w_5} \\ & \frac{4}{24^2 6^4} (\gamma^{u_0 u_1}) (\gamma^{u_2 u_3}) (\gamma^{v_0 \dots v_4 x_0}) (\gamma^{w_0 \dots w_4}{}_{x_0}) G_{u_0 u_1 v_0}{}^{y_0} G_{u_2 u_3 w_0 y_0} G_{v_1 \dots v_4} G_{w_1 \dots w_4} \\ & \frac{4}{24^3 6^4} (\gamma^{u_0 u_1}) (\gamma^{v_0 \dots v_4 y_0}) (\gamma^{w_0 \dots w_4 y_1}) (\gamma^{x_0 \dots x_3}{}_{y_0 y_1}) G_{u_0 u_1 v_0 w_0} G_{x_0 \dots x_3} G_{v_1 \dots v_4} G_{w_1 \dots w_4} \\ & \frac{1}{24^4 6^4} (\gamma^{u_0 \dots u_3}{}_{y_0 y_1}) (\gamma^{v_0 \dots v_3 y_0}{}_{z_0}) (\gamma^{w_0 \dots w_3 y_1}{}_{z_1}) (\gamma_{x_0 \dots x_3}{}^{z_0 z_1}) G_{u_0 \dots u_3} G_{v_0 \dots v_3} G_{w_0 \dots w_3} G_{x_0 \dots x_3}. \end{aligned}$$

Suppose now that the purely fermionic component of X_8 can be cast in the τ -exact form of Eq. (4.18). The right-hand side of Eq. (C2) would then take the form

$$\begin{aligned}
(\gamma_{a_1 a_2})_{\alpha_1 \alpha_2} X_{\alpha_3 \dots \alpha_{10}}^{(8)} &= (\gamma_{a_1 a_2})_{\alpha_1 \alpha_2} (\gamma^f)_{\alpha_3 \alpha_4} G_{f \alpha_5 \dots \alpha_{10}} \\
&= (\gamma^f)_{\alpha_1 \alpha_2} (3(\gamma_{[a_1 a_2]})_{\alpha_3 \alpha_4} G_{|f] \alpha_5 \dots \alpha_{10}} - 2(\gamma_{f a_1})_{\alpha_3 \alpha_4} G_{a_2 \alpha_5 \dots \alpha_{10}}) \\
&= (\gamma^f)_{\alpha_1 \alpha_2} (3(\gamma_{[a_1 a_2]})_{\alpha_3 \alpha_4} G_{|f] \alpha_5 \dots \alpha_{10}}),
\end{aligned}$$

which yields

$$K_{f a_1 a_2 \alpha_3 \dots \alpha_{10}} = 3(\gamma_{[a_1 a_2]})_{\alpha_3 \alpha_4} G_{|f] \alpha_5 \dots \alpha_{10}}.$$

In the following we examine whether $X_{0,8}$ can be τ exact. Since (C2) contains many different types of terms, it is useful to reduce this expression by simplifying every pair of γ -matrices whose bosonic indices contain contractions, using the decompositions in Appendix D. When applied to $(\text{tr}R^2)^2$, this method gives three terms of the form $\gamma^{(2)}\gamma^{(2)}\gamma^{(2)}\gamma^{(2)}$, $\gamma^{(2)}\gamma^{(2)}\gamma^{(2)}\gamma^{(6)}$, and $\gamma^{(2)}\gamma^{(2)}\gamma^{(6)}\gamma^{(6)}$, together with several manifestly τ -exact terms. Applied to $\text{tr}R^4$, this method gives several terms of the form previously encountered, plus some new terms of the form $\gamma^{(2)}\gamma^{(2)}\gamma^{(5)}\gamma^{(5)}$, which are equivalent to $\gamma^{(2)}\gamma^{(2)}\gamma^{(6)}\gamma^{(6)}$ by Hodge duality. In

order to compare $(\text{tr}R^2)^2$ with $\text{tr}R^4$, all the $\gamma^{(6)}\gamma^{(6)}$ terms must be converted into the form $\gamma^{(5)}\gamma^{(5)}$. This creates new γ -matrices with contracted bosonic indices, which are simplified as before using Appendix D. At the end of this process all the terms have the form $\gamma^{(2)}\gamma^{(2)}\gamma^{(2)}\gamma^{(2)}$, $\gamma^{(2)}\gamma^{(2)}\gamma^{(2)}\gamma^{(6)}$, or $\gamma^{(2)}\gamma^{(2)}\gamma^{(5)}\gamma^{(5)}$ contracted with G^4 (without any contractions among γ -matrices),

$$(\gamma^{a_1 a_2})(\gamma^{a_3 a_4})(\gamma^{a_5 a_6})(\gamma^{a_7 a_8})G_{a_1 a_2; a_3 a_4; a_5 a_6; a_7 a_8}^4, \quad (\text{C3})$$

$$(\gamma^{a_1 a_2})(\gamma^{a_3 a_4})(\gamma^{a_5 a_6})(\gamma^{a_7 \dots a_{12}})G_{a_1 a_2; a_3 a_4; a_5 a_6; a_7 \dots a_{12}}^4, \quad (\text{C4})$$

$$(\gamma^{a_1 a_2})(\gamma^{a_3 a_4})(\gamma^{a_5 \dots a_9})(\gamma^{a_{10} \dots a_{14}})G_{a_1 a_2; a_3 a_4; a_5 \dots a_9; a_{10} \dots a_{14}}^4, \quad (\text{C5})$$

up to manifestly τ -exact terms that we do not need to write out explicitly. In the above, $G_{a_1 a_2; \dots; a_7 a_8}^4$, $G_{a_1 a_2; \dots; a_7 \dots a_{12}}^4$, $G_{a_1 a_2; \dots; a_{10} \dots a_{14}}^4$ denote certain sums of G^4 terms with 8,4,2 indices contracted, respectively. More explicitly,

$$\begin{aligned}
G_{a_1 a_2; a_3 a_4; a_5 a_6; a_7 a_8}^4 &= \frac{7}{2^7 3^3} G_{a_1}{}^{efg} G_{a_2 a_7 a_8}{}^e G_{a_3 a_5 a_6}{}^h G_{a_4 fgh} + \dots, \\
G_{a_1 a_2; a_3 a_4; a_5 a_6; a_7 \dots a_{12}}^4 &= \frac{25}{2^9 3^4} G_{a_1 a_7}{}^{fg} G_{a_2 a_{11} a_{12} f} G_{a_3 a_4 a_9 a_{10}} G_{a_5 a_6 a_8 g} + \dots, \\
G_{a_1 a_2; a_3 a_4; a_5 \dots a_9; a_{10} \dots a_{14}}^4 &= \frac{1}{2^{11} 3^6} G_{a_1 a_2 a_{10}}{}^f G_{a_3 a_5 a_{11} f} G_{a_4 a_{12} a_{13} a_{14}} G_{a_6 a_7 a_8 a_9} + \dots,
\end{aligned} \quad (\text{C6})$$

where the ellipses stand for more than a hundred terms of this form. No obvious cancellations appear between these three types of terms at this point.

Let us further analyze how $X_{0,8}$ is decomposed into irreducible components. First, the product of four γ -matrices contains a symmetric product of eight spinor indices that can be decomposed as follows in irreps of B_5 :

$$(00001)^{\otimes_5 8} = \underbrace{1(00000) \oplus \dots \oplus 1(40000)}_{45 \text{ terms with multiplicity 1}} \oplus \underbrace{2(00004) \oplus 2(10002) \oplus 2(01002)}_{3 \text{ terms with multiplicity 2}}.$$

Each irrep on the right-hand side above corresponds to a γ -structure that can be thought of as a Clebsch-Gordan coefficient: the γ -structure corresponding to (00000) can be thought of as a Clebsch-Gordan coefficient from the scalar to $(00001)^{\otimes_5 8}$, etc.

Next, the product of four four forms G can be decomposed as follows in irreps of B_5 ,

$$(00010)^{\otimes_5 4} = \underbrace{4(00000) \oplus \dots \oplus 6(00004) \oplus \dots \oplus 3(40000)}_{95 \text{ terms, various multiplicities}},$$

and all 95 terms except (00006), (00008), (01006), and (10006) can be found in $(00001)^{\otimes_5 8}$. This analysis implies that the contraction of four γ -matrices with four four forms G can be decomposed into 51 γ -structures, each contracted with (multiple) G^4 terms corresponding to the same irrep of B_5 .

For example, the term (00000) in the decomposition of $(00001)^{\otimes_5 8}$ gives rise to a single γ -structure contracted with the four possible G^4 terms giving rise to a scalar. Explicitly we have

$$\begin{aligned}
&(\gamma^{e_1})(\gamma_{e_1})(\gamma^{e_2})(\gamma_{e_2})(\alpha_1 G_{a_1 \dots a_4} G^{a_1 \dots a_4} G_{b_1 \dots b_4} G^{b_1 \dots b_4} + \alpha_2 G_{a_1 a_2}{}^{b_1 b_2} G_{b_1 b_2 c_1 c_2} G^{c_1 c_2} G_{d_1 d_2} G^{d_1 d_2 a_1 a_2} \\
&+ \alpha_3 G_{a_1 b_1}{}^{c_1 c_2} G_{c_1 c_2 d_1 f_1} G^{a_1 d_1} G_{g_1 g_2} G^{g_1 g_2 b_1 f_1} + \alpha_4 G_{a_1}{}^{b_1 \dots b_3} G_{b_1 \dots b_3 c_1} G^{a_1} G_{d_1 \dots d_3} G^{c_1 d_1 \dots d_3}),
\end{aligned}$$

for some constants $\alpha_1, \dots, \alpha_4$. Similarly, the (00004) gives rise to the following term,

$$\begin{aligned}
 & (\beta_1(\gamma^{e_1})(\gamma_{e_1})(\gamma^{a_1 \dots a_6})(\gamma^{b_1 \dots b_6}) + \beta_2(\gamma^{[a_1]}(\gamma^{a_2 \dots a_6]})(\gamma^{[b_1]}(\gamma^{b_2 \dots b_6]})) \\
 & \times (\alpha_1 G_{a_1 a_2 b_1}{}^{e_1} G_{a_3 b_2 b_3 e_1} G_{a_4 \dots a_6}{}^{e_2} G_{b_4 \dots b_6 e_2} + \alpha_2 G_{a_1 a_2 b_1}{}^{e_1} G_{a_3 a_4 b_2}{}^{e_2} G_{a_5 b_3 b_4 e_1} G_{a_6 b_5 b_6 e_2} \\
 & + \alpha_3 G_{a_1 \dots a_4} G_{a_5 b_3 b_4}{}^{e_1} G_{a_6 b_5 b_6}{}^{e_2} G_{b_1 b_2 e_1 e_2} + \alpha_4 G_{a_1 b_1 b_2}{}^{e_1} G_{a_2 \dots a_5} G_{a_6 b_6 e_1 e_2} G_{b_3 \dots b_5 e_1} \\
 & + \alpha_5 G_{a_1 \dots a_4} G_{a_5 b_1}{}^{e_1 e_2} G_{a_6 b_2 e_1 e_2} G_{b_3 \dots b_6} + \alpha_6 G_{a_1 \dots a_3 b_3} G_{a_4 \dots a_6 b_2} G_{b_1 b_2}{}^{e_1 e_2} G_{b_5 b_6 e_1 e_2}),
 \end{aligned}$$

for some constants $\beta_1, \beta_2, \alpha_1, \dots, \alpha_6$. The 51 γ -structures involved in the decomposition of $X^{(8)}$ can all be found explicitly, and only three of them are not τ exact: (04000), (03002), and (02004). In other words, except for the structures corresponding to these three irreps all other γ -structures appearing in $X_{0,8}$ involve at least one contraction with a $\gamma^{(1)}$.

Going back to (C3), the $G_{a_1 a_2; \dots; a_7 a_8}^4$ term, by virtue of its contraction with the four γ -matrices, transforms in the symmetrized product of four Young diagrams \square , cf., Appendix F. Decomposing in irreducible representations of S_8 ,

$$\square \otimes_{S^4} \square \otimes_{S^4} \square \otimes_{S^4} \square = \square \oplus \square \oplus \square \oplus \square \oplus \underbrace{\square}_{YT1} \quad (5 \text{ terms}) .$$

At the same time $G_{a_1 a_2; \dots; a_7 a_8}^4$ admits a decomposition into modules of $B_5 \times S_8, \sum_R V_R \times R$, where V_R is the plethysm of the module $V = (10000)$ of B_5 with respect to the Young diagram R of S_8 . Moreover, only the plethysms corresponding to the right-hand side of (C7) appear in the decomposition of $G_{a_1 a_2; \dots; a_7 a_8}^4$ under $B_5 \times S_8$. On the other hand we can compute the module V_R corresponding to each R on the right-hand side of (C7), using [68], with the result that only the plethysm corresponding to $YT1$ contains (04000), while neither (02004) nor (03002) is contained in any of the plethysms corresponding to the Young diagrams on the right-hand side of (C7).

The $G_{a_1 a_2; \dots; a_7 \dots a_{12}}^4$ term of (C4) admits the following decomposition in irreps of S_{12} :

$$\square \otimes_{S^3} \square \otimes_{S^3} \square = \underbrace{\square}_{YT2} \oplus \dots \quad (16 \text{ terms}) .$$

Only the plethysms corresponding to the Young diagrams on the right-hand side of (C8) appear in the decomposition of $G_{a_1 a_2; \dots; a_7 \dots a_{12}}^4$ under $B_5 \times S_{12}$. On the other hand it can be shown that only the plethysm corresponding to $YT2$ contains (03002), while neither (04000) nor (02004) is contained in any of the plethysms corresponding to the Young diagrams on the right-hand side of (C8).

Finally, the $G_{a_1 a_2; \dots; a_{10} \dots a_{14}}^4$ term of (C5) admits the following decomposition in irreps of S_{14} :

$$\square \otimes_{S^2} \square \otimes_{S^2} \square = \underbrace{\square}_{YT3} \oplus \underbrace{\square}_{YT4} \oplus \dots \quad (23 \text{ terms}) .$$

Moreover, only the plethysms corresponding to the Young diagrams on the right-hand side of (C9) appear in the decomposition of $G_{a_1 a_2; \dots; a_{10} \dots a_{14}}^4$ under $B_5 \times S_{14}$. On the other hand it can be shown that only the plethysm corresponding to $YT3$ contains (02004); only the plethysm corresponding to $YT4$ contains (03002), while (04000) is not contained in any of the plethysms corresponding to the Young diagrams on the right-hand side of (C9).

Using the method of Appendix F, the γ -matrices in (C3) and (C5) can be projected respectively onto $YT1$ and $YT3$. The terms (C3), (C5) can thus be shown to vanish. Moreover, it can be seen that the cancellations are sensitive to the relative coefficient between $(\text{tr}R^2)^2$ and $\text{tr}R^4$ inside X_8 . In other words, it can be shown that $(\text{tr}R^2)^2$ and $\text{tr}R^4$ are not separately τ exact.

APPENDIX D: ELEVEN-DIMENSIONAL γ -MATRICES

In this section we give our conventions for the eleven-dimensional γ -matrices, and list a number of Fierz identities used in the analysis presented in the main text.

Hodge duality for γ -matrices is defined as follows,

$$\star\gamma^{(n)} = -(-1)^{\frac{1}{2}n(n-1)}\gamma^{(11-n)}, \quad (\text{D1})$$

where our definition of the Hodge operator reads

$$(\star S)_{a_1\dots a_k} = \frac{1}{(11-k)!}\epsilon_{a_1\dots a_k b_1\dots b_{11-k}}S_{b_1\dots b_{11-k}}.$$

The symmetry properties of the γ -matrices are given by

$$(\gamma^{a_1\dots a_n})_{\alpha\beta} = (-1)^{\frac{1}{2}(n-1)(n-2)}(\gamma^{a_1\dots a_n})_{\beta\alpha},$$

where $\gamma^{(0)}$ is identified with the charge conjugation matrix.

The following Fierz identities were used in the analysis. Antisymmetrization over the a_i and b_j indices is always understood, as well as symmetrization over all fermionic indices of the γ -matrices (which are suppressed here to avoid cluttering the notation),

$$\begin{aligned} &(\gamma^{a_1\dots a_5 e_1})(\gamma_{b_1\dots b_5 e_1}) = \\ &+120\delta_{b_1\dots b_5}^{a_1\dots a_5}(\gamma^{e_1})(\gamma_{e_1}) \\ &+1(\gamma^{a_1\dots a_5})(\gamma_{b_1\dots b_5}) \\ &-600\delta_{b_1\dots b_3}^{a_1\dots a_3}(\gamma_{e_1})(\gamma^{e_1 a_4 a_5}{}_{b_4 b_5}) \\ &+25\delta_{b_1}^{a_1}(\gamma_{e_1})(\gamma^{e_1 a_1\dots a_4}{}_{b_1\dots b_4}) \\ &-150\frac{1}{2}(\delta_{b_1}^{a_1}(\gamma^{a_2 a_3})(\gamma^{a_4 a_5}{}_{b_2\dots b_5}) + (a \leftrightarrow b)) \\ &+600\delta_{b_1\dots b_3}^{a_1\dots a_3}(\gamma^{a_4 a_5})(\gamma_{b_4 b_5}) \end{aligned}$$

$$\begin{aligned} &(\gamma^{a_1\dots a_3 e_1\dots e_3})(\gamma_{b_1\dots b_3 e_1\dots e_3}) = \\ &+36\delta_{b_1\dots b_3}^{a_1\dots a_3}(\gamma_{e_1})(\gamma^{e_1}) \\ &-108\delta_{b_1}^{a_1}(\gamma_{e_1})(\gamma^{e_1 a_2 a_3}{}_{b_2 b_3}) \\ &+216\delta_{b_1}^{a_1}(\gamma^{a_2 a_3})(\gamma_{b_2 b_3}) \\ &-36\frac{1}{2}((\gamma^{a_1})(\gamma^{a_2 a_3}{}_{b_1\dots b_3}) + (a \leftrightarrow b)) \\ &+324\delta_{b_1 b_2}^{a_1 a_2}(\gamma^{a_3})(\gamma_{b_3}) \end{aligned}$$

$$\begin{aligned} &(\gamma^{a_1 e_1\dots e_5})(\gamma_{b_1 e_1\dots e_5}) = \\ &+240\delta_{b_1}^{a_1}(\gamma_{e_1})(\gamma^{e_1}) \\ &+1680(\gamma^{a_1})(\gamma_{b_1}) \end{aligned}$$

$$\begin{aligned} &(\gamma^{a_1 a_2 e_1\dots e_3})(\gamma_{b_1 b_2 e_1\dots e_3}) = \\ &-36\delta_{b_1 b_2}^{a_1 a_2}(\gamma_{e_1})(\gamma^{e_1}) \\ &+24(\gamma_{e_1})(\gamma^{e_1 a_1 a_2}{}_{b_1 b_2}) \\ &-42(\gamma^{a_1 a_2})(\gamma_{b_1 b_2}) \\ &+168\delta_{b_1}^{a_1}(\gamma^{a_2})(\gamma_{b_2}) \end{aligned}$$

$$\begin{aligned} &(\gamma^{a_1\dots a_4 e_1})(\gamma_{b_1\dots b_4 e_1}) = \\ &+6\frac{1}{2}((\gamma^{a_1 a_2})(\gamma^{a_3 a_4}{}_{b_1\dots b_4}) + (a \leftrightarrow b)) \\ &-72\delta_{b_1 b_2}^{a_1 a_2}(\gamma^{a_3 a_4})(\gamma_{b_3 b_4}) \\ &-48\frac{1}{2}(\delta_{b_1}^{a_1}(\gamma^{a_2})(\gamma^{a_3 a_4}{}_{b_2\dots b_4}) + (a \leftrightarrow b)) \\ &+96\delta_{b_1\dots b_3}^{a_1\dots a_3}(\gamma^{a_4})(\gamma_{b_4}) \\ &-1(\gamma_{e_1})(\gamma^{e_1 a_1\dots a_4}{}_{b_1\dots b_4}) \\ &+72\delta_{b_1 b_2}^{a_1 a_2}(\gamma_{e_1})(\gamma^{e_1 a_3 a_4}{}_{b_3 b_4}) \\ &-24\delta_{b_1\dots b_4}^{a_1\dots a_4}(\gamma_{e_1})(\gamma_{e_1}) \end{aligned}$$

$$\begin{aligned} &(\gamma^{a_1\dots a_4 e_1 e_2})(\gamma_{b_1\dots b_4 e_1 e_2}) = \\ &-12\frac{1}{2}((\gamma^{a_1 a_2})(\gamma^{a_3 a_4}{}_{b_1\dots b_4}) + (a \leftrightarrow b)) \\ &+288\delta_{b_1 b_2}^{a_1 a_2}(\gamma^{a_3 a_4})(\gamma_{b_3 b_4}) \\ &-96\frac{1}{2}(\delta_{b_1}^{a_1}(\gamma^{a_2})(\gamma^{a_3 a_4}{}_{b_2\dots b_4}) + (a \leftrightarrow b)) \\ &+192\delta_{b_1\dots b_3}^{a_1\dots a_3}(\gamma^{a_4})(\gamma_{b_4}) \\ &+2(\gamma_{e_1})(\gamma^{e_1 a_1\dots a_4}{}_{b_1\dots b_4}) \\ &-144\delta_{b_1 b_2}^{a_1 a_2}(\gamma_{e_1})(\gamma^{e_1 a_3 a_4}{}_{b_3 b_4}) \\ &+48\delta_{b_1\dots b_4}^{a_1\dots a_4}(\gamma_{e_1})(\gamma_{e_1}) \end{aligned}$$

$$\begin{aligned} &(\gamma^{a_1 a_2 e_1\dots e_4})(\gamma_{b_1 b_2 e_1\dots e_4}) = \\ &+48\delta_{b_1 b_2}^{a_1 a_2}(\gamma_{e_1})(\gamma^{e_1}) \\ &-96(\gamma_{e_1})(\gamma^{e_1 a_1 a_2}{}_{b_1 b_2}) \\ &+168(\gamma^{a_1 a_2})(\gamma_{b_1 b_2}) \\ &+672\delta_{b_1}^{a_1}(\gamma^{a_1})(\gamma^{b_1}) \end{aligned}$$

$$\begin{aligned} &(\gamma^{e_1\dots e_6})(\gamma_{e_1\dots e_6}) = \\ &+4320(\gamma_{e_1})(\gamma^{e_1}) \end{aligned}$$

$$\begin{aligned} &(\gamma^{a_1 e_1\dots e_4})(\gamma_{b_1 e_1\dots e_4}) = (\gamma^{e_1\dots e_5})(\gamma_{e_1\dots e_5}) = \\ &-96\delta_{b_1}^{a_1}(\gamma_{e_1})(\gamma^{e_1}) -720(\gamma_{e_1})(\gamma^{e_1}) \\ &+336(\gamma^{a_1})(\gamma_{b_1}) \end{aligned}$$

$$\begin{aligned} &(\gamma^{a_1\dots a_3 e_1 e_2})(\gamma_{b_1\dots b_3 e_1 e_2}) = \\ &-24\delta_{b_1\dots b_3}^{a_1\dots a_3}(\gamma_{e_1})(\gamma^{e_1}) \\ &+36\delta_{b_1}^{a_1}(\gamma_{e_1})(\gamma^{e_1 a_2 a_3}{}_{b_2 b_3}) \\ &-54\delta_{b_1}^{a_1}(\gamma^{a_2 a_3})(\gamma_{b_2 b_3}) \\ &-12\frac{1}{2}((\gamma^{a_1})(\gamma^{a_2 a_3}{}_{b_1\dots b_3}) + (a \leftrightarrow b)) \\ &+108\delta_{b_1 b_2}^{a_1 a_2}(\gamma^{a_3})(\gamma_{b_3}) \end{aligned}$$

$$\begin{aligned}
& (\gamma^{a_1 e_1})(\gamma_{b_1 \dots b_4 e_1}) = \\
& +1(\gamma_{e_1})(\gamma^{e_1 a_1}_{b_1 \dots b_4}) \\
& +12\delta_{b_1}^{a_1}(\gamma_{b_2})(\gamma_{b_3 b_4})
\end{aligned}$$

$$\begin{aligned}
& (\gamma^{a_1 \dots a_4 e_1})(\gamma_{b_1 \dots b_5 e_1}) = \\
& -60\eta^{a_1 b_1}(\gamma^{a_2 a_3})(\gamma_{a_4 b_2 \dots b_5}) \\
& -60\eta_{a_1 b_1}(\gamma^{a_2})(\gamma^{a_3 a_4 b_2 \dots b_5}) \\
& -720\delta_{c_1 \dots c_3}^{a_1 \dots a_3}\eta^{c_1 b_1}\eta^{c_2 b_2}\eta^{c_3 b_3}(\gamma^{a_4})(\gamma_{b_4 b_5}) \\
& +240\delta_{c_1 \dots c_3}^{a_1 \dots a_3}\eta^{c_1 b_1}\eta^{c_2 b_2}\eta^{c_3 b_3}(\gamma^{a_4})(\gamma_{b_4 b_5}) \\
& +140\eta^{a_1 b_1}(\gamma^{[a_2})(\gamma^{a_3 a_4 b_2 \dots b_5]}) \\
& -120\delta_{c_1 c_2}^{a_1 a_2}\eta^{c_1 b_1}\eta^{c_2 b_2}(\gamma_{e_1})(\gamma^{e_1 a_3 a_4 b_3 \dots b_5}).
\end{aligned}$$

APPENDIX E: ELEVEN-DIMENSIONAL SUPERSPACE

In this section we review the properties of on-shell eleven-dimensional superspace at lowest order in the Planck length [33]. The theory thus obtained is equivalent to CJS supergravity [1].

The nonzero superfield components are as follows:

$$\begin{aligned}
G_{ab\alpha\beta} &= -i(\gamma_{ab})_{\alpha\beta}, \\
T_{\alpha\beta}{}^f &= -i(\gamma^f)_{\alpha\beta}, \\
T_{\alpha\alpha}{}^\beta &= -\frac{1}{36}\left((\gamma^{bcd})_{\alpha}{}^\beta G_{abcd} + \frac{1}{8}(\gamma_a{}^{bcde})_{\alpha}{}^\beta G_{bcde}\right), \\
R_{\alpha\beta ab} &= \frac{i}{6}\left((\gamma^{gh})_{\alpha\beta} G_{ghab} + \frac{1}{24}(\gamma_{ab}{}^{ghij})_{\alpha\beta} G_{ghij}\right), \\
R_{\alpha abc} &= \frac{i}{2}((\gamma_a T_{bc})_{\alpha} - 2(\gamma_{[b} T_{c]a})_{\alpha}). \tag{E1}
\end{aligned}$$

The action of the spinorial derivative on the superfields reads

$$\begin{aligned}
D_\alpha G_{abcd} &= 6i(\gamma_{[ab])_{\alpha\epsilon} T_{|cd]}{}^\epsilon, \\
D_\alpha R_{abcd} &= d_{[a|} R_{\alpha|b]cd} - T_{ab}{}^\epsilon R_{\epsilon acd} + 2T_{[a|\alpha}{}^\epsilon R_{\epsilon|b]cd}, \\
D_\alpha T_{ab}{}^\beta &= \frac{1}{4}R_{abcd}(\gamma^{cd})_{\alpha}{}^\beta - 2D_{[a} T_{b]\alpha}{}^\beta - 2T_{[a|\alpha}{}^\epsilon T_{|b]\epsilon}{}^\beta. \tag{E2}
\end{aligned}$$

The equations of motion for the field strengths G , R , and T are given by

$$(T_{a_1 a_2 a_3 a_4} + T_{a_1 a_3 a_2 a_4} + T_{a_2 a_1 a_3 a_4} + T_{a_2 a_3 a_1 a_4} + T_{a_3 a_1 a_2 a_4} + T_{a_3 a_2 a_1 a_4}),$$

and then antisymmetrizing over a_1 and a_4 ,

$$\begin{aligned}
& (\gamma^{a_1 e_1})(\gamma_{b_1 e_1}) = +1(\gamma^{a_1})(\gamma_{b_1}) \\
& -1\delta_{b_1}^{a_1}(\gamma_{e_1})(\gamma^{e_1})
\end{aligned}$$

$$\begin{aligned}
& (\gamma^{a_1 e_1})(\gamma_{b_1 \dots b_5 e_1}) = \\
& -6(\gamma^{[a_1})(\gamma_{b_1 \dots b_5]}) \\
& -5\eta^{a_1 b_1}(\gamma_{e_1})(\gamma^{e_1 b_2 \dots b_5}) \\
& +1(\gamma^{a_1})(\gamma_{b_1 \dots b_5})
\end{aligned}$$

$$\begin{aligned}
D^f G_{f a_1 a_2 a_3} &= -\frac{1}{1152}\epsilon_{a_1 a_2 a_3 b_1 \dots b_4 c_1 \dots c_4} G^{b_1 \dots b_4} G^{c_1 \dots c_4}, \\
(\gamma^a)_{\alpha\epsilon} T_{ab}{}^\epsilon &= 0, \\
R_{ab} - \frac{1}{2}\eta_{ab} R &= \frac{1}{12}\left(G_{afgh} G_b{}^{fgh} - \frac{1}{8}\eta_{ab} G_{fghi} G^{fghi}\right). \tag{E3}
\end{aligned}$$

APPENDIX F: TENSOR REPRESENTATION OF A YOUNG DIAGRAM

A Young diagram with n boxes, see [69] for a review, represents an irreducible representation of the symmetric group S_n . It is possible to give explicit expressions for Young diagrams in the form of tensors. The method is more easily understood using a specific example. Consider a tensor $T_{a_1 a_2 a_3 a_4}$ without any *a priori* symmetry properties, and let us construct its projection onto $\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$. Several symmetry operations have to be applied on the tensor, but the Young diagram does not state which indices correspond to its different boxes. First one must determine all the *standard tableaux*, i.e., all the Young diagrams with numbered boxes, with increasing numbers in all rows and columns. Different Young tableaux corresponding to the same Young diagram give equivalent but distinct representations of the symmetric group. The diagram $\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$ has three standard tableaux, $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \end{array}$, $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \end{array}$, and $\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 3 \end{array}$, to which correspond three tensors, $T^{(1)}$, $T^{(2)}$, and $T^{(3)}$, respectively.

To obtain the tensor corresponding to a given standard tableau, one must first symmetrize over the indices indicated in each row, and then antisymmetrize over the indices indicated in each column. For example, $(T^{(1)})_{a_1 a_2 a_3 a_4}$ is obtained by first symmetrizing over the indices a_1 , a_2 , and a_3 ,

$$\begin{aligned}
 (\Pi^{(1)}T)_{a_1a_2a_3a_4} &= (T^{(1)})_{a_1a_2a_3a_4} = \frac{1}{8}(T_{a_1a_2a_3a_4} + T_{a_1a_3a_2a_4} + T_{a_2a_1a_3a_4} + T_{a_2a_3a_1a_4} + T_{a_2a_3a_4a_1} + T_{a_2a_4a_3a_1} \\
 &+ T_{a_3a_1a_2a_4} + T_{a_3a_2a_1a_4} + T_{a_3a_2a_4a_1} + T_{a_3a_4a_2a_1} + T_{a_4a_2a_3a_1} + T_{a_4a_3a_2a_1}).
 \end{aligned}$$

The overall normalization above can be straightforwardly determined by imposing $\Pi^{(1)}\Pi^{(1)}T = \Pi^{(1)}T$, where $\Pi^{(1)}T = T^{(1)}$ is the projection of the tensor T onto the Young tableau $\begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}$.

For example, the tensors $T^{(1)}$ and $T^{(2)}$, associated with $\begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}$ and $\begin{smallmatrix} \square & \square & \square & \square \\ \square \end{smallmatrix}$, respectively, obey the following properties:

$$\begin{aligned}
 (T^{(1)})_{[ab|c|d]} &= 0 & (T^{(2)})_{[abc|d]} &= 0 \\
 (T^{(1)})_{[a|bc|d]} &= (T^{(1)})_{abcd} & (T^{(2)})_{[ab|cd]} &= (T^{(2)})_{abcd} \\
 (T^{(1)})_{a(bc)d} &= (T^{(1)})_{abcd} & (T^{(2)})_{ab(cd)} &= (T^{(2)})_{abcd}.
 \end{aligned}$$

More generally, each $T^{(i)}$ has exactly three independent orderings of indices, which can be taken to be $T_{a_1a_4a_3a_2}^{(1)}$, $T_{a_2a_1a_4a_3}^{(1)}$, and $T_{a_2a_3a_1a_4}^{(1)}$. Any symmetry operation on the indices of $T^{(i)}$ can be expressed as a linear combination of these three orderings, e.g.,

$$\begin{aligned}
 T_{a_1(a_2a_3a_4)}^{(1)} &= T_{a_1a_4a_3a_2}^{(1)} + \frac{1}{3}T_{a_2a_1a_4a_3}^{(1)} + \frac{1}{3}T_{a_2a_3a_1a_4}^{(1)} \\
 T_{[a_1a_2]a_3a_4}^{(1)} &= \frac{1}{2}T_{a_1a_4a_3a_2}^{(1)} + \frac{1}{2}T_{a_2a_1a_4a_3}^{(1)} + 0T_{a_2a_3a_1a_4}^{(1)}.
 \end{aligned}$$

A tensor T projected onto a nonstandard tableau can be expressed as a linear combination of the three standard ones. For example, it is straightforward (but tedious) to check that the projection onto the nonstandard tableau $\begin{smallmatrix} \square & \square & \square \\ \square & \square \end{smallmatrix}$ can be decomposed as

$$\begin{aligned}
 (\Pi^{(4)}T)_{a_1a_2a_3a_4} &= T_{a_1a_4a_3a_2}^{(1)} + T_{a_2a_1a_4a_3}^{(1)} + 0T_{a_2a_3a_1a_4}^{(1)} \\
 &+ 0T_{a_1a_4a_3a_2}^{(2)} - T_{a_2a_1a_4a_3}^{(2)} + T_{a_2a_3a_1a_4}^{(2)} \\
 &+ T_{a_2a_1a_3a_4}^{(3)} + 0T_{a_2a_1a_4a_3}^{(3)} - T_{a_2a_3a_1a_4}^{(3)}.
 \end{aligned} \tag{F1}$$

Every other tableau (corresponding to the same Young diagram $\begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}$) and any symmetry operation on the indices can be expressed as a linear combination of those nine elements. The automatization of general decompositions onto Young tableaux, such as the one above, has been implemented in the computer program [31].

More generally a tensor $T_{a_1a_2a_3a_4}$ without any *a priori* symmetry properties can be decomposed into ten Young tableaux,

$$\begin{smallmatrix} \square & \square & \square & \square \\ \square \end{smallmatrix}^{\otimes 4} = \begin{smallmatrix} \square & \square & \square & \square \\ \square \end{smallmatrix}^S \oplus 3 \begin{smallmatrix} \square & \square & \square \\ \square & \square \end{smallmatrix}^{T(1,2,3)} \oplus 2 \begin{smallmatrix} \square & \square & \square \\ \square & \square \end{smallmatrix}^{T'(1,2)} \oplus 3 \begin{smallmatrix} \square & \square & \square \\ \square & \square \end{smallmatrix}^{T''(1,2,3)} \oplus \begin{smallmatrix} \square & \square & \square & \square \\ \square \end{smallmatrix}^A, \tag{F2}$$

where $T^{(1)}$, $T^{(2)}$, and $T^{(3)}$ are the Young tableaux appearing on the right-hand side of (F1) above, and correspond to the term $3 \begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}$. The remaining Young tableaux in the decomposition can be explicitly constructed using the same method.

Consider now a tensor T with a symmetry structure given by, e.g., $\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix} \otimes \square$. The previous decomposition of $\square^{\otimes 4}$ can also be used to decompose T into its irreducible components. Indeed, a tensor with structure $\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix} \otimes \square$ can be viewed as a particular set of symmetry operations performed on the indices of a tensor without any symmetry (i.e., with structure $\square^{\otimes 4}$). Therefore T can be expressed as a linear combination of the tensors already used in the decomposition (F2).

The following example shows the decomposition of the symmetric product of two three forms H ,

$$T_{a_1\dots a_6} := H_{a_1a_2a_3}H_{a_4a_5a_6} \longrightarrow \begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}^{\otimes S^2} = \begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix} \oplus \begin{smallmatrix} \square & \square & \square & \square \\ \square \end{smallmatrix}.$$

There are five standard tableaux corresponding to each of the Young diagrams $\begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}$, $\begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}$. The tensors corresponding to these Young diagrams can be denoted by $T^{(1)}, \dots, T^{(5)}$ and $T'^{(1)}, \dots, T'^{(5)}$, respectively. In the particular example above, it can be shown that

$$\begin{aligned}
 H_{a_1a_2a_3}H_{a_4a_5a_6} &= T_{a_1a_2a_3a_4a_5a_6}^{(1)} + T_{a_1a_2a_3a_4a_5a_6}'^{(1)} \\
 &+ T_{a_1a_2a_3a_4a_6a_5}'^{(1)} - T_{a_1a_2a_3a_5a_6a_4}^{(1)},
 \end{aligned}$$

i.e., only the tensors $T^{(1)}$ and $T'^{(1)}$, corresponding to $\begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}$

and $\begin{smallmatrix} \square & \square & \square & \square \\ \square \end{smallmatrix}$, respectively, enter the decomposition.

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