PHYSICAL REVIEW D **95,** 026008 (2017)

# One-loop tests of the supersymmetric higher spin $AdS_4/CFT_3$ correspondence

Yi Pang,[1,*] Ergin Sezgin,[2,†] and Yaodong Zhu[2,‡]

[1]*Max-Planck-Insitut für Gravitationsphysik (Albert-Einstein-Institut) Am Mühlenberg 1, DE-14476 Potsdam, Germany*

[2]*George and Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M University, College Station, Texas 77843, USA*

(Received 15 November 2016; published 11 January 2017)

We compute one-loop free energy for $D=4$ Vasiliev higher spin gravities based on Konstein-Vasiliev algebras $hu(m;n|4)$, $ho(m;n|4)$, or $husp(m;n|4)$ and subject to higher spin-preserving boundary conditions, which are conjectured to be dual to the $U(N)$, $O(N)$ or $USp(N)$ singlet sectors, respectively, of free conformal field theories (CFTs) on the boundary of $AdS_4$. Ordinary supersymmetric higher spin theories appear as special cases of Konstein-Vasiliev theories, when the corresponding higher spin algebra contains $OSp(\mathcal{N}|4)$ as a subalgebra. In $AdS_4$ with an $S^3$ boundary, we use a regularization scheme for individual spins that employs their character such that the subsequent sum over all spins is finite, thereby avoiding the need for additional regularization. We find that the contribution of the infinite tower of bulk fermions vanishes. As a result, the free energy is the sum of those which arise in type A and type B models with internal symmetries, the known mismatch between the bulk and boundary free energies for type B model persists, and ordinary supersymmetric higher spin theories exhibit the mismatch as well. The only models that have a match are type A models with internal symmetries, corresponding to $n=0$. The matching requires identification of the inverse Newton constant $G_N^{-1}$ with $N$ plus a proper integer as was found previously for special cases. In $AdS_4$ with an $S^1\times S^2$ boundary, the bulk one-loop free energies match those of the dual free CFTs for arbitrary $m$ and $n$. We also show that a supersymmetric double-trace deformation of free CFT based on $OSp(1|4)$ does not contribute to the $\mathcal{O}(N^0)$ free energy, as expected from the bulk.

DOI: 10.1103/PhysRevD.95.026008

## I. INTRODUCTION

It has been known for some time that the conjectured holographic duals of higher spin (HS) gravities [1] can be as simple as free conformal field theories (CFTs) living on the boundary of anti-de Sitter spacetime. Moreover, it has also been noted that the duality is expected to arise in weakly coupled regimes of both bulk and boundary field theories. Therefore, one expects that higher spin AdS/CFT correspondence should be amenable to test order by order in perturbation theory.

Free CFTs arise in conjectured dualities in the context of parity invariant HS gravities in four dimensions subject to HS symmetry-preserving boundary conditions. There are two types of parity invariant Vasiliev HS gravities, known as type A and B [2]. In their simplest forms, they both contain an infinite tower of massless even spin fields, each occurring once. They differ from each other in the parity of the spin-0 field, which is parity even (odd) in type A (B) theory. It has been conjectured that type A theory with a $\Delta=1$ boundary condition imposed on the scalar is dual to the $O(N)$ singlet sector of $N$ free real scalars [3], while type B theory with a $\Delta=2$ boundary condition imposed on the pseudoscalar is dual to the $O(N)$ singlet sector of $N$ free Majorana fermions [2] (for earlier work in which HS holography involving CFTs with matrix valued free fields, see Ref. [4]). These are HS symmetry-preserving boundary conditions, with standard boundary conditions imposed on all other fields understood. The dual CFT can be altered by changing the boundary conditions imposed on the spin-0 field in such a way that they break HS symmetry. For instance, type A model with a $\Delta=2$ boundary condition on the scalar is conjectured to be dual to the critical $O(N)$ vector model [3], while the type B model with a $\Delta=1$ boundary condition imposed on the pseudoscalar is conjectured to be dual to the $O(N)$ Gross-Neveu model [2].

An important test of the holography is to match the free energy of the bulk theory with that of the CFT defined on the conformal boundary of the bulk geometry. Assuming the bulk HS theory possesses an action formulation, the partition function evaluated on Euclidean $AdS_4$ can be expanded in terms of $G_N$ as

$$F_{\rm bulk}=\frac{1}{G_N}F^{(0)}_{\rm bulk}+F^{(1)}_{\rm bulk}+G_NF^{(2)}_{\rm bulk}+\cdots. \quad (1.1)$$

*yi.pang@aei.mpg.de
†sezgin@physics.tamu.edu
‡yaodongmatt@physics.tamu.edu

 

When the bulk Euclidean $AdS_4$ is the hyperbolic space $H_4$ of which the conformal boundary is a round $S^3$, the free energy of the bulk HS theory should match with that of a free CFT on a round $S^3$. The free energy of a free CFT on $S^3$ takes the simple form [5]

$$F_{\mathrm{CFT}} = N F^{(0)}_{\mathrm{CFT}}, \tag{1.2}$$

where $F^{(0)}_{\mathrm{CFT}}$ is the free energy of a single component in the $U(N)$ or $O(N)$ vector model. The zeroth-order contribution $F^{(0)}_{\mathrm{bulk}}$ has not been computed so far due to the lack of an action for Vasiliev theory with all the required properties. We will return to this point in the conclusions. Matching $F_{\mathrm{bulk}}$ with $F_{\mathrm{CFT}}$ necessarily requires that $F_{\mathrm{bulk}}$ is proportional to $F^{(0)}_{\mathrm{CFT}}$ at each order in the small $G_N$ expansion and that $G_N$ is identified in terms of $N$ as

$$G_N^{-1} \to \gamma(N + \Delta N), \tag{1.3}$$

with $\gamma$ and $\Delta N$ being constants, and $\Delta N$ should be a fixed integer for a given bulk/boundary dual pair. Therefore, the higher-order quantum corrections affect simply the relation between $G_N$ and $N$. Assuming Fronsdal type quadratic action for the massless HS fields, one-loop computations have shown that these requirements are fulfilled in the conjectured duality between type A theory and the bosonic $O(N)$ vector model [6]. However, for the conjectured duality between type B theory and the fermionic $O(N)$ vector model [2], these requirements are not satisfied since $F^{(1)}_{\mathrm{bulk}}$ and $F^{(0)}_{\mathrm{CFT}}$ are not proportional to each other. Matching of the free energy was also found in the type A/critical $O(N)$ vector duality, but not in the type B$/O(N)$ Gross-Neveu duality. In the critical $O(N)$ vector model, the conformal dimensions of HS currents receive quantum corrections. The leading $1/N$ corrections are summarized in Ref. [7]. These anomalous dimensions of HS currents at $\mathcal{O}(1/N)$ should be compared with the one-loop corrections to the AdS energies of HS fields computed directly from the bulk HS theory. It would be interesting to check whether they match precisely.

The principal aim of this paper is to extend the one-loop tests by computing the free energies in a wider class of HS theories in four dimensions that are expected to be dual to free CFTs on the boundary of $AdS_4$. In particular, we wish to study the consequences of supersymmetry which combine type A and type B spectra of fields with an infinite tower of massless fermions. The underlying HS algebras, denoted by $hu(m;n|4)$, $ho(m;n|4)$, and $husp(m;n|4)$, and their representations were determined some time ago by Konstein and Vasiliev [8]. These representations are obtained from two-fold tensor products of bosonic and fermionic singleton representations of $SO(3,2)$ which also carry fundamental representations of classical Lie groups. Vasiliev equations for these theories are described in detail in Ref. [9]. Their spectral properties will be summarized in the next section. It suffices to mention here that generically their underlying HS algebras serve as infinite-dimensional supersymmetry algebras, and only in special cases, namely when $m = n = 2^k$ for some $k$ corresponds to the fundamental spinor representation of $O(\mathcal{N})$, they contain the $AdS_4$ superalgebra $OSp(\mathcal{N}|4)$, in which case the singletons in the boundary CFT are in the spinor representations of the $R$-symmetry group $SO(\mathcal{N})$.[1] We shall also consider the extension of these models by introduction of internal symmetry [9].

When the boundary of $AdS_4$ is $S^3$, we compute the one-loop free energy by using a regularization scheme for individual spins that employs their character such that the subsequent sum over all spins is finite. Thus, we avoid the need for additional regularization in summing over an infinite tower of HS fields. This approach has been utilized in Ref. [10] for the sum over all bosons. Here, we adapt the method for summing over the tower of HS fermions and the even and odd spin towers of HS fields separately. Furthermore, we find that the contribution of the infinite tower of fermionic fields to the free energy vanishes. Putting all results together, we find that the bulk free energy may match that of the dual free CFT only for type A models. Their spectra consist of bosonic fields arising from the tensor product of two bosonic singletons in the fundamental representation of classical Lie groups. The matching requires identification of the inverse Newton constant $G_N^{-1}$ with $N$ plus a proper integer as was found previously for special cases. Note that the mismatch in the free energy at one loop occurs in particular for type B models of which the spectra consist of bosonic fields arising from the tensor product of two spinor singletons in the fundamental representation of classical Lie groups.

When $AdS_4$ is written in the thermal AdS coordinates, with the boundary being $S^1 \times S^2$, we find that the bulk one-loop free energies match those of the dual free CFTs for generic Konstein-Vasiliev models.

The $\mathcal{N} = 1$ higher spin theory admits an $\mathcal{N} = 1$ mixed boundary condition which corresponds to adding a supersymmetric double-trace deformation in the free CFT. We show that such a double-trace deformation does not contribute to the $\mathcal{O}(N^0)$ free energy, compatible with the fact that imposing the mixed boundary condition does not change the bulk spectrum and therefore the bulk one-loop free energy remains the same.

The rest of the paper is organized as follows. In Sec. II, we review the spectra of HS gravities based on HS algebras

[1] In order to distinguish the notion of supersymmetry in generic Konstein-Vasiliev models, where supersymmetry is in the higher spin sense, from the special cases where $OSp(\mathcal{N}|4)$ arises as a subalgebra, we shall sometimes refer to the latter ones as "ordinary supersymmetric HS theories."

$hu(m;n|4)$, $ho(m;n|4)$, and $husp(m;n|4)$. In Sec. III, we compute the one-loop free energies of these theories in $AdS_4$ with an $S^3$ boundary, where we also consider the ordinary supersymmetric HS theories with internal symmetry. We adopt an alternate regularization scheme introduced in Ref. [10] in the bosonic sector, then generalize the method also to the fermionic sector. As mentioned above, this method gives rise to a convergent sum over the contributions of an infinite tower of HS fields, thereby avoiding the need for additional regularization. In Sec. IV, we compare the results obtained in the bulk with the corresponding ones in the boundary CFTs. In Sec. V, we implement the one-loop test to HS theories in thermal AdS with the dual CFTs on the boundary $S^1 \times S^2$. In Sec. VI, we study a possible mixed boundary condition for $\mathcal{N}=1$ higher spin theory and the effect on the free energy on the CFT side where a supersymmetric double-trace deformation is turned on. We summarize and comment on our results in Sec. VII and comment on possible ways to approach the problem of mismatch of free energies in type B and ordinary supersymmetric HS theories and their conjectured duals. We also comment on the action formulation proposed in Ref. [11] in the context of classical free energy in the bulk. The validity and detailed calculation of the alternate regularization method adopted in this paper are shown in the Appendix.

## II. KONSTEIN-VASILIEV AND SUPERSYMMETRIC HIGHER SPIN THEORIES

The group theoretical building blocks for the construction of the physical spectra of HS theories in $AdS_4$ are the singleton representations of $SO(3,2)$. There are two of them referred to as Di and Rac. Using the standard notation $D(E_0, s)$ for the discrete unitary representations of $sp(4;\mathbb{R}) \sim SO(3,2)$, where $E_0$ is the lowest energy and $s$ is the spin of the lowest weight state, Di refers to the $D(1,1/2)$, and Rac refers to the $D(1/2,0)$ representations. An important property these representations have is given by Flato-Fronsdal theorem which states that

$$\mathrm{Rac}\otimes\mathrm{Rac} = \sum_{s=0}^{\infty} D(1+s,s),$$
$$\mathrm{Di}\otimes\mathrm{Di} = D(2,0) + \sum_{s=1}^{\infty} D(1+s,s),$$
$$\mathrm{Di}\otimes\mathrm{Rac} = \sum_{s=0}^{\infty} D(3/2+s,1/2+s), \quad (2.1)$$

where $s=0,1,2,\ldots$. The representations $D(1+s,s)$ are massless spin $s$ fields, and $D(2,0)$ is a massless pseudoscalar field. To introduce internal symmetry, consider the singleton representations

$$S_+ := (\mathrm{Rac}, m) \oplus (\mathrm{Di}, n), \quad S_- := (\mathrm{Di}, m) \oplus (\mathrm{Rac}, n), \quad (2.2)$$

where $m$ labels the fundamental representations of $u(m)$ or $usp(m)$ or a vector representation of $so(m)$. It has been shown that the physical spectra of three types of HS theories, based on HS algebras denoted by $hu(m;n|4)$, $ho(m;n|4)$, and $husp(m;n|4)$, are obtained from the following tensor products of the singletons,

$$hu(m;n|4)\colon\ S_+ \otimes \bar{S}_+, \qquad hu(n;m|4)\colon\ S_- \otimes \bar{S}_-, \quad (2.3)$$

$$ho(m;n|4)\colon\ (S_+ \otimes S_+)_S, \qquad ho(n;m|4)\colon\ (S_- \otimes S_-)_S, \quad (2.4)$$

$$husp(m;n|4)\colon\ (S_+ \otimes S_+)_A,$$
$$husp(m;n|4)\colon\ (S_- \otimes S_-)_A, \quad (2.5)$$

where $(\cdot)_S$ and $(\cdot)_A$ stand for symmetric and antisymmetric tensor products, respectively. These algebras contain $u(m)\otimes u(n)$, $o(m)\otimes o(n)$, and $usp(m)\otimes usp(n)$ as maximal bosonic subalgebras. The resulting spectra are as follows [8],

$$\begin{array}{lll}
hu(m;n|4): & (m^2-1,1)\oplus(1,n^2-1)\oplus(1,1)\oplus(1,1) & s=0,1,2,3,\ldots \\
 & (m,\bar{n})\oplus(\bar{m},n) & s=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots \\
ho(m;n|4): & \left(\frac{1}{2}m(m-1),1\right)\oplus\left(1,\frac{1}{2}n(n-1)\right) & s=1,3,\ldots \\
 & \left(\frac{1}{2}m(m+1)-1,1\right)\oplus\left(1,\frac{1}{2}n(n+1)-1\right)\oplus(1,1)\oplus(1,1) & s=0,2,4,\ldots \\
 & (m,n) & s=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots \\
husp(m;n|4): & \left(\frac{1}{2}m(m+1),1\right)\oplus\left(1,\frac{1}{2}n(n+1)\right) & s=1,3,\ldots \\
 & \left(\frac{1}{2}m(m-1)-1,1\right)\oplus\left(1,\frac{1}{2}n(n-1)-1\right)\oplus(1,1)\oplus(1,1) & s=0,2,4,\ldots \\
 & (m,n) & s=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots,
\end{array} \quad (2.6)$$

where the dimensions of the representations are shown. While there are the isomorphisms $hu(m;n|4) \sim hu(n;m|4)$, $ho(m;n|4) \sim ho(n;m|4)$, and $husp(m;n|4) \sim husp(n;m|4)$, the corresponding spectra listed above form inequivalent representations since there are $\{m^2, m(m+1)/2, m(m-1)/2\}$ scalars in $D(1,0)$ representations, and $\{n^2, n(n+1)/2, n(n-1)/2\}$ scalars in $D(2,0)$ representations of $SO(3,2)$, in the cases of $hu(m;n|4)$, $ho(m;n|4)$, and $husp(m;n|4)$, respectively. The models with $mn > 0$ contain fermions and are based on HS algebras that are superalgebras in the sense that they involve bosonic and fermionic generators and graded commutators. However, unless $m = n = 2^{N/2-1}$ or $m = n = 2^{(N-1)/2}$, these algebras do not contain a finite-dimensional superalgebra and as such they are infinite-dimensional algebras. In the case of $m = n = 2^{N/2-1}$, the Rac and Di belong to left- and right-handed fundamental spinor representations of $SO(\mathcal{N})$, and we have the isomorphisms

$$shs^E(\mathcal{N}|4) \cong \begin{cases} hu(2^{\frac{\mathcal{N}}{2}-1}; 2^{\frac{\mathcal{N}}{2}-1}|4) & \mathcal{N} = 2 \bmod 4, \\ husp(2^{\frac{\mathcal{N}}{2}-1}; 2^{\frac{\mathcal{N}}{2}-1}|4) & \mathcal{N} = 4 \bmod 8, \\ ho(2^{\frac{\mathcal{N}}{2}-1}; 2^{\frac{\mathcal{N}}{2}-1}|4) & \mathcal{N} = 8 \bmod 8. \end{cases} \tag{2.7}$$

The HS superalgebra $shs^E(\mathcal{N}|4)$ contains the $\mathcal{N}$ extended $AdS_4$ superalgebra $OSp(\mathcal{N}|4)$ as a subalgebra. In the case of $m = n = 2^{(\mathcal{N}-1)/2}$, the Di and Rac belong to the $2^{(\mathcal{N}-1)/2}$ dimensional fundamental spinor representations of $SO(\mathcal{N})$, and we have the isomorphisms

$$shs^E(\mathcal{N}|4) \cong \begin{cases} ho(2^{(\mathcal{N}-1)/2}; 2^{(\mathcal{N}-1)/2}|4) & \mathcal{N} = 1 \bmod 8, \\ husp(2^{(\mathcal{N}-1)/2}; 2^{(\mathcal{N}-1)/2}|4) & \mathcal{N} = 5 \bmod 8. \end{cases} \tag{2.8}$$

As for the case of $\mathcal{N} = 3 \bmod 4$, it has been shown in Ref. [9] that it is equivalent to the case of $\mathcal{N} = 4 \bmod 4$. The $OSp(\mathcal{N}|4)$ supermultiplet content of the spectra described above can be determined in a straightforward way, but this information is not needed for the purposes of this paper.

The supersymmetric HS models described above can be extended by the introduction of internal symmetry. In this case, the Di and Rac representations not only carry the spinor representation of $SO(\mathcal{N})$ but also a fundamental representation of a classical Lie algebra. Working out their tensor products yields the spectrum of the expected dual HS theory, which can be found in Table 5 of Ref. [9].

## III. FREE ENERGIES OF KONSTEIN-VASILIEV HIGHER SPIN THEORIES IN $AdS_4$ WITH AN $S^3$ BOUNDARY

In this section, we shall compute the free energy of Konstein-Vasiliev HS theories in $AdS_4$ with an $S^3$ boundary, imposing the HS symmetry-preserving boundary conditions. Free energy of bosonic HS fields in $AdS_4$ has been studied in Refs. [6,12–14]. The regularization scheme that has been used in summing over the infinite tower of HS fields, however, is very complicated. Here, we employ a simpler alternate method which utilizes the character of the irreducible representation of $SO(2,3)$. As an important consequence, the regularized individual spin contributions are such that the subsequent sum over the infinite tower of higher spins is finite, thereby avoiding the need for additional regularization of this sum. This method was introduced in Ref. [10] to compute the one-loop free energy of massive HS fields but was not applied to the computation of the above free energies to exhibit the contributions of the infinite tower of odd and even spins separately. In what follows, we shall use the alternate method to compute these contributions separately. We then generalize the method and apply it to the computation in the bulk fermion sector in the subsequent subsection.

The one-loop correction to the free energy is defined as $F^{(1)} = -\log Z^{(1)}$ where $Z^{(1)}$ is the one-loop partition function. For HS theory with $n_S$ real scalars; $n_P$ pseudoscalars; $n_1$ copies of fields with $s = 1, 3, \ldots, \infty$; $n_2$ copies of fields with $s = 2, 4, \ldots, \infty$ fields; and $n_F$ copies of spin-$1/2, 3/2, \ldots, \infty$ fields, we have

$$\begin{aligned} F^{(1)}(n_S, n_P, n_1, n_2, n_F) = & \frac{1}{2} n_S \log \det{}_1 \mathcal{D}_B(1,0) + \frac{1}{2} n_P \log \det{}_2 \mathcal{D}_B(2,0) \\ & + \frac{1}{2} n_1 \sum_{k=0}^{\infty} [\log \det \mathcal{D}_B(2k+2, 2k+1) - \log \det \mathcal{D}_B(2k+3, 2k)] \\ & + \frac{1}{2} n_2 \sum_{k=1}^{\infty} [\log \det \mathcal{D}_B(2k+1, 2k) - \log \det \mathcal{D}_B(2k+2, 2k-1)] \\ & - \frac{1}{2} n_F \log \det \mathcal{D}_F\left(\frac{3}{2}, \frac{1}{2}\right) \\ & - \frac{1}{2} n_F \sum_{k=1}^{\infty} \left[\log \det \mathcal{D}_F\left(k+\frac{3}{2}, k+\frac{1}{2}\right) - \log \det \mathcal{D}_F\left(k+\frac{5}{2}, k-\frac{1}{2}\right)\right], \end{aligned} \tag{3.1}$$

where we have defined

$$\mathcal{D}_B(\Delta,s) = [-\nabla^2 + \Delta(\Delta-3) - s],$$
$$\mathcal{D}_F(\Delta,s) = \left[-\nabla^2 + \Delta(\Delta-3) + \frac{9}{4}\right]. \quad (3.2)$$

The negative contributions in the bosonic sector and the positive contributions in the fermionic sector are due to ghosts. In computing $\det_1$ and $\det_2$, the irregular ($\Delta_- = 1$) and regular ($\Delta_+ = 2$) boundary conditions are to be used.

For a differential operator of the form $\mathcal{D} = -\nabla^2 + X$, or $\mathcal{D} = -\nabla^2 + Y$, writing

$$-\log\ \det\mathcal{D} = \int_0^\infty \frac{dt}{t} K_{\mathcal{D}}(t), \quad K_{\mathcal{D}}(t) := \mathrm{Tr}[e^{-t\mathcal{D}}] \quad (3.3)$$

and defining the spectral zeta function

$$\zeta_{\mathcal{D}}(z) := \frac{1}{\Gamma(z)}\int_0^\infty dt\, t^{z-1} K_{\mathcal{D}}(t), \quad (3.4)$$

one finds the standard result [15]

$$-\log\ \det\mathcal{D} = \zeta_{\mathcal{D}}(0)\log(\ell^2\Lambda^2) + \zeta'_{\mathcal{D}}(0), \quad (3.5)$$

where $\ell$ is the AdS radius and $\Lambda$ is the renormalization scale. For fields of arbitrary spins in hyperbolic space $H_4$, the spectral zeta function technique has been developed in Refs. [16,17] to compute their one-loop effective potentials.

### A. Bosons

Upon Euclideanization of $\mathrm{AdS}_4$ to $H_4$, the boundary is $S^3$, and in this setting various free energies of the bosonic HS theory are given by

$$F^{(1)}_{\text{even}\,1} = -\frac{1}{2}\left[\zeta^B_{(1,0)}(0) + \sum_{s=2,4,\dots}^{\infty}(\zeta^B_{(s+1,s)}(0) - \zeta^B_{(s+2,s-1)}(0))\right]\log(\ell^2\Lambda^2) - \frac{1}{2}\left[\zeta^{B\prime}_{(1,0)}(0) + \sum_{s=2,4,\dots}^{\infty}(\zeta^{B\prime}_{(s+1,s)}(0) - \zeta^{B\prime}_{(s+2,s-1)}(0))\right],$$

$$F^{(1)}_{\text{even}\,2} = -\frac{1}{2}\left[\zeta^B_{(2,0)}(0) + \sum_{s=2,4,\dots}^{\infty}(\zeta^B_{(s+1,s)}(0) - \zeta^B_{(s+2,s-1)}(0))\right]\log(\ell^2\Lambda^2) - \frac{1}{2}\left[\zeta^{B\prime}_{(2,0)}(0) + \sum_{s=2,4,\dots}^{\infty}(\zeta^{B\prime}_{(s+1,s)}(0) - \zeta^{B\prime}_{(s+2,s-1)}(0))\right],$$

$$F^{(1)}_{\text{odd}} = -\frac{1}{2}\sum_{s=1,3,\dots}^{\infty}(\zeta^B_{(s+1,s)}(0) - \zeta^B_{(s+2,s-1)}(0))\log(\ell^2\Lambda^2) - \frac{1}{2}\sum_{s=1,3,\dots}^{\infty}(\zeta^{B\prime}_{(s+1,s)}(0) - \zeta^{B\prime}_{(s+2,s-1)}(0)), \quad (3.6)$$

where $F^{(1)}_{\text{even}\,1}$ and $F^{(1)}_{\text{even}\,2}$ denote the total free energy of all even spin fields $s = 0, 2, 4\cdots$, in which the scalar satisfies $\Delta = 1$ and $\Delta = 2$ boundary conditions, respectively, and $F^{(1)}_{\text{odd}}$ denotes the total free energy of all odd spin fields $s = 1, 3, 5\cdots$.

As stated earlier, we now employ a method simpler than those used previously, utilizing the character of the irreducible representation of $SO(2,3)$. The method is based on the observation that the spectral zeta function of a bosonic spin-$s$ field can be recast in the form [10]

$$\zeta^B_{(\Delta,s)}(z) = \frac{1}{\Gamma(z)}\int_0^\infty d\beta\left[\mu(z,\beta) + \nu(z,\beta)\frac{\partial^2}{\partial\alpha^2}\right]\chi_{\Delta,s}(\beta,\alpha)|_{\alpha=0}, \quad (3.7)$$

in which

$$\chi_{\Delta,s}(\beta,\alpha) = \frac{e^{-\beta(\Delta-\frac{3}{2})}\sin[(s+\frac{1}{2})\alpha]}{4\sinh\frac{\beta}{2}\sin\frac{\alpha}{2}(\cosh\beta - \cos\alpha)},$$
$$\mu(z,\beta) = \frac{1}{3}\sinh\frac{\beta}{2}\left[f_1(z,\beta)\left(-6 + \sinh^2\frac{\beta}{2}\right) + 4f_3(z,\beta)\sinh^2\frac{\beta}{2}\right],$$
$$\nu(z,\beta) = -4f_1(z,\beta)\sinh^3\frac{\beta}{2},$$
$$f_n(z,\beta) = \sqrt{\pi}\int_0^\infty du\, u^n\tanh(\pi u)\left(\frac{\beta}{2u}\right)^{z-\frac{1}{2}} J_{z-1/2}(u\beta), \quad (3.8)$$

where $\chi_{\Delta,s}(\beta,\alpha)$ is the character of a representation of $SO(3,2)$ labeled by $D(\Delta,s)$. Owing to the $e^{-\beta(\Delta-\frac{3}{2})}$ factor in the character, $\sum_s\zeta_{(\Delta,s)}(z)$ is convergent. Therefore, no regularization is needed in performing the sum over infinitely many spins. This is the desired feature for computing the one-loop free energy of HS theory where the summation over infinitely many spins is encountered. It was also noticed by Ref. [10] that, since the one-loop free energy depends only on $\zeta(0)$ and $\zeta'(0)$, an alternate zeta function $\tilde{\zeta}(z)$ is physically equivalent to the original $\zeta(z)$, provided that $\tilde{\zeta}(0) = \zeta(0)$, and $\tilde{\zeta}'(0) = \zeta'(0)$. Thus, for the convenience of calculation, one can in fact utilize an alternate zeta function which is physically equivalent to the original zeta function. For bosonic HS fields, one choice of the alternate zeta function takes the form [10]

$$\tilde{\zeta}^B_{(\Delta,s)}(z) = \frac{1}{\Gamma(2z)}\int_0^\infty d\beta\,\beta^{2z-1}\coth\frac{\beta}{2} \times\left[1 + \left(\sinh^2\frac{\beta}{2}\right)\partial^2_\alpha\right]\chi_{\Delta,s}(\beta,\alpha)\bigg|_{\alpha=0}. \quad (3.9)$$

The physical equivalence between the alternate spectral zeta function and the original one (3.7) is shown in the

Appendix. The total character of all even spin fields and that of all odd spin fields are computed as

$$\chi_{\mathrm{even}\,1}(\beta,\alpha)=\chi_{1,0}(\beta,\alpha)+\sum_{s=2,4,\ldots}(\chi_{s+1,s}(\beta,\alpha)-\chi_{s+2,s-1}(\beta,\alpha))=\frac{1+\cos\alpha+\cosh\beta+\cosh 2\beta}{4(\cos\alpha-\cosh\beta)^2(\cos\alpha+\cosh\beta)},\qquad(3.10)$$

$$\chi_{\mathrm{even}\,2}(\beta,\alpha)=\chi_{2,0}(\beta,\alpha)+\sum_{s=2,4,\ldots}(\chi_{s+1,s}(\beta,\alpha)-\chi_{s+2,s-1}(\beta,\alpha))=\frac{1+\cos\alpha+\cos 2\alpha+\cosh\beta}{4(\cos\alpha-\cosh\beta)^2(\cos\alpha+\cosh\beta)},\qquad(3.11)$$

$$\chi_{\mathrm{odd}}(\beta,\alpha)=\sum_{s=1,3,\ldots}(\chi_{s+1,s}(\beta,\alpha)-\chi_{s+2,s-1}(\beta,\alpha))=\frac{\cos\alpha+\cosh\beta+2\cos\alpha\cosh\beta}{4(\cos\alpha-\cosh\beta)^2(\cos\alpha+\cosh\beta)}.\qquad(3.12)$$

Substituting the results above into (3.9), we find

$$\begin{aligned}\tilde\zeta^B_{\mathrm{even},1}(z)&=\frac{1}{\Gamma(2z)}\int_0^\infty d\beta\,\beta^{2z-1}\frac{\cosh^2\beta}{4\sinh^3\beta},\\ \tilde\zeta^B_{\mathrm{even},2}(z)&=-\frac{1}{\Gamma(2z)}\int_0^\infty d\beta\,\beta^{2z-1}\frac{1+2\cosh\beta}{4\sinh^3\beta},\\ \tilde\zeta^B_{\mathrm{odd}}(z)&=-\tilde\zeta^B_{\mathrm{even}\,1}(z).\end{aligned}\qquad(3.13)$$

With the help of the following identities,

$$\begin{aligned}\frac{1}{\sinh^3\frac{\beta}{2}}&=\frac{2}{\beta^2}\frac{\partial^2}{\partial x^2}\frac{1}{\sinh\frac{\beta x}{2}}\Big|_{x=1}-\frac{1}{2\sinh\frac{\beta}{2}},\\ 4^{-z}\zeta\left(2z,\frac{a}{2}\right)&=\frac{1}{\Gamma(2z)}\int_0^\infty d\beta\,\beta^{2z-1}\frac{e^{-a\beta}}{1-e^{-2\beta}},\end{aligned}\qquad(3.14)$$

where $\zeta(a,b)$ is the Hurwitz zeta function, we finally obtain

$$\begin{aligned}\tilde\zeta^B_{\mathrm{even}\,1}(z)&=4^{-(2+z)}\left[3\zeta\left(2z,-\frac{1}{2}\right)+4\zeta\left(2z-2,-\frac{1}{2}\right)+8\zeta\left(2z-1,-\frac{1}{2}\right)+(4^z-1)\zeta(2z)+3(4^z-4)\zeta(2z-2)-4(4^z-2)\zeta(2z-1)\right],\\ \tilde\zeta^B_{\mathrm{even}\,2}(z)&=4^{-(1+z)}[-4\zeta(2z-2,0)-4\zeta(2z-1,0)+(4^z-1)\zeta(2z)-4^z\zeta(2z-2)+4\zeta(2z-1)].\end{aligned}\qquad(3.15)$$

By using the relation between $F^{(1)}$ and spectral zeta function, one arrives at the results

$$\begin{aligned}F^{(1)}_{\mathrm{even}\,1}&=\frac{1}{16}\left(2\log 2-\frac{3\zeta(3)}{\pi^2}\right),\\ F^{(1)}_{\mathrm{even}\,2}&=\frac{1}{16}\left(2\log 2-\frac{5\zeta(3)}{\pi^2}\right),\\ F^{(1)}_{\mathrm{odd}}&=-F^{(1)}_{\mathrm{even}\,1}.\end{aligned}\qquad(3.16)$$

Note that the potential logarithmic divergences in $F^{(1)}_{\mathrm{even}\,1}$ and $F^{(1)}_{\mathrm{even}\,2}$ have canceled out, and the above finite results are from $\tilde\zeta^{B\prime}(0)$ terms, in agreement with Ref. [6]. Furthermore, these results can be used as building blocks for the computation of the free energies of the Konstein-Vasiliev models we are interested in, thanks to the observation that for all those models discussed in Sec. II it is always the case that

$$n_2=n_S+n_P,\qquad(3.17)$$

where we recall that $n_2$ is number of copies of even fields with $s=2,4,\ldots\infty$; $n_S$ is the number of scalars; and $n_P$ is the number of pseudoscalars.

### B. Fermions

We now compute the one-loop free energy of all fermionic HS fields. The spectral zeta function of a spin-$s$ fermion field is given by

$$\zeta^F_{(\Delta,s)}(z)=\frac{1}{\Gamma(z)}\int_0^\infty d\beta\left[\mu(z,\beta)+\nu(z,\beta)\frac{\partial^2}{\partial\alpha^2}\right]\chi_{\Delta,s}(\beta,\alpha)\Big|_{\alpha=0},\qquad(3.18)$$

where

$$\begin{aligned}\chi_{\Delta,s}(\beta,\alpha)&=\frac{e^{-\beta(\Delta-\frac{3}{2})}\sin[(s+\frac{1}{2})\alpha]}{4\sinh\frac{\beta}{2}\sin\frac{\alpha}{2}(\cosh\beta-\cos\alpha)},\\ \mu(z,\beta)&=\frac{1}{3}\sinh\frac{\beta}{2}\left[f_1(z,\beta)\left(-6+\sinh^2\frac{\beta}{2}\right)+4f_3(z,\beta)\sinh^2\frac{\beta}{2}\right],\\ \nu(z,\beta)&=-4f_1(z,\beta)\sinh^3\frac{\beta}{2},\\ f_n(z,\beta)&=\sqrt{\pi}\int_0^\infty du\,u^n\coth(\pi u)\left(\frac{\beta}{2u}\right)^{z-\frac{1}{2}}J_{z-1/2}(u\beta).\end{aligned}\qquad(3.19)$$

To compute the one-loop free energy of all fermionic HS fields, we propose the following alternate spectral zeta function, which is much easier to use. The physical equivalence between the alternate spectral zeta function (3.20) and the original one (3.18) is shown in the Appendix,

$$\tilde{\zeta}^F_{(\Delta,s)}(z) = \frac{1}{\Gamma(2z)}\int_0^\infty d\beta\beta^{2z-1} \times \left[\frac{1}{4}\sinh\frac{\beta}{2} + \frac{1}{\sinh\frac{\beta}{2}} + \sinh\frac{\beta}{2}\partial_\alpha^2\right]\chi_{\Delta,s}(\beta,\alpha)\bigg|_{\alpha=0}. \tag{3.20}$$

The sum of characters of all fermionic HS fields is computed as

$$\chi_{\frac{3}{2},\frac{1}{2}}(\beta,\alpha) + \sum_{s=3/2}^\infty [\chi_{s+1,s}(\beta,\alpha) - \chi_{s+2,s-1}(\beta,\alpha)] = \frac{\cos\frac{\alpha}{2}\cosh\frac{\beta}{2}}{(\cos\alpha - \cosh\beta)^2}. \tag{3.21}$$

It is straightforward to check that

$$\left[\frac{1}{4}\sinh\frac{\beta}{2} + \frac{1}{\sinh\frac{\beta}{2}} + \left(\sinh\frac{\beta}{2}\right)\partial_\alpha^2\right] \times \left(\chi_{\frac{3}{2},\frac{1}{2}}(\beta,\alpha) + \sum_{s=3/2}^\infty [\chi_{s+1,s}(\beta,\alpha) - \chi_{s+2,s-1}(\beta,\alpha)]\right)\bigg|_{\alpha=0} = 0, \tag{3.22}$$

which indicates that the total one-loop free energy of fermionic HS fields in fact vanishes.

### C. Summary

For a Konstein-Vasiliev higher theory consisting of $n_S$ real scalars; $n_P$ pseudoscalars; $n_1$ copies of fields with $s = 1, 3, \ldots, \infty$; $n_2 = n_S + n_P$ copies of fields with $s = 2, 4, \ldots, \infty$ fields; and $n_F$ copies of spin-$1/2, 3/2, \ldots, \infty$ fields, we have

$$F^{(1)}(n_S, n_P, n_1, n_2, n_F) = \frac{\log 2}{8}(n_S + n_P - n_1) - \frac{\zeta(3)}{16\pi^2}(3n_S + 5n_P - 3n_1), \tag{3.23}$$

where we have used the relation $n_2 = n_S + n_P$. The values of $n_S$, $n_P$, and $n_1$ can be read off from (2.6) for various Konstein-Vasiliev models. Substituting them into the equation above, we obtain

$$hu(m;n|4)\colon\ F^{(1)}_{hu} = -\frac{\zeta(3)}{8\pi^2}n^2, \tag{3.24}$$

$$ho(m;n|4)\colon\ F^{(1)}_{ho} = \frac{\log 2}{8}(m+n) - \frac{\zeta(3)}{16\pi^2}(3m+4n+n^2), \tag{3.25}$$

$$husp(m;n|4)\colon\ F^{(1)}_{husp} = -\frac{\log 2}{8}(m+n) + \frac{\zeta(3)}{16\pi^2}(3m+4n-n^2). \tag{3.26}$$

The one-loop free energy of $husp(m;n|4)$ model is related to the one of the $ho(m;n|4)$ model via $m \to -m$, $n \to -n$. The ordinary supersymmetric HS models correspond to the cases $m = n = 2^{\frac{\mathcal{N}}{2}-1}$ for even $\mathcal{N}$ and $m = n = 2^{(\mathcal{N}-1)/2}$ for odd $\mathcal{N}$.

As for the ordinary supersymmetric HS models with internal symmetries, we recall that their spectra can be obtained by assigning fundamental representations of the internal symmetry group to the $OSp(\mathcal{N}|4)$ singletons and working out the their two-fold tensor products. The resulting spectra are provided in Table 5 of Ref. [9]. In particular, the number of fermions with $s = \frac{1}{2}\bmod 2$ and $s = \frac{3}{2}\bmod 2$ are the same. As a consequence, the contributions of the fermions to the one-loop free energy will continue to vanish since in (3.20) we found that fermions with each half-integer spin occurring once give a vanishing contribution. Consequently, the bulk free energy becomes the sum of free energies of type A and type B models with the desired internal symmetries, and both $\log 2$ and $\zeta(3)$ terms will show up in the one-loop free energy. This information is sufficient to perform the one-loop test by means of comparing the bulk and boundary free energies, as we shall see at the end of the next section.

## IV. FREE ENERGIES OF FREE CFTS ON $S^3$ AND COMPARISON

The free energies of free scalars and free fermions which are conformally coupled to $S^3$ have been studied in Ref. [5]. A conformally coupled free scalar and a free fermion on $S^3$ are described by the following two actions, respectively,

$$S_S = \frac{1}{2}\int d^3x\sqrt{g}\left[(\nabla\phi)^2 + \frac{3}{4L^2}\phi^2\right], \quad S_D = \frac{1}{2}\int d^3x\sqrt{g}\psi^\dagger(\mathrm{i}D\psi), \tag{4.1}$$

where $L$ is the radius of the round $S^3$. Free energies of the above two theories are defined as usual:

$$F_S = -\log Z_S = \frac{1}{2}\log\det[\Lambda^{-2}\mathcal{O}_S], \qquad \mathcal{O} = -\nabla^2 + \frac{3}{4L^2},$$
$$F_D = -\log Z_D = -\log\det[\Lambda^{-1}\mathcal{O}_D], \qquad \mathcal{O} = \mathrm{i}D. \tag{4.2}$$

Using the zeta function, $F_S$ and $F_D$ can be computed straightforwardly, and the results are [5]

$$F_S = \frac{1}{16}\left(2\log 2 - \frac{3\zeta(3)}{\pi^2}\right), \quad F_D = \frac{1}{8}\left(2\log 2 + \frac{3\zeta(3)}{\pi^2}\right). \tag{4.3}$$

Notice that the free energy of a Majorana fermion on $S^3$ is $\frac{1}{2}F_D$.

A bulk HS theory is conjectured to be dual to a free vector model when the boundary conditions of the bulk fields preserve the HS symmetry [3,4], which is the case here. Assuming the bulk HS theory possesses an action, its free energy associated with $AdS_4$ should have the form displayed in (1.1) where $G_N$ is the Newton's constant. In cases where the boundary of $AdS_4$ is $S^3$, the bulk free energy should be compared with that of a free vector model on $S^3$ order by order in $1/N$ expansion. Hence, the comparison requires an identification between $G_N$ and $N$. It was suggested by Ref. [6] that in general the relation between $G_N$ and $N$ is of the form given in (1.3) where $\gamma$ and $\Delta N$ are constants and especially $\Delta N$ should be an integer. The basic fields in the vector model constitute a vector in the fundamental representation of a classical Lie group, which can be $U(N)$, $O(N)$, or $USp(N)$ in our cases. The free energy of a free vector model can be computed exactly and be put in the form[2]

$$F_{\rm CFT} = N F^{(0)}_{\rm CFT}, \tag{4.4}$$

where we use $F^{(0)}_{\rm CFT}$ to denote the contribution of a single component in the vector. For $F_{\rm bulk}$ to match with $F_{\rm CFT}$, it is clear that the bulk free energy at each order in $G_N$ expansion should all be proportional to $F^{(0)}_{\rm CFT}$.

Various one-loop tests of HS holography have been carried out in the literature [6,12]. For instance, the non-minimal type A model is conjectured to be dual to the $U(N)$ singlet sector of $N$ complex scalars. When HS symmetry is preserved by the boundary condition, $F^{(1)}_{\rm bulk}$ was found to be zero, indicating that $G_N^{-1}$ is identified with $N$ at one-loop order. For the minimal A model, the conjectured dual CFT is the $O(N)$ singlet sector of $N$ real scalars. In this case, $F^{(1)}_{\rm bulk}$ is equal to $F_S$, the free energy of a real free scalar (4.3). Thus, matching the bulk and boundary free energies at one-loop order requires $G_N^{-1}$ being identified with $N-1$. The $husp(2;0|4)$ Vasiliev theory is conjectured to be dual to the $USp(N)$ singlet sector of $N$ complex scalars, and $F^{(1)}_{\rm bulk}$ is equal to $-F_S$. Therefore, for $husp(2;0|4)$ higher spin theory, $G_N^{-1}$ is identified with $N+1$ at one-loop order.

In this section, we consider the cases in which the bulk HS symmetry is preserved by the boundary condition, thus the CFT duals are certain singlet sectors of free CFTs composed by free scalars and free fermions. For the $hu(m;n|4)$ theory, the dual CFT consists of $Nm$ complex free scalars $\phi^{ia}$; $i=1,2,\ldots N$; $a=1,2,\ldots m$; and $Nn$ Dirac fermions $\psi^{ir}$, $r=1,2,\ldots n$. The $m^2$ $\Delta=1$ scalars and $n^2$ $\Delta=2$ pseudoscalars correspond to the operators

$$\bar\phi_{ia}\phi^{ib}, \qquad \bar\psi_{ia}\psi^{ib}. \tag{4.5}$$

The free energy of this theory is given by

$$F_{\rm CFT} = N F^{(0)}_{\rm CFT}, \qquad F^{(0)}_{\rm CFT} = 2mF_S + nF_D, \tag{4.6}$$

where $F_S$ and $F_D$ are given in (4.3).

For the $ho(m;n|4)$ theory, the dual CFT consists of $Nm$ real free scalars $\phi^{ia}$; $i=1,2,\ldots N$; $a=1,2,\ldots m$; and $Nn$ majorana fermions $\psi^{ir}$, $r=1,2,\ldots n$. The $m^2$ $\Delta=1$ scalar fields and $n^2$ $\Delta=2$ pseudoscalars correspond to the operators

$$\phi^{ia}\phi^{jb}\delta_{ij}, \qquad \bar\psi^{ia}\psi^{jb}\delta_{ij}. \tag{4.7}$$

The free energy is given by

$$F_{\rm CFT} = N F^{(0)}_{\rm CFT}, \qquad F^{(0)}_{\rm CFT} = mF_S + \frac{1}{2}nF_D. \tag{4.8}$$

For the $husp(m;n|4)$ theory, the dual CFT consists of $Nm$ complex free scalars $\phi^{ia}$; $i=1,2,\ldots N$; $a=1,2,\ldots m$; and $Nn$ Dirac fermions $\psi^{ir}$, $r=1,2,\ldots n$, subject to the symplectic reality condition. The $m^2$ $\Delta=1$ scalar fields and $n^2$ $\Delta=2$ pseudoscalars correspond to the operators

$$\phi^{ia}\phi^{jb}\Omega_{ij}, \qquad \bar\psi^{ia}\psi^{jb}\Omega_{ij}, \tag{4.9}$$

where $\Omega_{ij}$ is the $USp(N)$ invariant tensor. The free energy of this theory is given by

$$F_{\rm CFT} = N F^{(0)}_{\rm CFT}, \qquad F^{(0)}_{\rm CFT} = mF_S + \frac{1}{2}nF_D. \tag{4.10}$$

Since supersymmetric HS theories can be mapped to special cases of Konstein-Vasiliev models, we will not give separate discussions on them.

As discussed before, duality between the bulk HS theory and boundary free CFT may be achieved only if $F^{(1)}_{\rm bulk}$ is proportional to $F^{(0)}_{\rm CFT}$. Using (3.23), (4.3), (4.6), (4.8), and (4.10), we find that this requirement amounts to

$$(m+n)(3n_S+5n_P-3n_1) = 3(m-n)(n_S+n_P-n_1), \tag{4.11}$$

obtained by setting the ratios of $\log 2$ and $\xi(3)$ dependent terms equal to each other. Taking the values of $n_S$, $n_P$, and $n_1$ from (2.6), these ratios for the bulk sides can be read off from (3.24), (3.25), and (3.26) in terms of $m$ and $n$. One can show that for all three Konstein-Vasiliev models, the only solution to the equation above is given by $n=0$, which implies bosonic type A models. In this case, the $\log 2$ and $\zeta(3)$ dependent terms arise in the same ratio as of a single real scalar field, and we have the result

$$F^{(1)}_{hu(m;0|4)} = 0, \quad F^{(1)}_{ho(m;0|4)} = mF_S, \quad F^{(1)}_{ho(m;0|4)} = -mF_S. \tag{4.12}$$

[2]Strictly speaking, the bulk HS theory is dual to the $U(N)$, $O(N)$, or $USp(N)$ singlet sector of a free CFT. The partition function of a free CFT on $S^3$ is evaluated in the vacuum which is already a singlet state under the corresponding symmetry group in each case. Thus, imposing the singlet constraint should not affect the free energy.

Therefore, assuming that $F^{(0)}_{\rm bulk} = F^{(0)}_{\rm CFT}$, the bulk and boundary free energies match with each other, provided that

$$\begin{aligned} hu(m;0|4)&:\ G_N^{-1} \to N, \\ ho(m;0|4)&:\ G_N^{-1} \to N-1, \\ husp(m;0|4)&:\ G_N^{-1} \to N+1. \end{aligned} \tag{4.13}$$

The holographic dictionaries relating $G_N$ to $N$ in various HS models have been put forward in Ref. [6] via testing the holography of $hu(1;0|4)$, $ho(1;0|4)$, and $husp(2;0|4)$ models at one-loop level. Here, we have extended the validity of these holographic mappings to $hu(m;0|4)$, $ho(m;0|4)$, and $husp(m;0|4)$ Konstein-Vasiliev models. We see that the inclusion of the infinite tower of bulk fermions does not cure the problem with the mismatch of the free energies in the type B model, which corresponds to the case in which $m = 0$ and $n \neq 0$, and its conjectured dual.

Finally, we consider the ordinary supersymmetric models with internal symmetry discussed earlier, the spectra of which are given in Table 5 of Ref. [9]. In Sec. III, we found that the contributions of the bulk fermions give vanishing contributions to one-loop free energy and consequently the bulk one-loop free energy becomes the sum of the ones of type A and type B models with the desired internal symmetries. In particular, there is still a nonvanishing $\zeta(3)$ term. On the other hand, it is easy to show that the $\zeta(3)$ dependent terms on the CFT side vanish. Therefore, we conclude the problem of free energy mismatch will persist in ordinary supersymmetric HS theories with internal symmetry.

## V. ONE-LOOP FREE ENERGIES OF SUPERSYMMETRIC HIGHER SPIN THEORIES IN AdS$_4$ WITH $S^1_\beta \times S^2$ BOUNDARY

In thermal AdS$_4$, the one-loop free energy of the bulk theory takes the form [13]

$$F^{(1)}_{\rm bulk} = F(\beta)_{\rm bulk} + \beta E_{c\,{\rm bulk}} + a_{\rm bulk}\log\Lambda, \tag{5.1}$$

where $\beta$ is the period of the imaginary time, $F(\beta)_{\rm bulk}$ is the thermal free energy which can be computed by taking the log of the thermal partition function as $F(\beta)_{\rm bulk} \equiv \beta^{-1}\log Z_{\rm bulk}$ with $Z_{\rm bulk} \equiv {\rm tr}\, e^{-\beta H_{\rm bulk}}$, and $a_{\rm bulk}$ is the anomaly coefficient related to the Seeley coefficient. The trace denotes the sum over all HS particle states. $a_{\rm bulk}$ is proportional to the integral of local curvature invariants and should be the same for AdS$_4$ with an $S^3$ boundary and for the thermal AdS$_4$. Thus, after summing over spins, the total $a_{\rm bulk}$ should vanish as shown in previous sections. $E_{c\,{\rm bulk}}$ is the one-loop contribution to the Casimir energy which can be extracted from the thermal free energy in a standard way [cf. Eqs. (5.5) and (5.6)].

The free energy of the $U(N)$, $O(N)$, or $USp(N)$ singlet sector of a free vectorial CFT on $S^1_\beta \times S^2$ takes a similar form,

$$F_{\rm CFT} = F^{\rm singlet}(\beta)_{\rm CFT} + \beta E_{c\,{\rm CFT}} + a_{\rm CFT}\log\Lambda, \tag{5.2}$$

in which $F(\beta)_{\rm CFT}$ is the free energy of the subsector in Hilbert space consisting of only the states that are invariant under the required symmetry group. The Casimir energy $E_{c{\rm CFT}}$ is given by $NE_0$, where $E_0$ is the Casimir energy of a single conformally invariant free field on $S^1_\beta \times S^2$. The anomaly coefficient $a_{\rm CFT}$ vanishes on $S^1_\beta \times S^2$, which is conformally flat and has vanishing Euler number. Therefore, there are no logarithmic divergent terms on both the bulk and the boundary sides. There remains comparison of the thermal part of the free energies and the Casimir energies on both sides. The thermal parts of the free energies are expected to match since, by definition, the bulk and boundary thermal partition functions which give rise to the corresponding thermal free energies are both equal to the character of the HS algebra associated with the spectrum of the HS theory. The comparison between the bulk and boundary Casimir energies, however, is not straightforward, since, different from $E_{c\,{\rm bulk}}$, the Casimir energy on the CFT side is not directly related to the thermal free energy of the singlet sector through (5.5). Holographic matching of the free energies at $\mathcal{O}(N^0)$ demands that $E_{c\,{\rm bulk}}$ is an integer times the Casimir energy of a single conformally invariant free field on $S^1_\beta \times S^2$.

In this section, we first study the one-loop free energy of Konstein-Vasiliev theory in thermal AdS$_4$ with an $S^1_\beta \times S^2$ boundary. We then compare the bulk result with the free energy of the corresponding dual CFT at $\mathcal{O}(N^0)$. Recall that there exist generalizations of $d > 4$ Vasiliev theory which are dual to the $U(N)$ or $O(N)$ singlet sector of free scalars or fermions [18]. The free energy of this type of HS theory in thermal AdS$_d$ has been calculated in Ref. [13] and compared with $O(N^0)$ term in the free energy of the large $N$ $U(N)$ or $O(N)$ vectorial free CFT. It was found that the matching of the free energy implies shifts in the relation between $G_N^{-1}$ and $N$ at leading order by an integer.

Different from Ref. [13] where the bulk theories are purely bosonic, in our case the bulk theory includes also fermionic HS fields. Accordingly, the dual CFT consists of both scalars and fermions. In particular, the fermionic HS fields are dual to the bilinear conserved currents built out of both scalars and fermions. State operator correspondence then implies the existence of scalar-fermion mixed states in the Hilbert space that are singlet under the required symmetry group. These scalar-fermion mixed states contribute to the thermal free energy of the singlet sector nontrivially, which means that the $F^{\rm singlet}(\beta)$ for a CFT involving both scalars and fermions cannot be obtained by a simple sum of the $F^{\rm singlet}(\beta)$'s of a pure-scalar CFT and of a pure-fermion CFT.

Below, we start with the computation of the free energies in Konstein-Vasiliev models, which include supersymmetric HS theories as special cases. The story is far more elaborate in higher dimensions. In particular, we refer the reader to Refs. [19–21] for the case of five dimensions and Ref. [22] for the case of seven dimensions.

### A. Bulk side

As stated earlier, the one-loop free energy of a massless field in thermal $AdS_4$ has the structure displayed in (5.1) with the vanishing log divergence. $F(\beta)$ can be obtained from the grand canonical partition function as

$$\text{For bosons:}\ F(\beta)_{\rm bulk} = -\sum_{m=1}^{\infty}\frac{1}{m}\mathcal{Z}(m\beta), \tag{5.3}$$

$$\text{For fermions:}\ F(\beta)_{\rm bulk} = \sum_{m=1}^{\infty}\frac{(-1)^m}{m}\mathcal{Z}(m\beta). \tag{5.4}$$

Here, $\mathcal{Z}(\beta)$ is the one-particle canonical partition function. The Casimir energy $E_{c\,{\rm bulk}}$ can be obtained from the energy $\zeta$-function as

$$E_{c\,{\rm bulk}} = \pm\frac{1}{2}\zeta_E(-1), \tag{5.5}$$

where $\pm$ correspond to bosonic and fermionic cases, respectively. The energy $\zeta$-function is related to the one-particle partition function by a Mellin transform:

$$\zeta_E(z) = \frac{1}{\Gamma(z)}\int_0^\infty d\beta\beta^{z-1}\mathcal{Z}(\beta). \tag{5.6}$$

In $d=4$, the thermal one-particle partition function for a scalar field is given by

$$\mathcal{Z}_0^{(\Delta)} = \frac{q^\Delta}{(1-q)^3}\quad \Delta > \frac{1}{2}, \tag{5.7}$$

where $\Delta$ is the AdS energy and $q=e^{-\beta}$ [23]. The thermal one-particle partition function for an $s\geq\frac{1}{2}$ massless field takes the form

$$\mathcal{Z}_s(\beta) = \frac{q^{s+1}}{(1-q)^3}[2s+1-(2s-1)q]. \tag{5.8}$$

From the results derived in Ref. [13], we deduce the useful formulas,[3]

$$\begin{aligned}
F^{(1)}_{\rm even\,1} &= F(\beta)_{\rm even\,1} = -\sum_{m=1}^{\infty}\frac{1}{m}\mathcal{Z}_{\rm even\,1}(m\beta),\\
\mathcal{Z}_{\rm even\,1}(\beta) &= \frac{1}{2}\frac{q(1+q)^2}{(1-q)^4} + \frac{1}{2}\frac{q(1+q^2)}{(1-q^2)^2}\\
&= \frac{1}{2}[\tilde{\mathcal{Z}}_0(\beta)]^2 + \frac{1}{2}\tilde{\mathcal{Z}}_0(2\beta),\\
F^{(1)}_{\rm even\,2} &= F(\beta)_{\rm even\,2} = -\sum_{m=1}^{\infty}\frac{1}{m}\mathcal{Z}_{\rm even\,2}(m\beta),\\
\mathcal{Z}_{\rm even\,2}(\beta) &= \frac{2q^2}{(1-q)^4} - \frac{q^2}{(1-q^2)^2} = \frac{1}{2}[\tilde{\mathcal{Z}}_{\frac{1}{2}}(\beta)]^2 - \frac{1}{2}\tilde{\mathcal{Z}}_{\frac{1}{2}}(2\beta),\\
F^{(1)}_{\rm odd\,1} &= F(\beta)_{\rm odd} = -\sum_{m=1}^{\infty}\frac{1}{m}\mathcal{Z}_{\rm odd}(m\beta),\\
\mathcal{Z}_{\rm odd}(\beta) &= \frac{1}{2}\frac{q(1+q)^2}{(1-q)^4} - \frac{1}{2}\frac{q(1+q^2)}{(1-q^2)^2}\\
&= \frac{1}{2}[\tilde{\mathcal{Z}}_0(\beta)]^2 - \frac{1}{2}\tilde{\mathcal{Z}}_0(2\beta),
\end{aligned} \tag{5.9}$$

where for later convenience we express the results in terms of the characters $\tilde{\mathcal{Z}}_0(\beta)$ and $\tilde{\mathcal{Z}}_{\frac{1}{2}}(\beta)$ of the conformally coupled free scalar and the free real fermion which realize the spin-0 and spin-$\frac{1}{2}$ singleton representations of the $SO(3,2)$, respectively,

$$\tilde{\mathcal{Z}}_0(\beta) = \frac{q^{\frac{1}{2}}(1+q)}{(1-q)^2},\qquad \tilde{\mathcal{Z}}_{\frac{1}{2}}(\beta) = \frac{2q}{(1-q)^2}. \tag{5.10}$$

By using (5.5) and (5.6), one can show that $\mathcal{Z}_{\rm even\,1}(\beta)$, $\mathcal{Z}_{\rm even\,2}(\beta)$, and $\mathcal{Z}_{\rm odd}(\beta)$ all lead to vanishing Casimir energy [13].[4] Therefore, we simply dropped the $E_c$ term in (5.9). Also, one should note that

$$\frac{1}{2}[\tilde{\mathcal{Z}}_{\frac{1}{2}}(\beta)]^2 + \frac{1}{2}\tilde{\mathcal{Z}}_{\frac{1}{2}}(2\beta) = \frac{1}{2}[\tilde{\mathcal{Z}}_0(\beta)]^2 - \frac{1}{2}\tilde{\mathcal{Z}}_0(2\beta). \tag{5.11}$$

For all the fermionic fields, we find that the total one-particle canonical partition function is given by

$$\begin{aligned}
\mathcal{Z}^F(\beta) &= \sum_{s=\frac{1}{2}}^{\infty}\frac{q^{s+1}}{(1-q)^3}[2s+1-(2s-1)q]\\
&= \frac{2q^{\frac{3}{2}}(1+q)}{(1-q)^4} = \tilde{\mathcal{Z}}_0(\beta)\tilde{\mathcal{Z}}_{\frac{1}{2}}(\beta).
\end{aligned} \tag{5.12}$$

Using the total one-particle canonical partition function, we can construct the energy $\zeta$-function for fermions:

[3] In the rest of this subsection, the thermal free energies and partition functions refer to those of the bulk theory.

[4] A similar technique using the $SO(3,2)$ character has been applied to compute the one-loop free energy of HS theories constructed using higher-order singletons [24] in thermal $AdS_4$, where the vanishing of Casimir energy was also observed.

$$
\begin{aligned}
\zeta_E^F(z) &= \frac{1}{\Gamma(z)}\int_0^\infty d\beta \beta^{z-1}\frac{2e^{-\frac{3}{2}\beta}(1+e^{-\beta})}{(1-e^{-\beta})^4} \\
&= 2\sum_{n=1}^{\infty}\binom{n+2}{3}\left[\left(n+\frac{1}{2}\right)^{-z}+\left(n+\frac{3}{2}\right)^{-z}\right] \\
&= \frac{1}{8}\zeta\left(z,\frac{5}{2}\right)-\frac{1}{12}\zeta\left(z-1,\frac{5}{2}\right)-\frac{1}{2}\zeta\left(z-2,\frac{5}{2}\right) \\
&\quad +\frac{1}{3}\zeta\left(z-3,\frac{5}{2}\right)-\frac{1}{8}\zeta\left(z,\frac{3}{2}\right)-\frac{1}{12}\zeta\left(z-1,\frac{3}{2}\right) \\
&\quad +\frac{1}{2}\zeta\left(z-2,\frac{3}{2}\right)+\frac{1}{3}\zeta\left(z-3,\frac{3}{2}\right). \qquad (5.13)
\end{aligned}
$$

This vanishes at $z=-1$. Therefore, the total Casimir energy for fermionic HS fields vanishes in thermal $AdS_4$ as well, and the correspoding one-loop free energy is simply

$$
F^{(1)F} = F(\beta)^F_{\rm bulk} = \sum_{m=1}^{\infty}\frac{(-1)^m}{m}\mathcal{Z}^F(m\beta). \qquad (5.14)
$$

Summarizing the results above and using the spectra given in (2.6), we find that the one-loop free energies for generic Konstein-Vasiliev HS theories are given by

$$
hu(m;n|4):\ F^{(1)}_{hu} = -\sum_{k=1}^{\infty}\frac{1}{k}[m\tilde{\mathcal{Z}}_0(k\beta)+n(-)^{k+1}\tilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)]^2, \qquad (5.15)
$$

$$
\begin{aligned}
ho(m;n|4):\ F^{(1)}_{ho} &= -\sum_{k=1}^{\infty}\frac{1}{2k}([m\tilde{\mathcal{Z}}_0(k\beta)+n(-)^{k+1}\tilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)]^2 \\
&\quad + m\tilde{\mathcal{Z}}_0(2k\beta)-n\tilde{\mathcal{Z}}_{\frac{1}{2}}(2k\beta)), \qquad (5.16)
\end{aligned}
$$

$$
\begin{aligned}
husp(m;n|4):\ F^{(1)}_{husp} &= -\sum_{k=1}^{\infty}\frac{1}{2k}([m\tilde{\mathcal{Z}}_0(k\beta) \\
&\quad + n(-)^{k+1}\tilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)]^2 \\
&\quad - m\tilde{\mathcal{Z}}_0(2k\beta)+n\tilde{\mathcal{Z}}_{\frac{1}{2}}(2k\beta)). \qquad (5.17)
\end{aligned}
$$

The free energy of $husp(m;n|4)$ theory can be obtained from that of the $ho(m;n|4)$ theory by $m\to -m$, $n\to -n$.

### B. CFT side and comparison

In this section, we calculate the partition function of the singlet sector of free CFTs on $S^1_\beta\times S^2$. We closely follow the technique developed in Refs. [25,26]. The partition function of a CFT on $S^1_\beta\times S^2$ is equal to the thermal partition function due to the vanishing of Casimir energy [24] and logarithmic divergence. Therefore, we have

$$
Z(\beta) = \sum_{i\in \text{physical states}} q^{E_i}, \qquad q=e^{-\beta}, \qquad (5.18)
$$

where the physical states are restricted to be the singlet states of $U(N)$, $O(N)$, or $USp(N)$ for our purpose. We have also used the fact that there is no nontrivial chemical potential in the system. The thermal partition functions of the $U(N)$ and $O(N)$ singlet sectors of free scalar and free fermion theories have been studied in Refs. [13,27]. We generalize their results to the cases with both scalars and fermions. We first consider the $U(N)$ singlet sector of a free CFT with $Nm$ complex free scalars and $Nn$ Dirac fermions. As shown in Refs. [13,27], the thermal partition function can be expressed as a path integral localized on the eigenvalues of the $U(N)$ matrix,

$$
\begin{aligned}
Z_{U(N)}(\beta) &= e^{-F(\beta)_{U(N)}} = \int\prod_{i=1}^{N}d\alpha_i e^{-S(\alpha_1,\ldots\alpha_N)}, \\
S(\alpha_1,\ldots\alpha_N) &= -\frac{1}{2}\sum_{i\neq j=1}^{N}\log\sin^2\frac{\alpha_i-\alpha_j}{2}+2\sum_{i=1}^{N}f_\beta(\alpha_i), \\
f_\beta(\alpha) &= \sum_{k=1}^{N}c_k(\beta)\cos(k\alpha), \\
c_k(\beta) &= -\frac{1}{k}[m\tilde{\mathcal{Z}}_0(k\beta)+n(-)^{k+1}\tilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)], \qquad (5.19)
\end{aligned}
$$

where the matter contents affect the effective action through $c_k(\beta)$. In the large $N$ limit, the integral over $\alpha_i$ can be replaced by the path integral over the eigenvalue density $\rho(\alpha)$, $\alpha\in(-\pi,\pi)$. $\rho(\alpha)$ satisfies the standard normalization

$$
\int_{-\pi}^{\pi}d\alpha\rho(\alpha)=1. \qquad (5.20)
$$

The effective action in terms of $\rho(\alpha)$ takes the form

$$
\begin{aligned}
S(\rho) &= N^2\int d\alpha d\alpha' K(\alpha-\alpha')\rho(\alpha)\rho(\alpha') \\
&\quad +2N\int d\alpha\rho(\alpha)f_\beta(\alpha), \\
K(\alpha-\alpha') &= -\frac{1}{2}\log(2-2\cos\alpha),\ \ f_\beta(\alpha)=\sum_{k=1}^{N}c_k(\beta)\cos(k\alpha). \qquad (5.21)
\end{aligned}
$$

Integrating out $\rho$, one obtains

$$
\begin{aligned}
F(\beta)_{U(N)} &= -\sum_{k=1}^{\infty}k[c_k(\beta)]^2 \\
&= -\sum_{k=1}^{\infty}\frac{1}{k}[m\tilde{\mathcal{Z}}_0(k\beta)+n(-)^{k+1}\tilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)]^2, \qquad (5.22)
\end{aligned}
$$

which coincides with one-loop free energy for $hu(m;n|4)$ higher spin theory (5.15). Next, we study the $O(N)$ singlet sector of a free CFT with $Nm$ real free scalars and $Nn$ Majorana fermions. This is a generalization of the results in Ref. [13], where the free CFT consists of only scalars or fermions. It is suggested in Ref. [13] that one can choose $N$ to be even, namely $N = 2\mathrm{N}$ for simplicity in large $N$. The difference between even $N$ and odd $N$ cases is at the next order in $1/N$ expansion. The free energy of the $O(2\mathrm{N})$ singlet sector of a free CFT with $Nm$ real free scalars and $Nn$ Majorana fermions can again be written as a path integral over the eigenvalues of the $O(N)$ matrix. The effective potential of the $O(N)$ singlet sector is given by [13]

$$S(\alpha_1,\ldots\alpha_\mathrm{N}) = -\frac{1}{2}\sum_{i\neq j=1}^{\mathrm{N}}\log\sin^2\frac{\alpha_i-\alpha_j}{2} - \frac{1}{2}\sum_{i\neq j=1}^{\mathrm{N}}\log\sin^2\frac{\alpha_i+\alpha_j}{2} + 2\sum_{i=1}^{\mathrm{N}} f_\beta(\alpha_i), \tag{5.23}$$

where $f_\beta$ is the same as the one in (5.19). The effective potential for the $O(N)$ singlet sector differs from that of the $U(N)$ by the $\log\sin^2\alpha$ terms which come from the Van der Monde determinant or the Haar measure. In the large $N$ limit, the path integral over $\alpha_i$ can again be recast into an integral over the eigenvalue density $\rho(\alpha)$. After integrating out $\rho$, one obtains

$$\begin{aligned} F(\beta)_{O(N)} &= -\sum_{k=1}^{\infty}\frac{k}{2}\left([c_k(\beta)]^2 - \frac{2}{k}c_{2k}(\beta)\right) \\ &= -\sum_{k=1}^{\infty}\frac{1}{2k}([m\tilde{\mathcal{Z}}_0(k\beta) + n(-)^{k+1}\tilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)]^2 \\ &\quad + m\tilde{\mathcal{Z}}_0(2k\beta) - n\tilde{\mathcal{Z}}_{\frac{1}{2}}(2k\beta)), \end{aligned} \tag{5.24}$$

which matches the one-loop free energy of $ho(m;n|4)$ HS theory in (5.16). In the last case, we consider the $USp(N)$ singlet sector of a free CFT with $Nm$ complex free scalars $\phi^{ia}$; $i = 1,2,\ldots N$; $a = 1,2,\ldots m$; and $Nn$ Dirac fermions subject to the symplectic real condition. Since $N$ is even in this case, we denote $N$ by $2\mathrm{N}$. The effective potential of the $USp(N)$ singlet sector takes the form

$$\begin{aligned} S(\alpha_1,\ldots\alpha_\mathrm{N}) = &-\frac{1}{2}\sum_{i\neq j=1}^{\mathrm{N}}\log\sin^2\frac{\alpha_i-\alpha_j}{2} - \frac{1}{2}\sum_{i,j=1}^{\mathrm{N}}\log\sin^2\frac{\alpha_i+\alpha_j}{2} \\ &-\frac{1}{2}\sum_{i=1}^{\mathrm{N}}\log\sin^2\alpha_i + 2\sum_{i=1}^{\mathrm{N}} f_\beta(\alpha_i). \end{aligned} \tag{5.25}$$

In the large $N$ limit, the path integral over $\alpha_i$ can be evaluated by using the same technique as before. The free energy of the $USp(N)$ singlet sector of a free CFT is obtained as

$$\begin{aligned} F(\beta)_{USp(N)} &= -\sum_{k=1}^{\infty}\frac{k}{2}\left([c_k(\beta)]^2 + \frac{2}{k}c_{2k}(\beta)\right) \\ &= -\sum_{k=1}^{\infty}\frac{1}{2k}([m\tilde{\mathcal{Z}}_0(k\beta) + n(-)^{k+1}\tilde{\mathcal{Z}}_{\frac{1}{2}}(k\beta)]^2 \\ &\quad - m\tilde{\mathcal{Z}}_0(2k\beta) + n\tilde{\mathcal{Z}}_{\frac{1}{2}}(2k\beta)), \end{aligned} \tag{5.26}$$

which matches the one-loop free energy of $husp(m;n|4)$ HS theory in (5.17).

## VI. MIXED BOUNDARY CONDITIONS IN BULK AND INTERACTING $\mathcal{N} = 1$ SCFT

In $\mathcal{N} = 1$ HS theory, the $OSp(1|4)$ invariant boundary conditions are given in Ref. [2].[5] To describe this, we write the boundary behavior ($\rho \to 0$) of the complex scalar $\phi = A + \mathrm{i}B$ as

$$A = \rho\alpha_+ + \rho^2\beta_+, \qquad B = \rho\alpha_- + \rho^2\beta_- \tag{6.1}$$

and define the 3D, $\mathcal{N} = 1$ superfields

$$\Phi_- = \alpha_- + \mathrm{i}\bar{\theta}\eta_- - \frac{\bar{\theta}\theta}{2\mathrm{i}}\beta_+, \quad \Phi_+ = \alpha_+ + \mathrm{i}\bar{\theta}\eta_+ + \frac{\bar{\theta}\theta}{2\mathrm{i}}\beta_-. \tag{6.2}$$

The boundary conditions preserving $OSp(1|4)$ take the form

$$\Phi_- = \lambda\Phi_+, \tag{6.3}$$

where $\lambda$ is an arbitrary real number. In terms of the new scalar fields, we have

$$A' = \sin\vartheta A - \cos\vartheta B, \qquad B' = \cos\vartheta A + \sin\vartheta B, \tag{6.4}$$

where $\tan\vartheta = \lambda$, and the boundary condition (6.3) is equivalent to

$$\alpha'_+ = 0, \qquad \beta'_- = 0. \tag{6.5}$$

The linearized bulk scalar field equations would remain the same form under the $SO(2)$ rotation, and thus the newly defined scalar fields $A'$ and $B'$ possess the same Feffer-Graham expansion as the original scalar fields $A$ and $B$. The boundary condition (6.5) implies that near the boundary

$$A' = \rho^2\beta'_+, \qquad B' = \rho\alpha'_-. \tag{6.6}$$

Therefore, in computing the one-loop free energy, $A'$ should have $\Delta = 2$, while $B'$ should have $\Delta = 1$, which does not affect the $\mathcal{N} = 1$ HS spectrum and the corresponding one-loop calculation. On the CFT side, the boundary condition (6.3) implies the $\mathcal{N} = 1$ free CFT being deformed by a supersymmetric double-trace term,

[5] Here, we correct a sign error in the result given by Ref. [2].

$$\Delta S = \frac{\lambda}{2}\int d^3x d^2\theta \mathcal{O}^2, \quad (6.7)$$

where $\mathcal{O}$ is given by

$$\mathcal{O} = \frac{1}{\sqrt{N}}W^2, \qquad W = \varphi + \mathrm{i}\bar{\theta}\psi + \frac{\bar{\theta}\theta}{2\,\mathrm{i}}f. \quad (6.8)$$

We compute the difference between the free energy of the deformed CFT and that of the free CFT, following the procedure adopted in Refs. [5,28]. Denoting the partition function of the free CFT by $Z_0$, we calculate

$$\Delta F = -\log\frac{Z}{Z_0}. \quad (6.9)$$

Using the Hubbard-Stratonovich transformation, we have

$$\frac{Z}{Z_0} = \frac{1}{\int D\Sigma \exp(\frac{1}{2\lambda}\int dz'\Sigma^2)} \times \int D\Sigma\left\langle \exp\left[\int dz\left(\frac{1}{2\lambda}\Sigma^2 + \Sigma\mathcal{O}\right)\right]\right\rangle_0, \quad (6.10)$$

where $\Sigma$ is an auxiliary superfield and $z$ denotes the supercoordinate. In the large $N$ limit, the higher point functions of $\mathcal{O}$ are suppressed. This allows us to write

$$\left\langle \exp\left[\int dz\Sigma\mathcal{O}\right]\right\rangle_0 = \exp\left[\frac{1}{2}\left\langle\left(\int dz\Sigma\mathcal{O}\right)^2\right\rangle_0 + o(1/N)\right]. \quad (6.11)$$

Note that $\Sigma$ and $\mathcal{O}$ are single-trace operators of $\mathcal{N}=1$ superfields, say $M$ and $W$, respectively, each with component fields $A^i$, $\lambda^i$, $B^i$ and $\phi^i$, $\psi^i$, $f^i$, where $B$ and $f$ are auxiliary fields and the index $i$ stands for the representation of $O(N)$. The component fields obey the following superconformal transformations,

$$\delta A = \frac{1}{4}\xi\lambda \qquad \delta\phi = \frac{1}{4}\xi\psi \quad (6.12)$$

$$\delta\lambda = \partial A\xi - \frac{1}{4}B\xi + A\eta \quad \delta\psi = \partial\phi\xi - \frac{1}{4}f\xi + \phi\eta \quad (6.13)$$

$$\delta B = -\xi\nabla\lambda \qquad \delta f = -\xi\nabla\psi, \quad (6.14)$$

where $\xi$ and $\eta$ are spinors satisfying the conformal Killing spinor equation $\nabla_\mu\xi = \gamma_\mu\eta$.

Integrating out the spinor coordinates $\theta$ and $\bar{\theta}$, we obtain

$$\int dz\frac{1}{2\lambda}\Sigma^2 = \frac{1}{\lambda}\int dx^3\sqrt{g}\left(B^iA^iA^jA^j + \frac{1}{2}\lambda^i\lambda^iA^jA^j + \lambda^i\lambda^jA^iA^j\right) = \frac{1}{\lambda}\int dx^3\sqrt{g}(\Sigma_2\Sigma_1 + \Sigma_{3/2}\Sigma_{3/2}), \quad (6.15)$$

$$\int dz\Sigma\mathcal{O} = \int dx^3\sqrt{g}\left(f^i\phi^iA^jA^j + \frac{1}{2}\psi^i\psi^iA^jA^j + B^iA^i\phi^j\phi^j + \frac{1}{2}\lambda^i\lambda^i\phi^j\phi^j + 2\psi^i\lambda^j\phi^iA^j\right) = \int dx^3\sqrt{g}(\mathcal{O}_2\Sigma_1 + \Sigma_2\mathcal{O}_1 + 2\mathcal{O}_{3/2}\Sigma_{3/2}), \quad (6.16)$$

where we defined

$$\Sigma_1 = A^iA^i, \quad \mathcal{O}_1 = \phi^i\phi^i, \quad \Sigma_{3/2} = A^i\lambda^i, \quad \mathcal{O}_{3/2} = \phi^i\psi^i,$$
$$\Sigma_2 = B^iA^i + \frac{1}{2}\lambda^i\lambda^i, \qquad \mathcal{O}_2 = f^i\phi^i + \frac{1}{2}\psi^i\psi^i, \quad (6.17)$$

with the lower indices labeling the dimension of the single-trace operators.

With the above preparation, the second factor of (6.10) at large $N$ is

$$\begin{aligned}\int D\Sigma\exp\left[\frac{1}{2\lambda}\int dz\Sigma^2 + \frac{1}{2}\left\langle\left(\int dz\Sigma\mathcal{O}\right)^2\right\rangle_0\right] &= \int D\Sigma\exp\left[\frac{1}{\lambda}\int dx^3\sqrt{g}(\Sigma_2\Sigma_1 + \Sigma_{3/2}\Sigma_{3/2}) + \frac{1}{2}\left\langle\left(\int dx^3\sqrt{g}(\mathcal{O}_2\Sigma_1 + \Sigma_2\mathcal{O}_1 + 2\mathcal{O}_{3/2}\Sigma_{3/2})\right)^2\right\rangle_0\right] \\ &= \int D\Sigma\exp\left[\frac{1}{\lambda}\int dV(\Sigma_2\Sigma_1 + \Sigma_{3/2}\Sigma_{3/2}) + \frac{1}{2}\int\int dVdV'(\Sigma_1(x)\Sigma_1(x')\langle\mathcal{O}_2(x)\mathcal{O}_2(x')\rangle_0 \right.\\ &\left. + \Sigma_2(x)\Sigma_2(x')\langle\mathcal{O}_1(x)\mathcal{O}_1(x')\rangle_0 + 4\Sigma_{3/2}(x)\Sigma_{3/2}(x')\langle\mathcal{O}_{3/2}(x)\mathcal{O}_{3/2}(x')\rangle_0)\right],\end{aligned} \quad (6.18)$$

where $dV \equiv dx^3\sqrt{g}$ and we dropped vanishing terms in the two-point function to reach the last line.

The integral in (6.10) then becomes Gaussian, which integrates to give

$$\frac{Z}{Z_0}=\frac{\det(\mathbb{1}+2\lambda\langle\mathcal{O}_{3/2}\mathcal{O}_{3/2}\rangle_0)}{\{\det(\frac{\lambda}{2}\langle\mathcal{O}_2\mathcal{O}_2\rangle_0)\det(\frac{\lambda}{2}\langle\mathcal{O}_1\mathcal{O}_1\rangle_0)\det(\mathbb{1}-(\frac{\lambda}{4}\langle\mathcal{O}_2\mathcal{O}_2\rangle_0)^{-1}(\frac{\lambda}{4}\langle\mathcal{O}_1\mathcal{O}_1\rangle_0)^{-1})\}^{\frac{1}{2}}}. \tag{6.19}$$

At $\lambda\to\infty$, the change of the free energy compared to the free theory is

$$\Delta F=-\log\frac{Z}{Z_0}$$
$$=-\mathrm{tr}\,\log(2\langle\mathcal{O}_{3/2}\mathcal{O}_{3/2}\rangle_0)+\frac{1}{2}\mathrm{tr}\,\log\left(\frac{1}{2}\langle\mathcal{O}_2\mathcal{O}_2\rangle_0\right)$$
$$+\frac{1}{2}\mathrm{tr}\,\log\left(\frac{1}{2}\langle\mathcal{O}_1\mathcal{O}_1\rangle_0\right). \tag{6.20}$$

The two-point functions $\langle\mathcal{O}_1\mathcal{O}_1\rangle_0$ and $\langle\mathcal{O}_2\mathcal{O}_2\rangle_0$ can be expanded in terms of scalar harmonics on $S^3$ [28],

$$\langle\mathcal{O}_\Delta(x)\mathcal{O}_\Delta(x')\rangle_0=\sum_{\ell m}g_\ell^\Delta Y^*_{\ell m}(x)Y_{\ell m}(x'), \tag{6.21}$$

where $g_\ell^\Delta$ is given by

$$g_\ell^\Delta=R^{3-2\Delta}\pi^{\frac{3}{2}}2^{3-\Delta}\frac{\Gamma(\frac{3}{2}-\Delta)}{\Gamma(\Delta)}\frac{\Gamma(\ell+\Delta)}{(3+\ell-\Delta)}. \tag{6.22}$$

Since the harmonics satisfy orthonormal relations, we have

$$\int\sqrt{g}d^3y\langle\mathcal{O}_2(x)\mathcal{O}_2(y)\rangle_0\langle\mathcal{O}_1(y)\mathcal{O}_1(x')\rangle_0$$
$$=\sum_{\ell m}g_\ell^{\Delta=2}g_\ell^{\Delta=1}Y^*_{\ell m}(x)Y_{\ell m}(x'). \tag{6.23}$$

It is straightforward to see that $g_\ell^{\Delta=2}g^{\Delta=1}$ is independent of $\ell$, and therefore according to Ref. [28], $\mathrm{tr}\,\log\langle\mathcal{O}_2\mathcal{O}_2\rangle_0+\mathrm{tr}\,\log\langle\mathcal{O}_1\mathcal{O}_1\rangle_0$ does not contribute to $\Delta F$.

Similarly, for the fermionic two-point function, it is shown in Ref. [5] that $\mathrm{tr}\,\log\langle\mathcal{O}_{3/2}\mathcal{O}_{3/2}\rangle_0$ is also zero. Therefore, in the IR, there is no modification to the free energy given by the double-trace deformation.

When $\lambda$ is small, one can apply perturbation theory to compute $\Delta F$ induced by the deformation. As shown in Ref. [5], the change of free energy caused by the deformation is proportional to the beta function of the deformation coupling. The deformation appearing here is exactly marginal in the $N\to\infty$ limit, which implies that the beta function of the coupling constant is suppressed by $1/N$. Thus, at small coupling, it can also be seen that the deformation does not affect the $\mathcal{O}(N^0)$ free energy. In summary, although we have not computed the free energy of the deformed theory for arbitrary $\lambda$, the vanishing of $\Delta F$ at $\mathcal{O}(N^0)$ in both the strong and weak coupling limits provides strong evidence that $\Delta F$ does not receive $\mathcal{O}(N^0)$ contribution from the supersymmetric double-trace deformation, which is exactly marginal in the $N\to\infty$ limit.

## VII. CONCLUSIONS

We have carried out a one-loop test of the conjectured dualities between Konstein-Vasiliev HS theories in $AdS_4$ with $S^3$ and $S^1_\beta\times S^2$ boundaries. These theories are based on the HS algebras $hu(m;n|4)$, $ho(m;n|4)$, and $husp(m;n|4)$ which contain $u(m)\oplus u(n)$, $o(m)\oplus o(n)$, and $usp(m)\oplus usp(n)$ as bosonic subalgebras. Generically, these HS algebras can be interpreted as infinite-dimensional supersymmetry algebras, and they do not contain the extended $AdS_4$ superalgebra $OSp(\mathcal{N}|4)$ as a subalgebra. They do so only in the special case of $m=n=2^{\frac{\mathcal{N}}{2}-1}$ for even $\mathcal{N}$ or $2^{(\mathcal{N}-1)/2}$ for odd $\mathcal{N}$. Our results for the free energies extend previous ones [6,12,13] by the inclusion of fermionic bulk degrees of freedom. In computing the one-loop free energies of bosonic and fermionic HS fields in $AdS_4$ with an $S^3$ boundary, we have adopted the modified spectral zeta function method suggested by Ref. [10], thereby reproducing the one-loop free energy for bosonic HS fields in a much simpler way without the ambiguities encountered in Refs. [6,12]. We also find that the total one-loop free energy of an infinite tower of bulk fermionic fields vanishes.

Matching the bulk fields with boundary operators suggests that the possible CFT duals of Konstein-Vasiliev theories based on $hu(m;n|4)$, $ho(m;n|4)$, and $husp(m;n|4)$, and subject to HS symmetry-preserving boundary conditions, are, respectively, the $U(N)$, $O(N)$, and $USp(N)$ singlet sectors of free scalars and free fermions vector representations of the bosonic subalgebras conformally coupled to $S^3$. We find that the free energy of the HS theory may match with that of the free CFT only when the bulk theories are $hu(m;0|4)$, $ho(m;0|4)$, and $husp(m;0|4)$ Konstein-Vasiliev theories and with identifications $G_N^{-1}=\gamma(N+\Delta N)$ with suitable integers $\Delta N$. These are generalized type A theories with bosonic scalars on the boundary and bosonic bulk HS fields containing even parity scalars. Thus, in particular, the free energies for generalized type B models with fermions on the $S^3$ boundary and bosonic HS fields including odd parity scalar fields do not match. The mismatch in the case of $m=0$, $n=1$ corresponding to the simplest type B model has already been noted in Ref. [6] where the one-loop free energy $F^{(1)}=-\zeta(3)/(8\pi^2)$ obtained in the bulk does not agree with the free energy of Dirac fermions on the $S^3$ boundary. We have also calculated the free energies of Konstein-Vasiliev theories in $\mathrm{AdS}_4$ with an $S^1_\beta \times S^2$ boundary. In this case, we find that the free energies of all three families of Konstein-Vasiliev theories match those of the conjectured dual free CFTs.

Turning to the problem of mismatch in free energies of the type B model and its conjectured dual, one may have to take into account the issue of how to impose the $O(N)$ invariance condition on the CFT side. A natural way of implementing it is to gauge the $O(N)$ symmetry by means of a vector gauge field with a level $k$ Chern-Simons (CS) kinetic term. This term breaks parity, but the result for the free energy of the parity invariant model can be obtained in a limit in which the CS gauge field decouples. It has been suggested in Ref. [6] that as the fermions coupled to CS on the boundary give rise to a shift in the level $k$, it may not be justified to obtain the result for parity-preserving case by a naive subtraction of a CS contribution from the free energy on the CFT side. However, one expects that this effect becomes irrelevant in the decoupling limit in which $k \to \infty$. In fact, we have examined the procedure of decoupling CS in the large $k$ limit by evaluating the $S^3$ free energies for the Aharony-Bergman-Jafferis (ABJ) model based on $U(N)_k \times U(1)_{-k}$ [29,30] and a few $\mathcal{N} = 3$ CS matter theories in which the matter sector consists of fundamental hypermultiplets [31–33]. After subtracting the contribution from the pure CS term, we indeed obtain the free energies of free vector models. Therefore, the puzzle of the free energy mismatch in type B remains unresolved, and its solution requires deeper understanding of HS/vector model holography. In this context, it has been suggested by Ref. [34] and explored further in Ref. [35] that the vectorlike limit of the ABJ model based on $U(N)_k \times U(M)_{-k}$ is given by

$$N, k \to \infty \quad \text{with} \quad \lambda \equiv \frac{N}{k} \quad \text{and} \quad M \text{ finite.} \tag{7.1}$$

In this limit, the ABJ theory effectively behaves like a $\mathcal{N} = 6$ CS gauged vector model with $U(M)$ flavor symmetry [34]. Its bulk dual is conjectured to be the parity violating $\mathcal{N} = 6$ $U(M)$ gauged Vasiliev theory [34]. The parity violating angle $\theta_0$ is conjectured to be related to the CFT 't Hooft coupling by $\theta_0 = \pi\lambda/2$ [34].[6]

Turning to the question of free energy in the parity invariant HS theory, we may first keep $\lambda$ finite and consider the limit $\lambda \to 0$ that is required for the parity invariant limit at the end.[7] Different from the parity-preserving HS theories, in the $\mathcal{N} = 6$ parity violating HS theory, a mixed boundary condition needs to be imposed on the bulk $U(1)$ gauge field in order to preserve the $\mathcal{N} = 6$ supersymmetry [34]. The one-loop determinant of the bulk $U(1)$ gauge field with mixed boundary conditions should contain a $\log N$ term [36]. It was argued in Ref. [35] that the $\log N$ term can be fully captured by the $1/N$ correction to the anomalous dimension of the spin-0 ghost with the $\Delta_-$ boundary condition. This correction has not been computed on the bulk side. However, with the assumption that it is nonvanishing and parametrized by an undetermined constant, the resulting bulk one-loop free energy has been computed in Ref. [35]. Comparing this result with the free energy of ABJ theory in the vectorlike limit (7.1), with the free energy of pure $U(M)$ CS subtracted, the matching of the $\log N$ terms present in the free energies on both sides leads to the identification [35]

$$G_N = \frac{\gamma}{N} \frac{\pi\lambda}{\sin(\pi\lambda)}, \tag{7.2}$$

where $\gamma$ is an undetermined constant. On the other hand, an exact expression for $G_N$ has been obtained from the correlation function for two stress tensors on the CFT side in Ref. [37]. Comparing the relevant terms in these expressions for $G_N$, one deduces that $\gamma = 2/\pi$. Assuming the stated value of $\gamma$, in the limit $\lambda \to 0$, required for obtaining the parity invariant HS theory, one finds the relation $G_N = 2/(N\pi)$ which differs from the one that appears in the HS/free vector model holography by a factor of $\pi$. This is due to the fact that, while we assume that $F^{(0)}_{\mathrm{bulk}} = F^{(0)}_{\mathrm{CFT}}$ in the HS/free vector model holography, the example of HS/ABJ holography seems to suggest that $F^{(0)}_{\mathrm{bulk}} = F^{(0)}_{\mathrm{CFT}}/\gamma$. The above approach may seem to resolve the free energy problem in the type B model; however, a more rigorous computation of the one-loop free energy of the bulk $U(1)$ gauge field with mixed boundary conditions is needed to justify this value of $\gamma$. Furthermore, beyond the $\log N$ dependence, the terms of higher order in $1/N$ have not been compared in the matching of the free energies. These issues clearly deserve further study.

Another interesting future direction is to consider HS/free matrix model holography. In this case, the corresponding bulk HS theory contains infinitely many massive HS fields in addition to the usual massless ones. Recently, a preliminary one-loop test of HS/free matrix model

[6] Besides the Newton constant which is small in the limit described above, there is also a bulk 't Hooft coupling $g^2_{\mathrm{bulk}} M \sim M/N \ll 1$. String theory emerges when $M/N \sim 1$. Due to strong interactions, the HS particles form $U(M)$ singlet states which are described by the color neutral string states. Since the M theory circle $R_{11} \sim (M/k^5)^{1/6}$ shrinks and $\sqrt{\alpha'}/R_{\mathrm{AdS}} \sim (k/M)^{1/4} \to \infty$, this is a type IIA string in the high energy limit. The $\mathcal{N} = 6$ parity violating $U(M)$ gauged Vasiliev theory can be perceived as a deconfinement phase of a type IIA string when $M/N \ll 1$, in which the string states fragment into HS particles colored under $U(M)$ [34].

[7] There are subtleties regarding the $\lambda \to 0$ limit having to do with the subtraction of the free energy coming from the CS term, which may correspond to subtraction of an open string sector in the bulk [34].

holography was carried out in Ref. [10]. A dual pair considered in Ref. [10] consists of a free scalar field, namely the bosonic singleton Rac, in the adjoint representation of $SU(N)$ and a HS theory in $AdS_4$ of which the spectrum can be constructed from the two-, three-, and four-fold tensor products of the Rac. The bulk fields are dual to the single-trace of product of multiple Racs. The one-loop free energies of the bulk fields belonging to the first few Regge trajectories were computed in Ref. [10]. The one-loop free energy of the first trajectory comprised of massless HS fields is equal to that of a real conformally coupled scalar; however, such a feature ceases to exist for higher trajectories. It is possible that after summing over all trajectories, the total bulk free energy may possess a nice property. But such a difficult task has not been completed. It is also possible that supersymmetry may provide simplifications, as we recall that in $AdS_5$, the long multiplet of $SU(2,2|4)$ gives rise to vanishing one-loop free energy [19]. It should be noted that the matrix phase of the ABJ model based on $U(N)_k \times U(M)_{-k}$ with $M \sim N$ has conserved HS currents emerging in the limit $\lambda \to 0$, which implies the presence of massless HS particles in the spectrum of a type IIA string. Thus, in the regime

$$M \sim N, \qquad \lambda = N/k \to 0, \tag{7.3}$$

the duality between a IIA string on $AdS_4 \times \mathbb{CP}^3$ and ABJ theory may provide an example of HS/free matrix model duality [34] if the contribution from the CS term in the CFT can be simply subtracted. For the string theory interpretation of this limit, we refer the reader to Ref. [38]. The point we wish to stress here is that there are two regimes of type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ which remarkably give two different supersymmetric HS theories, one of which is expected to be dual to a vector model and the other to a matrix model on the boundary of $AdS_4$, and that the puzzle we have encountered in the one-loop test of holography by computing the free energies in the case of the vector model remains to be investigated thoroughly in the case of the matrix model.

A complete matching of the free energies on both sides requires the knowledge of $F^{(0)}_{\rm bulk}$ which can only be computed from the full action for HS theory. There exists an action that takes the form of a Chern-Simons action in a generalized spacetime of the form $\mathcal{M}_9 = \mathcal{X}_5 \times \mathcal{Z}_4$ where $\mathcal{Z}_4$ is a twistor space with no boundary, and the spacetime $\mathcal{M}_4$ resides on an open region of the boundary of $\mathcal{X}_5$ [11]. The action contains Lagrange multiplier master fields, but they do not propagate to produce unwanted degrees of freedom. What remains to be done is to add suitable HS invariant deformations that reside on the boundary of $\mathcal{M}_9$, which are highly restricted and for which candidates have been proposed [11], and to construct a boundary action that resides on the boundary of asymptotically $AdS_4$ spacetime $\mathcal{M}_4$ which has not been constructed so far. These are needed for obtaining the field equations through an appropriate variational principle, and once they are constructed, the full action can be quantized in a path integral approach, and the Feynman rules can be derived, even though the action does not have the traditional form consisting of an infinite sum of Einstein-Hilbert term and powers of curvature tensors and their derivatives. It remains to be seen whether the result for the one-loop free energy computed in this fashion agrees with that obtained under the assumption that the quadratic action for the HS fluctuations around $AdS_4$ has the standard Fronsdal form with two derivatives. In particular, it would be interesting to determine if the mismatch in the free energies encountered in the type B and ordinary supersymmetric HS theories and their conjectured duals may find a resolution in a computation based on the action discussed above.

## ACKNOWLEDGMENTS

We thank S. Giombi, I. Klebanov, E. Skvortsov, P. Sundell, A. Tseytlin, M. Vasiliev, and X. Yin for discussions. Y. P. would like to thank Nordita Institute, Beijing Normal University, and Sun Yet-sen University for hospitality during various stages of this work. Y. P. and E. S. would also like to thank Munich Institute for Astro- and Particle Physics for providing a wonderful work environment. The work of E. S. is supported in part by NSF Grant No. PHY-1214344. Y. P. is supported by the Alexander von Humboldt fellowship.

*Note added.*—After this paper appeared on the arXiv, two related papers [39,40] appeared the following day, where the vanishing of the contribution from the bulk fermions to the one-loop free energy has also been shown. It is worth noting that the regularization scheme we use for the individual spins is such that the subsequent sum over the infinite tower of higher spins is finite, unlike the method used in Refs. [39,40] where an additional regularization is needed to perform this sum.

## APPENDIX: COMPARISON OF $\zeta_{(\Delta,s)}(z)$ WITH $\tilde{\zeta}_{(\Delta,s)}(z)$

In this section, we will show that the alternate spectral zeta function is physically equivalent to the original spectral zeta in computing the one-loop free energy of HS fields.

### 1. Bosonic case

For bosonic HS fields, the physical equivalence of the alternate spectral zeta function and the original spectral zeta function has been studied in Ref. [10] in the case of summing over all integer spins. The crucial point is that for a given HS field labeled by $(\Delta, s)$, the difference between the alternate spectral zeta function and the original zeta function can be expressed as a contour integral encircling $\beta = 0$ [10],

$$\begin{aligned}\tilde{\zeta}^B_{(\Delta,s)}(z) - \zeta^B_{(\Delta,s)}(z) &= \frac{1}{3}\left(s+\frac{1}{2}\right)\nu^2\left[\frac{1}{6}\nu^2 - \left(s+\frac{1}{2}\right)^2\right] \\ &= \frac{z}{2\pi i}\oint d\beta \frac{2\sinh^3\frac{\beta}{2}}{\beta^3} \\ &\quad \times \left(\frac{8}{3\beta^2} + \frac{2}{\sinh^2\frac{\beta}{2}} - \frac{1}{3} + 4\partial_\alpha^2\right)\chi_{\Delta,s}(\beta,\alpha)\bigg|_{\alpha=0} + \mathcal{O}(z^2).\end{aligned} \quad \text{(A1)}$$

It has been shown in Ref. [10] that upon summing over all integer spins, the contour integral vanishes. We have also checked that this is also true for summing over all even spins or odd spins separately.

### 2. Fermionic case

For fermionic HS fields, we will elaborate on the physical equivalence of the alternate spectral zeta function and the original spectral zeta function which has not been studied elsewhere. For a fermionic HS field labeled by $(\Delta, s)$, the original spectral zeta function is given by [17]

$$\zeta^F_{(\Delta,s)}(z) = \frac{2s+1}{6}\int_0^\infty du \frac{u\coth(\pi u)[u^2 + (s+\frac{1}{2})^2]}{(u^2+\nu^2)^z}, \quad \text{(A2)}$$

where $\nu = \Delta - \frac{3}{2}$ in $D = 4$. Using the following identities,

$$\begin{aligned}&\left(s+\frac{1}{2}\right)\left[u^2 + \left(s+\frac{1}{2}\right)^2\right] \\ &\quad = \left(u^2\frac{d}{d\alpha} - \frac{d^3}{d\alpha^3}\right)\sin\left[\left(s+\frac{1}{2}\right)\alpha\right]\bigg|_{\alpha=0}, \\ &\frac{1}{(u^2+\nu^2)^z} = \frac{\sqrt{\pi}}{\Gamma(z)}\int_0^\infty d\beta e^{-\beta\nu}\left(\frac{\beta}{2u}\right)^{z-\frac{1}{2}} J_{z-1/2}(u\beta),\end{aligned} \quad \text{(A3)}$$

one can recast the spectral zeta function as in (3.18). The alternate spectral zeta function proposed in (3.20) can be computed exactly,

$$\begin{aligned}\tilde{\zeta}^F_{(\Delta,s)}(z) &= (2s+1)\left(\frac{1}{32} - \frac{s(s+1)}{24}\right)\frac{1}{\Gamma(2z)} \\ &\quad \times \int_0^\infty d\beta\beta^{2z-1}e^{-\nu\beta}\frac{1}{\sinh^2\frac{\beta}{2}} \\ &\quad + \frac{2s+1}{16}\frac{1}{\Gamma(2z)}\int_0^\infty d\beta\beta^{2z-1}e^{-\nu\beta}\frac{1}{\sinh^4\frac{\beta}{2}} \\ &= \frac{2s+1}{24}[\nu((2s+1)^2 - 4\nu^2)\zeta(2z,\nu) \\ &\quad + 4\zeta(2z-3,\nu) - 12\nu\zeta(2z-2,\nu) \\ &\quad + (12\nu^2 - 4s(s+1) - 1)\zeta(2z-1,\nu)],\end{aligned} \quad \text{(A4)}$$

from which we see that $\tilde{\zeta}^F_{(\Delta,s)}(0)$ matches $\zeta^F_{(\Delta,s)}(0)$. The latter takes the form

$$\begin{aligned}\zeta^F_{(\Delta,s)}(0) &= \frac{s+\frac{1}{2}}{6}\left[\frac{\nu^4}{2} - \left(s+\frac{1}{2}\right)^2\nu^2\right] \\ &\quad + \frac{1}{3}(2s+1)\left[\frac{1}{240} + \frac{(s+\frac{1}{2})^2}{24}\right].\end{aligned} \quad \text{(A5)}$$

It is easier to obtain this result of $\zeta^F_{(\Delta,s)}(0)$ using (A2) than (3.18). Next, we compute the first derivative of $\tilde{\zeta}^F_{(\Delta,s)}(z)$ at $z = 0$, which is given by

$$\begin{aligned}\tilde{\zeta}^{F\prime}_{(\Delta,s)}(0) &= \frac{2s+1}{12}[\nu((2s+1)^2 - 4\nu^2)\zeta'(0,\nu) + 4\zeta'(-3,\nu) \\ &\quad - 12\nu\zeta'(-2,\nu) + (12\nu^2 - 4s(s+1) - 1)\zeta'(-1,\nu)].\end{aligned} \quad \text{(A6)}$$

After some algebra, we obtain the difference between $\tilde{\zeta}^{F\prime}_{(\Delta,s)}(0)$ and $\zeta^{F\prime}_{(\Delta,s)}(0)$,

$$\tilde{\zeta}^{F\prime}_{(\Delta,s)}(0) - \zeta^{F\prime}_{(\Delta,s)}(0) = -\frac{1}{24}(2s+1)^3\nu^2 + \frac{2s+1}{9}\nu^4. \quad \text{(A7)}$$

The technique involved in the calculation is analogous to the bosonic case, and we refer readers to Appendix B of Ref. [10] for more details. This result can again be converted to a contour integral of $\beta$ circling $\beta = 0$,

$$\begin{aligned}\tilde{\zeta}^{F\prime}_{(\Delta,s)}(0) - \zeta^{F\prime}_{(\Delta,s)}(0) &= 2\pi i\oint d\beta\frac{2\sinh^3\frac{\beta}{2}}{\beta^3} \\ &\quad \times \left(\frac{32}{3\beta^2} + \frac{2}{\sinh^2\frac{\beta}{2}} - \frac{1}{3} + 4\partial_\alpha^2\right)\chi_{\Delta,s}(\beta,\alpha).\end{aligned} \quad \text{(A8)}$$

From (3.21), one can see that the total character of the fermionic sector including the contributions of all physical fermionic higher fields and their ghosts gives rise to an even function of $\beta$ which has a vanishing contour integral. Therefore, we have shown that in computing the one-loop free energy of the whole fermionic sector, the alternate spectral zeta function is physically equivalent to the original one.

[1] M. A. Vasiliev, More on equations of motion for interacting massless fields of all spins in $(3+1)$-dimensions, Phys. Lett. B **285**, 225 (1992).

[2] E. Sezgin and P. Sundell, Holography in 4D (super) higher spin theories and a test via cubic scalar couplings, J. High Energy Phys. 07 (2005) 044.

[3] I. R. Klebanov and A. M. Polyakov, AdS dual of the critical $O(N)$ vector model, Phys. Lett. B **550**, 213 (2002).

[4] E. Sezgin and P. Sundell, Massless higher spins and holography, Nucl. Phys. **B644**, 303 (2002).

[5] I. R. Klebanov, S. S. Pufu, and B. R. Safdi, $F$-theorem without supersymmetry, J. High Energy Phys. 10 (2011) 038.

[6] S. Giombi and I. R. Klebanov, One loop tests of higher spin AdS/CFT, J. High Energy Phys. 12 (2013) 068.

[7] E. D. Skvortsov, On (Un)Broken Higher-Spin Symmetry in Vector Models, arXiv:1512.05994.

[8] S. E. Konstein and M. A. Vasiliev, Extended higher spin superalgebras and their massless representations, Nucl. Phys. **B331**, 475 (1990).

[9] E. Sezgin and P. Sundell, Supersymmetric higher spin theories, J. Phys. A **46**, 214022 (2013).

[10] J. B. Bae, E. Joung, and S. Lal, One-loop test of free SU$(N)$ adjoint model holography, J. High Energy Phys. 04 (2016) 061.

[11] N. Boulanger, E. Sezgin, and P. Sundell, 4D Higher Spin Gravity with Dynamical Two-Form as a Frobenius–Chern–Simons Gauge Theory, arXiv:1505.04957.

[12] S. Giombi, I. R. Klebanov, and B. R. Safdi, Higher spin $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ at one loop, Phys. Rev. D **89**, 084004 (2014).

[13] S. Giombi, I. R. Klebanov, and A. A. Tseytlin, Partition functions and Casimir energies in higher spin $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$, Phys. Rev. D **90**, 024048 (2014).

[14] S. Giombi, TASI Lectures on the Higher Spin - CFT duality, arXiv:1607.02967.

[15] S. W. Hawking, Zeta function regularization of path integrals in curved space-time, Commun. Math. Phys. **55**, 133 (1977).

[16] R. Camporesi, Zeta function regularization of one loop effective potentials in anti-de Sitter space-time, Phys. Rev. D **43**, 3958 (1991).

[17] R. Camporesi and A. Higuchi, Arbitrary spin effective potentials in anti-de Sitter space-time, Phys. Rev. D **47**, 3339 (1993).

[18] M. A. Vasiliev, Nonlinear equations for symmetric massless higher spin fields in (A)dS(d), Phys. Lett. B **567**, 139 (2003).

[19] M. Beccaria and A. A. Tseytlin, Higher spins in $\mathrm{AdS}_5$ at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT, J. High Energy Phys. 11 (2014) 114.

[20] M. Beccaria and A. A. Tseytlin, Vectorial $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality for spin-one boundary theory, J. Phys. A **47**, 492001 (2014).

[21] J. B. Bae, E. Joung, and S. Lal, On the holography of free Yang-Mills, J. High Energy Phys. 10 (2016) 074.

[22] M. Beccaria, G. Macorini, and A. A. Tseytlin, Supergravity one-loop corrections on $\mathrm{AdS}_7$ and $\mathrm{AdS}_3$, higher spins and AdS/CFT, Nucl. Phys. **B892**, 211 (2015).

[23] G. W. Gibbons, M. J. Perry, and C. N. Pope, Partition functions, the Bekenstein bound and temperature inversion in anti-de Sitter space and its conformal boundary, Phys. Rev. D **74**, 084009 (2006).

[24] T. Basile, X. Bekaert, and N. Boulanger, Flato-Fronsdal theorem for higher-order singletons, J. High Energy Phys. 11 (2014) 131.

[25] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, The Hagedorn-deconfinement phase transition in weakly coupled large $N$ gauge theories, Adv. Theor. Math. Phys. **8**, 603 (2004).

[26] H. J. Schnitzer, Confinement/deconfinement transition of large $N$ gauge theories with $N(f)$ fundamentals: $N(f)/N$ finite, Nucl. Phys. **B695**, 267 (2004).

[27] S. H. Shenker and X. Yin, Vector Models in the Singlet Sector at Finite Temperature, arXiv:1109.3519.

[28] S. S. Gubser and I. R. Klebanov, A universal result on central charges in the presence of double trace deformations, Nucl. Phys. **B656**, 23 (2003).

[29] A. Kapustin, B. Willett, and I. Yaakov, Exact results for Wilson loops in superconformal Chern-Simons theories with matter, J. High Energy Phys. 03 (2010) 089.

[30] H. Awata, S. Hirano, and M. Shigemori, The partition function of ABJ theory, Prog. Theor. Exp. Phys. **2013**, 53B04 (2013).

[31] M. Marino, Lectures on localization and matrix models in supersymmetric Chern-Simons-matter theories, J. Phys. A **44**, 463001 (2011).

[32] D. R. Gulotta, C. P. Herzog, and T. Nishioka, The ABCDEF's of Matrix Models for Supersymmetric Chern-Simons Theories, J. High Energy Phys. 04 (2012) 138.

[33] M. Mezei and S. S. Pufu, Three-sphere free energy for classical gauge groups, J. High Energy Phys. 02 (2014) 037.

[34] C. M. Chang, S. Minwalla, T. Sharma, and X. Yin, ABJ Triality: From Higher Spin Fields to Strings, J. Phys. A **46**, 214009 (2013).

[35] S. Hirano, M. Honda, K. Okuyama, and M. Shigemori, ABJ theory in the higher spin limit, J. High Energy Phys. 08 (2016) 174.

[36] S. Giombi, I. R. Klebanov, S. S. Pufu, B. R. Safdi, and G. Tarnopolsky, AdS description of induced higher-spin gauge theory, J. High Energy Phys. 10 (2013) 016.

[37] M. Honda, Identification of bulk coupling constant in higher spin/ABJ correspondence, J. High Energy Phys. 08 (2015) 110.

[38] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, $N=6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals, J. High Energy Phys. 10 (2008) 091.

[39] M. Gunaydin, E. D. Skvortsov, and T. Tran, Exceptional $F(4)$ higher-spin theory in AdS(6) at one-loop and other tests of duality, J. High Energy Phys. 11 (2016) 168.

[40] S. Giombi, I. R. Klebanov, and Z. M. Tan, The ABC of Higher-Spin AdS/CFT, arXiv:1608.07611.