Six-dimensional Landau-Ginzburg-Wilson theory

J. A. Gracey and R. M. Simms

Theoretical Physics Division, Department of Mathematical Sciences, University of Liverpool,
P.O. Box 147, Liverpool L69 3BX, United Kingdom

We renormalize the six-dimensional cubic theory with an $O(N) \times O(m)$ symmetry at three loops in the modified minimal subtraction ($\overline{\text{MS}}$) scheme. The theory lies in the same universality class as the four-dimensional Landau-Ginzburg-Wilson model. As a check we show that the critical exponents derived from the three-loop renormalization group functions at the Wilson-Fisher fixed point are in agreement with the large N d-dimensional critical exponents of the underlying universal theory. Having established this connection we analyze the fixed point structure of the perturbative renormalization group functions to estimate the location of the conformal window of the $O(N) \times O(2)$ model.

I. INTRODUCTION

Scalar quantum field theories have been the subject of intense interest in recent years in the context of trying to develop our understanding of conformal field theories in dimensions greater than two using established concepts [1–7] in a modern application [8–11]. One of the main aspects of this activity is in finding the conformal window of a theory where there are nontrivial fixed points of the β-function. In this window one in principle has a theory where ideas for extending Zamolodchikov's c-theorem [12] to higher dimensions can be explored, for example, as well as other properties of strictly two-dimensional conformal field theories. One of the first examples of a quantum field theory with a conformal window was that derived from the Banks-Zaks fixed point in quantum chromodynamics (QCD) [13,14]. For a range of the number of quark flavors, N_f, there is a nontrivial fixed point at a nonzero value of the strong coupling constant. This is established from two-loop perturbation theory and the upper bound of the window follows from the one-loop term. However, the lower bound has not been unambiguously resolved partly because near this lower end the value of the critical coupling constant is not small and the perturbative approximation appears to break down. Nonperturbative lattice studies have yet to provide definitive data which determine the location of the lower bound precisely. Until relatively recently these two methods of perturbative analysis of β-functions and lattice studies of the breakdown of the chiral limit were generally the main techniques to access the conformal windows of a theory. A third approach was developed several years ago which falls under the banner of the conformal bootstrap technique [8–11]. It is a numerical analysis of the operator content of a scalar field theory and exploits the decoupling of operators from the spectrum in the limit to a fixed point. This allows one to obtain accurate values for scaling dimensions [10]. To date one main application has been to study scalar field theories across different spacetime dimensions. Recent examples include trying to find the conformal window for the $O(N)$ scalar ϕ^3 theory in five dimensions [15–19]. Originally Ma found that the conformal window was located at $N = 1038$ [20] in strictly six dimensions but the higher order terms in the ϵ in $d = 6 - 2\epsilon$ dimensions were computed to three [21] and four [22] loops. Using summation techniques the bound in five dimensions was reduced but not to the low values indicated by, for example, conformal bootstrap analyses [17]. While such agreement between techniques has yet to be resolved for other quantities in the conformal window, such as estimates of critical exponents, there is very strong overlap in the values which suggest these complementary methods do provide a solid insight into the properties of these scalar quantum field theories.

One upshot of the reopening of studies in higher dimensional theories has been that higher order perturbative results have been computed beyond a few loops. For instance, the three-loop results of [23,24] from 30 years ago for six-dimensional ϕ^3 theory were only extended to four loops recently [22]. Equally the five-loop renormalization group functions of four-dimensional ϕ^4 theory from the mid-1990s has now been extended to six loops in [25,26] and to seven loops for the field anomalous dimension [27]. Given the interest in the conformal bootstrap and its potential application to nonscalar theories or to scalar theories with symmetry other than $O(N)$ it is worthwhile to provide higher-loop perturbative results to complement recent [28] and future bootstrap studies in various dimensions. Therefore the aim of this article is to renormalize the six-dimensional extension of the Landau-Ginzburg-Wilson (LGW) model to three loops. In effect this is a ϕ^3 type theory but endowed with an $O(N) \times O(m)$ symmetry. It has applications in condensed matter problems such as randomly dilute spin models [29,30]. A conformal bootstrap analysis has recently been provided from the conformal bootstrap technology [28], and one aim is to provide

data to complement similar bootstrap analyses in the future. In addition analysis for the $O(4) \times O(2)$ theory, which describes the chiral phase transition in two-flavor QCD, has been discussed in [31]. A second motivation is to continue exploring the tower of theories across the dimensions which are in the same universality class at the Wilson-Fisher fixed point [32]. The LGW model with an $O(N) \times O(m)$ is a new example to continue this investigation which we will carry out in depth here. The extension of the four-dimensional $O(N) \times O(m)$ symmetric theory of [30] to six dimensions is termed the ultraviolet completion of the theory. Once a theory has been constructed in a fixed dimension with the same symmetries as its lower-dimensional counterpart the ϵ expansion of the critical exponents at the fixed point can be used to access nonperturbative fixed point properties in the lower dimensional partner. This ultraviolet-infrared connection was recognized earlier in [33,34] but its power is being exploited in current analyses. To connect theories across dimensions requires a technique beyond perturbation theory since coupling constants become dimensionful outside their critical dimension. The interpolating expansion parameter has to be dimensionless and in the context of theories with an $O(N)$ symmetry the parameter is $1/N$ where N is regarded as large. Earlier work by Vasil'ev *et al.* [35–37] provided critical exponents to three orders in $1/N$ for the universal theory in d dimensions with an $O(N)$ symmetry. In such exponents the origin of the d dependence is not from dimensional regularization. Rather large N Feynman integrals within the formalism of [35–37] are computed with analytic regularization. So the d dependence in the exponents is a true reflection of the properties and structure of the universal theory in any dimension. Indeed any perturbative expansion of a critically evaluated renormalization group function in the large N expansion agrees with the exponents of [35–37]. For the $O(N) \times O(m)$ extension this is also the case due to the computations of [38] in the large N expansion and explicit four-dimensional perturbation theory [39–45]. Moreover, such a check will also be important in establishing the correctness of our three-loop $\overline{\text{MS}}$ results for the six-dimensional theory in the same universality class prior to analyzing the renormalization group functions at criticality for a variety of values of N and m. We will do this both in the fixed dimension of six as well as within the ϵ expansion. Moreover, we will draw similar conclusions to others [28], where finding the specific location of the conformal window is not straightforward.

The paper is organized as follows. We construct the six-dimensional version of the theory with an $O(N) \times O(m)$ symmetry which is in the same universality class as the four-dimensional Landau-Ginzburg-Wilson model in Sec. II. The necessary large N analysis which will allow us to confirm this is also reviewed in that section. The main three-loop renormalization group functions are given in Sec. III together with a summary of the technology required

to determine them. From these results we derive the large N critical exponents in Sec. IV in order to compare with known $O(1/N^2)$ exponents. Having verified agreement with known results we discuss the search for a conformal window in Sec. V. A detailed fixed point analysis at three loops for a variety of values of N is provided in Sec. VI. Concluding observations are given in Sec. VII. Two appendixes are provided. The first gives the remaining renormalization group functions which were not displayed in the main text for space reasons, including the mass mixing matrix. The other appendix provides the full spectrum of fixed points for $N = 1000$ as a complete example of the rich structure of this model.

II. BACKGROUND

As we will be considering the six-dimensional model with $O(N) \times O(m)$ symmetry which is in the same universality class as the four-dimensional theory with the same symmetry we begin by recalling the relevant aspects of the latter theory. In this case the Lagrangian involves a quartic interaction for a scalar field ϕ^{ia} where $1 \leq i \leq N$ and $1 \leq a \leq m$. Consequently the Lagrangian is [38]

$$L^{(4)} = \frac{1}{2} \partial^\mu \phi^{ia} \partial_\mu \phi^{ia} + \frac{\bar{g}_1}{4!} (\phi^{ia} \phi^{ia})^2$$
$$+ \frac{\bar{g}_2}{4!} [(\phi^{ia} \phi^{ib})^2 - (\phi^{ia} \phi^{ia})^2] \quad (2.1)$$

where $\bar{g}_i$ are the couplings of the respective interactions. This version of the Landau-Ginzburg-Wilson theory is not the most useful for developing the large N expansion or indeed for seeing the connection with lower and higher dimensional theories. Instead it is better to reformulate $L^{(4)}$ in terms of cubic interactions by introducing a set of auxiliary fields $\tilde{\sigma}$ and $\tilde{T}^{ab}$ where the latter is antisymmetric and traceless in its $O(m)$ indices. Then $L^{(4)}$ becomes [38]

$$L^{(4)} = \frac{1}{2} \partial^\mu \phi^{ia} \partial_\mu \phi^{ia} + \frac{1}{2} \tilde{\sigma} \phi^{ia} \phi^{ia} + \frac{1}{2} \tilde{T}^{ab} \phi^{ia} \phi^{ib}$$
$$- \frac{3\tilde{\sigma}^2}{2\tilde{g}_1} - \frac{3\tilde{T}^{ab} \tilde{T}^{ab}}{2\tilde{g}_2} \quad (2.2)$$

where we have introduced $\tilde{g}_1 = \bar{g}_1 + (m-1)\tilde{g}_2/m$ and $\bar{g}_2 = \tilde{g}_2$ [29,30]. Here the coupling constants appear within the quadratic part of the Lagrangian which is the first step in constructing the critical exponents using the large N methods of [35,36]. However, for perturbative computations it is more appropriate for the couplings to appear with the actual interactions. So using a simple rescaling $L^{(4)}$ becomes

$$L^{(4)} = \frac{1}{2}\partial^\mu\phi^{ia}\partial_\mu\phi^{ia} + \frac{1}{2}\sigma^2 + \frac{1}{2}T^{ab}T^{ab} + \frac{1}{2}g_1\sigma\phi^{ia}\phi^{ia}$$
$$+ \frac{1}{2}g_2T^{ab}\phi^{ia}\phi^{ib}. \qquad (2.3)$$

In this formulation one can build the equivalent six-dimensional theory based on the dimensionalities of the fields and ensure that the Lagrangian is renormalizable. In d dimensions the ϕ^{ia} field has dimensions $(\frac{1}{2}d - 1)$ while σ and T^{ab} are both dimension 2. Clearly (2.3) is renormalizable in four dimensions. The key to constructing the six-dimensional extension is the retention of the two basic interactions of ϕ^{ia} with the auxiliary fields. This means that the dimensionalities of all three fields are preserved at the connecting Wilson-Fisher fixed point in d dimensions. However, the equivalent Lagrangian in six dimensions, $L^{(6)}$, has to have dimension 6 in order to retain a dimensionless action. Therefore the auxiliary field sector of (2.3) has to be replaced. This leads to

$$L^{(6)} = \frac{1}{2}\partial^\mu\phi^{ia}\partial_\mu\phi^{ia} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial^\mu T^{ab}\partial_\mu T^{ab}$$
$$+ \frac{1}{2}g_1\sigma\phi^{ia}\phi^{ia} + \frac{1}{6}g_2\sigma^3 + \frac{1}{2}g_3T^{ab}\phi^{ia}\phi^{ib}$$
$$+ \frac{1}{2}g_4\sigma T^{ab}T^{ab} + \frac{1}{6}g_5T^{ab}T^{ac}T^{bc} \qquad (2.4)$$

as the ultraviolet completion which is renormalizable in six dimensions and which should be equivalent to (2.3) in four dimensions at the Wilson-Fisher fixed point. As in previous extensions there are more interactions but also the σ and T^{ab} fields now cease being auxiliary fields and become propagating with fundamental propagators. The additional interactions which depend solely on σ and T^{ab} are referred to as spectators since they are only present in the critical dimension. The interactions with couplings g_1 and g_3 are the core ones which are present at the Wilson-Fisher fixed point through all the dimensions. They seed the universal theory in the sense that they determine the canonical dimensions of the fields. Thereby they induce the structure of the spectator interactions in each critical dimension by requiring renormalizability. Although our focus is primarily in relation to critical theories one could include masses for the three basic fields which would produce

$$L_m^{(6)} = L^{(6)} - \frac{1}{2}m_1^2\phi^{ia}\phi^{ia} - \frac{1}{2}m_2^2\sigma^2 - \frac{1}{2}m_3^2T^{ab}T^{ab} \qquad (2.5)$$

where m_i are the masses. Similar terms can be added to $L^{(4)}$.

To appreciate properties of these scalar models with an $O(N) \times O(m)$ symmetry we recall relevant properties of the β-functions of (2.3) which have been computed to

several loop orders [35,38]. However, for the present purposes it is sufficient to quote the results to two loops which are

$$\beta_1(\bar{g}_1, \bar{g}_2) = \frac{1}{2}(d-4)\bar{g}_1 + \frac{(mN+8)}{6}\bar{g}_1^2$$
$$- \frac{1}{3}(m-1)(N-1)\bar{g}_2\left(\bar{g}_1 - \frac{\bar{g}_2}{2}\right)$$
$$- \frac{1}{6}(3mN+14)\bar{g}_1^3$$
$$+ (m-1)(N-1)\left(\frac{11}{9}\bar{g}_1^2 - \frac{13}{12}\bar{g}_1\bar{g}_2 + \frac{5}{18}\bar{g}_2^2\right)\bar{g}_2$$
$$+ O(\bar{g}_i^4) \qquad (2.6)$$

and

$$\beta_2(\bar{g}_1, \bar{g}_2) = \frac{1}{2}(d-4)\bar{g}_2 + 2\bar{g}_1\bar{g}_2 + \frac{1}{6}(m+N-8)\bar{g}_2^2$$
$$- \frac{1}{18}(5mN+82)\bar{g}_1^2\bar{g}_2$$
$$+ \frac{1}{9}[5mN - 11(m+N) + 53]\bar{g}_1\bar{g}_2^2$$
$$- \frac{1}{36}[13mN - 35(m+N) + 99]\bar{g}_2^3$$
$$+ O(\bar{g}_i^4) \qquad (2.7)$$

where the order symbol is understood to mean any combination of the two coupling constants. For (2.3) [35,38], there are several fixed points which are the free field Gaussian fixed point, a point corresponding to the Heisenberg model and two where both critical couplings are nonzero. The fixed point corresponding to the Heisenberg case corresponds to $\bar{g}_1 \neq 0$ and $\bar{g}_2 = 0$ irrespective of whether m is set to unity or not. In the case when $m \neq 1$ the parameter m always appears as a multiplier of N. For the two fixed points where both critical couplings are nonzero one is known as the chiral stable (CS) fixed point and the other as the antichiral unstable (AU) one. For the Heisenberg fixed point in the context of the $(\bar{g}_1, \bar{g}_2)$-plane it is actually a saddle point and so it is unstable to perturbations in the $\bar{g}_2$ direction. In the reduction to the single coupling $O(N)$ scalar theory the Heisenberg fixed point would be stable.

As we will be using the large N results [38,46] to compare our six-dimensional perturbative results it is worthwhile to recall some of those results as well as give a perspective on the fixed point structure. In the large N method of [35,36] the critical exponents such as η and ω are computed by analyzing the skeleton Schwinger-Dyson equations at criticality. At that point the propagators and Green's functions obey scaling law type forms where the powers are in effect the critical exponents. If one expands

the exponent η, for example, in powers of $1/N$ where N is large,

$$\eta = \sum_{i=1}^{\infty} \frac{\eta_i}{N^i} \tag{2.8}$$

then each term, η_i, can be deduced from evaluating the relevant Feynman diagrams at each order of the $1/N$ expansion. While such diagrams are divergent they are *analytically* regularized which means that the solution for each η_i and the other exponents are determined as functions of the spacetime dimension d. Therefore these exponents, to as many orders in $1/N$ as they can be computed, correspond to the exponents of the universal quantum field theory which underlies the Wilson-Fisher fixed point in d dimensions. Thus when the exponents are expanded in powers of ϵ, where $d = D - 2\epsilon$ and D is the critical dimension of a specific theory, then the ϵ expansion will agree with the same expansion of the corresponding renormalization group function at the same fixed point. For theories such as (2.3) and (2.4) which are in the same universality class the large N critical exponents computed in [38,46] reflect the three nontrivial fixed points noted above. The different solutions for the Heisenberg, AU and CS cases emerge from simple conditions which are best seen in the Lagrangian formulation involving the fields σ and T^{ab}. These can be summarized by the vector (σ, T^{ab}) so that the Heisenberg fixed point is $(\sigma, 0)$; AU is $(0, T^{ab})$; and CS is (σ, T^{ab}) where a zero entry in the vector means the corresponding field is absent at that fixed point. In other words in the large N construction the critical exponents for a particular fixed point are determined by including only those nonzero fields in the vector in the skeleton Schwinger-Dyson expansion.

If we define the scaling dimensions of the fields ϕ^{ia}, σ and T^{ab} by α, β and γ respectively then

$$\alpha = \mu - 1 + \frac{1}{2}\eta, \qquad \beta = 2 - \eta - \chi,$$

$$\gamma = 2 - \eta - \chi_T, \tag{2.9}$$

where $d = 2\mu$, define the respective anomalous dimensions with η corresponding to that of ϕ^{ia}. The exponents χ and χ_T correspond to the respective vertex anomalous dimensions of σ and T^{ab} with ϕ^{ia}. These interactions are present in the universal theory. By contrast the spectator interactions, which involve only these two fields with themselves and not ϕ^{ia}, are present in the different forms in the theory in each critical spacetime dimension D. For completeness it is worth noting the leading large N critical exponent expressions [35,36,38],

$$\eta_1^{\mathrm{H}} = -\frac{4\Gamma(2\mu - 2)}{\Gamma(2 - \mu)\Gamma(\mu - 1)\Gamma(\mu - 2)\Gamma(\mu + 1)m}$$

$$\eta_1^{\mathrm{CS}} = -\frac{2(m+1)\Gamma(2\mu - 2)}{\Gamma(\mu + 1)\Gamma(\mu - 1)\Gamma(\mu - 2)\Gamma(2 - \mu)}$$

$$\eta_1^{\mathrm{AU}} = -\frac{2(m-1)(m+2)\Gamma(2\mu - 2)}{m\Gamma(\mu + 1)\Gamma(\mu - 1)\Gamma(\mu - 2)\Gamma(2 - \mu)}$$

$$\chi_1^{\mathrm{H}} = -\frac{\mu(4\mu - 5)\eta_1^{\mathrm{H}}}{(\mu - 2)}, \qquad \chi_1^{\mathrm{CS}} = -\frac{\mu(4\mu - 5)\eta_1^{\mathrm{CS}}}{(\mu - 2)}$$

$$\chi_{T,1}^{\mathrm{CS}} = -\frac{\mu[(2\mu - 3)m + (4\mu - 5)]\eta_1^{\mathrm{CS}}}{(\mu - 2)(m + 1)}$$

$$\chi_{T,1}^{\mathrm{AU}} = -\frac{\mu(m-2)[(m+4)(2\mu - 3) + 1]\eta_1^{\mathrm{AU}}}{(m-1)(m+2)(\mu - 2)}. \tag{2.10}$$

Higher order corrections are available in [35–38,46]. For the four-dimensional theory the exponents corresponding to the critical slope of the β-functions have also been determined, [46], which also give an insight into the stability of each fixed point. With

$$\omega = (\mu - 2) + \sum_{i=1}^{\infty} \frac{\omega_i}{N^i} \tag{2.11}$$

then [46]

$$\omega_{+1}^{\mathrm{Heis}} = -\frac{4(2\mu - 1)^2\Gamma(2\mu - 2)}{\Gamma(2 - \mu)\Gamma(\mu - 1)\Gamma(\mu - 2)\Gamma(\mu + 1)mN}$$

$$\omega_{+1}^{\mathrm{AU}} = -\left[2\mu^2 - 3\mu - 1 + \frac{\mu(m-2)[2\mu - 5 - 2(m+4)(2\mu - 3)]}{(m-1)(m+2)}\right]\frac{\eta_1^{\mathrm{AU}}}{N}$$

$$\omega_{\pm1}^{\mathrm{CS}} = \frac{(2\mu - 1)\eta_1^{\mathrm{CS}}}{2(m+1)(\mu - 2)N}[m(\mu - 1)(\mu - 4) + (2\mu^2 - 7\mu + 4)$$

$$\pm \mu[(m^2 - 1)(\mu - 1)^2 + 2(m-1)(2\mu - 3)(\mu - 1) + (5\mu - 8)^2]^{\frac{1}{2}}] \tag{2.12}$$

where the $\pm$ sign corresponds to two solutions in the CS case due to the presence of the two fields σ and T^{ab}. For the other two fixed points there is only one solution since there is in effect only one coupling constant relevant at these respective points. These large N exponents in essence appear to provide a more fundamental insight into the critical point structure of the underlying universal theory in the large N expansion. Although it is worth emphasizing that these results are useful for checking explicit perturbative expressions it is the critical point structure of the $O(N) \times O(m)$ theory for finite N which is our main focus.

In addition to the wave function and coupling constant renormalization we will also consider the renormalization of the three masses in (2.5) and determine the mixing matrix of anomalous dimensions to three loops. However, the comparison of the exponents derived from this matrix to the corresponding large N critical exponents is rather subtle which derives from the underlying operator. This is apparent in comparing the structure of the quadratic terms in σ and T^{ab} in (2.2), (2.3) and (2.5). In four dimensions the quadratic terms are present to implement the auxiliary field formulation of the quartic interaction. By contrast in six dimensions these fields have no auxiliary interpretation and the quadratic parts have to appear with a mass in order to have a consistent dimensionality. In terms of (2.2) the couplings g_1 and g_2 are not dimensionless away from four dimensions and can be interpreted as mass scales in higher dimensions. In other words at criticality the exponents ω at each of the three fixed points will be related to the mass anomalous dimensions of σ and T^{ab} computed in perturbation theory using (2.4) and then evaluated at criticality. In reality it is not a direct relation since one is dealing with a mass mixing matrix. Instead one compares the appropriate exponent ω with the eigen-anomalous dimension of the mixing matrix at criticality. The situation for the mass exponent of ϕ^{ia} is slightly different. In perturbation theory the three mass operators have the same canonical dimension and hence the mixing matrix is 3×3. In the large N expansion the canonical dimension of $\frac{1}{2}\phi^{ia}\phi^{ia}$ differs from the other two mass operators and the critical exponent associated with the ϕ^{ia} mass operator is not related to an ω exponent. Instead like in the $O(N)$ scalar theory the ϕ^{ia} mass dimension is given by the anomalous dimension of the σ field. In other words it is proportional to the sum of η and χ.

III. RESULTS

We now turn to the derivation of the various renormalization group functions for (2.4) which builds essentially on the method developed in [22] for the $O(N)$ case but with a minor caveat in the computation of several β-functions. The general procedure is to use an automated Feynman diagram approach where all the graphs are generated electronically using the QGRAF package, [47]. With indices appended to

this output and Feynman rules substituted, the resulting scalar integrals are integrated by applying the integration by parts algorithm developed by Laporta, [48]. This reduces all the integrals to a basic set of what are termed master integrals whose ϵ expansion is substituted at the final step. To implement the Laporta algorithm we have used the REDUZE version [49,50] and the masters given in [51]. In [22] we checked that the three-loop masters were consistent with the known four-dimensional masters by applying the Tarasov method [52,53]. This is a way of relating d-dimensional Feynman integrals to $(d + 2)$-dimensional ones. For all the renormalization group functions we determine we use the method of [54] to implement the renormalization in an automatic way. The Feynman diagrams are all evaluated as functions of the bare parameters, such as the coupling constants, and then these are replaced by their renormalized counterparts which involved the as yet undetermined counterterms. The counterterms are then chosen to render the appropriate Green's function finite with reference to a particular scheme which throughout will be the $\overline{\text{MS}}$ scheme. All the computations we carry out could not be possible without the use of the symbolic manipulation language FORM and its threaded version TFORM [55,56].

With the $O(N) \times O(m)$ symmetry the Feynman rules for the propagators and vertices involving the field T^{ab} have an associated color tensor. In other words the T^{ab} propagator will involve the tensor [38]

$$P^{abcd} = \frac{1}{2}\left[\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc} - \frac{2}{m}\delta^{ab}\delta^{cd}\right] \qquad (3.1)$$

which satisfies the trace properties

$$P^{abcc} = P^{aacd} = 0,$$
$$P^{abcb} = \frac{(m-1)(m+2)}{2m}\delta^{ac}. \qquad (3.2)$$

It also satisfies the projector relations

$$P^{abpq}P^{pqcd} = P^{abcd},$$
$$P^{abpq}P^{cpdq} = \frac{(m-2)}{2m}P^{abcd}. \qquad (3.3)$$

Equipped with this the Feynman rule for the triple T^{ab} vertex involves the rank 6 color tensor

$$P_3^{abcdef} = P^{abpq}P^{cdpr}P^{efqr}. \qquad (3.4)$$

Consequently

$$P_3^{abcdpq} P^{efpq} = P_3^{abcdef},$$

$$P_3^{abcded} = \frac{(m-2)(m+4)}{4m} P^{abce}$$

$$P_3^{abpqrs} P_3^{cdpqrs} = \frac{(m-2)(m+4)}{4m} P^{abcd} \qquad (3.5)$$

for instance. Encoding these within a FORM module allows the group theory evaluation of the higher-loop graphs to proceed more efficiently.

The procedure we used to compute the large number of Feynman diagrams for the most part follows that described in [22] to which we refer the reader for the more technical aspects and focus on the amendments we made here to renormalize (2.5) at three loops. One useful technique which was exploited in [22] was that in addition to the wave function renormalization constants, the coupling constant and mass renormalization constants could be determined by purely evaluating the two-point functions of each field. This was because each basic scalar propagator $1/k^2$ could be replaced by

$$\frac{\delta^{ij}}{k^2} \mapsto \frac{\delta^{ij}}{k^2} + \frac{\lambda_1 \delta^{ij}}{(k^2)^2} + \frac{\lambda_2 g d^{ijk_e}}{(k^2)^2} \qquad (3.6)$$

where the parameters λ_1 and λ_2 tag the mass operator insertion and three-point vertex insertion, both at zero momentum. The group structure of the general cubic theory is included in the final term and k_e is a fixed index corresponding to the external leg of that vertex. In performing this replacement one truncates the expansion at the linear term in λ_i as this reproduces all the relevant graphs for the respective mass operator and vertex renormalizations with one nullified external leg. This expansion does not lead to any problems in six dimensions as a $1/(k^2)^2$

TABLE I. Number of Feynman diagrams for each two- and three-point function.

Green's function	One loop	Two loops	Three loops	Total
$\phi^{ia}\phi^{jb}$	2	23	514	539
$\sigma\sigma$	3	19	343	365
$T^{ab}T^{cd}$	3	27	589	619
$T^{ab}\phi^{ia}\phi^{jb}$	5	137	4984	5126
$T^{ab}T^{cd}T^{ef}$	5	155	5857	6017
Total	18	361	12287	12666

propagator is infrared safe unlike in four dimensions. We have recalled this procedure partly because we have exploited it to minimize the amount of computations we need to perform and equally because it is not fully applicable to renormalizing (2.4) since it misses out on certain graphs which involve the $\sigma T^{ab} T^{ab}$ vertex. While we used it for the mass mixing matrix for (2.5) the replacement does not generate all the vertex graphs for the renormalization of the couplings g_3 and g_5. Instead for the associated three-point Green's functions we had to generate all the Feynman diagrams separately using QGRAF and evaluate them with one nullified external vertex. While tedious there were no major difficulties. To gauge the size of the overall renormalization which was carried out, the number of graphs we computed for each Green's function is given in Table I.

The results of our computations are the renormalization group functions. As we will mainly focus our analysis on the $O(N) \times O(2)$ theory we record these, partly because of that but also due to space consideration, but note that the full $O(N) \times O(m)$ expressions are provided in the associated data file [57]. First, the anomalous dimensions for the three fields are

$$\gamma_\phi(g_i)|_{m=2} = -\frac{1}{6}[g_1^2 + g_3^2]$$

$$+ \frac{1}{432}[-22Ng_1^4 + 26g_1^4 + 48g_1^3 g_2 - 11g_1^2 g_2^2 + 52g_1^2 g_3^2 - 22g_1^2 g_4^2 + 144g_1 g_3^2 g_4 - 11Ng_3^4 - 22g_3^4 - 22g_3^2 g_4^2]$$

$$+ \frac{1}{31104}[52N^2 g_1^6 - 464Ng_1^6 + 5184\zeta_3 g_1^6 - 9064g_1^6 + 5292Ng_1^5 g_2 - 3264g_1^5 g_2 - 772Ng_1^4 g_2^2 + 5184\zeta_3 g_1^4 g_2^2$$

$$- 11762g_1^4 g_2^2 + 40Ng_1^4 g_3^2 + 15552\zeta_3 g_1^4 g_3^2 - 27192g_1^4 g_3^2 + 104Ng_1^4 g_4^2 + 236g_1^4 g_4^2 + 942g_1^3 g_2^3 - 3264g_1^3 g_2 g_3^2$$

$$+ 2388g_1^3 g_2 g_4^2 + 5292Ng_1^3 g_3^2 g_4 - 9792g_1^3 g_3^2 g_4 - 504g_1^3 g_4^3 + 327g_1^2 g_2^4 + 118g_1^2 g_2^2 g_3^2 - 772g_1^2 g_2^2 g_4^2$$

$$+ 10368\zeta_3 g_1^2 g_2 g_3^2 g_4 - 23760g_1^2 g_2 g_3^2 g_4 + 2904g_1^2 g_2 g_4^3 - 736Ng_1^2 g_3^4 + 2304g_1^2 g_3^4 - 1648Ng_1^2 g_3^2 g_4^2$$

$$+ 20736\zeta_3 g_1^2 g_3^2 g_4^2 - 47048g_1^2 g_3^2 g_4^2 - 144g_1^2 g_4^4 + 1194g_1 g_2^2 g_3^2 g_4 - 756g_1 g_2 g_3^2 g_4^2 + 5292Ng_1 g_3^4 g_4$$

$$+ 1944g_1 g_3^4 g_4 + 6408g_1 g_3^2 g_4^3 - 412g_2^2 g_3^2 g_4^2 + 1452g_2 g_3^2 g_4^3 + 13N^2 g_3^6 - 1282Ng_3^6 + 5184\zeta_3 g_3^6 - 9844g_3^6$$

$$- 360Ng_3^4 g_4^2 - 3724g_3^4 g_4^2 - 144g_3^2 g_4^4] + O(g_i^8)$$

$$\gamma_\sigma(g_i)|_{m=2} = \frac{1}{12}[-2Ng_1^2 - g_2^2 - 2g_4^2]$$

$$+ \frac{1}{432}[4Ng_1^4 + 96Ng_1^3g_2 - 22Ng_1^2g_2^2 + 4Ng_1^2g_3^2 + 96Ng_1g_3^2g_4 + 13g_2^4 - 22g_2^2g_4^2 + 96g_2g_4^3 - 22Ng_3^2g_4^2 + 4g_4^4]$$

$$+ \frac{1}{62208}[-11048N^2g_1^6 + 10368\zeta_3Ng_1^6 - 17120Ng_1^6 + 4608N^2g_1^5g_2 + 2112Ng_1^5g_2 + 12N^2g_1^4g_2^2$$

$$+ 25920\zeta_3Ng_1^4g_2^2 - 53292Ng_1^4g_2^2 + 20736\zeta_3Ng_1^4g_3^2 - 34240Ng_1^4g_3^2 - 824Ng_1^4g_4^2 - 3120Ng_1^3g_2^3$$

$$- 2688Ng_1^3g_2g_3^2 + 4608Ng_1^3g_2g_4^2 + 11712Ng_1^3g_3^2g_4 - 20448Ng_1^3g_4^3 + 1904Ng_1^2g_2^4 - 392Ng_1^2g_2^2g_3^2$$

$$+ 24Ng_1^2g_2^2g_4^2 + 31104\zeta_3Ng_1^2g_2g_3^2g_4 - 66672Ng_1^2g_2g_3^2g_4 + 4608Ng_1^2g_2g_4^3 - 5524N^2g_1^2g_3^4 - 3776Ng_1^2g_3^4$$

$$+ 41472\zeta_3Ng_1^2g_3^2g_4^2 - 77824Ng_1^2g_3^2g_4^2 - 824Ng_1^2g_4^4 + 5808Ng_1g_2^2g_3^2g_4 - 12240Ng_1g_2g_3^2g_4^2 + 2304N^2g_1g_3^4g_4$$

$$- 672Ng_1g_3^4g_4 + 4992Ng_1g_3^2g_4^3 + 2592\zeta_3g_2^6 - 5195g_2^6 + 1904g_2^4g_4^2 - 3120g_2^3g_4^3 - 1648Ng_2^2g_3^2g_4^2$$

$$+ 25920\zeta_3g_2^2g_4^4 - 53280g_2^2g_4^4 + 4776Ng_2g_3^2g_4^3 + 6720g_2g_4^5 + 6N^2g_3^4g_4^2 - 8408Ng_3^4g_4^2 + 680Ng_3^2g_4^4$$

$$+ 10368\zeta_3g_4^6 - 28168g_4^6] + O(g_i^8)$$

$$\gamma_T(g_i)|_{m=2} = \frac{1}{12}[-Ng_3^2 - 2g_4^2] + \frac{1}{432}[2Ng_1^2g_3^2 - 22Ng_1^2g_4^2 + 96Ng_1g_3^2g_4 - 11g_2^2g_4^2 + 48g_2g_4^3 - 22Ng_3^4 - 11Ng_3^2g_4^2 + 4g_4^4]$$

$$+ \frac{1}{31104}[-206N^2g_1^4g_3^2 + 2592\zeta_3Ng_1^4g_3^2 - 4280Ng_1^4g_3^2 + 52N^2g_1^4g_4^2 - 196Ng_1^4g_4^2 + 1200Ng_1^3g_2g_3^2$$

$$+ 2904Ng_1^3g_2g_4^2 + 1152N^2g_1^3g_3^2g_4 - 1344Ng_1^3g_3^2g_4 - 504Ng_1^3g_4^3 - 103Ng_1^2g_2^2g_3^2 - 772Ng_1^2g_2^2g_4^2$$

$$+ 5184\zeta_3Ng_1^2g_2g_3^2g_4 - 9576Ng_1^2g_2g_3^2g_4 + 2388Ng_1^2g_2g_4^3 - 2556N^2g_1^2g_3^4 + 2168Ng_1^2g_3^4 - 46N^2g_1^2g_3^2g_4^2$$

$$+ 15552\zeta_3Ng_1^2g_3^2g_4^2 - 33836Ng_1^2g_3^2g_4^2 + 340Ng_1^2g_4^4 + 576Ng_1g_2^2g_3^2g_4 - 5364Ng_1g_2g_3^2g_4^2 + 576N^2g_1g_3^4g_4$$

$$+ 8376Ng_1g_3^4g_4 + 2496Ng_1g_3^2g_4^3 + 327g_2^4g_4^2 + 942g_2^3g_4^3 - 23Ng_2^2g_3^2g_4^2 + 5184\zeta_3g_2^2g_4^4 - 12534g_2^2g_4^4$$

$$+ 576Ng_2g_3^2g_4^3 + 2028g_2g_4^5 - 412N^2g_3^6 + 2592\zeta_3Ng_3^6 - 5354Ng_3^6 + 13N^2g_3^4g_4^2 - 2152Ng_3^4g_4^2$$

$$- 36Ng_3^2g_4^4 + 5184\zeta_3g_4^6 - 9476g_4^6] + O(g_i^8) \tag{3.7}$$

where ζ_z is the Riemann zeta function and the argument of the functions represents all five coupling constants. The five β-functions are of similar form and we note that

$$\beta_1(g_i)|_{m=2} = \frac{1}{24}[-2Ng_1^3 + 8g_1^3 + 12g_1^2g_2 - g_1g_2^2 + 8g_1g_3^2 - 2g_1g_4^2 + 12g_3^2g_4]$$

$$+ \frac{1}{864}[-172Ng_1^5 - 536g_1^5 + 264Ng_1^4g_2 - 360g_1^4g_2 - 22Ng_1^3g_2^2 - 628g_1^3g_2^2 + 4Ng_1^3g_3^2 - 1072g_1^3g_3^2 + 40g_1^3g_4^2$$

$$- 24g_1^2g_2^3 - 240g_1^2g_2g_3^2 + 168g_1^2g_2g_4^2 + 96Ng_1^2g_3^2g_4 - 600g_1^2g_3^2g_4 - 216g_1^2g_4^3 + 13g_1g_2^4 - 22g_1g_2^2g_4^2$$

$$- 648g_1g_2g_3^2g_4 + 96g_1g_2g_4^3 - 88Ng_1g_3^4 + 16g_1g_3^4 - 22Ng_1g_3^2g_4^2 - 1256g_1g_3^2g_4^2 + 4g_1g_4^4 - 108g_2g_3^2g_4^2$$

$$+ 84Ng_3^4g_4 - 24g_3^4g_4 + 60g_3^2g_4^3]$$

$$+ \frac{1}{124416}[14648N^2g_1^7 + 259200\zeta_3Ng_1^7 - 81376Ng_1^7 + 20736\zeta_3g_1^7 + 251360g_1^7 - 144N^2g_1^6g_2$$

$$- 311040\zeta_3Ng_1^6g_2 + 249408Ng_1^6g_2 + 186624\zeta_3g_1^6g_2 + 18000g_1^6g_2 + 12N^2g_1^5g_2^2 + 25920\zeta_3Ng_1^5g_2^2$$

$$- 107980Ng_1^5g_2^2 - 41472\zeta_3g_1^5g_2^2 + 358480g_1^5g_2^2 + 20736\zeta_3Ng_1^5g_3^2 - 106848Ng_1^5g_3^2 + 62208\zeta_3g_1^5g_3^2$$

$$+ 754080g_1^5g_3^2 + 23496Ng_1^5g_4^2 - 15712g_1^5g_4^2 - 9120Ng_1^4g_2^3 + 124416\zeta_3g_1^4g_2^3 + 97776g_1^4g_2^3 + 7488Ng_1^4g_2g_3^2$$

$$+ 248832\zeta_3g_1^4g_2g_3^2 + 59712g_1^4g_2g_3^2 - 4896Ng_1^4g_2g_4^2 - 20736g_1^4g_2g_4^2 - 186624\zeta_3Ng_1^4g_3^2g_4 + 160704Ng_1^4g_3^2g_4$$

$$+ 435456\zeta_3g_1^4g_3^2g_4 - 29424g_1^4g_3^2g_4 + 6624Ng_1^4g_4^3 - 50688g_1^4g_4^3 + 1904Ng_1^3g_2^4 + 62208\zeta_3g_1^3g_2^4 + 9960g_1^3g_2^4$$

$$- 392Ng_1^3g_2^2g_3^2 + 158032g_1^3g_2^2g_3^2 + 24Ng_1^3g_2^2g_4^2 - 44032g_1^3g_2^2g_4^2 + 31104\zeta_3Ng_1^3g_2g_3^2g_4 - 98352Ng_1^3g_2g_3^2g_4$$

$$- 82944\zeta_3g_1^3g_2g_3^2g_4 + 655776g_1^3g_2g_3^2g_4 + 4608Ng_1^3g_2g_4^3 - 124416\zeta_3g_1^3g_2g_4^3 + 17664g_1^3g_2g_4^3 - 5524N^2g_1^3g_3^4$$

$$+ 373248\zeta_3 N g_1^3 g_3^4 - 27552 N g_1^3 g_3^4 - 124416\zeta_3 g_1^3 g_3^4 + 218016 g_1^3 g_3^4 + 41472\zeta_3 N g_1^3 g_3^2 g_4^2 - 62336 N g_1^3 g_3^2 g_4^2$$

$$- 165888\zeta_3 g_1^3 g_3^2 g_4^2 + 924160 g_1^3 g_3^2 g_4^2 - 824 N g_1^3 g_4^4 + 248832\zeta_3 g_1^3 g_4^4 + 43968 g_1^3 g_4^4 - 31104\zeta_3 g_1^2 g_2^5$$

$$+ 33612 g_1^2 g_2^5 - 8352 g_1^2 g_2^3 g_3^2 - 6000 g_1^2 g_2^3 g_4^2 + 5808 N g_1^2 g_2^2 g_3^2 g_4 + 124416\zeta_3 g_1^2 g_2^2 g_3^2 g_4 + 142128 g_1^2 g_2^2 g_3^2 g_4$$

$$- 10656 g_1^2 g_2^2 g_3^3 - 93312\zeta_3 N g_1^2 g_2 g_3^4 + 100488 N g_1^2 g_2 g_3^4 - 36480 g_1^2 g_2 g_3^4 - 20688 N g_1^2 g_2 g_3^2 g_4^2$$

$$+ 373248\zeta_3 g_1^2 g_2 g_3^2 g_4^2 + 275568 g_1^2 g_2 g_3^2 g_4^2 - 186624\zeta_3 g_1^2 g_2 g_4^4 + 245616 g_1^2 g_2 g_4^4 + 2304 N^2 g_1^2 g_3^4 g_4$$

$$- 248832\zeta_3 N g_1^2 g_3^4 g_4 + 128688 N g_1^2 g_3^4 g_4 + 248832\zeta_3 g_1^2 g_3^4 g_4 - 201120 g_1^2 g_3^4 g_4 - 16128 N g_1^2 g_3^2 g_4^3$$

$$+ 248832\zeta_3 g_1^2 g_3^2 g_4^3 + 183840 g_1^2 g_3^2 g_4^3 - 16128 g_1^2 g_4^5 + 2592\zeta_3 g_1 g_2^6 - 5195 g_1 g_2^6 + 1904 g_1 g_2^4 g_4^2$$

$$- 27072 g_1 g_2^3 g_3^2 g_4 - 3120 g_1 g_2^3 g_4^3 - 1648 N g_1 g_2^2 g_3^2 g_4^2 + 186624\zeta_3 g_1 g_2^2 g_3^2 g_4^2 + 60176 g_1 g_2^2 g_3^2 g_4^2$$

$$+ 25920\zeta_3 g_1 g_2^2 g_4^4 - 53280 g_1 g_2^2 g_4^4 - 42912 N g_1 g_2 g_3^4 g_4 - 124416\zeta_3 g_1 g_2 g_3^4 g_4 + 36288 g_1 g_2 g_3^4 g_4$$

$$+ 4776 N g_1 g_2 g_3^2 g_4^3 + 62208\zeta_3 g_1 g_2 g_3^2 g_4^3 + 60672 g_1 g_2 g_3^2 g_4^3 + 6720 g_1 g_2 g_4^5 + 6424 N^2 g_1 g_3^6 + 32144 N g_1 g_3^6$$

$$+ 82944\zeta_3 g_1 g_3^6 + 72416 g_1 g_3^6 + 6 N^2 g_1 g_3^4 g_4^2 - 45512 N g_1 g_3^4 g_4^2 + 209504 g_1 g_3^4 g_4^2 + 680 N g_1 g_3^2 g_4^4$$

$$+ 124416\zeta_3 g_1 g_3^2 g_4^4 - 85632 g_1 g_3^2 g_4^4 + 10368\zeta_3 g_1 g_4^6 - 28168 g_1 g_4^6 + 11808 g_2^3 g_3^2 g_4^2 - 62208\zeta_3 g_2^2 g_3^2 g_4^3$$

$$+ 63744 g_2^2 g_3^2 g_4^3 + 3600 N g_2 g_3^4 g_4^2 - 62208\zeta_3 g_2 g_3^4 g_4^2 + 70272 g_2 g_3^4 g_4^2 - 43488 g_2 g_3^2 g_4^4 - 1188 N^2 g_3^6 g_4$$

$$+ 31848 N g_3^6 g_4 - 62208\zeta_3 g_3^6 g_4 + 165840 g_3^6 g_4 - 2208 N g_3^4 g_4^3 + 124416\zeta_3 g_3^4 g_4^3 + 23136 g_3^4 g_4^3$$

$$- 62208\zeta_3 g_3^2 g_4^5 + 118320 g_3^2 g_4^5] + O(g_i^9). \tag{3.8}$$

The remaining expressions are given in Appendix A where the mixing matrix of mass anomalous dimensions is also provided. One test of the expressions we have computed is that the double and triple poles of all the underlying renormalization constants correctly emerge as predicted by the renormalization group formalism. Equally we have checked the limit back to the pure $O(N)$ theory where the $O(m)$ indices are completely passive and found agreement with [21]. The final checks which we have derived from the comparison with the large N exponents we devote to the next section.

IV. LARGE N ANALYSIS

Equipped with the explicit forms of the renormalization group functions we are in a position to check them against the large N critical exponents for each of the three fixed points. In order to do this we follow the prescription introduced in [21] and define scaled coupling constants by

$$g_1 = i\sqrt{\frac{12\epsilon}{mN}}x, \qquad g_2 = i\sqrt{\frac{12\epsilon}{mN}}y, \qquad g_3 = i\sqrt{\frac{12\epsilon}{N}}z,$$

$$g_4 = i\sqrt{\frac{12\epsilon}{mN}}t, \qquad g_5 = i\sqrt{\frac{12\epsilon}{N}}w. \tag{4.1}$$

With these we can deduce the location of each fixed point in a large N expansion where each coefficient of the power of $1/N$ is a function of ϵ, having set $d = 6 - 2\epsilon$. Each of the three fixed points is defined by different field content and therefore for the Heisenberg and AU only several of the coupling constants are nonzero. From the respective β-functions we find

$$x = 1 + \left(22 - \frac{155}{3}\epsilon + \frac{1777}{36}\epsilon^2\right)\frac{1}{mN}$$

$$+ \left(726 - 3410\epsilon + \frac{29093}{9}\epsilon^2 - 4680\zeta_3\epsilon^2\right)\frac{1}{m^2 N^2}$$

$$+ O\left(\epsilon^3; \frac{1}{N^3}\right)$$

$$y = 6 + \left(972 - 1290\epsilon + \frac{2781}{2}\epsilon^2\right)\frac{1}{mN}$$

$$+ (412596 - 1036020\epsilon + 1083644\epsilon^2 - 628560\zeta_3\epsilon^2)$$

$$\times \frac{1}{m^2 N^2} + O\left(\epsilon^3; \frac{1}{N^3}\right)$$

$$z = t = w = 0 \tag{4.2}$$

for the Heisenberg case which is consistent with [21] where the order symbol represents the truncation point for the two independent expansions. For AU we have

$$z = 1 + \left(11 - 40\frac{1}{m} + \frac{7}{2}m + \frac{299}{3}\frac{1}{m}\epsilon - \frac{155}{6}\epsilon - \frac{25}{3}m\epsilon - \frac{3829}{36}\frac{1}{m}\epsilon^2 + \frac{1777}{72}\epsilon^2 + \frac{80}{9}m\epsilon^2 \right)\frac{1}{N}$$

$$+ \left(-\frac{477}{2} + 2400\frac{1}{m^2} - 1320\frac{1}{m} + \frac{231}{2}m + \frac{147}{8}m^2 - 14180\frac{1}{m^2}\epsilon + 6959\frac{1}{m}\epsilon + 1949\epsilon \right.$$

$$- \frac{2555}{4}m\epsilon - 150m^2\epsilon + \frac{38755}{18}\frac{1}{m^2}\epsilon^2 - 20664\zeta_3\frac{1}{m^2}\epsilon^2 - \frac{136469}{36}\frac{1}{m}\epsilon^2 + 10296\zeta_3\frac{1}{m}\epsilon^2$$

$$\left. - \frac{19919}{24}\epsilon^2 + 1242\zeta_3\epsilon^2 + \frac{123919}{144}m\epsilon^2 - 576\zeta_3 m\epsilon^2 + \frac{23695}{72}m^2\epsilon^2 \right)\frac{1}{N^2} + O\left(\epsilon^3; \frac{1}{N^3} \right)$$

$$w = 6 + \left(486 - 3240\frac{1}{m} + 81m + 5178\frac{1}{m}\epsilon - 645\epsilon - 150m\epsilon - \frac{12105}{2}\frac{1}{m}\epsilon^2 + \frac{2781}{4}\epsilon^2 + 180m\epsilon^2 \right)\frac{1}{N}$$

$$+ \left(-118071 + 4874400\frac{1}{m^2} - 1417320\frac{1}{m} + 32283m + \frac{10161}{4}m^2 - 14470680\frac{1}{m^2}\epsilon \right.$$

$$+ 3945054\frac{1}{m}\epsilon + 464454\epsilon - \frac{186915}{2}m\epsilon - 10830m^2\epsilon + 16668989\frac{1}{m^2}\epsilon^2$$

$$- 7556976\zeta_3\frac{1}{m^2}\epsilon^2 - \frac{8635273}{2}\frac{1}{m}\epsilon^2 + 2238192\zeta_3\frac{1}{m}\epsilon^2 - \frac{2355195}{4}\epsilon^2 + 236196\zeta_3\epsilon^2$$

$$\left. + \frac{915527}{8}m\epsilon^2 - 42120\zeta_3 m\epsilon^2 + \frac{76709}{4}m^2\epsilon^2 \right)\frac{1}{N^2} + O\left(\epsilon^3; \frac{1}{N^3} \right) \tag{4.3}$$

and, finally,

$$x = 1 + \left(11 + 11m - \frac{155}{6}\epsilon - \frac{155}{6}m\epsilon + \frac{1777}{72}\epsilon^2 + \frac{1777}{72}m\epsilon^2 \right)\frac{1}{N} + \left(\frac{1563}{2} + 63m - \frac{237}{2}m^2 - \frac{6855}{2}\epsilon - \frac{835}{2}m\epsilon \right.$$

$$\left. + 435\epsilon m^2 + \frac{35345}{9}\epsilon^2 - 2646\zeta_3\epsilon^2 + \frac{4085}{72}m\epsilon^2 - 1602\zeta_3 m\epsilon^2 - \frac{54101}{72}m^2\epsilon^2 - 432\zeta_3 m^2\epsilon^2 \right)\frac{1}{N^2} + O\left(\epsilon^3; \frac{1}{N^3} \right)$$

$$y = 6 + \left(486 + 486m - 645\epsilon - 645m\epsilon + \frac{2781}{4}\epsilon^2 + \frac{2781}{4}m\epsilon^2 \right)\frac{1}{N}$$

$$+ \left(248949 + 133398m + 30249m^2 - 660675\epsilon - 317175m\epsilon - 58170m^2\epsilon + \frac{1419565}{2}\epsilon^2 - 354780\zeta_3\epsilon^2 \right.$$

$$\left. + \frac{1289545}{4}m\epsilon^2 - 215460\zeta_3 m\epsilon^2 + \frac{205901}{4}m^2\epsilon^2 - 58320\zeta_3 m^2\epsilon^2 \right)\frac{1}{N^2} + O\left(\epsilon^3; \frac{1}{N^3} \right)$$

$$z = 1 + \left(11 + \frac{7}{2}m - \frac{155}{6}\epsilon - \frac{25}{3}m\epsilon + \frac{1777}{72}\epsilon^2 + \frac{80}{9}m\epsilon^2 \right)\frac{1}{N}$$

$$+ \left(\frac{1563}{2} + \frac{231}{2}m + \frac{147}{8}m^2 - \frac{6855}{2}\epsilon - \frac{2555}{4}m\epsilon - 150m^2\epsilon + \frac{35345}{9}\epsilon^2 - 2646\zeta_3\epsilon^2 + \frac{123919}{144}m\epsilon^2 \right.$$

$$\left. - 576\zeta_3 m\epsilon^2 + \frac{23695}{72}m^2\epsilon^2 \right)\frac{1}{N^2} + O\left(\epsilon^3; \frac{1}{N^3} \right)$$

$$t = 6 + \left(486 + 216m - 645\epsilon - 315m\epsilon + \frac{2781}{4}\epsilon^2 + \frac{1407}{4}m\epsilon^2 \right)\frac{1}{N} + \left(248949 + 65988m + 7389m^2 - 660675\epsilon \right.$$

$$- 168030m\epsilon - 19545m^2\epsilon + \frac{1419565}{2}\epsilon^2 - 354780\zeta_3\epsilon^2 + 183756m\epsilon^2 - 99900\zeta_3 m\epsilon^2$$

$$\left. + 25357m^2\epsilon^2 - 11664\zeta_3 m^2\epsilon^2 \right)\frac{1}{N^2} + O\left(\epsilon^3; \frac{1}{N^3} \right)$$

$$
\begin{aligned}
w = 6 &+ \left(486 + 81m - 645\epsilon - 150m\epsilon + \frac{2781}{4}\epsilon^2 + 180m\epsilon^2 \right)\frac{1}{N} \\
&+ \left(248949 + 32283m + \frac{10161}{4}m^2 - 660675\epsilon - \frac{186915}{2}m\epsilon - 10830m^2\epsilon + \frac{1419565}{2}\epsilon^2 - 354780\zeta_3\epsilon^2 \right. \\
&\left. + \frac{915527}{8}m\epsilon^2 - 42120\zeta_3 m\epsilon^2 + \frac{76709}{4}m^2\epsilon^2 \right)\frac{1}{N^2} + O\left(\epsilon^3; \frac{1}{N^3} \right)
\end{aligned}
\tag{4.4}
$$

for CS where all the couplings are active. With these particular values at each of the three fixed points we find agreement with the known large N exponents [35–38,46] out to $O(\epsilon^3)$. This includes the mass mixing matrix. However, the comparison with the mass dimension exponents is not straightforward since one has to compare with the anomalous dimensions of the eigenvalues of the mass mixing matrix $\gamma_{ij}(g_k)$ evaluated at each critical point. For instance, at AU the exponent ω_{+1}^{AU} is in precise agreement with the critical eigen-anomalous dimension. Equally at CS the exponents $\eta + \chi$ and the linear combination $\omega_{+1}^{CS} + \omega_{-1}^{CS}$ are also in exact correspondence with the $O(\epsilon^3)$ terms of the eigen-anomalous dimensions. These nontrivial large N checks at each of the three fixed points on the three-loop $\overline{\text{MS}}$ renormalization group functions provide confidence that our perturbative computation is correct.

V. CONFORMAL WINDOW SEARCH

One of our aims is to find the conformal window for (2.4). Given the nature of the renormalization group equations computed at three loops it transpires that pinning down the actual range of the conformal window is not straightforward. A similar observation was made in [28] for the four-dimensional $O(N) \times O(3)$ case using the conformal bootstrap method. For the pure $O(N)$ case [21], which has two coupling constants unlike our five here, the conformal window was determined by solving the equations

$$
\beta_1(g_i) = \beta_2(g_i) = 0, \qquad \det\left(\frac{\partial\beta_i}{\partial g_j} \right) = 0 \tag{5.1}
$$

where $i = 1$ and 2. As the generalization of these equations to five couplings is

$$
\beta_1(g_i) = \beta_2(g_i) = \ldots = \beta_5(g_i) = 0 \tag{5.2}
$$

together with the Hessian it turned out that our computer resources were not sufficient to solve the complete system numerically in general. Instead we have resorted to an alternative strategy which could equally well have been applied to the pure $O(N)$ theory. One observation of [21,22] with respect to the conformal window in the $O(N)$ case was the nature of the fixed point spectrum above and below a conformal window boundary. At leading order the main

window boundary is at $N_{cr} = 1038$ [20] for $O(N)$. Above this value of N_{cr} there are fixed points with real couplings. By contrast below this point there are no real fixed points. Given this distinguishing property we have solved the equations (5.2) for fixed values of N and then analyzed the stability properties of the real solutions. The stability of a fixed point is determined by finding the eigenvalues of the stability matrix $\mathcal{S}$ at each real fixed point in turn for the chosen value of N where $\mathcal{S}$ is defined by

$$
\mathcal{S} = \left(\frac{\partial\beta_i}{\partial g_j} \right). \tag{5.3}
$$

Specifically if all the eigenvalues are negative then this signifies ultraviolet (UV) stability, while if all eigenvalues are positive then that fixed point would be UV unstable and consequently infrared (IR) stable. Obtaining a mixed signature indicates that the fixed point is a saddle point. In the situation where the eigenvalues are zero, we can only conclude that the fixed point is marginal and beyond the linear approximation. We did not find any such cases for the values of N analyzed. While this may appear to be a tedious process for finding the conformal window boundary it turned out to be relatively quick since one can narrow the search area by a process of sectioning.

To illustrate the process we focus for the moment on the $O(N) \times O(2)$ theory. First, given the fact that there are more couplings in (2.4) the criteria defining the window boundary differ slightly from the properties of the $O(N)$ case. In order to define this we need to introduce a descriptive syntax which derives partly from the nature of the fixed points which emerge and the structure of the four-dimensional $O(N) \times O(m)$ coupling constant plane. In (2.2) there were three nontrivial fixed points designated Heisenberg, antichiral unstable and chiral stable and they were associated with different combinations of the fields σ and T^{ab} that were active or not at a fixed point. Moreover with fewer couplings in four dimensions each type of fixed point had a definite stability which led to the notation AU or CS aside from the Heisenberg solution which was necessarily a saddle point. In our conformal window analysis of (2.4) we will retain our AU and CS syntax as well as Heisenberg but use it to represent the field content only. So, for instance, indicating an AU fixed point will mean that only interactions involving the T^{ab} field are present while a

CS type of fixed point will correspond to all interactions of (2.4) being active. This readjustment in syntax is necessary since, as will become clear, the fixed point structure is much richer than that of the six-dimensional $O(N)$ ϕ^3 theory and (2.2). So we will refer to Heisenberg, AU and CS types of solutions. Illustrating this with the coupling vector $(g_1, g_2, g_3, g_4, g_5)$ their characteristic critical coupling constant patterns respectively are $(x, y, 0, 0, 0)$; $(0, 0, z, 0, w)$; and (x, y, z, t, w) where we mean that x, y, z, t and w are nonzero in these patterns. For simplicity we have omitted the constant of proportionality given in (4.1). It is important to appreciate that for the Heisenberg, AU and CS patterns the actual fixed point which is present could actually be stable or unstable and not be related to the U or S of the label type. In one respect the emergence of these patterns within the perturbative context, where we are now working, should not be surprising as the fixed N analysis has to at least contain the Heisenberg, AU and CS large N solutions. With this syntax for (2.4) we can now give our criteria for the conformal window boundary. From the analysis we have carried out we regard a window boundary to be where there is a change in the number of a particular pattern of fixed points such as CS. We note that as in the $O(N)$ case various fixed point solutions are connected to each other via symmetries [21], and so we focus on a representative fixed point of each such class in the discussion. We also find a large number of fixed points with complex and purely imaginary values which may indicate nonunitarity solutions or even the existence of a limit cycle. In our discussions in this and the next section we will focus only on the real solutions for the critical couplings as they lead to clear stability properties.

As a first stage to our search strategy it is best to summarize the analysis for the upper boundary we found which was $N = 1105$ when $m = 2$. For the case of $N = 1106$ we have three CS type fixed points. One of these is UV stable which is at

$$x = 1.024331 + 0.602917\epsilon - 618.493720\epsilon^2 + O(\epsilon^3)$$
$$y = 10.027831 - 224.568795\epsilon + 204744.131100\epsilon^2 + O(\epsilon^3)$$
$$z = 1.014679 + 0.242004\epsilon - 259.254500\epsilon^2 + O(\epsilon^3)$$
$$t = 8.413935 - 122.062932\epsilon + 110001.339800\epsilon^2 + O(\epsilon^3)$$
$$w = 7.750728 - 86.093662\epsilon + 77109.596670\epsilon^2 + O(\epsilon^3).$$
$$(5.4)$$

The corresponding critical exponents are

$$\gamma_\phi^* = 0.002810\epsilon - 0.003531\epsilon^2 - 2.095198\epsilon^3 + O(\epsilon^4)$$
$$\gamma_\sigma^* = 1.158724\epsilon - 2.828644\epsilon^2 + 2307.673939\epsilon^3 + O(\epsilon^4)$$
$$\gamma_T^* = 1.093583\epsilon - 1.472805\epsilon^2 + 1165.028293\epsilon^3 + O(\epsilon^4).$$
$$(5.5)$$

The other two CS style fixed points are saddle points at

$$x = 1.023546 - 0.790738\epsilon + 618.557767\epsilon^2 + O(\epsilon^3)$$
$$y = 10.288220 + 238.034889\epsilon - 204695.170900\epsilon^2 + O(\epsilon^3)$$
$$z = 1.014350 - 0.341297\epsilon + 259.727356\epsilon^2 + O(\epsilon^3)$$
$$t = 8.553710 + 126.145941\epsilon - 109987.441000\epsilon^2 + O(\epsilon^3)$$
$$w = 7.848666 + 87.779203\epsilon - 77103.604170\epsilon^2 + O(\epsilon^3)$$
$$(5.6)$$

and

$$x = -0.869900 - 0.200484\epsilon - 0.868576\epsilon^2 + O(\epsilon^3)$$
$$y = 20.723963 + 8.470150\epsilon - 14.322290\epsilon^2 + O(\epsilon^3)$$
$$z = 1.011451 - 0.019282\epsilon + 0.058843\epsilon^2 + O(\epsilon^3)$$
$$t = -4.381299 - 2.162646\epsilon - 6.897939\epsilon^2 + O(\epsilon^3)$$
$$w = 5.927808 + 0.692949\epsilon + 3.355853\epsilon^2 + O(\epsilon^3). \quad (5.7)$$

In addition there are three Heisenberg fixed points, one of which is UV stable at

$$x = 1.010040 - 0.023705\epsilon + 0.020596\epsilon^2 + O(\epsilon^3)$$
$$y = 6.557735 - 0.940183\epsilon + 0.810426\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \qquad t = 0, \qquad w = 0 \qquad (5.8)$$

with critical exponents

$$\gamma_\phi^* = 0.000922\epsilon - 0.001777\epsilon^2 - 0.000152\epsilon^3 + O(\epsilon^4)$$
$$\gamma_\sigma^* = 1.039622\epsilon - 0.075355\epsilon^2 - 0.008779\epsilon^3 + O(\epsilon^4)$$
$$\gamma_T^* = 0. \qquad (5.9)$$

The other two fixed points are saddle points and are located at

$$x = 0.979414 - 0.003228\epsilon + 0.071572\epsilon^2 + O(\epsilon^3)$$
$$y = 17.380571 + 10.947386\epsilon + 21.645075\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \qquad t = 0, \qquad w = 0 \qquad (5.10)$$

and

$$x = -0.857078 - 0.208350\epsilon - 0.632470\epsilon^2 + O(\epsilon^3)$$
$$y = 19.745752 + 9.661778\epsilon - 2.588019\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \qquad t = 0, \qquad w = 0. \qquad (5.11)$$

There was one AU fixed point which is UV stable at

$$x = 0, \quad y = 0, \quad t = 0$$
$$z = 0.998197 + 0.006635\epsilon - 0.008935\epsilon^2 + O(\epsilon^3)$$
$$w = 5.367450 + 0.851212\epsilon - 1.446454\epsilon^2 + O(\epsilon^3) \quad (5.12)$$

with critical exponents

$$\gamma_\phi^* = 0.001802\epsilon - 0.003273\epsilon^2 - 0.000708\epsilon^3 + O(\epsilon^4)$$
$$\gamma_\sigma^* = 0$$
$$\gamma_T^* = 0.996396\epsilon + 0.006664\epsilon^2 + 0.002605\epsilon^3 + O(\epsilon^4).$$
$$(5.13)$$

For values of N above 1106 the same pattern and number of Heisenberg, AU and CS fixed points emerge with the same stability structure. By contrast for $N = 1105$ a different style of solution emerges. This is first seen in the CS type of fixed points in that we have only *one* such fixed point which is at

$$x = -0.869887 - 0.200513\epsilon - 0.868979\epsilon^2 + O(\epsilon^3)$$
$$y = 20.715552 + 8.465518\epsilon - 14.330113\epsilon^2 + O(\epsilon^3)$$
$$z = 1.011461 - 0.019297\epsilon + 0.058911\epsilon^2 + O(\epsilon^3)$$
$$t = -4.380955 - 2.163247\epsilon - 6.901395\epsilon^2 + O(\epsilon^3)$$
$$w = 5.927669 + 0.693620\epsilon + 3.359563\epsilon^2 + O(\epsilon^3). \quad (5.14)$$

More crucially it is a saddle point. In other words there is no stable CS fixed point. So given this change in pattern we regard $N = 1105$ as the bound for the conformal window in six dimensions. It is instructive to provide the picture for the other types of fixed points for $N = 1105$. There are also three Heisenberg fixed points. The UV stable one is

$$x = 1.010049 - 0.023726\epsilon + 0.020611\epsilon^2 + O(\epsilon^3)$$
$$y = 6.558394 - 0.941587\epsilon + 0.811596\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \qquad t = 0, \qquad w = 0 \qquad (5.15)$$

with critical exponents

$$\gamma_\phi^* = 0.000923\epsilon - 0.001779\epsilon^2 - 0.000152\epsilon^3 + O(\epsilon^4)$$
$$\gamma_\sigma^* = 1.039662\epsilon - 0.075439\epsilon^2 - 0.008783\epsilon^3 + O(\epsilon^4)$$
$$\gamma_T^* = 0 \qquad (5.16)$$

while the other two fixed points are saddle points at

$$x = -0.857055 - 0.208383\epsilon - 0.632604\epsilon^2 + O(\epsilon^3)$$
$$y = 19.736951 + 9.657499\epsilon - 2.589415\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \qquad t = 0, \qquad w = 0 \qquad (5.17)$$

and

$$x = 0.979447 - 0.003297\epsilon + 0.071496\epsilon^2 + O(\epsilon^3)$$
$$y = 17.371128 + 10.944494\epsilon + 21.644028\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \qquad t = 0, \qquad w = 0. \qquad (5.18)$$

The one AU fixed point is UV stable and is located at

$$x = 0, \quad y = 0, \quad t = 0$$
$$z = 0.998195 + 0.006641\epsilon - 0.008942\epsilon^2 + O(\epsilon^3)$$
$$w = 5.367025 + 0.851662\epsilon - 1.447623\epsilon^2 + O(\epsilon^3) \quad (5.19)$$

with critical exponents

$$\gamma_\phi^* = 0.001803\epsilon - 0.003276\epsilon^2 - 0.000709\epsilon^3 + O(\epsilon^4)$$
$$\gamma_\sigma^* = 0$$
$$\gamma_T^* = 0.996393\epsilon + 0.006670\epsilon^2 + 0.002608\epsilon^3 + O(\epsilon^4).$$
$$(5.20)$$

For $N < 1105$ we applied our algorithm of section searching for changes in fixed point patterns but found no further boundaries. However the structure for certain fixed values of N will be recorded later for completeness. One observation on our window analysis is that the boundary at $N = 1105$ is not dissimilar to the leading order value of $N_{cr} = 1038$ [20] for the $O(N)$ case. In [21,22] the $O(\epsilon^3)$ corrections to N_{cr} were computed and by using resummation methods a value of N_{cr} around 400 was found for the five-dimensional theory. Clearly applying our section search method cannot be readily extended beyond the leading order which is for the strictly six-dimensional theory. Instead solving (5.2) simultaneously with $\det(\mathcal{S}) = 0$ would be the way to extract such corrections but was beyond the range of our computational tools.

We close this section by briefly discussing a different tack for gaining more insight into the conformal window problem for $O(N) \times O(m)$. As is apparent from the $O(N) \times O(2)$ case the change in nature of the fixed points indicates a boundary. Moreover different types of (real) solutions emerge. Therefore, for the general $O(N) \times O(m)$ case we searched for the conformal window for the AU pattern of couplings. In other words we set $x = y = t = 0$ at the outset for a selection of values of m and solved (5.1). Included in this is the equation for the Hessian which allows us to determine the critical value of N defining the window boundary, which we will denote by $N_{cr}^{(m)}$ for this AU pattern, without having to do a section search. To get a perspective on our results we have provided the leading order value for $N_{cr}^{(m)}$ for various m in Table II. As $m \to \infty$ we found that $N_{cr}^{(m)}$ asymptotes to a straight line. While this is only a partial picture for the situation for $m > 2$ one thing is evident which is that in six dimensions when $m \geq 5$ there should be a change in pattern for AU type fixed points for a fixed N search akin to that illustrated in our section-based search for $m = 2$. This is in addition to the change in pattern for the other style of solutions. The solution given in Table II for $m = 4$ reflects that there was no solution rather than an exact value of zero. Although we have recorded 0 in

the table for that reason it does appear to be consistent with the monotonic increase in $N_{cr}^{(m)}$ with m. Since we are able to solve (5.1) the three-loop corrections to the leading order values in Table II have been determined. We found

$$N_{cr}^{(1)} = -2946.134605 + 3951.961993\epsilon + 2676.699839\epsilon^2 + O(\epsilon^3)$$

$$z = 1.006955 - 0.008027\epsilon + 0.012574\epsilon^2 + O(\epsilon^3)$$

$$w = 8.952176 - 0.933006\epsilon + 1.840946\epsilon^2 + O(\epsilon^3)$$

$$N_{cr}^{(2)} = -1087.488959 + 1415.172128\epsilon + 261.248651\epsilon^2 + O(\epsilon^3)$$

$$z = 1.001844 - 0.004332\epsilon + 0.005483\epsilon^2 + O(\epsilon^3)$$

$$w = 9.000046 - 1.261448\epsilon - 1.365084\epsilon^2 + O(\epsilon^3)$$

$$N_{cr}^{(3)} = -410.145045 + 439.505646\epsilon + 1591.300276\epsilon^2 + O(\epsilon^3)$$

$$z = 0.988129 + 0.002686\epsilon + 0.093273\epsilon^2 + O(\epsilon^3)$$

$$w = 9.206805 - 2.755615\epsilon + 36.244925\epsilon^2 + O(\epsilon^3)$$

$$N_{cr}^{(5)} = 216.767170 - 419.773422\epsilon + 25581.601520\epsilon^2 + O(\epsilon^3)$$

$$z = 1.094548 - 0.073610\epsilon - 9.071267\epsilon^2 + O(\epsilon^3)$$

$$w = 8.708936 + 1.332420\epsilon - 267.527508\epsilon^2 + O(\epsilon^3)$$

$$N_{cr}^{(6)} = 421.682453 - 774.149504\epsilon + 8084.140233\epsilon^2 + O(\epsilon^3)$$

$$z = 1.053874 - 0.038490\epsilon - 0.487837\epsilon^2 + O(\epsilon^3)$$

$$w = 8.724938 + 0.659976\epsilon - 46.386543\epsilon^2 + O(\epsilon^3)$$

$$N_{cr}^{(10)} = 992.309977 - 1796.450905\epsilon + 1605.099447\epsilon^2 + O(\epsilon^3)$$

$$z = 1.035563 - 0.025317\epsilon + 0.018625\epsilon^2 + O(\epsilon^3)$$

$$w = 8.766117 + 0.281651\epsilon + 0.445942\epsilon^2 + O(\epsilon^3)$$

$$N_{cr}^{(20)} = 1999.619696 - 3823.678958\epsilon - 645.564678\epsilon^2 + O(\epsilon^3)$$

$$z = 1.032648 - 0.021711\epsilon + 0.036581\epsilon^2 + O(\epsilon^3)$$

$$w = 8.770214 + 0.294322\epsilon + 6.987749\epsilon^2 + O(\epsilon^3)$$

$$N_{cr}^{(30)} = 2887.855771 - 5724.541609\epsilon - 1656.156005\epsilon^2 + O(\epsilon^3)$$

$$z = 1.032739 - 0.020634\epsilon + 0.038671\epsilon^2 + O(\epsilon^3)$$

$$w = 8.766733 + 0.353761\epsilon + 8.295335\epsilon^2 + O(\epsilon^3)$$

$$N_{cr}^{(40)} = 3746.323521 - 7599.677475\epsilon - 2431.924312\epsilon^2 + O(\epsilon^3)$$

$$z = 1.032998 - 0.020013\epsilon + 0.039622\epsilon^2 + O(\epsilon^3)$$

$$w = 8.763914 + 0.398295\epsilon + 8.889004\epsilon^2 + O(\epsilon^3)$$

$$N_{cr}^{(50)} = 4592.876982 - 9466.056881\epsilon - 3115.903624\epsilon^2 + O(\epsilon^3)$$

$$z = 1.033232 - 0.019588\epsilon + 0.040210\epsilon^2 + O(\epsilon^3)$$

$$w = 8.761832 + 0.430906\epsilon + 9.233526\epsilon^2 + O(\epsilon^3)$$

$$(5.21)$$

for a selection of m where the respective critical couplings have been displayed. This is an important point. While we have provided values for $N_{cr}^{(m)}$ in (5.21) other solutions were found for each m. In [21,22] there were three solutions but the small $N_{cr}^{(m)}$ solutions were discarded because they were negative or had complex critical couplings. We have followed the same reasoning here. The negative values for $N_{cr}^{(m)}$ are in keeping with similar negative solutions for the eight-dimensional ultraviolet completion of the $O(N)$ sequence of theories [58]. We have also excluded from this AU analysis values of $N_{cr}^{(m)}$ which have large critical couplings as such values are clearly outside the perturbative approximation we are using.

VI. FIXED POINT ANALYSIS

In this section we present a fixed point analysis for a variety of specific values of N. This includes the determination of those fixed points which are stable or otherwise, in order to give a flavor of the fixed point spectrum away from $N = 1105$. In addition we will indicate the potential for another conformal window boundary for non-CS type fixed points. While we focus on a selection of values of N, for a reader interested in exploring the solution space further the complete set of renormalization group functions for arbitrary m can be analyzed. They are available in the attached data file. We will begin by looking at $N = 1000$ and then proceed to lower values of N. For $N = 1000$ we have one CS fixed type point as expected which is a saddle point:

TABLE II. Leading order value of $N_{cr}^{(m)}$ for the conformal window for different values of m.

m	1	2	3	4	5	6	10	20	30	40	50
$N_{cr}^{(m)}$	−2946.1	−1087.5	−410.2	0	216.8	421.7	992.3	1999.6	2887.9	3746.3	4592.9

$$x = -0.868555 - 0.203744\epsilon - 0.915849\epsilon^2 + O(\epsilon^3)$$
$$y = 19.811433 + 7.966436\epsilon - 15.205442\epsilon^2 + O(\epsilon^3)$$
$$z = 1.012581 - 0.020942\epsilon + 0.066905\epsilon^2 + O(\epsilon^3)$$
$$t = -4.342552 - 2.231269\epsilon - 7.303390\epsilon^2 + O(\epsilon^3)$$
$$w = 5.911324 + 0.770705\epsilon + 3.795188\epsilon^2 + O(\epsilon^3). \quad (6.1)$$

We also have three Heisenberg fixed points, one of which is UV stable at

$$x = 1.011102 - 0.026162\epsilon + 0.022238\epsilon^2 + O(\epsilon^3)$$
$$y = 6.637801 - 1.117476\epsilon + 0.962982\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \qquad t = 0, \qquad w = 0 \qquad (6.2)$$

with the critical exponents

$$\gamma_\phi^* = 0.001022\epsilon - 0.001981\epsilon^2 - 0.000145\epsilon^3 + O(\epsilon^4)$$
$$\gamma_\sigma^* = 1.044358\epsilon - 0.085562\epsilon^2 - 0.009090\epsilon^3 + O(\epsilon^4)$$
$$\gamma_T^* = 0. \qquad (6.3)$$

The other two fixed points

$$x = -0.854446 - 0.212078\epsilon - 0.647751\epsilon^2 + O(\epsilon^3)$$
$$y = 18.789145 + 9.197094\epsilon - 2.733818\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \qquad t = 0, \qquad w = 0 \qquad (6.4)$$

and

$$x = 0.983210 - 0.011253\epsilon + 0.063259\epsilon^2 + O(\epsilon^3)$$
$$y = 16.345805 + 10.658027\epsilon + 21.524495\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \qquad t = 0, \qquad w = 0 \qquad (6.5)$$

are saddle points. Again we also have one AU fixed point which is UV stable which is located at

$$x = 0, \qquad y = 0, \qquad t = 0$$
$$z = 0.998006 + 0.007339\epsilon - 0.009793\epsilon^2 + O(\epsilon^3)$$
$$w = 5.318846 + 0.901757\epsilon - 1.582315\epsilon^2 + O(\epsilon^3) \quad (6.6)$$

giving the critical exponents

$$\gamma_\phi^* = 0.001992\epsilon - 0.003615\epsilon^2 - 0.000789\epsilon^3 + O(\epsilon^4)$$
$$\gamma_\sigma^* = 0$$
$$\gamma_T^* = 0.996016\epsilon + 0.007374\epsilon^2 + 0.003032\epsilon^3 + O(\epsilon^4).$$
$$(6.7)$$

To illustrate the full spectrum of fixed points for a particular value of N we have provided the remaining fixed points for $N = 1000$ in Appendix B. In addition to other real

solutions which do not fit the Heisenberg, AU or CS pattern we record the complex solutions for completeness there.

Next examining the value of $N = 600$ in order to illustrate a change in the fixed point pattern, we have one CS saddle point solution at

$$x = -0.862204 - 0.222460\epsilon - 1.255729\epsilon^2 + O(\epsilon^3)$$
$$y = 15.871131 + 5.764114\epsilon - 20.128689\epsilon^2 + O(\epsilon^3)$$
$$z = 1.020205 - 0.031243\epsilon + 0.133767\epsilon^2 + O(\epsilon^3)$$
$$t = -4.133094 - 2.641545\epsilon - 10.197488\epsilon^2 + O(\epsilon^3)$$
$$w = 5.794456 + 1.264133\epsilon + 7.071880\epsilon^2 + O(\epsilon^3). \quad (6.8)$$

In addition there are three Heisenberg fixed points with the one at

$$x = 1.018022 - 0.037843\epsilon + 0.001985\epsilon^2 + O(\epsilon^3)$$
$$y = 7.507506 - 4.490389\epsilon + 9.490485\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \qquad t = 0, \qquad w = 0 \qquad (6.9)$$

being UV stable, giving

$$\gamma_\phi^* = 0.001727\epsilon - 0.003465\epsilon^2 - 0.000010\epsilon^3 + O(\epsilon^4)$$
$$\gamma_\sigma^* = 1.083337\epsilon - 0.195953\epsilon^2 + 0.089727\epsilon^3 + O(\epsilon^4)$$
$$\gamma_T^* = 0 \qquad (6.10)$$

while the other two fixed points are saddle points at

$$x = -0.839313 - 0.232926\epsilon - 0.736639\epsilon^2 + O(\epsilon^3)$$
$$y = 14.602366 + 7.174642\epsilon - 3.193826\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \qquad t = 0, \qquad w = 0 \qquad (6.11)$$

and

$$x = 1.007039 - 0.068389\epsilon + 0.063230\epsilon^2 + O(\epsilon^3)$$
$$y = 11.302398 + 11.995559\epsilon + 13.765855\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \qquad t = 0, \qquad w = 0. \qquad (6.12)$$

There is also one AU fixed point which is UV stable at

$$x = 0, \qquad y = 0, \qquad t = 0$$
$$z = 0.996683 + 0.012238\epsilon - 0.015305\epsilon^2 + O(\epsilon^3)$$
$$w = 5.033838 + 1.165363\epsilon - 2.476601\epsilon^2 + O(\epsilon^3) \quad (6.13)$$

with exponents

$$\gamma_\phi^* = 0.003311\epsilon - 0.005969\epsilon^2 - 0.001383\epsilon^3 + O(\epsilon^4)$$

$$\gamma_\sigma^* = 0$$

$$\gamma_T^* = 0.993377\epsilon + 0.012333\epsilon^2 + 0.006784\epsilon^3 + O(\epsilon^4). \quad (6.14)$$

We have recorded this spectrum to contrast it with that for $N = 519$. So for this value we then have one CS fixed point which is a saddle point at

$$x = -0.860686 - 0.228611\epsilon - 1.395472\epsilon^2 + O(\epsilon^3)$$

$$y = 14.937476 + 5.238428\epsilon - 21.686769\epsilon^2 + O(\epsilon^3)$$

$$z = 1.023093 - 0.0347571\epsilon + 0.165366\epsilon^2 + O(\epsilon^3)$$

$$t = -4.069923 - 2.780163\epsilon - 11.373918\epsilon^2 + O(\epsilon^3)$$

$$w = 5.750244 + 1.435664\epsilon + 8.435080\epsilon^2 + O(\epsilon^3). \quad (6.15)$$

However, by contrast, we have one Heisenberg fixed point which is a saddle point at

$$x = -0.834431 - 0.239444\epsilon - 0.765623\epsilon^2 + O(\epsilon^3)$$

$$y = 13.591847 + 6.689873\epsilon - 3.246649\epsilon^2 + O(\epsilon^3)$$

$$z = 0, \qquad t = 0, \qquad w = 0. \quad (6.16)$$

In addition there is one UV stable AU type fixed point at

$$x = 0, \quad y = 0, \quad t = 0$$

$$z = 0.996169 + 0.014150\epsilon - 0.017238\epsilon^2 + O(\epsilon^3)$$

$$w = 4.941667 + 1.239792\epsilon - 2.803418\epsilon^2 + O(\epsilon^3) \quad (6.17)$$

giving critical exponents

$$\gamma_\phi^* = 0.003824\epsilon - 0.006875\epsilon^2 - 0.0016281\epsilon^3 + O(\epsilon^4)$$

$$\gamma_\sigma^* = 0$$

$$\gamma_T^* = 0.992352\epsilon + 0.014277\epsilon^2 + 0.008615\epsilon^3 + O(\epsilon^4). \quad (6.18)$$

So between $N = 519$ and $N = 600$ the behavior of a Heisenberg type fixed point changes. This seems to indicate that a conformal window type region exists with respect to the Heisenberg structure and thus there is a new window between 519 and 600. However, its actual location is not of major significance in the context of (2.4) as this in effect corresponds to the original Heisenberg model with no T^{ab} field.

The final case we consider in detail in our excursion through fixed values of N is $N = 2$. It is of potential interest since for this value in a variety of models a supersymmetric solution emerged [21,59,60]. We have three CS fixed points, all of which are saddle points at

$$x = -0.454392 - 1.128422\epsilon - 10.883437\epsilon^2 + O(\epsilon^3)$$

$$y = 0.673205 + 1.783387\epsilon + 15.854883\epsilon^2 + O(\epsilon^3)$$

$$z = 0.318954 + 0.395758\epsilon + 3.102196\epsilon^2 + O(\epsilon^3)$$

$$t = 0.379850 + 0.510247\epsilon + 4.361634\epsilon^2 + O(\epsilon^3). \quad (6.19)$$

The value for the coupling w has not been provided with the others as a novel feature emerged for this set. It transpired that there were three fixed points with the same x, y, z and t values but differing only in the w value. Therefore, we note these values separately as

$$w \in \{0.717916 + 0.824313\epsilon + 5.193907\epsilon^2 + O(\epsilon^3),$$
$$-0.267715 - 0.020190\epsilon + 0.685452\epsilon^2 + O(\epsilon^3),$$
$$-0.450201 - 0.479646\epsilon - 3.632751\epsilon^2 + O(\epsilon^3)\}. \quad (6.20)$$

There was one Heisenberg fixed point, which is a saddle point:

$$x = -0.470736 - 0.737444\epsilon - 5.708527\epsilon^2 + O(\epsilon^3)$$

$$y = 0.762184 + 0.999917\epsilon + 6.174478\epsilon^2 + O(\epsilon^3)$$

$$z = 0, \qquad t = 0, \qquad w = 0. \quad (6.21)$$

In addition we found one AU fixed point which is UV stable:

$$x = 0, \qquad y = 0, \qquad t = 0$$

$$z = 0.577350 + 1.507526\epsilon + 19.533564\epsilon^2 + O(\epsilon^3)$$

$$w = 0.800625 + 1.806817\epsilon + 27.377665\epsilon^2 + O(\epsilon^3) \quad (6.22)$$

with critical exponents

$$\gamma_\phi^* = \gamma_T^* = 0.333333\epsilon + 1.333333\epsilon^2 + 22.148148\epsilon^3 + O(\epsilon^4)$$

$$\gamma_\sigma^* = 0. \quad (6.23)$$

One property of the emergent supersymmetric solutions found in earlier work [21,59,60] was that critical couplings were equivalent. For this AU solution a different feature is apparent which is that the exponents of ϕ^{ia} and T^{ab} are equal.

VII. DISCUSSION

We have provided a comprehensive three-loop analysis of the extension of the four-dimensional Landau-Ginzburg-Wilson $O(N) \times O(m)$ symmetric theory to six dimensions. One aspect of our study was to investigate the ultraviolet completion beyond four dimensions. The main interest previously had been in $O(N)$ symmetric theories and our

extension (2.4) fits in with the vision of how to proceed. Briefly this requires a common interaction which seeds the theories in various even dimensions along the thread of Wilson-Fisher fixed points in d dimensions. Each fixed dimension Lagrangian is required to be renormalizable in its critical dimension which requires the addition of matter-field-independent extra interactions. With the increase in dimension the number of these so-called spectator interactions increases. For (2.4) overall there are five interactions, each with its own coupling. One reassuring aspect of our computations is the verification that the three-loop renormalization group functions are consistent with the large N critical exponents of [38]. These exponents are determined in the underlying universal theory and as $1/N$ is a dimensionless coupling constant it transcends a specific dimension. In other words the exponents contain information on the respective renormalization group functions in *all* the theories connected via the Wilson-Fisher fixed point thread. Indeed verifying that our three-loop perturbative results were consistent with the results of [38] was an important check.

One consequence of the larger number of coupling constants is a richer spectrum of fixed points for specific values of N and m. While our analysis of this concentrated on $m = 2$ we do not expect that the general picture of fixed points differs conceptually for higher values of m. Instead the boundary values will be at different values of N as our AU study for various values of m illustrated. Our $m = 2$ analysis was similar to the $O(N)$ case of [21] with real and complex critical couplings with the latter corresponding to nonunitary theories. However, for real solutions we were able to isolate fixed points which had a structure in keeping with the phase plane in the four-dimensional model. In other words there are Heisenberg, AU and CS type

solutions which depend on which combination of σ and T^{ab} fields is active and their stability was studied for certain values of N. One of the main areas of interest in the $O(N)$ and $O(N) \times O(m)$ symmetric theories is whether there is a fixed point in the five-dimensional theory and if so what is the conformal window. In [15] a bootstrap study indicated that this was not an easy exercise from the lower-dimensional point of view unless one was examining AU type coupling patterns. Our investigation left us with a similar point of view. However, we were able to narrow down the leading order value of the window for CS solutions for $m = 2$. By contrast we were able to solve the AU set of equations and found that a window exists above $m = 5$. However, we have provided the full data from our renormalization group functions which can be mined for future studies for other values of m.

ACKNOWLEDGMENTS

APPENDIX A: REMAINING RENORMALIZATION GROUP FUNCTIONS

For completeness we record the remaining β-functions for the $O(N) \times O(2)$ theory for comparison with (3.8) as well as various other renormalization group functions. The β-functions are

$$\beta_2(g_i)|_{m=2} = \frac{1}{8}[8Ng_1^3 - 2Ng_1^2g_2 + 3g_2^3 - 2g_2g_4^2 + 8g_4^3]$$

$$+ \frac{1}{288}[-48Ng_1^5 - 644Ng_1^4g_2 - 120Ng_1^3g_2^2 - 48Ng_1^3g_3^2 + 62Ng_1^2g_2^3 + 4Ng_1^2g_2g_3^2 - 648Ng_1^2g_3^2g_4$$

$$+ 96Ng_1g_2g_3^2g_4 - 216Ng_1g_3^2g_4^2 - 125g_2^5 + 62g_2^3g_4^2 - 120g_2^2g_4^3 - 22Ng_2g_3^2g_4^2 - 644g_2g_4^4 + 84Ng_3^2g_4^3 - 48g_4^5]$$

$$+ \frac{1}{41472}[110784N^2g_1^7 + 68448Ng_1^7 - 153896N^2g_1^6g_2 + 10368\zeta_3Ng_1^6g_2 + 118816Ng_1^6g_2 + 45216N^2g_1^5g_2^2$$

$$+ 124416\zeta_3Ng_1^5g_2^2 + 50592Ng_1^5g_2^2 + 136896Ng_1^5g_3^2 + 1920Ng_1^5g_4^2 - 3156N^2g_1^4g_2^3 + 88128\zeta_3Ng_1^4g_2^3$$

$$+ 255780Ng_1^4g_2^3 + 20736\zeta_3Ng_1^4g_2g_3^2 + 111200Ng_1^4g_2g_3^2 - 65912Ng_1^4g_2g_4^2 + 116928Ng_1^4g_3^2g_4$$

$$+ 108864Ng_1^4g_4^3 - 41472\zeta_3Ng_1^3g_2^4 - 17376Ng_1^3g_2^4 - 45888Ng_1^3g_2^2g_3^2 + 45216Ng_1^3g_2^2g_4^2 + 248832\zeta_3Ng_1^3g_2g_3^2g_4$$

$$+ 37632Ng_1^3g_2g_3^2g_4 - 175968Ng_1^3g_2g_3^2 + 55392N^2g_1^3g_3^4 + 10560Ng_1^3g_3^4 + 269760Ng_1^3g_3^2g_4^2 + 108864Ng_1^3g_4^4$$

$$- 12544Ng_1^2g_2^5 + 4264Ng_1^2g_2^3g_3^2 - 6312Ng_1^2g_2^3g_4^2 - 155520\zeta_3Ng_1^2g_2^2g_3^2g_4 + 174384Ng_1^2g_2^2g_3^2g_4$$

$$+ 45216Ng_1^2g_2^2g_4^3 - 5524N^2g_1^2g_2g_3^4 - 3776Ng_1^2g_2g_3^4 + 476928\zeta_3Ng_1^2g_2g_3^2g_4^2 + 181376Ng_1^2g_2g_3^2g_4^2$$

$$- 65912Ng_1^2g_2g_4^4 - 71424N^2g_1^2g_3^4g_4 - 25632Ng_1^2g_3^4g_4 + 273936Ng_1^2g_3^2g_4^3 + 1920Ng_1^2g_4^5 - 14064Ng_1g_2^3g_3^2g_4$$

$+ 3312Ng_1g_2^2g_3^2g_4^2 + 2304N^2g_1g_2g_3^4g_4 - 672Ng_1g_2g_4^4 - 124416\zeta_3Ng_1g_2g_3^2g_4^3 + 30912Ng_1g_2g_3^2g_4^3$

$+ 20304N^2g_1g_3^4g_4^2 - 35424Ng_1g_3^4g_4^2 - 61344Ng_1g_3^2g_4^4 + 12960\zeta_3g_2^7 + 33085g_2^7 - 12544g_2^5g_4^2 - 41472\zeta_3g_2^4g_4^3$

$- 17376g_2^4g_4^3 + 5648Ng_2^3g_3^2g_4^2 + 88128\zeta_3g_2^3g_4^4 + 252624g_2^3g_4^4 - 24600Ng_2^2g_3^2g_4^3 + 124416\zeta_3g_2^2g_4^5$

$+ 95808g_2^2g_4^5 + 6N^2g_2g_3^4g_4^2 - 8408Ng_2g_3^4g_4^2 - 4360Ng_2g_3^2g_4^4 + 10368\zeta_3g_2g_4^6 - 35080g_2g_4^6 - 1584N^2g_3^4g_4^3$

$+ 61248Ng_3^4g_4^3 - 1776Ng_3^2g_4^5 + 179232g_4^7] + O(g_i^9)$

$$\beta_3(g_i)|_{m=2} = \frac{g_3}{24}[8g_1^2 + 24g_1g_4 - Ng_3^2 - 4g_3^2 - 2g_4^2]$$

$+ \frac{g_3}{864}[40Ng_1^4 - 536g_1^4 - 120g_1^3g_2 + 168Ng_1^3g_4 - 480g_1^3g_4 + 20g_1^2g_2^2 - 648g_1^2g_2g_4 - 214Ng_1^2g_3^2 + 56g_1^2g_3^2$

$- 22Ng_1^2g_4^2 - 1256g_1^2g_4^2 + 84g_1g_2^2g_4 - 216g_1g_2g_4^2 + 180Ng_1g_3^2g_4 + 24g_1g_3^2g_4 + 120g_1g_4^3 - 11g_2^2g_4^2 + 48g_2g_4^3$

$- 44Ng_3^4 - 476g_3^4 - 11Ng_3^2g_4^2 - 260g_3^2g_4^2 + 4g_4^4]$

$+ \frac{g_3}{62208}[-688N^2g_1^6 - 22480Ng_1^6 + 10368\zeta_3g_1^6 + 125680g_1^6 - 10440Ng_1^5g_2 + 62208\zeta_3g_1^5g_2 - 11856g_1^5g_2$

$- 1584N^2g_1^5g_4 - 62208\zeta_3Ng_1^5g_4 + 40176Ng_1^5g_4 + 62208\zeta_3g_1^5g_4 + 41712g_1^5g_4 + 1312Ng_1^4g_2^2$

$- 20736\zeta_3g_1^4g_2^2 + 74048g_1^4g_2^2 - 45216Ng_1^4g_2g_4 + 127440g_1^4g_2g_4 + 6562N^2g_1^4g_3^2 + 189216\zeta_3Ng_1^4g_3^2$

$+ 4032Ng_1^4g_3^2 - 93312\zeta_3g_1^4g_3^2 + 131280g_1^4g_3^2 + 52N^2g_1^4g_4^2 - 40452Ng_1^4g_4^2 + 158032g_1^4g_4^2 - 11436g_1^3g_2^3$

$+ 5712Ng_1^3g_2^2g_4 + 62208\zeta_3g_1^3g_2^2g_4 + 52752g_1^3g_2^2g_4 - 31104\zeta_3Ng_1^3g_2g_3^2 + 15456Ng_1^3g_2g_3^2 - 16176g_1^3g_2g_3^2$

$- 8184Ng_1^3g_2g_4^2 + 62208g_1^3\zeta_3g_2g_4^2 + 80808g_1^3g_2g_4^2 + 360N^2g_1^3g_3^2g_4 - 186624\zeta_3Ng_1^3g_3^2g_4 + 212856Ng_1^3g_3^2g_4$

$+ 62208\zeta_3g_1^3g_3^2g_4 + 25680g_1^3g_3^2g_4 - 2712Ng_1^3g_4^3 + 124416\zeta_3g_1^3g_4^3 + 80592g_1^3g_4^3 - 204g_1^2g_2^4 - 31104\zeta_3g_1^2g_2^3g_4$

$+ 9360g_1^2g_2^3g_4 + 3281Ng_1^2g_2^2g_3^2 + 2828g_1^2g_2^2g_3^2 - 772Ng_1^2g_2^2g_4^2 + 124416\zeta_3g_1^2g_2^2g_4^2 - 5168g_1^2g_2^2g_4^2$

$+ 5184\zeta_3Ng_1^2g_2g_3^2g_4 + 6984Ng_1^2g_2g_3^2g_4 - 41472\zeta_3g_1^2g_2g_3^2g_4 + 25056g_1^2g_2g_3^2g_4 + 2388Ng_1^2g_2g_4^3$

$+ 62208\zeta_3g_1^2g_2g_4^3 - 9312g_1^2g_2g_4^3 + 828N^2g_1^2g_3^4 - 59016Ng_1^2g_3^4 + 186624\zeta_3g_1^2g_3^4 - 15744g_1^2g_3^4 - 46N^2g_1^2g_3^2g_4^2$

$+ 15552\zeta_3Ng_1^2g_3^2g_4^2 - 56812Ng_1^2g_3^2g_4^2 - 20736\zeta_3g_1^2g_3^2g_4^2 + 280544g_1^2g_3^2g_4^2 + 340Ng_1^2g_4^4 + 32352g_1^2g_4^4$

$- 1716g_1g_2^4g_4 - 13320g_1g_2^3g_4^2 + 180Ng_1g_2^2g_3^2g_4 + 8364g_1g_2^2g_3^2g_4 - 62208\zeta_3g_1g_2^2g_4^3 + 96912g_1g_2^2g_4^3$

$- 7740Ng_1g_2g_3^2g_4^2 - 62208\zeta_3g_1g_2g_3^2g_4^2 + 76536g_1g_2g_3^2g_4^2 - 22320g_1g_2g_4^4 + 180N^2g_1g_3^4g_4 - 17184Ng_1g_3^4g_4$

$+ 121056g_1g_3^4g_4 + 5520Ng_1g_3^2g_4^3 + 62208\zeta_3g_1g_3^2g_4^3 + 129984g_1g_3^2g_4^3 - 62208\zeta_3g_1g_4^5 + 56112g_1g_4^5 + 327g_2^4g_4^2$

$+ 942g_2^3g_4^3 - 23Ng_2^2g_3^2g_4^2 + 2560g_2^2g_3^2g_4^2 + 5184\zeta_3g_2^2g_4^4 - 12534g_2^2g_4^4 + 576Ng_2g_3^2g_4^3 - 24312g_2g_3^2g_4^4$

$+ 2028g_2g_4^5 - 386N^2g_3^6 + 2592\zeta_3Ng_3^6 + 18434Ng_3^6 + 10368\zeta_3g_3^6 + 23512g_3^6 + 13N^2g_3^4g_4^2 - 8416Ng_3^4g_4^2$

$+ 93312\zeta_3g_3^4g_4^2 - 70304g_3^4g_4^2 - 36Ng_3^2g_4^4 + 62208\zeta_3g_3^2g_4^4 + 9648g_3^2g_4^4 + 5184\zeta_3g_4^6 - 9476g_4^6] + O(g_i^9)$

$$\beta_4(g_i)|_{m=2} = \frac{1}{24}[-2Ng_1^2g_4 + 12Ng_1g_3^2 - g_2^2g_4 + 12g_2g_4^2 - 2Ng_3^2g_4 + 6g_4^3]$$

$+ \frac{1}{864}[4Ng_1^4g_4 + 96Ng_1^3g_2g_4 - 72Ng_1^3g_3^2 - 216Ng_1^3g_4^2 - 22Ng_1^2g_2^2g_4 - 324Ng_1^2g_2g_3^2 + 168Ng_1^2g_2g_4^2$

$- 1288Ng_1^2g_3^2g_4 + 40Ng_1^2g_4^3 - 216Ng_1g_2g_3^2g_4 + 144Ng_1g_3^4 - 36Ng_1g_3^2g_4^2 + 13g_2^4g_4 - 24g_2^3g_4^2 - 650g_2^2g_4^3$

$+ 42Ng_2g_3^2g_4^2 - 96g_2g_4^4 - 152Ng_3^4g_4 + 40Ng_3^2g_4^3 - 708g_4^5]$

$+ \frac{1}{124416}[-11048N^2g_1^6g_4 + 10368\zeta_3Ng_1^6g_4 - 17120Ng_1^6g_4 + 4608N^2g_1^5g_2g_4 + 2112Ng_1^5g_2g_4$

$+ 57312N^2g_1^5g_3^2 + 102672Ng_1^5g_3^2 + 27072N^2g_1^5g_4^2 - 43200Ng_1^5g_4^2 + 12N^2g_1^4g_2^2g_4 + 25920\zeta_3Ng_1^4g_2^2g_4$

$- 53292Ng_1^4g_2^2g_4 - 32544N^2g_1^4g_2g_3^2 + 58464Ng_1^4g_2g_3^2 - 4752N^2g_1^4g_2g_4^2 - 186624\zeta_3Ng_1^4g_2g_4^2$

$+ 250368Ng_1^4g_2g_4^2 - 143672N^2g_1^4g_3^2g_4 + 31104\zeta_3Ng_1^4g_3^2g_4 + 239520Ng_1^4g_3^2g_4 - 1376N^2g_1^4g_4^3$

$$+ 248832\zeta_3 N g_1^4 g_4^3 + 44520 N g_1^4 g_4^3 - 3120 N g_1^3 g_3^2 g_4 + 79200 N g_1^3 g_2^2 g_3^3 - 10656 N g_1^3 g_2^2 g_4^3 + 27072 N^2 g_1^3 g_2 g_3^2 g_4$$

$$+ 248832\zeta_3 N g_1^3 g_2 g_3^2 g_4 - 15168 N g_1^3 g_2 g_3^2 g_4 - 124416\zeta_3 N g_1^3 g_2 g_4^3 + 22272 N g_1^3 g_2 g_4^3 + 108864 N^2 g_1^3 g_3^3$$

$$+ 8640 N g_1^3 g_3^4 + 31680 N^2 g_1^3 g_3^2 g_4^2 + 248832\zeta_3 N g_1^3 g_3^2 g_4^2 + 82656 N g_1^3 g_3^2 g_4^2 - 44064 N g_1^3 g_4^4 + 1904 N g_1^2 g_2^4 g_4$$

$$+ 10944 N g_1^2 g_2^3 g_3^2 - 6000 N g_1^2 g_2^3 g_4^2 + 186624\zeta_3 N g_1^2 g_2^2 g_3^2 g_4 + 67740 N g_1^2 g_2^2 g_3^2 g_4 - 44008 N g_1^2 g_2^2 g_4^3$$

$$- 19440 N^2 g_1^2 g_2 g_3^4 - 62208\zeta_3 N g_1^2 g_2 g_3^4 + 62352 N g_1^2 g_2 g_3^4 - 1584 N^2 g_1^2 g_2 g_3^2 g_4^2 + 82944\zeta_3 N g_1^2 g_2 g_3^2 g_4^2$$

$$+ 528600 N g_1^2 g_2 g_3^2 g_4^2 - 25632 N g_1^2 g_2 g_4^4 - 106612 N^2 g_1^2 g_3^4 g_4 + 33696 N g_1^2 g_3^4 g_4 - 1768 N^2 g_1^2 g_3^2 g_4^3$$

$$+ 41472\zeta_3 N g_1^2 g_3^2 g_4^3 + 635376 N g_1^2 g_3^2 g_4^3 + 7784 N g_1^2 g_4^5 - 25344 N g_1 g_2^3 g_3^2 g_4 - 62208\zeta_3 N g_1 g_2^2 g_3^2 g_4^2$$

$$+ 85656 N g_1 g_2^2 g_3^2 g_4^2 + 6768 N^2 g_1 g_2 g_3^4 g_4 - 124416\zeta_3 N g_1 g_2 g_3^4 g_4 + 44928 N g_1 g_2 g_3^4 g_4 - 177120 N g_1 g_2 g_3^2 g_4^3$$

$$+ 31104 N^2 g_1 g_3^6 - 62208\zeta_3 N g_1 g_3^6 + 162576 N g_1 g_3^6 + 24912 N^2 g_1 g_3^4 g_4^2 + 124416\zeta_3 N g_1 g_3^4 g_4^2 - 11232 N g_1 g_3^4 g_4^2$$

$$- 124416\zeta_3 N g_1 g_3^2 g_4^4 + 3312 N g_1 g_3^2 g_4^4 + 2592\zeta_3 g_2^6 g_4 - 5195 g_2^6 g_4 - 31104\zeta_3 g_2^5 g_4^2 + 33612 g_2^5 g_4^2$$

$$+ 62208\zeta_3 g_2^4 g_4^3 + 11864 g_2^4 g_4^3 + 2592 N g_2^3 g_3^2 g_4^2 + 124416\zeta_3 g_2^3 g_4^4 + 88656 g_2^3 g_4^4 - 16788 N g_2^2 g_3^2 g_4^3$$

$$- 15552\zeta_3 g_2^2 g_4^5 + 250512 g_2^2 g_4^5 - 396 N^2 g_2 g_3^4 g_4^2 + 15072 N g_2 g_3^4 g_4^2 + 16680 N g_2 g_3^2 g_4^4 - 124416\zeta_3 g_2 g_4^6$$

$$+ 267264 g_2 g_4^6 - 13168 N^2 g_3^6 g_4 + 72576\zeta_3 N g_3^6 g_4 - 4856 N g_3^6 g_4 - 1130 N^2 g_3^4 g_4^3 + 124416\zeta_3 N g_3^4 g_4^3$$

$$+ 88584 N g_3^4 g_4^3 - 41896 N g_3^2 g_4^5 + 279936\zeta_3 g_4^7 + 184632 g_4^7] + O(g_i^9)$$

$$\beta_5(g_i)|_{m=2} = \frac{1}{8}[4N g_3^3 - N g_3^2 g_5 + 10 g_4^2 g_5 - 3 g_5^3]$$

$$+ \frac{1}{1152}[-96 N g_1^2 g_3^3 + 8 N g_1^2 g_3^2 g_5 + 248 N g_1^2 g_4^2 g_5 - 2592 N g_1 g_3^3 g_4 - 480 N g_1 g_3^2 g_4 g_5 + 124 g_2^2 g_4^2 g_5$$

$$- 1248 g_2 g_4^3 g_5 + 336 N g_3^5 + 344 N g_3^4 g_5 - 432 N g_3^3 g_4^2 + 324 N g_3^3 g_5^2 + 292 N g_3^2 g_4^2 g_5 - 126 N g_3^2 g_5^3 - 2864 g_4^4 g_5$$

$$+ 2340 g_4^2 g_5^3 - 513 g_5^5]$$

$$+ \frac{1}{165888}[3840 N^2 g_1^4 g_3^3 + 136896 N g_1^4 g_3^3 - 1648 N^2 g_1^4 g_3^2 g_5 + 20736\zeta_3 N g_1^4 g_3^2 g_5 - 34240 N g_1^4 g_3^2 g_5$$

$$- 5920 N^2 g_1^4 g_4^2 g_5 + 17056 N g_1^4 g_4^2 g_5 - 19008 N g_1^3 g_2 g_3^3 + 9600 N g_1^3 g_2 g_3^2 g_5 - 56256 N g_1^3 g_2 g_4^2 g_5$$

$$- 130176 N^2 g_1^3 g_3^3 g_4 + 581760 N g_1^3 g_3^3 g_4 + 63360 N^2 g_1^3 g_3^2 g_4 g_5 - 183552 N g_1^3 g_3^2 g_4 g_5 - 165888\zeta_3 N g_1^3 g_4^3 g_5$$

$$+ 30528 N g_1^3 g_4^3 g_5 + 1920 N g_1^2 g_2^2 g_3^3 - 824 N g_1^2 g_2^2 g_3^2 g_5 + 16672 N g_1^2 g_2^2 g_4^2 g_5 + 311040 N g_1^2 g_2 g_3^3 g_4$$

$$- 207360\zeta_3 N g_1^2 g_2 g_3^2 g_4 g_5 + 161856 N g_1^2 g_2 g_3^2 g_4 g_5 - 107616 N g_1^2 g_2 g_4^3 g_5 + 217728 N^2 g_1^2 g_3^5 - 119616 N g_1^2 g_3^5$$

$$- 98208 N^2 g_1^2 g_3^4 g_5 - 37952 N g_1^2 g_3^4 g_5 + 27072 N^2 g_1^2 g_3^3 g_4^2 + 995328\zeta_3 N g_1^2 g_3^3 g_4^2 + 21120 N g_1^2 g_3^3 g_4^2$$

$$+ 64800 N g_1^2 g_3^3 g_5^2 - 6704 N^2 g_1^2 g_3^2 g_4^2 g_5 + 622080\zeta_3 N g_1^2 g_3^2 g_4^2 g_5 + 1466912 N g_1^2 g_3^2 g_4^2 g_5 - 6984 N g_1^2 g_3^2 g_5^3$$

$$- 114592 N g_1^2 g_4^4 g_5 + 74448 N g_1^2 g_4^2 g_5^3 - 65088 N g_1 g_2^2 g_3^3 g_4 + 31680 N g_1 g_2^2 g_3^2 g_4 g_5 + 497664\zeta_3 N g_1 g_2 g_3^3 g_4^2$$

$$+ 300672 N g_1 g_2 g_3^3 g_4^2 - 42912 N g_1 g_2 g_3^2 g_4^2 g_5 - 220608 N^2 g_1 g_3^5 g_4 - 248832\zeta_3 N g_1 g_3^5 g_4 - 45504 N g_1 g_3^5 g_4$$

$$+ 58752 N^2 g_1 g_3^4 g_4 g_5 - 497664\zeta_3 N g_1 g_3^4 g_4 g_5 - 250944 N g_1 g_3^4 g_4 g_5 + 248832\zeta_3 N g_1 g_3^3 g_4^3 + 1087488 N g_1 g_3^3 g_4^3$$

$$- 559872\zeta_3 N g_1 g_3^3 g_4 g_5^2 - 909792 N g_1 g_3^3 g_4 g_5^2 - 497664\zeta_3 N g_1 g_3^2 g_4^3 g_5 - 14592 N g_1 g_3^2 g_4^3 g_5$$

$$+ 186624\zeta_3 N g_1 g_3^2 g_4 g_5^3 - 72576 N g_1 g_3^2 g_4 g_5^3 - 4248 g_2^4 g_4^2 g_5 - 82944\zeta_3 g_2^3 g_4^3 g_5 - 38544 g_2^3 g_4^3 g_5$$

$$+ 13536 N g_2^2 g_3^3 g_4^2 - 3352 N g_2^2 g_3^2 g_4^2 g_5 + 290304\zeta_3 g_2^2 g_4^4 g_5 + 549360 g_2^2 g_4^4 g_5 + 37224 g_2^2 g_4^2 g_5^3 - 82944\zeta_3 N g_2 g_3^3 g_4^3$$

$$- 76032 N g_2 g_3^3 g_4^3 - 6336 N g_2 g_3^2 g_4^3 g_5 + 497664\zeta_3 g_2 g_4^5 g_5 + 444768 g_2 g_4^5 g_5 - 311040\zeta_3 g_2 g_4^3 g_5^3 - 387504 g_2 g_4^3 g_5^3$$

$$+ 6816 N^2 g_3^7 - 124416\zeta_3 N g_3^7 + 275712 N g_3^7 + 64528 N^2 g_3^6 g_5 + 207360\zeta_3 N g_3^6 g_5 + 15920 N g_3^6 g_5$$

$$+ 27072 N^2 g_3^5 g_4^2 - 248832\zeta_3 N g_3^5 g_4^2 + 87936 N g_3^5 g_4^2 - 30456 N^2 g_3^5 g_5^2 + 279936\zeta_3 N g_3^5 g_5^2 - 5184 N g_3^5 g_5^2$$

$$- 4648 N^2 g_3^4 g_4^2 g_5 - 62208\zeta_3 N g_3^4 g_4^2 g_5 - 167936 N g_3^4 g_4^2 g_5 + 2376 N^2 g_3^4 g_5^3 + 139968\zeta_3 N g_3^4 g_5^3 + 199728 N g_3^4 g_5^3$$

$$
\begin{aligned}
&- 12096 N g_3^3 g_4^4 + 186624 \zeta_3 N g_3^3 g_4^2 g_5^2 + 16848 N g_3^3 g_4^2 g_5^2 - 77760 \zeta_3 N g_3^3 g_5^4 + 19116 N g_3^3 g_5^4 - 96864 N g_3^2 g_4^4 g_5 \\
&+ 90360 N g_3^2 g_4^2 g_5^3 - 28026 N g_3^2 g_5^5 + 373248 \zeta_3 g_4^6 g_5 + 1293344 g_4^6 g_5 - 933120 \zeta_3 g_4^4 g_5^3 - 1361376 g_4^4 g_5^3 \\
&+ 629856 \zeta_3 g_4^2 g_5^5 + 760428 g_4^2 g_5^5 - 104976 \zeta_3 g_5^7 - 137295 g_5^7] + O(g_i^9).
\end{aligned}
\tag{A1}
$$

The elements of the mass mixing matrix are

$$
\begin{aligned}
\gamma_{11}(g_i)|_{m=2} =\ & \frac{1}{3}[g_1^2 + g_3^2] \\
&+ \frac{1}{216}[-44 N g_1^4 - 134 g_1^4 - 30 g_1^3 g_2 + 5 g_1^2 g_2^2 - 268 g_1^2 g_3^2 + 10 g_1^2 g_4^2 - 90 g_1 g_3^2 g_4 - 22 N g_3^4 + 4 g_3^4 + 10 g_3^2 g_4^2] \\
&+ \frac{1}{15552}[3212 N^2 g_1^6 + 31104 \zeta_3 N g_1^6 - 8032 N g_1^6 + 2592 \zeta_3 g_1^6 + 31420 g_1^6 - 15552 \zeta_3 N g_1^5 g_2 + 4518 N g_1^5 g_2 \\
&+ 15552 \zeta_3 g_1^5 g_2 - 2964 g_1^5 g_2 + 7852 N g_1^4 g_2^2 - 5184 \zeta_3 g_1^4 g_2^2 + 18512 g_1^4 g_2^2 - 9076 N g_1^4 g_3^2 + 7776 \zeta_3 g_1^4 g_3^2 \\
&+ 94260 g_1^4 g_3^2 + 3040 N g_1^4 g_4^2 - 1964 g_1^4 g_4^2 - 2859 g_1^3 g_2^3 + 15552 \zeta_3 g_1^3 g_2 g_3^2 - 2964 g_1^3 g_2 g_3^2 - 1578 g_1^3 g_2 g_4^2 \\
&- 15552 \zeta_3 N g_1^3 g_3^2 g_4 + 4518 N g_1^3 g_3^2 g_4 + 46656 \zeta_3 g_1^3 g_3^2 g_4 - 8892 g_1^3 g_3^2 g_4 - 4140 g_1^3 g_4^3 - 51 g_1^2 g_2^4 - 982 g_1^2 g_2^2 g_3^2 \\
&+ 328 g_1^2 g_2^2 g_4^2 - 10368 \zeta_3 g_1^2 g_2 g_3^2 g_4 + 38988 g_1^2 g_2 g_3^2 g_4 - 1032 g_1^2 g_2 g_4^3 + 46656 \zeta_3 N g_1^2 g_3^4 - 2972 N g_1^2 g_3^4 \\
&- 15552 \zeta_3 g_1^2 g_3^4 + 27252 g_1^2 g_3^4 + 12664 N g_1^2 g_3^2 g_4^2 - 20736 \zeta_3 g_1^2 g_3^2 g_4^2 + 74048 g_1^2 g_3^2 g_4^2 + 312 g_1^2 g_4^4 - 789 g_1 g_2^2 g_3^2 g_4 \\
&- 6210 g_1 g_2 g_3^2 g_4^2 - 15552 \zeta_3 N g_1 g_3^4 g_4 + 4518 N g_1 g_3^4 g_4 + 15552 \zeta_3 g_1 g_3^4 g_4 - 17928 g_1 g_3^4 g_4 - 10944 g_1 g_3^2 g_4^3 \\
&+ 250 g_2^2 g_3^2 g_4^2 - 516 g_2 g_3^2 g_4^3 + 803 N^2 g_3^6 + 4018 N g_3^6 + 10368 \zeta_3 g_3^6 + 9052 g_3^6 + 4686 N g_3^4 g_4^2 + 8854 g_3^4 g_4^2 \\
&+ 312 g_3^2 g_4^4] + O(g_i^8)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{12}(g_i)|_{m=2} =\ & N g_1^2 + \frac{N}{12}[-2 g_1^4 - 18 g_1^3 g_2 - 3 g_1^2 g_2^2 - 2 g_1^2 g_3^2 - 18 g_1 g_3^2 g_4 - 3 g_3^2 g_4^2] \\
&+ \frac{N}{864}[2308 N g_1^6 + 1426 g_1^6 - 1984 N g_1^5 g_2 + 1822 g_1^5 g_2 + 282 N g_1^4 g_2^2 + 864 \zeta_3 g_1^4 g_2^2 + 1430 g_1^4 g_2^2 + 2852 g_1^4 g_3^2 \\
&+ 40 g_1^4 g_4^2 + 864 \zeta_3 g_1^3 g_2^3 + 1420 g_1^3 g_2^3 + 2020 g_1^3 g_2 g_3^2 - 904 g_1^3 g_2 g_4^2 + 1426 g_1^3 g_3^2 g_4 + 1512 g_1^3 g_4^3 - 21 g_1^2 g_2^4 \\
&- 300 g_1^2 g_2^2 g_3^2 + 282 g_1^2 g_2^2 g_4^2 + 1728 \zeta_3 g_1^2 g_2 g_3^2 g_4 + 1260 g_1^2 g_2 g_3^2 g_4 - 1080 g_1^2 g_2 g_4^3 + 1154 N g_1^2 g_3^4 + 220 g_1^2 g_3^4 \\
&+ 4060 g_1^2 g_3^2 g_4^2 + 756 g_1^2 g_4^4 - 864 \zeta_3 g_1 g_2^2 g_3^2 g_4 + 828 g_1 g_2^2 g_3^2 g_4 + 3456 \zeta_3 g_1 g_2 g_3^2 g_4^2 + 1332 g_1 g_2 g_3^2 g_4^2 - 992 N g_1 g_3^4 g_4 \\
&- 356 g_1 g_3^4 g_4 + 1796 g_1 g_3^2 g_4^3 + 324 g_2^2 g_3^2 g_4^2 - 432 g_2 g_3^2 g_4^3 + 141 N g_3^4 g_4^2 - 246 g_3^4 g_4^2 + 66 g_3^2 g_4^4] + O(g_i^8)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{13}(g_i)|_{m=2} =\ & \frac{1}{2} N g_3^2 + \frac{N}{24}[-2 g_1^2 g_3^2 - 6 g_1^2 g_4^2 - 36 g_1 g_3^2 g_4 + 4 g_3^4 - 3 g_3^2 g_4^2] \\
&+ \frac{N}{1728}[796 N g_1^4 g_3^2 + 1426 g_1^4 g_3^2 + 376 N g_1^4 g_4^2 - 600 g_1^4 g_4^2 - 198 g_1^3 g_2 g_3^2 - 1728 \zeta_3 g_1^3 g_2 g_4^2 + 1656 g_1^3 g_2 g_4^2 \\
&- 1984 N g_1^3 g_3^2 g_4 + 4040 g_1^3 g_3^2 g_4 + 3456 \zeta_3 g_1^3 g_4^3 + 576 g_1^3 g_4^3 + 20 g_1^2 g_2^2 g_3^2 + 836 g_1^2 g_2^2 g_4^2 + 2160 g_1^2 g_2 g_3^2 g_4 \\
&- 864 g_1^2 g_2 g_4^3 + 1512 N g_1^2 g_3^4 + 120 g_1^2 g_3^4 + 188 N g_1^2 g_3^2 g_4^2 + 3456 \zeta_3 g_1^2 g_3^2 g_4^2 + 1660 g_1^2 g_3^2 g_4^2 + 132 g_1^2 g_4^4 \\
&- 452 g_1 g_2^2 g_3^2 g_4 + 1728 \zeta_3 g_1 g_2 g_3^2 g_4^2 + 1800 g_1 g_2 g_3^2 g_4^2 - 992 N g_1 g_3^4 g_4 - 226 g_1 g_3^4 g_4 + 3592 g_1 g_3^2 g_4^3 + 47 g_2^2 g_3^2 g_4^2 \\
&- 540 g_2 g_3^2 g_4^3 + 432 N g_3^6 - 864 \zeta_3 g_3^6 + 2258 g_3^6 + 94 N g_3^4 g_4^2 + 780 g_3^4 g_4^2 + 822 g_3^2 g_4^4] + O(g_i^8)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{21}(g_i)|_{m=2} =\ & \frac{g_1^2}{2} + \frac{1}{72}[14 N g_1^4 - 20 g_1^4 - 54 g_1^3 g_2 - 2 g_1^2 g_2^2 - 20 g_1^2 g_3^2 + 14 g_1^2 g_4^2 - 54 g_1 g_3^2 g_4 - 9 g_3^2 g_4^2] \\
&+ \frac{1}{10368}[-396 N^2 g_1^6 - 15552 \zeta_3 N g_1^6 + 17596 N g_1^6 + 5184 \zeta_3 g_1^6 + 3476 g_1^6 - 9792 N g_1^5 g_2 + 17532 g_1^5 g_2 \\
&- 500 N g_1^4 g_2^2 + 10368 \zeta_3 g_1^4 g_2^2 + 10054 g_1^4 g_2^2 + 848 N g_1^4 g_3^2 + 10368 \zeta_3 g_1^4 g_3^2 + 6952 g_1^4 g_3^2 - 792 N g_1^4 g_4^2
\end{aligned}
$$

$$- 676g_1^4g_4^2 + 5184\zeta_3g_1^3g_2^3 + 864g_1^3g_2^3 + 13824g_1^3g_2g_3^2 - 3888g_1^3g_2g_4^2 - 2640Ng_1^3g_3^2g_4 + 24948g_1^3g_3^2g_4$$

$$- 10368\zeta_3g_1^3g_4^3 + 7344g_1^3g_4^3 - 2592\zeta_3g_1^2g_2^4 + 2801g_1^2g_2^4 - 696g_1^2g_2^2g_3^2 - 500g_1^2g_2^2g_4^2 + 10368\zeta_3g_1^2g_2g_3^2g_4$$

$$+ 12744g_1^2g_2g_3^2g_4 - 5904g_1^2g_2g_4^3 - 7776\zeta_3Ng_1^2g_3^4 + 8374Ng_1^2g_3^4 - 3040g_1^2g_3^4 - 704Ng_1^2g_3^2g_4^2 + 20736\zeta_3g_1^2g_3^2g_4^2$$

$$+ 17600g_1^2g_3^2g_4^2 - 5184\zeta_3g_1^2g_4^4 + 10532g_1^2g_4^4 - 2256g_1g_2^2g_3^2g_4 + 15552\zeta_3g_1g_2g_3^2g_4^2 + 7128g_1g_2g_3^2g_4^2$$

$$- 3576Ng_1g_3^4g_4 - 10368\zeta_3g_1g_3^4g_4 + 3024g_1g_3^4g_4 + 5184\zeta_3g_1g_3^2g_4^3 - 3672g_1g_3^2g_4^3 + 984g_2^2g_3^2g_4^2$$

$$- 5184\zeta_3g_2g_3^2g_4^3 + 4968g_2g_3^2g_4^3 + 300Ng_3^4g_4^2 - 5184\zeta_3g_3^4g_4^2 + 5856g_3^4g_4^2 - 672g_3^2g_4^4] + O(g_i^8)$$

$$\gamma_{22}(g_i)|_{m=2} = \frac{1}{12}[-2Ng_1^2 + 5g_2^2 - 2g_4^2]$$

$$+ \frac{1}{216}[-160Ng_1^4 - 60Ng_1^3g_2 + 52Ng_1^2g_2^2 + 2Ng_1^2g_3^2 + 48Ng_1g_3^2g_4 - 97g_2^4 + 52g_2^2g_4^2$$

$$- 60g_2g_4^3 - 11g_3^2g_4^2N - 160g_4^4]$$

$$+ \frac{1}{62208}[-82472N^2g_1^6 + 10368\zeta_3Ng_1^6 + 55600Ng_1^6 + 45216N^2g_1^5g_2 + 124416\zeta_3Ng_1^5g_2 - 28128Ng_1^5g_2$$

$$- 4740N^2g_1^4g_2^2 + 57024\zeta_3Ng_1^4g_2^2 + 308076Ng_1^4g_2^2 + 20736\zeta_3Ng_1^4g_3^2 + 38480Ng_1^4g_3^2 - 33368Ng_1^4g_4^2$$

$$- 62208\zeta_3Ng_1^3g_2^3 - 22992Ng_1^3g_2^3 - 45888Ng_1^3g_2g_3^2 + 45216Ng_1^3g_2g_4^2 + 124416\zeta_3Ng_1^3g_3^2g_4 + 24672Ng_1^3g_3^2g_4$$

$$- 98208Ng_1^3g_4^3 - 19768Ng_1^2g_2^4 + 6592Ng_1^2g_2^2g_3^2 - 9480Ng_1^2g_2^2g_4^2 - 155520\zeta_3Ng_1^2g_2g_3^2g_4 + 174384Ng_1^2g_2g_3^2g_4$$

$$+ 45216Ng_1^2g_2g_4^3 - 5524N^2g_1^2g_3^4 - 3776Ng_1^2g_3^4 + 259200\zeta_3Ng_1^2g_3^2g_4^2 + 51776Ng_1^2g_3^2g_4^2 - 33368Ng_1^2g_4^4$$

$$- 24000Ng_1g_2^2g_3^2g_4 + 3312Ng_1g_2g_3^2g_4^2 + 2304N^2g_1g_3^4g_4 - 672Ng_1g_3^4g_4 - 62208\zeta_3Ng_1g_3^2g_4^3 + 17952Ng_1g_3^2g_4^3$$

$$+ 18144\zeta_3g_2^6 + 52225g_2^6 - 19768g_2^4g_4^2 - 62208\zeta_3g_2^3g_4^3 - 22992g_2^3g_4^3 + 9296g_2^2g_3^2g_4^2 + 57024\zeta_3g_2^2g_4^4$$

$$+ 303336g_2^2g_4^4 - 24600Ng_2g_3^2g_4^3 + 124416\zeta_3g_2g_4^5 + 17088g_2g_4^5 + 6N^2g_3^4g_4^2 - 8408Ng_3^4g_4^2 - 1840Ng_3^2g_4^4$$

$$+ 10368\zeta_3g_4^6 - 26872g_4^6] + O(g_i^8)$$

$$\gamma_{23}(g_i)|_{m=2} = \frac{g_4^2}{2} + \frac{1}{144}[-54Ng_1^2g_3^2 + 28Ng_1^2g_4^2 - 36Ng_1g_3^2g_4 - 4g_2^2g_4^2 - 108g_2g_4^3 + 7Ng_3^2g_4^2 - 12g_4^4]$$

$$+ \frac{1}{10368}[-2712N^2g_1^4g_3^2 + 6060Ng_1^4g_3^2 - 396N^2g_1^4g_4^2 - 5184\zeta_3Ng_1^4g_4^2 + 10928Ng_1^4g_4^2 + 6480Ng_1^3g_2g_3^2$$

$$- 5904Ng_1^3g_2g_4^2 + 2256N^2g_1^3g_3^2g_4 + 20736\zeta_3Ng_1^3g_3^2g_4 - 14400Ng_1^3g_3^2g_4 - 10368\zeta_3Ng_1^3g_4^3 + 7344Ng_1^3g_4^3$$

$$+ 912Ng_1^2g_2^2g_3^2 - 500Ng_1^2g_2^2g_4^2 + 15552\zeta_3Ng_1^2g_2g_3^2g_4 + 8424Ng_1^2g_2g_3^2g_4 - 3888Ng_1^2g_2g_4^3 - 1620N^2g_1^2g_3^4$$

$$- 5184\zeta_3Ng_1^2g_3^4 + 5196Ng_1^2g_3^4 - 132N^2g_1^2g_3^2g_4^2 - 2592\zeta_3Ng_1^2g_3^2g_4^2 + 37030Ng_1^2g_3^2g_4^2 - 1468Ng_1^2g_4^4$$

$$- 2112Ng_1g_2^2g_3^2g_4 - 5184\zeta_3Ng_1g_2g_3^2g_4^2 + 5616Ng_1g_2g_3^2g_4^2 + 564N^2g_1g_3^4g_4 - 10368\zeta_3Ng_1g_3^4g_4$$

$$+ 3744Ng_1g_3^4g_4 - 8064Ng_1g_3^2g_4^3 - 2592\zeta_3g_2^4g_4^2 + 2801g_2^4g_4^2 + 5184\zeta_3g_2^3g_4^3 + 864g_2^3g_4^3 + 216Ng_2^2g_3^2g_4^2$$

$$+ 10368\zeta_3g_2^2g_4^4 + 9554g_2^2g_4^4 - 1188Ng_2g_3^2g_4^3 + 7740g_2g_4^5 - 33N^2g_3^4g_4^2 + 1256Ng_3^4g_4^2 + 632Ng_3^2g_4^4$$

$$- 10368\zeta_3g_4^6 + 20676g_4^6] + O(g_i^8)$$

$$\gamma_{31}(g_i)|_{m=2} = \frac{g_3^2}{2} + \frac{1}{72}[-20g_1^2g_3^2 - 18g_1^2g_4^2 - 108g_1g_3^2g_4 + 7Ng_3^4 - 2g_3^4 + 5g_3^2g_4^2]$$

$$+ \frac{1}{10368}[-5184\zeta_3Ng_1^4g_3^2 + 9404Ng_1^4g_3^2 + 5184\zeta_3g_1^4g_3^2 + 3476g_1^4g_3^2 + 2256Ng_1^4g_4^2 - 1464g_1^4g_4^2$$

$$+ 3708g_1^3g_2g_3^2 - 5184g_1^3g_2g_4^2 - 7152Ng_1^3g_3^2g_4 + 27648g_1^3g_3^2g_4 + 20736\zeta_3g_1^3g_4^3 + 3456g_1^3g_4^3 - 374g_1^2g_2^2g_3^2$$

$$+ 5016g_1^2g_2^2g_4^2 + 10368\zeta_3g_1^2g_2g_3^2g_4 + 9504g_1^2g_2g_3^2g_4 - 10368\zeta_3g_1^2g_2g_4^3$$

$$+ 9936g_1^2g_2g_4^3 - 10368\zeta_3Ng_1^2g_3^4 + 7768Ng_1^2g_3^4 + 10368\zeta_3g_1^2g_3^4 - 4808g_1^2g_3^4 - 1760Ng_1^2g_3^2g_4^2$$

$$+ 20736\zeta_3g_1^2g_3^2g_4^2 + 22616g_1^2g_3^2g_4^2 - 1344g_1^2g_4^4 - 2280g_1g_2^2g_3^2g_4 + 9072g_1g_2g_3^2g_4^2 - 6216Ng_1g_3^4g_4$$

$$+ 11556 g_1 g_3^4 g_4 + 10368 \zeta_3 g_1 g_3^2 g_4^3 - 7344 g_1 g_3^2 g_4^3 + 344 g_2^2 g_3^2 g_4^2 - 2952 g_2 g_3^2 g_4^3 - 99 N^2 g_3^6 + 2654 N g_3^6$$

$$- 5184 \zeta_3 g_3^6 + 13820 g_3^6 - 184 N g_3^4 g_4^2 + 10368 \zeta_3 g_3^4 g_4^2 + 1928 g_3^4 g_4^2 - 5184 \zeta_3 g_3^2 g_4^4 + 9860 g_3^2 g_4^4] + O(g_i^8)$$

$$\gamma_{32}(g_i)|_{m=2} = g_4^2 + \frac{1}{24}[-18 N g_1^2 g_3^2 - 12 N g_1 g_3^2 g_4 - 6 g_2^2 g_4^2 - 36 g_2 g_4^3 + 7 N g_3^2 g_4^2 - 4 g_4^4]$$

$$+ \frac{1}{864}[1010 N g_1^4 g_3^2 + 756 N g_1^4 g_4^2 + 1728 \zeta_3 N g_1^3 g_2 g_3^2 - 900 N g_1^3 g_2 g_3^2 - 1080 N g_1^3 g_2 g_4^2 + 1560 N g_1^3 g_3^2 g_4$$

$$+ 1512 N g_1^3 g_4^3 - 432 \zeta_3 N g_1^2 g_2^2 g_3^2 + 846 N g_1^2 g_2^2 g_3^2 + 282 N g_1^2 g_2^2 g_4^2 + 2592 \zeta_3 N g_1^2 g_2 g_3^2 g_4 + 2268 N g_1^2 g_2 g_3^2 g_4$$

$$- 904 N g_1^2 g_2 g_4^3 - 496 N^2 g_1^2 g_3^4 - 178 N g_1^2 g_3^4 + 3911 N g_1^2 g_3^2 g_4^2 + 40 N g_1^2 g_4^4 - 216 N g_1 g_2^2 g_3^2 g_4$$

$$- 1728 \zeta_3 N g_1 g_2 g_3^2 g_4^2 + 792 N g_1 g_2 g_3^2 g_4^2 + 282 N^2 g_1 g_3^4 g_4 - 492 N g_1 g_3^4 g_4 - 1344 N g_1 g_3^2 g_4^3 - 21 g_2^4 g_4^2$$

$$+ 864 \zeta_3 g_2^3 g_4^3 + 1420 g_2^3 g_4^3 - 204 N g_2^2 g_3^2 g_4^2 + 864 \zeta_3 g_2^2 g_4^4 + 1712 g_2^2 g_4^4 - 70 N g_2 g_3^2 g_4^3 - 162 g_2 g_4^5$$

$$- 33 N^2 g_3^4 g_4^2 + 1276 N g_3^4 g_4^2 - 37 N g_3^2 g_4^4 + 3734 g_4^6] + O(g_i^8)$$

$$\gamma_{33}(g_i)|_{m=2} = \frac{1}{12}[-N g_3^2 + 4 g_4^2] + \frac{1}{432}[2 N g_1^2 g_3^2 + 20 N g_1^2 g_4^2 - 12 N g_1 g_3^2 g_4 + 10 g_2^2 g_4^2 - 60 g_2 g_4^3 - 76 N g_3^4 + 31 N g_3^2 g_4^2 - 356 g_4^4]$$

$$+ \frac{1}{31104}[-206 N^2 g_1^4 g_3^2 + 2592 \zeta_3 N g_1^4 g_3^2 - 4280 N g_1^4 g_3^2 - 344 N^2 g_1^4 g_4^2 + 968 N g_1^4 g_4^2 + 1200 N g_1^3 g_2 g_3^2$$

$$- 2064 N g_1^3 g_2 g_4^2 + 4536 N^2 g_1^3 g_3^2 g_4 - 12144 N g_1^3 g_3^2 g_4 - 8280 N g_1^3 g_4^3 - 103 N g_1^2 g_2^2 g_3^2 + 656 N g_1^2 g_2^2 g_4^2$$

$$- 10368 \zeta_3 N g_1^2 g_2 g_3^2 g_4 + 5328 N g_1^2 g_2 g_3^2 g_4 - 3156 N g_1^2 g_2 g_4^3 - 7416 N^2 g_1^2 g_3^4 + 13436 N g_1^2 g_3^4 - 442 N^2 g_1^2 g_3^2 g_4^2$$

$$+ 113644 N g_1^2 g_3^2 g_4^2 + 2152 N g_1^2 g_4^4 + 2268 N g_1 g_2^2 g_3^2 g_4 - 7308 N g_1 g_2 g_3^2 g_4^2 + 3960 N^2 g_1 g_3^4 g_4$$

$$+ 31104 \zeta_3 N g_1 g_3^4 g_4 - 16680 N g_1 g_3^4 g_4 - 31104 \zeta_3 N g_1 g_3^2 g_4^3 - 15216 N g_1 g_3^2 g_4^3 - 102 g_2^4 g_4^2 - 5718 g_2^3 g_4^3$$

$$- 221 N g_2^2 g_3^2 g_4^2 - 10368 \zeta_3 g_2^2 g_4^4 + 52728 g_2^2 g_4^4 + 1080 N g_2 g_3^2 g_4^3 + 3108 g_2 g_4^5 - 3292 N^2 g_3^6 + 18144 \zeta_3 N g_3^6$$

$$- 1214 N g_3^6 - 284 N^2 g_3^4 g_4^2 + 31104 \zeta_3 N g_3^4 g_4^2 + 24248 N g_3^4 g_4^2 - 10644 N g_3^2 g_4^4 + 67392 \zeta_3 g_4^6 + 53200 g_4^6] + O(g_i^8).$$

$$(A2)$$

APPENDIX B: COMPLEX SOLUTIONS

In this appendix we provide the remaining spectrum of fixed points for $N = 1000$ and $m = 2$ as an example of the mix of purely real, purely imaginary or fully complex fixed points. As in the main text we exclude solutions related by symmetries. First, there are five sets of complex fixed points of the CS type. The first is at

$$x = (0.030988i - 0.117059) + (0.225504 - 0.026447i)\epsilon + (0.141512 + 0.0220556i)\epsilon^2 + O(\epsilon^3)$$

$$y = (10.616979i + 7.410016) + (-0.306607 + 5.10124i)\epsilon + (-3.30970 + 5.02141i)\epsilon^2 + O(\epsilon^3)$$

$$z = (0.041381i + 0.950398) + (0.073839 - 0.000086i)\epsilon + (-0.076593 + 0.240769i)\epsilon^2 + O(\epsilon^3)$$

$$t = (-3.598246i + 9.770473) + (2.741560 - 4.671500i)\epsilon + (17.262953 - 21.70028i)\epsilon^2 + O(\epsilon^3)$$

$$w \in \{(12.52843i - 8.437892) + (-0.081315 + 6.488068i)\epsilon + (79.099646 + 19.016683i)\epsilon^2 + O(\epsilon^3),$$

$$(-8.157796i + 0.215249) + (10.458651 + 2.563357i)\epsilon + (-13.448748 + 143.623375i)\epsilon^2 + O(\epsilon^3),$$

$$(-4.370634i + 8.222643) + (-1.841641 - 7.930799i)\epsilon + (5.191836 - 47.804171i)\epsilon^2 + O(\epsilon^3)\}$$

$$(B1)$$

where we have grouped three solutions together given that the only difference is in the location of the w coupling. The remainder are

$$x = (-0.003823i + 1.005167) + (-0.054618 + 0.012486i)\epsilon + (0.018976 + 0.143820i)\epsilon^2 + O(\epsilon^3)$$
$$y = (2.003185i + 18.658117) + (5.392312 + 1.086468i)\epsilon + (14.851433 - 19.827819i)\epsilon^2 + O(\epsilon^3)$$
$$z = (0.008575i + 1.025642) + (-0.085717 - 0.009151i)\epsilon + (-0.006651 - 0.017253i)\epsilon^2 + O(\epsilon^3)$$
$$t = (-4.466842i + 8.052554) + (4.210604 + 3.034436i)\epsilon + (8.671652 + 12.119131i)\epsilon^2 + O(\epsilon^3)$$
$$w \in \{(15.494072i - 6.618774) + (6.586489 - 2.039527i)\epsilon + (160.338535 - 22.392846i)\epsilon^2 + O(\epsilon^3),$$
$$(-2.718385i + 6.100016) + (3.357728 + 1.279947i)\epsilon + (12.015204 + 7.083669i)\epsilon^2 + O(\epsilon^3),$$
$$(-12.775687i + 0.518757) + (0.842674 + 1.030180i)\epsilon + (56.581857 + 57.905307i)\epsilon^2 + O(\epsilon^3)\} \tag{B2}$$

$$x = (0.050822i - 0.763449) + (-0.342230 - 0.090513i)\epsilon + (-1.048864 - 0.055624i)\epsilon^2 + O(\epsilon^3)$$
$$y = (7.435673i + 12.90737) + (11.071087 + 3.990671i)\epsilon + (10.916564 + 32.815621i)\epsilon^2 + O(\epsilon^3)$$
$$z = (0.0450638i + 0.849646) + (0.197056 - 0.055744i)\epsilon + (0.452791 + 0.136376i)\epsilon^2 + O(\epsilon^3)$$
$$t = (-1.982473i + 12.903491) + (6.044935 - 1.720892i)\epsilon + (6.597369 - 13.346172i)\epsilon^2 + O(\epsilon^3)$$
$$w \in \{(6.421453i - 11.600963) + (-2.494744 + 2.659224i)\epsilon + (12.554273 + 26.208510i)\epsilon^2 + O(\epsilon^3),$$
$$(-3.154923i + 14.29153) + (7.238628 - 2.639826i)\epsilon + (4.840108 - 21.122141i)\epsilon^2 + O(\epsilon^3),$$
$$(-3.266530i - 2.690567) + (1.337926 + 0.955634i)\epsilon + (33.890419 + 26.868026i)\epsilon^2 + O(\epsilon^3)\} \tag{B3}$$

$$x = (-0.555294i - 1.513381) + (-23.114452 + 24.950148i)\epsilon + (3858.041480 + 2762.998578i)\epsilon^2 + O(\epsilon^3)$$
$$y = (34.90363i - 21.842603) + (890.252852 + 680.087930i)\epsilon + (74550.859506 - 131881.754919i)\epsilon^2 + O(\epsilon^3)$$
$$z = (0.549418i + 1.214497) + (24.255203 - 14.435870i)\epsilon + (-2470.779282 - 3096.675176i)\epsilon^2 + O(\epsilon^3)$$
$$t = (-29.55207i + 15.18796) + (-850.256009 - 539.584442i)\epsilon + (-56438.610729 + 124039.167626i)\epsilon^2 + O(\epsilon^3)$$
$$w \in \{(42.91263i - 23.15191) + (1127.488695 + 813.413116i)\epsilon + (82884.915451 - 172830.465074i)\epsilon^2 + O(\epsilon^3),$$
$$(0.409667i + 1.270019) + (35.031670 + 22.655513i)\epsilon + (1486.251233 - 4603.586046i)\epsilon^2 + O(\epsilon^3),$$
$$(-43.32229i + 21.88189) + (-1155.605341 - 813.415117i)\epsilon + (-84174.229119 + 179594.341857i)\epsilon^2$$
$$+ O(\epsilon^3)\} \tag{B4}$$

and

$$x = (-0.006320i - 1.030086) + (0.107095 + 0.043240i)\epsilon + (-0.036554 + 0.122859i)\epsilon^2 + O(\epsilon^3)$$
$$y = (2.138785i - 9.592803) + (-5.961631 - 11.215238i)\epsilon + (-22.616924 - 62.894519i)\epsilon^2 + O(\epsilon^3)$$
$$z = (0.002767i + 1.0176108) + (-0.059037 - 0.021032i)\epsilon + (0.043635 - 0.043068i)\epsilon^2 + O(\epsilon^3)$$
$$t = (1.198860i - 8.255330) + (-2.187227 - 7.108184i)\epsilon + (-8.358400 - 32.699994i)\epsilon^2 + O(\epsilon^3)$$
$$w \in \{(13.36369i - 4.799020) + (10.845688 - 1.270978i)\epsilon + (139.104337 - 28.027224i)\epsilon^2 + O(\epsilon^3),$$
$$(-12.50645i - 2.865408) + (-1.424629 - 4.030314i)\epsilon + (67.742495 + 17.073470i)\epsilon^2 + O(\epsilon^3),$$
$$(-0.857238i + 7.664428) + (1.116387 + 5.387238i)\epsilon + (4.573636 + 22.217349i)\epsilon^2 + O(\epsilon^3)\}. \tag{B5}$$

There were several sets where some of the fixed points were either real or imaginary in addition to one being fully complex since we found the solutions

$x = 0.114419i - 0.553587i\epsilon + 6.740110i\epsilon^2 + O(\epsilon^3)$

$y = 22.486625i - 31.203475i\epsilon + 603.274688i\epsilon^2 + O(\epsilon^3)$

$z = 1.070456 - 0.504440\epsilon + 9.518801\epsilon^2 + O(\epsilon^3)$

$t = -11.601013i + 36.808088i\epsilon - 722.589642i\epsilon^2$
$\quad + O(\epsilon^3)$

$w \in \{(22.427674i - 1.617314)$
$\quad + (1.687428 - 38.335633i)\epsilon$
$\quad + (255.539793 + 718.517860i)\epsilon^2 + O(\epsilon^3),$
$\quad 3.234627 + 8.891244\epsilon - 144.067750\epsilon^2$
$\quad + O(\epsilon^3)\}$ \hfill (B6)

and

$x = -0.868555 - 0.203744\epsilon - 0.915849\epsilon^2 + O(\epsilon^3)$

$y = 19.811433 + 7.966436\epsilon - 15.205442\epsilon^2 + O(\epsilon^3)$

$z = 1.012581 - 0.020942\epsilon + 0.066905\epsilon^2 + O(\epsilon^3)$

$t = -4.342552 - 2.231269\epsilon - 7.303390\epsilon^2 + O(\epsilon^3)$

$w = (15.014668i - 2.955662) + (4.805741 + 1.917129i)\epsilon$
$\quad + (111.835693 - 52.975405i)\epsilon^2 + O(\epsilon^3).$ \hfill (B7)

This completes the set of all CS type solutions in addition to those in Sec. VI.

For the remaining solutions we found that at least one of the critical couplings was zero. First we grouped those solutions where the couplings were either real or imaginary. We found

$x = 0, \quad t = 0$

$y = 14.907120i + 11.502407i\epsilon - 10.399304i\epsilon^2 + O(\epsilon^3)$

$z = 0.998006 + 0.007339\epsilon - 0.009793\epsilon^2 + O(\epsilon^3)$

$w \in \{(15.559942i - 2.659423) + (4.519271 + 2.244641i)\epsilon$
$\quad + (110.184708 - 62.342815i)\epsilon^2 + O(\epsilon^3),$
$\quad 5.318846 + 0.901757\epsilon - 1.582315\epsilon^2 + O(\epsilon^3)\}$ \hfill (B8)

and

$x = 0, \quad z = 0, \quad t = 0$

$y = 14.907120i + 11.502407i\epsilon - 10.399304i\epsilon^2 + O(\epsilon^3)$

$w \in \{10.540926 + 8.344899\epsilon - 16.563109\epsilon^2 + O(\epsilon^3), 0\}.$
\hfill (B9)

Included in the first set is a complex w coupling. However, the second solution of each set is an example which is similar to pure ϕ^3 theory when its coupling is purely imaginary. That particular $O(N)$ model described the Lee-Yang edge singularity problem [61]. Also the solutions of (B9) correspond to

the Lagrangian without a ϕ^{ia} field. In the case of the only nonzero coupling y this is the pure cubic theory involving only the σ field. The remaining solutions with any complex roots all have a vanishing critical z coupling and are

$x = (0.102517i - 1.069257) + (-0.187674 - 0.485282i)\epsilon$
$\quad + (6.388101 + 3.227418i)\epsilon^2 + O(\epsilon^3)$

$y = (8.388316i + 4.182323)$
$\quad + (-31.690792 - 0.719078i)\epsilon$
$\quad + (394.496262 - 256.273660i)\epsilon^2 + O(\epsilon^3)$

$z = 0$

$t = (-12.684990i - 5.204227)$
$\quad + (23.207111 + 4.566306i)\epsilon$
$\quad + (-270.175902 + 154.130691i)\epsilon^2 + O(\epsilon^3)$

$w \in \{(13.394284i + 8.214396)$
$\quad + (-30.663870 - 17.742257i)\epsilon$
$\quad + (479.083975 - 270.124833i)\epsilon^2 + O(\epsilon^3), 0\}$ \hfill (B10)

and

$x = 0, \quad z = 0$

$y = (12.918644i + 4.311494)$
$\quad + (2.006636 + 9.961474i)\epsilon$
$\quad + (-0.372460 - 5.181172i)\epsilon^2 + O(\epsilon^3)$

$t = (9.765251i - 1.446829)$
$\quad + (-0.575941 + 7.513693i)\epsilon$
$\quad + (-1.501827 - 7.061748i)\epsilon^2 + O(\epsilon^3)$

$w \in \{(7.382858i - 3.189516)$
$\quad + (-1.150870 + 6.001629i)\epsilon$
$\quad + (-0.334518 - 6.569977i)\epsilon^2 + O(\epsilon^3), 0\}.$ \hfill (B11)

The remainder of the solutions are real but interesting patterns emerge in several cases. First we record the fixed points where there is no pairing with another set. There were four such cases. Of all the real solutions we record we found only the first two correspond to stable fixed points which are

$x = 1.011102 - 0.026162\epsilon + 0.022238\epsilon^2 + O(\epsilon^3)$

$y = 6.637801 - 1.117476\epsilon + 0.962982\epsilon^2 + O(\epsilon^3)$

$z = 0, \quad t = 0$

$w = 10.540926 + 8.344899\epsilon - 16.563109\epsilon^2$
$\quad + O(\epsilon^3)$ \hfill (B12)

and

$$x = 0, \quad y = 0, \quad z = 0, \quad t = 0$$
$$w = 10.540926 + 8.344899\epsilon - 16.563109\epsilon^2 + O(\epsilon^3). \tag{B13}$$

The final solution corresponds to the pure T^{ab} theory when $m = 2$ but the $(0, 0, 0, 0, w)$ structure could be analyzed in isolation for arbitrary m. However, the stability of these two solutions, in contrast to the remaining real solutions which are not stable, appears to be driven by the vanishing of the couplings g_3 and g_4. In this case there is no interaction whatsoever between the pair of fields $\{\phi^{ia}, \sigma\}$ and T^{ab} which is apparent from (2.4). In other words one is dealing with a partitioned Lagrangian and the coupling constant space is also partitioned. So the stability here is a reflection of the stability of the two separate Lagrangians. In the second of these two solutions the situation is effectively trivial since it reflects that one of the two Lagrangians is a free field theory which has a zero β-function. That such solutions representing the sum of independent Lagrangians emerge similar to (B9) ought not to come as a surprise and should be regarded as an internal consistency in our analysis of solutions. The remaining unpaired solutions are

$$x = -0.854446 - 0.212078\epsilon - 0.647751\epsilon^2 + O(\epsilon^3)$$
$$y = 18.789145 + 9.197094\epsilon - 2.733818\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \quad t = 0$$
$$w = 10.540926 + 8.344899\epsilon - 16.563109\epsilon^2 + O(\epsilon^3) \tag{B14}$$

and

$$x = 0.983210 - 0.011253\epsilon + 0.063259\epsilon^2 + O(\epsilon^3)$$
$$y = 16.345805 + 10.658027\epsilon + 21.524495\epsilon^2 + O(\epsilon^3)$$
$$z = 0, \quad t = 0$$
$$w = 10.540926 + 8.344899\epsilon - 16.563109\epsilon^2 + O(\epsilon^3) \tag{B15}$$

which together with the other real solutions correspond to saddle point structures. For the paired solutions we have

$$x = 0.986386 + 0.006824\epsilon + 0.023890\epsilon^2 + O(\epsilon^3)$$
$$y = 3.882413 + 3.856888\epsilon + 1.139387\epsilon^2 + O(\epsilon^3)$$
$$z = 0$$
$$t = -6.810006 + 3.229867\epsilon + 5.755896\epsilon^2 + O(\epsilon^3)$$
$$w \in \{13.726061 + 3.729174\epsilon - 29.707955\epsilon^2 + O(\epsilon^3), 0\} \tag{B16}$$

and

$$x = 0, \quad z = 0$$
$$y = 17.222217 + 37.874140\epsilon + 404.846200\epsilon^2 + O(\epsilon^3)$$
$$t = -13.657731 - 33.103835\epsilon - 376.517213\epsilon^2 + O(\epsilon^3)$$
$$w \in \{20.542650 + 42.608899\epsilon + 533.442168\epsilon^2$$
$$+ O(\epsilon^3), 0\}. \tag{B17}$$

The final three pairings exhibit a novel feature in that in each set the critical x and t couplings are equal. This is clear since we found

$$x = -0.854046 - 0.211934\epsilon - 0.647273\epsilon^2 + O(\epsilon^3)$$
$$y = 18.789012 + 9.196991\epsilon - 2.731101\epsilon^2 + O(\epsilon^3)$$
$$z = 0$$
$$t = -0.854046 - 0.211934\epsilon - 0.647273\epsilon^2 + O(\epsilon^3)$$
$$w \in \{-10.598432 + 8.327096\epsilon - 16.655991\epsilon^2$$
$$+ O(\epsilon^3), 0\} \tag{B18}$$

$$x = 0.982680 - 0.011165\epsilon + 0.063307\epsilon^2 + O(\epsilon^3)$$
$$y = 16.347740 + 10.655223\epsilon + 21.515017\epsilon^2 + O(\epsilon^3)$$
$$z = 0$$
$$t = 0.982680 - 0.011165\epsilon + 0.063307\epsilon^2 + O(\epsilon^3)$$
$$w \in \{10.616993 + 8.279650\epsilon - 16.867303\epsilon^2 + O(\epsilon^3), 0\} \tag{B19}$$

and

$$x = 1.010586 - 0.0261234\epsilon + 0.022211\epsilon^2 + O(\epsilon^3)$$
$$y = 6.633618 - 1.114926\epsilon + 0.960725\epsilon^2 + O(\epsilon^3)$$
$$z = 0$$
$$t = 1.010586 - 0.026124\epsilon + 0.022211\epsilon^2 + O(\epsilon^3)$$
$$w \in \{10.621358 + 8.274362\epsilon - 16.895995\epsilon^2 + O(\epsilon^3), 0\}. \tag{B20}$$

While our focus here was on the $O(1000) \times O(2)$ theory it represents a snapshot of the spectrum of potential solutions for the general symmetry group. What has emerged are real fixed points in addition to the Heisenberg, AU and CS types which were motivated by the four-dimensional theory.

[1] G. Mack and A. Salam, Ann. Phys. (N.Y.) 53, 174 (1969).
[2] S. Ferrara, A. F. Grillo, and R. Gatto, Lett. Nuovo Cimento 2, 1363 (1971).
[3] S. Ferrara, A. F. Grillo, G. Parisi, and R. Gatto, Nucl. Phys. B49, 77 (1972).
[4] S. Ferrara, A. F. Grillo, and R. Gatto, Ann. Phys. (N.Y.) 76, 161 (1973).
[5] S. Ferrara, A. F. Grillo, R. Gatto, and G. Parisi, Nuovo Cimento Soc. Ital. Fis. 19A, 667 (1974).
[6] S. Ferrara, A. F. Grillo, and R. Gatto, Phys. Rev. D 9, 3564 (1974).
[7] A. M. Polyakov, Zh. Eksp. Teor. Fiz. 66, 23 (1974).
[8] F. A. Dolan and H. Osborn, Nucl. Phys. B599, 459 (2001).
[9] R. Rattazzi, V. S. Rychkov, and E. Tonni, J. High Energy Phys. 12 (2008) 031.
[10] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Phys. Rev. D 86, 025022 (2012).
[11] F. Kos, D. Poland, and D. Simmons-Duffin, J. High Energy Phys. 06 (2014) 091.
[12] A. B. Zamolodchikov, JETP Lett. 43, 730 (1986).
[13] W. E. Caswell, Phys. Rev. Lett. 33, 244 (1974).
[14] T. Banks and A. Zaks, Nucl. Phys. B196, 189 (1982).
[15] Y. Nakayama and T. Ohtsuki, Phys. Lett. B 734, 193 (2014).
[16] J.-B. Bae and S.-J. Rey, arXiv:1412.6549.
[17] S. M. Chester, S. S. Pufu, and R. Yacoby, Phys. Rev. D 91, 086014 (2015).
[18] A. Eichhorn, L. Janssen, and M. M. Scherer, Phys. Rev. D 93, 125021 (2016).
[19] K. Kamikado and T. Kanazawa, arXiv:1604.04830.
[20] E. Ma, Prog. Theor. Phys. 54, 1828 (1975).
[21] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, Phys. Rev. D 91, 045011 (2015).
[22] J. A. Gracey, Phys. Rev. D 92, 025012 (2015).
[23] O. F. de Alcantara Bonfim, J. E. Kirkham, and A. J. McKane, J. Phys. A 13, L247 (1980).
[24] O. F. de Alcantara Bonfim, J. E. Kirkham, and A. J. McKane, J. Phys. A 14, 2391 (1981).
[25] D. V. Batkovich, K. G. Chetyrkin, and M. V. Kompaniets, Nucl. Phys. B906, 147 (2016).
[26] M. V. Kompaniets and E. Panzer, Proc. Sci., LL2016 (2016) 038.
[27] O. Schnetz, arXiv:1606.08598.
[28] Yu. Nakayama and T. Ohtsuki, Phys. Rev. D 89, 126009 (2014).
[29] A. Pelissetto and E. Vicari, Phys. Rev. B 62, 6393 (2000).
[30] P. Calabrese, A. Pelissetto, and E. Vicari, Phys. Rev. B 67, 024418 (2003).
[31] Yu. Nakayama and T. Ohtsuki, Phys. Rev. D 91, 021901 (2015).
[32] K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972).
[33] A. J. McKane, D. J. Wallace, and R. K. P. Zia, Phys. Lett. 65B, 171 (1976).
[34] A. J. McKane, J. Phys. G 3, 1165 (1977).
[35] A. N. Vasil'ev, Y. M. Pismak, and J. R. Honkonen, Theor. Math. Phys. 46, 104 (1981).
[36] A. N. Vasil'ev, Y. M. Pismak, and J. R. Honkonen, Theor. Math. Phys. 47, 465 (1981).
[37] A. N. Vasil'ev, Y. M. Pismak, and J. R. Honkonen, Theor. Math. Phys. 50, 127 (1982).
[38] A. Pelissetto, P. Rossi, and E. Vicari, Nucl. Phys. B607, 605 (2001).
[39] H. Kawamura, Phys. Rev. B 38, 4916 (1988).
[40] H. Kawamura, Phys. Rev. B 42, 2610 (1990).
[41] H. Kawamura, J. Phys. Soc. Jpn. 59, 2305 (1990).
[42] H. Kawamura, J. Phys. Soc. Jpn. 61, 1299 (1992).
[43] H. Kawamura, J. Phys.: Condens. Matter 10, 4707 (1998).
[44] A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. B 63, 140414 (2001).
[45] A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. B 65, 020403 (2001).
[46] J. A. Gracey, Nucl. Phys. B644, 433 (2002).
[47] P. Nogueira, J. Comput. Phys. 105, 279 (1993).
[48] S. Laporta, Int. J. Mod. Phys. A 15, 5087 (2000).
[49] C. Studerus, Comput. Phys. Commun. 181, 1293 (2010).
[50] A. von Manteuffel and C. Studerus, arXiv:1201.4330.
[51] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Nucl. Phys. B837, 186 (2010).
[52] O. V. Tarasov, Phys. Rev. D 54, 6479 (1996).
[53] O. V. Tarasov, Nucl. Phys. B502, 455 (1997).
[54] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B 303, 334 (1993).
[55] J. A. M. Vermaseren, arXiv:math-ph/0010025.
[56] M. Tentyukov and J. A. M. Vermaseren, Comput. Phys. Commun. 181, 1419 (2010).
[57] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.95.025029 for file with electronic version of the renormalization group functions.
[58] J. A. Gracey, Phys. Rev. D 93, 025025 (2016).
[59] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, J. High Energy Phys. 09 (2015) 076.
[60] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, Prog. Theor. Exp. Phys. (2016) 12C105.
[61] M. E. Fisher, Phys. Rev. Lett. 40, 1610 (1978).