Noncommutative geometrical origin of the energy-momentum dispersion relation

A. Watcharangkool[*] and M. Sakellariadou[†]

*Department of Physics, King's College London, University of London, Strand WC2R 2LS,
London, United Kingdom*

We investigate a link between the energy-momentum dispersion relation and the spectral distance in the context of a Lorentzian almost-commutative spectral geometry, defined by the product of Minkowski spacetime and an internal discrete noncommutative space. Using the causal structure, the almost-commutative manifold can be identified with a pair of four-dimensional Minkowski spacetimes embedded in a five-dimensional Minkowski geometry. Considering fermions traveling within the light cone of the ambient five-dimensional spacetime, we then derive the energy-momentum dispersion relation.

I. INTRODUCTION

The framework of noncommutative geometry (NCG) offers a generalization to the notion of Riemannian geometry, replacing manifolds with algebras of bounded operators on Hilbert spaces [1]. The formalism was first used for commutative C^* algebras, and then was extended to *spaces* characterized by a noncommutative algebra of coordinates. Extending all basic geometric notions from ordinary manifolds to noncommutative spaces is a fundamental aspect of noncommutative geometry. In such a framework, all information about a physical system is encoded within the algebra of operators in a Hilbert space, with the action expressed in terms of a generalized Dirac operator. Following this approach, all fundamental forces in physics can be considered on an equal footing, namely as curvature on a noncommutative manifold, leading to a purely geometric explanation for the Standard Model of particle physics [2]. In addition, this approach implies an equivalent formulation for the distance on a manifold, defined as a set of pure states of a commutative C^* algebra. For example, on a manifold where points are identical to pure states of commutative C^* algebra, the geodesic distance between points on the manifold is completely determined by spectral data of a Dirac operator

$$d(x, y) = \sup\{|\omega_x(f) - \omega_y(f)| : f \in A, \|[-i\nabla\!\!\!\!/, f]\| \leq 1\}, \quad (1)$$

where A is a commutative pre-C^* algebra, $\omega_{x,y}$ are pure states of the algebra defined by $\omega_x(f) := f(x)$, and $-i\nabla\!\!\!\!/$ is the Dirac operator associated with the spin connection, playing the role of the inverse of the line element ds (where $ds = \sqrt{g_{\mu\nu}dx^\mu dx^\nu}$). Equation (1) above is known as the

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[*]apimook.watcharangkool@kcl.ac.uk
[†]mairi.sakellariadou@kcl.ac.uk

spectral distance formula or *Connes's distance formula*. As a distance function between pure states, the above expression makes perfect sense when one generalizes the commutative algebra to a noncommutative one; however, the physical meaning of this quantity is not clear in the noncommutative regime. It has been shown [3] that in an almost-commutative manifold, the spectral distance resembles the geodesic distance in a higher-dimension manifold, but extracting the physical meaning of this result is nontrivial.

An important issue of NCG is the lack of its Lorentzian version, which is the geometry of our physical spacetime. Strictly speaking, there is no particle physics model from NCG, but a model inspired by NCG. To investigate the energy-momentum dispersion relation, which is obtained in the framework of a relativistic theory, one may have to include the notion of causal structure into the geometry. Thus, in what follows, we will incorporate generic features about Lorentzian noncommutative geometry [4–8].

The rest of this paper is organized as follows: In Sec. II, we discuss some general properties of the spectral triple and the spectral distance formula. In Sec. III, we state the definition of a Lorentzian spectral triple, which will be used throughout this paper, and elaborate on the notion of causal structure. In Sec. IV, we investigate the link between the distance formula and the energy-momentum dispersion relation. We conclude in Sec. V.

II. ALMOST-COMMUTATIVE GEOMETRY AND DISTANCE FORMULA

A. Spectral triples

The spectral triple is a collection of data $(A, \mathcal{H}, D)$, where A is a dense subalgebra of a C^* algebra (pre-C^* algebra) acting as a subalgebra of bounded operators on a Hilbert space $\mathcal{H}$, and D is a Dirac operator (densely defined self-adjoint operator with compact resolvent). It can be seen as a generalized notion of geometry: if A is a unital

commutative algebra—namely, if we have a commutative spectral triple—then one can reconstruct the compact Riemannian spin manifold M, such that $A \simeq C^\infty(M)$ [9]. It is this duality between a commutative C^* algebra and the algebra of smooth functions on a Riemannian manifold that inspired the notion of noncommutative geometry: given a noncommutative algebra A, one may think of a non-commutative geometry as a space X for which A is the coordinate algebra.

In addition, one considers a real structure J and a grading operator γ (we refer the reader to Ref. [10] for details), which are crucial for the construction of spin manifold and obtaining the Standard Model of high-energy physics from noncommutative spectral geometry.

Let $M \times F$, where M is a four-dimensional Riemannian spin manifold and F an internal noncommutative space, define an almost-commutative manifold. Its spectral triple $(A, \mathcal{H}, D)$ is given by the algebra

$$C^\infty(M) \otimes A_F := C^\infty(M) \otimes \left(\bigoplus_{k=1}^n A_k \right), \qquad (2)$$

with finite-dimensional algebra (not necessarily commutative) A_F, Hilbert space $L^2(M, S) \otimes \mathcal{H}_F$, and Dirac operator $-i\slashed{\nabla} \otimes \mathrm{Id}_F + \gamma^5 \otimes D_F$, where H_F is a finite-dimensional Hilbert space and D_F a self-adjoint matrix (Dirac operator).

Choosing appropriately the algebra of the internal space F as

$$A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_4(\mathbb{C}), \qquad (3)$$

and applying the spectral action, which is basically the trace of the heat kernel of the Dirac operator, one obtains an effective description of the Standard Model [11].

B. Inner fluctuations

The symmetry in an almost-commutative manifold is the automorphism group of the algebra

$$\mathrm{Diff}(M \times F) := \mathrm{Aut}(C^\infty(M, A_F)), \qquad (4)$$

since the diffeomorphism group, which is the symmetry group on a manifold, is isomorphic to the automorphism of the algebra of smooth functions, $\mathrm{Diff}(M) \simeq \mathrm{Aut}(C^\infty(M))$. Being interested in the automorphism that would lead to the symmetries of the Standard Model, let us consider the inner automorphism α_u, characterized by a unitary element of the algebra

$$\alpha_u(a) \mapsto uau^*, \qquad (5)$$

where $u \in \mathcal{U}(A)$. Since the unitary equivalence is an important element for the physics of the Standard Model, we need to incorporate it in the spectral action. To do so, we define an algebra $B := \alpha_u(A) \simeq A$ as a unitary

equivalent algebra, and find its corresponding spectral triple $(B, \mathcal{H}', D')$, which involves the notion of Morita equivalence. The Morita equivalence between two C^* algebras B and A implies the existence of a projective right C^* module $\mathcal{E}$ (we refer the reader to Ref. [10] for more details on the C^* module), such that

$$B = \mathrm{End}_A(\mathcal{E}). \qquad (6)$$

Note that, in the case where the algebra has both left and right action on the Hilbert space, the definition of Morita equivalence requires a bimodule.

Since that algebra is the $\mathrm{End}_A(\mathcal{E})$, the natural choice for the Hilbert space of the new triple is $\mathcal{H}' := \mathcal{E} \otimes_A \mathcal{H}$, and it remains to choose the Dirac operator. Suppose there exists a Hermitian connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_D^1$ satisfying the conditions

$$\nabla(\xi a) = (\nabla \xi)a + \xi \otimes da, \quad \forall \, \xi \in \mathcal{E}, \quad a \in A, \quad (7)$$

$$d\langle \xi, \eta \rangle_A = \langle \xi, \nabla \eta \rangle_A - \langle \nabla \xi, \eta \rangle_A, \quad \forall \, \xi, \quad \eta \in \mathcal{E}, \quad (8)$$

where $da := [D, a]$, Ω_D^1 is the algebra of one-forms, and $\langle \cdot, \cdot \rangle_A : \mathcal{E} \times \mathcal{E} \to A$ denotes the Hermitian product. Then the Dirac operator can be defined by

$$D'(\xi \otimes \eta) = \xi \otimes D\eta + (\nabla \xi)\eta. \qquad (9)$$

For $B := \alpha_u(A) \simeq A$, we have $\mathcal{E} = A$; hence the Dirac operator is

$$D'(1_A \otimes \eta) = 1_A \otimes D\eta + (d1_A)\eta. \qquad (10)$$

When $d1_A = [D, 1_A] \neq 0$, the Dirac operator D' is $D' = D + \mathcal{B}$, where $\mathcal{B}$ is a self-adjoint element of $\Omega_D^1(A)$ and plays the role of gauge potential. Given the charge conjugation operator, the Dirac operator reads

$$D' = D + \mathcal{B} + \epsilon' J \mathcal{B} J^{-1}, \qquad (11)$$

called the *inner fluctuation*, with J a real structure (an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$), and the number $\epsilon' \in \{-1, 1\}$ a function of $n \bmod 8$.

C. Spectral distance formula

We have previously seen the spectral distance formula in the case of a commutative spectral triple, where elements of the algebra are just smooth functions. Since the formula is defined purely from spectral data, it is still valid for a noncommutative spectral triple. Hence,

$$d(\omega, \omega') = \sup\{|\omega(a) - \omega'(a)| : a \in A, \|[D, a]\| \leq 1\}, \qquad (12)$$

where $\omega, \omega' \in \mathcal{P}(A)$ are pure states of the algebra A, having in mind a generalized notion of points. Note that, although the distance formula exists, the notion of distance between any two pure states is well defined only when $d(\omega, \omega') < \infty$. Even though we consider a spectral triple in which the formula (12) gives finite distance, the meaning of the distance between pure states in an abstract noncommutative space is still quite difficult to understand. Nevertheless, in the case of an almost-commutative manifold, its pure states are isomorphic to the points on the product space, i.e. $\mathcal{P}(A) \cong M \times F$ [11]. In the case that F is a finite space, the geodesic distance squared between (x, e_i) and (y, e_j) for $e_i, e_j \in F$ is given by [3]

$$d^2(x \times e_i, y \times e_j) = d_M^2(x, y) + d_F^2(e_i, e_j), \quad (13)$$

where $d_M(x, y)$ is the geodesic distance on M and $d_F(e_i, e_j)$ stands for the shortest distance between internal states e_i and e_j. This Pythagorean theorem allows one to embed the almost-commutative manifold $M \times F$ in an $(n + 1)$-dimensional Riemannian manifold $M \times \mathbb{R}$. The metric of the almost-commutative manifold inherited from the ambient $(n + 1)$-dimensional manifold is

$$g_{ab} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & 1/d^2(e_i, e_j) \end{pmatrix}, \quad (14)$$

where $a, b \in \{0, 1, 2, 3, 4\}$ (namely, they refer to the almost-commutative manifold), and Greek indices $\mu, \nu \in \{0, 1, 2, 3\}$. The physical meaning of the Dirac operator, as discussed earlier, implies

$$ds^{-2}|_{M \times F} = D^2 = -\not\nabla^2 + D_F^2, \quad (15)$$

and hence D satisfies the Pythagorean theorem.

For the simple model of a two-sheet space $M \times \{0, 1\}$ with discrete spectral triple $(A_F, \mathcal{H}_F, D_F)$, given by

$$A_F = \mathbb{C} \oplus \mathbb{C}, \quad \mathcal{H}_F = \mathbb{C}^2, \quad D_F = \begin{pmatrix} 0 & m \\ m^* & 0 \end{pmatrix}, \quad (16)$$

where $m \in \mathbb{C}$ is a nonzero complex parameter, we have $d_F(0, 1) = 1/|m|$. So in this case $D_F^2 = |m|^2 \mathbb{1}_2$. Note that although $|m|$ is a constant in the two-sheet space, it can be a function of $x \in M$ if one considers an almost-commutative space with inner fluctuations.

In what follows, we restrict our study to the two-sheet space, since it was shown in Ref. [3] that if the internal space of an almost-commutative manifold is discrete, then one can reduce the distance formula in an almost-commutative manifold into that of a two-sheet geometry.

III. LORENTZIAN SPECTRAL TRIPLE

Although noncommutative geometry has been applied to a relativistic theory like the Standard Model, the definition

of a Lorentzian spectral triple remains an open question, the reason mainly being the lack of a manifold reconstruction theorem analogous to Connes's reconstruction theorem for a commutative spectral triple [9]. Nevertheless, there are a few similar definitions of Lorentzian spectral triples in the literature [4–7]. In this paper we adopt the definition proposed by Ref. [4], which will be sufficient to define a causal structure. Moreover, for a commutative case that is constructed from a globally hyperbolic manifold, one can define a distance formula (which will be defined in the next section) similar to the spectral distance formula. The Lorentzian version of the spectral distance formula was proposed in Ref. [12]; it was proved that the formula leads to the geodesic distance in Minkowski space.

Definition 1.—Lorentzian spectral triple.—A Lorentzian spectral triple is given by $(A, \tilde{A}, \mathcal{H}, D, \mathcal{J})$, where
   (1) A is a nonunital dense $*$ subalgebra of a C^* algebra, and $\tilde{A}$ is its preferred unitalization.
   (2) $\mathcal{H}$ is a Krein space with an indefinite product $(\cdot, \cdot)$.
   (3) $\mathcal{J}$ is a bounded self-adjoint symmetry operator, $\mathcal{J} = \mathcal{J}^*, \mathcal{J}^2 = 1$, commuting with A. The role of $\mathcal{J}$—dubbed *fundamental symmetry* or *signature operator*—is to turn the Krein space $\mathcal{H}$ into a Hilbert space. Note that $\mathcal{H}_{\mathcal{J}}$ is the same space as $\mathcal{H}$ with a positive definite inner product $\langle \cdot, \cdot \rangle := (\cdot, \mathcal{J} \cdot)$, hence a Hilbert space.
   (4) D is a densely defined operator on $\mathcal{H}_{\mathcal{J}}$, such that
      (a) $D = -\mathcal{J}D^*\mathcal{J} =: -D^+$; i.e. it is Krein anti-self-adjoint on $\mathcal{H}$.
      (b) $\forall\, a \in \tilde{A}, [D, a]$ extends to a bounded operator on $\mathcal{H}_{\mathcal{J}}$.
      (c) $\forall\, a \in A, a(1 + \langle D \rangle)^{-1/2}$ is compact on $\mathcal{H}_{\mathcal{J}}$, where $\langle D \rangle^2 := \frac{1}{2}(DD^* + D^*D)$.
   (5) There exists a densely defined self-adjoint operator $\mathcal{T}$ with $\mathrm{Dom}D \cap \mathrm{Dom}\mathcal{T}$ dense in $\mathcal{H}_{\mathcal{J}}$, such that
      (a) $(1 + \mathcal{T}^2)^{-1/2} \in \tilde{A}$.
      (b) $\mathcal{J} = -N[D, \mathcal{T}]$ for some positive element $N \in \tilde{A}$.

Let us consider the Lorentzian spectral triple [8]

$$(C_0^\infty(M), C_b^\infty(M), L^2(M, S), -i\not\nabla), \quad (17)$$

where M is a globally hyperbolic Lorentzian manifold with signature $(-, +, +, +)$, $C_0^\infty(M)$ is the algebra of smooth functions vanishing at infinity, and $C_b^\infty(M)$ is for the space of smooth bounded functions on the manifold. The Krein $L^2(M, S)$ is the space of the squared integrable smooth sections of the spinor bundle. The Dirac operator is defined by $-i\not\nabla := -i\gamma^\mu \nabla_\mu$, where ∇_μ is the spin connection on M. Note that we choose the representation of the gamma matrices such that

$$(\gamma^0)^* = -\gamma^0, \qquad (\gamma^k)^* = \gamma^k, \quad (18)$$

where $k = 1, 2, 3$, and satisfy the relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4. \tag{19}$$

The fundamental symmetry $\mathcal{J}$ can be derived from the lapse function N and the global time function $\mathcal{T}$, as follows: For a globally hyperbolic Lorentzian manifold M, there exists a global smooth time function $\mathcal{T}$ on M such that the line element of the manifold M splits as

$$ds^2 = -Nd\mathcal{T}^2 + ds_\mathcal{T}^2, \tag{20}$$

where $ds_\mathcal{T}^2$ is the line element on the Cauchy hypersurface $\Sigma_\mathcal{T}$ at constant time $\mathcal{T}$ and N is the lapse function. The fundamental symmetry in terms of N and $\mathcal{T}$ is $\mathcal{J} = -N[D, \mathcal{T}] = iN\gamma^0$, a condition that guarantees the Lorentzian signature.

To include a causal structure in the algebra, one defines a set of real-valued functions which are nondecreasing along a future-directed causal curve:

$$\mathcal{C} = \{f \in C_b^\infty(M)\colon f(x) \leq f(y) \quad \text{iff } x \preceq y, \quad \forall \, x, y \in M\}. \tag{21}$$

The set $\mathcal{C}$ is called the *causal cone*, and its elements are *smooth bounded causal functions*. In a globally hyperbolic spacetime (M, g), the geodesic distance coincides with the Lorentzian distance function [13]

$$d(x, y) = \inf\{f(y) - f(x)|f \in \mathcal{C},$$
$$\text{ess sup} g(\nabla f, \nabla f) \leq -1, \quad \forall \, x, y \in M \quad \text{with} \quad x \preceq y\}. \tag{22}$$

In the following, we highlight the definition of the causal cone expressed in terms of the spectral triple [8,12].

Proposition 1.—Let $(A, \tilde{A}, \mathcal{H}, D, \mathcal{J})$ be a commutative Lorentzian spectral triple constructed from a globally hyperbolic manifold. Then $f \in \tilde{A}$ is a causal function iff

$$(\psi, [D, f]\psi) \leq 0, \quad \forall \, \psi \in \mathcal{H}. \tag{23}$$

This can be generalized to a noncommutative spectral triple by replacing A with a noncommutative algebra [8].

For simplicity, let us consider a Minkowski spacetime, denoted by $\mathcal{M}$, as the globally hyperbolic spacetime. In a four-dimensional Minkowski spacetime, any two points $x, y \in \mathcal{M}$ can be connected by a spacelike curve, i.e. a curve $\gamma\colon [0, 1] \to \mathcal{M}$ such that $g(\dot{\gamma}, \dot{\gamma}) > 0$ along the curve. However, some of these points can also be connected by a causal curve, i.e. $g(\dot{\gamma}, \dot{\gamma}) \leq 0$ everywhere along the curve; these points are called causally related and are denoted by $x \preceq y$.

Consider two points x, y in the Minkowski four-dimensional spacetime $\mathcal{M}$, with signature $(-, +, +, +)$, connecting through a curve γ. We define the *extremal length squared* as

$$L^2(x, y) := \begin{cases} -\sup\{l(\gamma)^2 := (\int_\gamma \sqrt{-g(\dot{\gamma}, \dot{\gamma})}d\tau)^2 | g(\dot{\gamma}, \dot{\gamma}) \leq 0\}, & x \preceq y \\ \sup\{l(\gamma)^2 := (\int_\gamma \sqrt{g(\dot{\gamma}, \dot{\gamma})}d\tau)^2 | g(\dot{\gamma}, \dot{\gamma}) > 0\}, & x \npreceq y. \end{cases} \tag{24}$$

Since Minkowski spacetime is flat, $L^2(x, y) = -(x_0 - y_0)^2 + \|\mathbf{x} - \mathbf{y}\|^2$, which is zero or negative for two causally related points and strictly positive otherwise. Notice that, using $L^2(x, y)$ above, we can differentiate between points which are connected by a null curve and those which are not causally related. However, the distance defined by

$$d(x, y) = \begin{cases} \sqrt{-L^2(x, y)}, & x \preceq y \\ 0, & x \npreceq y \end{cases} \tag{25}$$

vanishes for both spacelike and lightlike separation.

IV. ENERGY-MOMENTUM DISPERSION RELATION FOR ALMOST-COMMUTATIVE SPECTRAL GEOMETRY

In the previous section, we have seen that the commutative Lorentzian spectral triple $(C_0^\infty(\mathcal{M}), C_b^\infty(\mathcal{M}), L^2(S, \mathcal{M}), \eth)$ yields a spectral distance equivalent to the

geodesic distance for Minkowski spacetime. Next, we shall define a distance function for an almost-commutative geometry—namely, the product of this Lorentzian spectral triple with a finite spectral triple—and examine the implications of the proposed distance function definition for relativistic particles.

A. Causal structure and distance

Consider a two-sheet space, defined by the tensor product of a commutative Lorentzian spectral triple and a discrete spectral triple $(A_F, \mathcal{H}_F, D_F)$, as in Eq. (16). Following Ref. [4], one can define a causal structure on the space of states $\mathcal{S}(\tilde{A})$ of the two-sheet space, using only the spectral data of the almost-commutative manifold; we highlight the procedure below.

Definition 2.—Let $\mathcal{C} = \{a \in \tilde{A} | a = a^*, (\psi, \times[D, a]\psi) \leq 0, \forall \psi \in \mathcal{H}\}$, such that $\text{span}_{\mathbb{C}}(\mathcal{C}) = \tilde{A}$. Two states $\omega, \omega' \in \mathcal{S}(\tilde{A})$ are causally related; i.e. $\omega \preceq \omega'$ iff for any $a \in \mathcal{C}$, one has

$$\omega(a) \leq \omega'(a). \tag{26}$$

Let us denote by $\mathcal{P}(A)$ the set of pure states of the algebra A, defined as the union of $\mathcal{M}_0 := \mathcal{M} \times \{0\}$ and $\mathcal{M}_1 := \mathcal{M} \times \{1\}$, hence the name of two-sheet spacetime. Thus, one may think of having two sheets of four-dimensional Minkowski spacetimes embedded in a five-dimensional one. Since we are interested in the causal relation between points on $\mathcal{M}_0$ and $\mathcal{M}_1$, we consider a particular type of mixed states $\omega_{x,\xi} \in \mathcal{N}(A) := \mathcal{M} \times [0,1] \subset \mathcal{S}(A)$ defined by

$$\omega_{x,\xi}(a \oplus b) = \xi a(x) + (1 - \xi)b(x) \tag{27}$$

for $a, b \in C_0^\infty(\mathcal{M})$. Such states $\omega_{x,\xi}$ can be considered as covering the area between the two sheets. The pure states in $\mathcal{M}(A)$ can be recovered with the choice $\xi = 0$ or $\xi = 1$.

Theorem 1.—The two states $\omega_{x,\xi}, \omega_{y,\eta} \in \mathcal{N}(A)$ are causally related if and only if $x \preceq y$ on $\mathcal{M}$ and

$$l(\gamma) \geq \frac{|\arcsin\sqrt{\eta} - \arcsin\sqrt{\xi}|}{|m|}, \tag{28}$$

where $l(\gamma)$ represents the length of a causal curve γ going from x to y on the manifold $\mathcal{M}$.

The above theorem [4] implies that if the discrete Dirac operator is trivial, i.e. $m = 0$, the causal relation holds only when $\xi = \eta$. Hence, the extremal length squared between two points $(x, 0), (y, 0) \in \mathcal{M}_0$ is

$$L^2(x, y) = -\sup_\gamma l^2(\gamma) = -(x_0 - y_0)^2 + \|\mathbf{x} - \mathbf{y}\|^2, \tag{29}$$

where γ denotes a causal curve.

If $m \neq 0$, any two points $(x, 0) \in \mathcal{M}_0$ and $(y, 1) \in \mathcal{M}_1$ are causally related iff there is a causal curve γ connecting x and y such that

$$l(\gamma) \geq \frac{\pi}{2|m|}, \tag{30}$$

implying

$$-\sup_\gamma l^2(\gamma) + \frac{\pi^2}{4|m|^2} \leq 0. \tag{31}$$

For any $(x, i), (y, j) \in \mathcal{M} \times \{0, 1\}$ with $i, j \in \{0, 1\}$, we define

$$L_m^2[(x, i), (y, j)] = \begin{cases} \frac{4}{\pi^2} L^2(x, y) + \frac{1}{|m|^2}, & i \neq j \\ \frac{4}{\pi^2} L^2(x, y), & i = j. \end{cases} \tag{32}$$

One notices that Eq. (32) is the Lorentzian version of the Pythagorean theorem [Eq. (13)].

From Eq. (24), we see that the above defined function, which we also call *extremal length squared* on $\mathcal{M} \times \{0, 1\}$,

is negative semidefinite when the points (x, i) and (y, j) are causally related, and positive otherwise. Combining the definition (32) and Theorem 1, one obtains a criterion for any two points (pure states) to be causally related.

Proposition 2.—The pure states (x, i) and (y, j), defined on an almost-commutative manifold, are said to be causally related if and only if $x \preceq y$ on $\mathcal{M}$ and

$$L_m^2[(x, i), (y, j)] \leq 0. \tag{33}$$

We will refer to the above condition as the *causal structure*.

One notices that the causal structure of the two-sheet space is exactly the same as that of a pair of four-dimensional Minkowski spacetimes embedded in a five-dimensional one ($\mathcal{M}_5 := \mathcal{M} \times [0, 1]$), with $1/|m|$ denoting the separation between the two 4-dimensional manifolds. The metric of the five-dimensional Minkowski spacetime $\mathcal{M}_5$ reads

$$g_{ab} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & 1/|m|^2 \end{pmatrix}, \tag{34}$$

where μ, ν are the spacetime indices in Minkowski spacetime, which, being flat, is denoted by $\eta_{\mu\nu}$. The metric (34) can be seen as a Wick-rotated version of (14).

Using metric (34), any two points in the two-sheet spacetime are causally related, provided they are causally related in $(\mathcal{M}_5, g)$. The line element in $\mathcal{M}_5$ is

$$\begin{aligned} ds^2 = g_{ab} dx^a dx^b &= \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{|m|^2} dx_F^2 \\ &= ds_{\mathcal{M}}^2 + ds_F^2, \end{aligned} \tag{35}$$

where dx_F is the infinitesimal of the interval [0, 1].

Making the appropriate choice for the Dirac operator $\mathcal{D}$ in M_5, such that

$$\mathcal{D}^2 = -\slashed{\nabla}^2 - |m|^2 \frac{\partial^2}{\partial x_F^2}, \tag{36}$$

the spectral distance expression (22) for a globally hyperbolic manifold, implies the geodesic expression as the one derived from the metric (34). To specify our notation, let us remark that $\mathcal{D}$ is defined by Eq. (36), whereas D will refer to the Dirac operator as defined for an almost-commutative manifold.

The Lorentzian version of the spectral distance formula is still applicable on the two-sheet space, since it is a submanifold of $\mathcal{M}_5$. Note that, to recover the D^2 operator as defined for an almost-commutative Lorentzian manifold, one chooses the boundary condition for a spinor in a five-dimensional Minkowski space such that for any $\phi \in L^2(M_5, S)$,

$$(\mathcal{D}^2 \phi)|_{\mathcal{M} \times \{0,1\}} = D^2 \phi|_{\mathcal{M} \times \{0,1\}} = (-\slashed{\nabla}^2 + |m|^2)\phi|_{\mathcal{M} \times \{0,1\}}. \tag{37}$$

B. Dirac operator and dispersion relation

Let us investigate the relation between distance for a two-sheet space and Dirac operator. To proceed, one needs to define the notion of parallel transport for such a manifold.

Definition 3.—Let $\mathcal{M} \times \{0, 1\}$ be a two-sheet space. A spinor field $\psi \in L^2(\mathcal{M}) \otimes \mathbb{C}^2$ is parallel transporting between $\mathcal{M}_i$ and $\mathcal{M}_j$ (which form the two-sheet spacetime) if there exists a spinor field $\phi \in L^2(\mathcal{M}_5, S)$, such that $\phi(y, j)$ is the parallel transport of $\phi(x, i)$, for $(x, i), (y, j) \in \mathcal{M}_5$, and

$$(\mathcal{D}^2\phi)|_{\mathcal{M}\times\{0,1\}} = D^2\phi|_{\mathcal{M}\times\{0,1\}} = D^2\psi. \quad (38)$$

Note that, if the spinor ϕ exists, then its uniqueness is guaranteed by the uniqueness of the solution of the differential equation (geodesic equation in this case).

Definition 4.—A parallel transporting spinor field $\psi \in L^2(\mathcal{M}) \otimes \mathbb{C}^2$, with $(\psi, \psi) \neq 0$, is called *causal* if

$$\frac{(D\psi, D\psi)}{(\psi, \psi)} \geq 0, \quad (39)$$

and is *harmonic* if the equality holds. Otherwise, the spinor is *noncausal*.

In the following, we will relate the definition for a causal spinor to the causal structure, Eq. (33), in the case of an almost-commutative geometry.

Proposition 3.—Let $\psi \in L^2(\mathcal{M}) \otimes \mathbb{C}^2, (\psi, \psi) \neq 0$ be a parallel transporting spinor field between $\mathcal{M}_i$ and $\mathcal{M}_j$. The geodesic of the spinor connecting any two points (x, i) and (y, j) is null iff the spinor field is harmonic.

Proof:—To prove this proposition, one has in principle to consider different cases. In the following, we will draw the proof for $i = 0, j = 1$. The other cases can be shown trivially.

First, suppose ψ is a parallel transporting spinor field between $\mathcal{M}_0$ and $\mathcal{M}_1$. For any $(x, 0), (y, 1) \in \mathcal{M}_5$, there is a spinor $\phi \in L^2(\mathcal{M}_5, S)$ such that $\phi(y, j)$ is the parallel transport of $\phi(x, i)$.

(a) If the geodesic for $\phi(t, \mathbf{x}, x_F)$ is null, then its line element is also null, i.e.

$$dt^2 = |d\mathbf{x}|^2 + \frac{1}{|m|^2} dx_F^2. \quad (40)$$

Since dt^2 and $|d\mathbf{x}|^2 + \frac{1}{|m|^2} dx_F^2$ are infinitesimal in Euclidean space, one can write

$$\frac{\partial^2 \phi}{\partial t^2} = \left(\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + |m|^2 \frac{\partial^2}{\partial x_F^2} \right) \phi. \quad (41)$$

The restriction of Eq. (41) onto the two-sheet space reads

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\phi}{\partial t^2}\bigg|_{\mathcal{M}\times\{0,1\}} = \left(\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + |m|^2 \frac{\partial^2}{\partial x_F^2} \right) \phi\bigg|_{\mathcal{M}\times\{0,1\}}$$

$$= \left(\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + |m|^2 \right) \psi, \quad (42)$$

using Eq. (38). Therefore,

$$(D\psi, D\psi) = (\psi, D^+D\psi) = -(\phi|_{\mathcal{M}\times\{0,1\}}, \mathcal{D}^2\phi|_{\mathcal{M}\times\{0,1\}})$$

$$= -(\phi|_{\mathcal{M}\times\{0,1\}}, \{-\nabla^2 + D_F^2\}\phi|_{\mathcal{M}\times\{0,1\}}) = 0, \quad (43)$$

where we have used that Dirac operator is Krein anti-self-adjoint. (b) Conversely, assuming that the spinor on the two-sheet space is harmonic,

$$0 = (D\psi, D\psi) = (\mathcal{D}\phi|_{\mathcal{M}\times\{0,1\}}, \mathcal{D}\phi|_{\mathcal{M}\times\{0,1\}})$$

$$= -\left(\phi|_{\mathcal{M}\times\{0,1\}}, \left\{ -\nabla^2 - |m|^2 \frac{\partial^2}{\partial x_F^2} \right\} \phi|_{\mathcal{M}\times\{0,1\}} \right). \quad (44)$$

Consider an inner product $(,)_5$ on $L^2(\mathcal{M}_5, S)$ as

$$(\mathcal{D}\phi, \mathcal{D}\phi)_5$$

$$= -\left(\phi, \left\{ -\nabla^2 - |m|^2 \frac{\partial^2}{\partial x_F^2} \right\} \phi \right)_5$$

$$= -\int_1^0 dx_F \left(\phi(x_F), \left\{ -\nabla^2 - |m|^2 \frac{\partial^2}{\partial x_F^2} \right\} \phi(x_F) \right). \quad (45)$$

Then, using Eq. (44) and the fact that the norm of a spinor is preserved along a geodesic, the inner product (45) vanishes, implying

$$\frac{\partial^2\phi(x)}{\partial t^2} = \left(\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + |m|^2 \frac{\partial^2}{\partial x_F^2} \right) \phi(x) \quad (46)$$

at every point on the geodesic. The inverse of $\frac{\partial^2}{\partial t^2}$ and of $\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + |m|^2 \frac{\partial^2}{\partial x_F^2}$ give a line element, which is null; therefore, the geodesic is itself null.

Let us note that in this study we restrict ourselves to the case of harmomic spinors, the reason being that we want to investigate their implications for the dispersion relation. The next proposition will show that harmonic spinors yield the energy-momentum dispersion relation, meaning that they can be interpreted as physical matter fields.

Proposition 4.—Let X be a compact subset of $\mathcal{M}$, and let $(A, \tilde{A}, \mathcal{H}, D)$ be the product of the Lorentzian spectral triple $(C^\infty(X), L^2(X, S), -i\partial)$ and the finite spectral triple $(A_F, \mathcal{H}_F, D_F)$. The eigenspinors Ψ_n of the Dirac operator,

with $(\Psi_n, \Psi_n) \neq 0$, are harmonic iff their eigenvalues satisfy the energy-momentum dispersion relation.

Proof:—Let $\Psi_n := \psi_p \otimes e_i \in \mathrm{Dom}D$ be a normalized eigenspinor of D, where ψ_p and e_i are eigenstates of $\partial\!\!\!/^2$ and D_F^2, respectively. Note that we choose the compact set $X \subset \mathcal{M}$ so that $\psi_p = \xi_p e^{i(-Et + \mathbf{p} \cdot \mathbf{x})}$ for ξ_p, a constant spinor, is square integrable. We will distinguish two cases—namely, whether $D_F^2 e_i = 0$ vanishes or not. (a) $D_F^2 e_i = 0$:

$$(D\Psi_n, D\Psi_n) = (\psi_p \otimes e_i, D^+ D\psi_p \otimes e_i) = (\psi_p, \partial\!\!\!/^2 \psi_p)(e_i, e_i) = (E^2 - \mathbf{p}^2)(\psi_p, \psi_p)$$
$$\Rightarrow \frac{(D\Psi_n, D\Psi_n)}{(\Psi_n, \Psi_n)} = E^2 - \mathbf{p}^2, \tag{47}$$

where $-E^2$ denotes the eigenvalue of the $\partial^2/\partial t^2$ operator, and $-p_i^2$ stands for the eigenvalue of $\partial^2/\partial x_i^2$. (**p** denotes a three-vector.)

The rhs of Eq. (47) is the energy-momentum dispersion relation for a massless fermion iff $(D\Psi_n, D\Psi_n) = 0$; i.e. Ψ_n is harmonic. (b) $D_F^2 e_i \neq 0$:

$$(D\Psi_n, D\Psi_n) = (\psi_p \otimes e_i, D^+ D\psi_p \otimes e_i) = (E^2 - \mathbf{p}^2)(\psi_p, \psi_p)(e_i, e_i) - m_i^2(\psi_p, \psi_p)(e_i, e_i)$$
$$\Rightarrow \frac{(D\Psi_n, D\Psi_n)}{(\Psi_n, \Psi_n)} = E^2 - \mathbf{p}^2 - m_i^2. \tag{48}$$

Correspondingly, the rhs of Eq. (48) is the energy-momentum dispersion relation for a massive fermion iff Ψ_n is harmonic.

Combining Propositions 2, 3, and 4 with Eq. (32), one may argue that the energy-momentum dispersion relation has its origin in the geometric construction of the almost-commutative manifold. Due to the causal relation between the two sheets, one may interpret this statement as the interaction between a fermion on one sheet and an anti-fermion on the other one.

To highlight the validity of Proposition 4 in the case of inner fluctuations of the Dirac operator, we will consider below a simple toy model—namely, electroweak theory with massless neutrinos.

C. A toy model: Electroweak theory with massless neutrinos

Consider the electroweak theory and assume neutrinos to be massless. To explain this theory in the context of almost-commutative spectral geometry, let us take the product of a Lorentzian spectral triple $(C_0^\infty(\mathcal{M}), L^2(\mathcal{M}, S), -i\partial\!\!\!/)$ with a finite spectral triple for the electroweak theory [11]. The spectral triple for the discrete (internal) space F is given by the algebra A_F, the Hilbert space $\mathcal{H}_F$, and the Dirac operator D_F:

$$A_F = \mathbb{C} \oplus \mathbb{H}, \tag{49}$$

$$\mathcal{H}_F = \mathcal{H}_l \oplus \mathcal{H}_{\bar{l}}, \tag{50}$$

$$D_F = \begin{pmatrix} 0 & Y^* & 0 & 0 \\ Y & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{Y}^* \\ 0 & 0 & \bar{Y} & 0 \end{pmatrix}, \tag{51}$$

where Y is a 2×2 mass matrix

$$Y = \begin{pmatrix} 0 & 0 \\ 0 & m_e \end{pmatrix}, \tag{52}$$

with m_e a complex parameter.

Assuming all inner fluctuations to vanish, apart from those of the scalar field Φ, the fluctuated Dirac operator for the almost-commutative manifold is

$$D_\Phi = -i\partial\!\!\!/ \otimes \mathbb{I}_F + \gamma^5 \otimes \Phi, \tag{53}$$

with

$$\Phi = D_F + a[D_F, b] + J_F a[D_F, b]J_F^*$$
$$= \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}, \tag{54}$$

for $a, b \in C_0^\infty(\mathcal{M}, A_F)$ and

$$\phi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{m}_e h_2 & \bar{m}_e(h_1 + 1) \\ 0 & -m_e \bar{h}_2 & 0 & 0 \\ 0 & m_e(\bar{h}_1 + 1) & 0 & 0 \end{pmatrix}, \tag{55}$$

where h_1, h_2 are complex functions. The trace of Φ^2 is given by

$$\mathrm{Tr}\Phi^2 = 2|m_e|^2|\varphi|^2, \tag{56}$$

where $\varphi := (h_1 + 1, h_2)$ is a doublet. Assuming φ undergoes symmetry breaking and denoting by v the new VEV,

we can choose $\varphi = (v + h, 0)$, where h is a small fluctuation around the vacuum.

To derive the dispersion relation, we will need D_Φ^2, given by

$$D_\Phi^2 = -\partial^2 \otimes \mathbb{I}_F + \gamma^\mu \gamma^5 \otimes \partial_\mu \Phi + \gamma^5 \gamma^\mu \otimes \partial_\mu \Phi + \mathbb{I}_4 \otimes \Phi^2$$
$$= -\partial^2 \otimes \mathbb{I}_F + \mathbb{I}_4 \otimes \Phi^2, \tag{57}$$

where we have used $\{\gamma^5, \gamma^\mu\} = 0$. We denote the basis of $\mathcal{H}_l$ and $\mathcal{H}_{\bar{l}}$ by $\{\nu_R, e_R, \nu_L, e_L\}$ and $\{\bar{\nu}_R, \bar{e}_R, \bar{\nu}_L, \bar{e}_L\}$, respectively.

The dispersion relation associated with harmonic eigenspinors $\psi_p \otimes e_L$ and $\psi_p \otimes \nu_L$ (the same result can be obtained for right-handed particles and antiparticles) can be found as follows:

$$(\psi_p \otimes e_L, D_\Phi^2 \psi_p \otimes e_L) = 0. \tag{58}$$

However,

$$(\psi_p \otimes e_L, D_\Phi^2 \psi_p \otimes e_L)$$
$$= (\psi_p, -\partial^2 \psi_p)(e_L, e_L) + (\psi_p, \psi_p)(e_L, \Phi^2 e_L)$$
$$= (-E^2 + \mathbf{p}^2)(\psi_p, \psi_p)(e_L, e_L)$$
$$\quad + \|m_e\|^2 (v^2 + 2vh + h^2)(\psi_p, \psi_p)(e_L, e_L)$$
$$= -E^2 + p^2 + \|m_e\|^2 (v^2 + 2vh + h^2). \tag{59}$$

Hence,

$$E^2 = p^2 + \|m_e\|^2 (v^2 + 2vh + h^2). \tag{60}$$

Since the fluctuation is small, we have $E^2 \sim p^2 + \|m_e\|^2 v^2$, which corresponds to the case (b) in the proof of Proposition 4. Similarly, the harmonic spinor $\psi_p \otimes \nu_L$ yields

$$E^2 = p^2, \tag{61}$$

corresponding to case (a) of the proof in Proposition 4.

V. CONCLUSIONS

In the context of almost-commutative spectral geometry, spectral distance between a pair of pure states in $M \times F$ was shown to be related to the infinitesimal distance ds^2 between two points in M and the distance between internal states in F, via the Pythagorean theorem [3]. Such a relation was shown [14] also to be valid for $1/ds^2$. For the latter case, one may observe a similarity between the Pythagorean theorem and the energy-momentum dispersion relation, implying a geometric origin of the dispersion relation.

To confirm the above observation, one has to reformulate the inverse distance, given by the inverse of the Dirac operator, in the context of Lorentzian almost-commutative spectral geometry. Following Ref. [4], one can write down the spectral triple for a Lorentzian almost-commutative manifold, and get the corresponding Dirac operator.

Having the Lorentzian Dirac operator, we are able to calculate the distance for a two-sheet manifold and define the notion of a causal structure for such a geometry. We were then able to show that the causal structure on a flat almost-commutative space can be identified with the causal structure on the five-dimensional Minkowski space with the metric

$$g_{ab} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & 1/|m|^2 \end{pmatrix}.$$

We have then suggested that spinors may be classified as causal, harmonic, or noncausal. The condition satisfied by harmonic spinors propagating in an almost-commutative manifold is equivalent to the causal relation, as suggested in Ref. [4]. We have further shown that a spinor is harmonic if and only if it satisfies the energy-momentum dispersion relation.

We have hence shown the geometric origin of the dispersion relation in the context of almost-commutative spectral geometry.

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