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Aether field in extra dimensions: Stefan-Boltzmann law and Casimir effect at finite temperature

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The Lorentz and *CPT* symmetries are not violated at the highest laboratory energies available. However these symmetries may be violated at Planck scale. A particular development is to investigate the breakdown of Lorentz and *CPT* symmetries by introducing an aether field that exhibits nonzero vacuum expectation value along the fifth dimension. The interactions of the aether field with scalar, electromagnetic, and fermions fields are analyzed. The Stefan-Boltzmann law and Casimir effect at finite temperature are calculated using the Thermo Field Dynamics formalism.

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I. INTRODUCTION

Lorentz and *CPT* invariance are fundamental symmetries of the Standard Model. Theories at high energies available in laboratory have been investigated for possible Lorentz and *CPT* violations with negative results. String theories are a particular example where Lorentz and *CPT* symmetries breaking has been analyzed [1]. These ideas led to the consideration of the Standard Model at very high energies (essentially infinite energies), i.e., the construction of the Standard Model Extension that includes a part that is Lorentz invariant and a part that is not invariant under Lorentz and *CPT* symmetry [2–4].

A particular mechanism to introduce spontaneous Lorentz breaking is to consider the existence of a vector field with nonzero vacuum expectation value [5,6]. This vector field has a fixed norm and a preferred direction at each point in space-time. A model with spontaneous Lorentz breaking in five dimensions has been developed [7,8]. It provides a different way to consider Lorentz violation in extra dimensions. This model adds a Lorentz-violating vector field, $u^a = (0,0,0,0,v)$, called the aether field, with nonzero expectation value aligned along the fifth dimension. It ensures that Lorentz invariance is preserved in four dimensions.

The aether field interacts with any matter field. All fields that interact with the aether field exhibit a violation of the Lorentz symmetry. Its interaction with other fields modifies their dispersion relations. For example this leads to different spacings in Kaluza-Klein towers of each field [7]. The role of the aether field for the stability of the extra dimension has been discussed [9,10]. The Casimir energy as a mechanism for stability in five-dimensional models has been considered [10,11].

The Casimir effect is the interaction between two parallel conducting plates [12]. The plates modify the quantum vacuum, and as a result the plates are attracted toward each other. Initially this effect was predicted for the electromagnetic field. However it can be defined for any quantum field. The first experimental observation was carried out by Sparnaay [13]. At present a high degree of accuracy has been achieved [14,15]. Our objective is to calculate the Casimir effect for the aether field interacting with various fields: scalar, electromagnetic and fermions at finite temperature.

The temperature effect may be calculated by three equivalents methods: (i) Matsubara formalism [16] is based on a substitution of time, t, by a complex time, $i\tau$. (ii) Closed time path formalism [17] is a real time formalism. This procedure leads to a doubling of the degrees of freedom of the system. And the Green functions are represented by a two-dimensional matrix structure. (iii) Another real time formalism is the Thermo Field Dynamics (TFD) [18–22]. This approach consists of two ingredients: (a) the doubling of the original Fock space, composed of the original and a fictitious space and (b) the Bogoliubov transformation. The Bogoliubov transformation is a rotation between two spaces, original and fictitious. Here the TFD formalism is chosen to calculate finite temperature effects.

This paper is organized as follows. In Sec. II, some details of the aether field are presented. In Sec. III, a brief introduction to TFD is given. In Secs. IV, V and VI, interactions between the aether field with scalar, electromagnetic and fermions fields are considered, respectively. For each case the Stefan-Boltzmann law and the Casimir effect at zero and finite temperature are

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calculated. In Sec. VII, some concluding remarks are discussed.

II. LORENTZ VIOLATION IN FIVE DIMENSIONS

The spontaneously broken Lorentz symmetry is written in terms of an aether field, u^a. The aether is a spacelike 5-vector with a nonvanishing expectation value. A five-dimensional flat space-time with coordinates $x^a = (x^\mu, y)$, with $\mu = 0, 1, 2, 3$, is considered. The fifth dimension is compactified on a circle. An antisymmetric tensor B_{ab} is defined in terms of the field u^a,

$$B_{ab} = \nabla_a u_b - \nabla_b u_a. \tag{1}$$

Then the action is

$$S = M_* \int d^5x\sqrt{g}\left[-\frac{1}{4}B_{ab}B^{ab} - \lambda(u_a u^a - v^2) + \sum_i \mathcal{L}_i\right], \tag{2}$$

where g is a metric determinant and M_* is a scaling parameter. Here $\mathcal{L}_i$ represents various interaction terms that couple the aether field to the matter field. A Lagrange multiplier, λ, enforces the constraint

$$u^a u_a = v^2. \tag{3}$$

For the case $\mathcal{L}_i = 0$, the field equation for u^a is

$$\nabla_a B^{ab} + v^{-2} u^b u_c \nabla_d B^{cd} = 0. \tag{4}$$

This equation is solved for any case for which $B_{ab} = 0$. A possible solution is

$$u^a = (0, 0, 0, 0, v), \tag{5}$$

where the aether is a spacelike vector field which has a nonvanishing component exclusively along the extra direction. This choice preserves the Lorentz invariance in the four-dimensional space. Furthermore the energy-momentum tensor associated with the aether field,

$$T^{ab} = B^{ac}B_c^b - \frac{1}{4}g^{ab}B_{cd}B^{cd} + v^{-2}u^a u^b u_c \nabla_d B^{cd}, \tag{6}$$

vanishes when $B^{ab} = 0$. This work is based on the solution given by Eq. (5).

III. THERMO FIELD DYNAMICS

Here a brief introduction to the TFD formalism is presented. In TFD the Fock space $\mathcal{S}$ is doubled, $\mathcal{S}_T = \mathcal{S} \otimes \tilde{\mathcal{S}}$, where $\tilde{\mathcal{S}}$ is the dual Fock space. Another fundamental ingredient in TFD is the Bogoliubov transformation which introduces thermal effects through a rotation between tilde ($\tilde{\mathcal{S}}$) and nontilde ($\mathcal{S}$) operators.

Using arbitrary operator $\mathcal{O}$ and $\tilde{\mathcal{O}}$ in Fock space $\mathcal{S}$ and $\tilde{\mathcal{S}}$, respectively, the Bogoliubov transformation is

$$\begin{pmatrix} \mathcal{O}(\alpha) \\ \xi\tilde{\mathcal{O}}^\dagger(\alpha) \end{pmatrix} = \mathcal{U}(\alpha)\begin{pmatrix} \mathcal{O}(k) \\ \xi\tilde{\mathcal{O}}^\dagger(k) \end{pmatrix}, \tag{7}$$

where $\xi = -1$ for bosons and $\xi = +1$ for fermions. The Bogoliubov transformation, $\mathcal{U}(\alpha)$, is defined as

$$\mathcal{U}(\alpha) = \begin{pmatrix} u(\alpha) & -w(\alpha) \\ \xi w(\alpha) & u(\alpha) \end{pmatrix}, \tag{8}$$

with $u^2(\alpha) + \xi w^2(\alpha) = 1$. Here a field theory on the topology $\Gamma_D^d = (\mathbb{S}^1)^d \times \mathbb{R}^{D-d}$ with $1 \leq d \leq D$, is considered. D are the space-time dimensions, and d is the number of compactified dimensions. This establishes a formalism in such way that any set of dimensions of the manifold $\mathbb{R}^D$ can be compactified, where the circumference of the nth $\mathbb{S}^1$ is specified by α_n. Then the α parameter is assumed as the compactification parameter defined by $\alpha = (\alpha_0, \alpha_1, \ldots\alpha_{D-1})$. The effect of temperature is described by the choice $\alpha_0 \equiv \beta$ and $\alpha_1, \ldots\alpha_{D-1} = 0$, where $\beta = 1/k_B T$ with k_B being the Boltzmann constant.

Any propagator in the TFD formalism is written in terms of the α-parameter. As an example consider the scalar field [19]. Then the propagator is

$$G_0^{(AB)}(x - x'; \alpha) = i\langle 0, \tilde{0}|\tau[\phi^A(x;\alpha)\phi^B(x';\alpha)]|0, \tilde{0}\rangle, \tag{9}$$

where

$$\phi(x;\alpha) = \mathcal{U}(\alpha)\phi(x)\mathcal{U}^{-1}(\alpha). \tag{10}$$

Here A and $B = 1$–2, and τ is the time ordering operator. In the thermal vacuum $|0(\alpha)\rangle = \mathcal{U}(\alpha)|0, \tilde{0}\rangle$, the propagator becomes

$$\begin{aligned} G_0^{(AB)}(x - x'; \alpha) &= i\langle 0(\alpha)|\tau[\phi^A(x)\phi^B(x')]|0(\alpha)\rangle, \\ &= i\int \frac{d^5k}{(2\pi)^5} e^{-ik(x-x')} G_0^{(AB)}(k;\alpha), \end{aligned} \tag{11}$$

where

$$G_0^{(AB)}(k;\alpha) = \mathcal{U}^{-1}(\alpha)G_0^{(AB)}(k)\mathcal{U}(\alpha), \tag{12}$$

with

$$G_0^{(AB)}(k) = \begin{pmatrix} G_0(k) & 0 \\ 0 & \xi G_0^*(k) \end{pmatrix}, \tag{13}$$

and

$$G_0(k) = \frac{1}{k^2 - m^2 + i\epsilon}, \tag{14}$$

where m is the scalar field mass.

The physical quantities are given by the nontilde variables. Then the physical Green function is

$$G_0^{(11)}(k;\alpha) = G_0(k) + \xi w^2(k;\alpha)[G_0^*(k) - G_0(k)], \tag{15}$$

where $w^2(k;\alpha)$ is the generalized Bogoliubov transformation [23] which is given as

$$w^2(k;\alpha) = \sum_{s=1}^{d} \sum_{\{\sigma_s\}} 2^{s-1} \sum_{l_{\sigma_1},\ldots,l_{\sigma_s}=1}^{\infty} (-\eta)^{s+\sum_{r=1}^{s} l_{\sigma_r}} \times \exp\left[-\sum_{j=1}^{s} \alpha_{\sigma_j} l_{\sigma_j} k^{\sigma_j}\right], \tag{16}$$

with d being the number of compactified dimensions, $\eta = 1(-1)$ for fermions (bosons), $\{\sigma_s\}$ denotes the set of all combinations with s elements, and k is the 5-momentum.

In this paper three different topologies are used:

(i) The topology $\Gamma_5^1 = \mathbb{S}^1 \times \mathbb{R}^4$, where $\alpha = (\beta, 0, 0, 0, 0)$. In this case the time axis is compactified in $\mathbb{S}^1$, with circumference β.

(ii) The topology Γ_5^1 with $\alpha = (0, 0, 0, i2d, 0)$, where the compactification along the coordinate z is considered.

(iii) The topology $\Gamma_5^2 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^3$ with $\alpha = (\beta, 0, 0, i2d, 0)$ is used. In this case the double compactification consists in one being the time and the other being along the coordinate z.

IV. SCALAR FIELD INTERACTING WITH AETHER FIELD

The Lagrangian for the massless scalar field interacting with the aether field is

$$\mathcal{L}_\phi = -\frac{1}{2}\partial_a\phi\partial^a\phi - \frac{1}{2\mu_\phi^2} u^a u^b \partial_a\phi\partial_b\phi, \tag{17}$$

where u^a is a spacelike 5-vector which has the form $u^a = (0, 0, 0, 0, v)$; the mass scale, μ_ϕ, is added for dimensional consistency; and the latin indices $a, b, c, \ldots$ run from 0 to 4.

To avoid divergences, the energy-momentum tensor,

$$T^{cd} = \frac{\partial\mathcal{L}_\phi}{\partial(\partial_c\phi)}\partial^d\phi - g^{cd}\mathcal{L}_\phi, \tag{18}$$

is written at different space-time points as

$$\begin{aligned} T^{cd}(x) = \lim_{x\to x'} \tau\Big\{ & -\partial^c\phi(x)\partial^d\phi(x') + \frac{1}{2} g^{cd}\partial_a\phi(x)\partial^a\phi(x') \\ & - \frac{1}{\mu_\phi^2}\Big[u^a u^c \partial_a\phi(x)\partial^d\phi(x') \\ & - \frac{1}{2} g^{cd} u^a u^b \partial_a\phi(x)\partial_b\phi(x')\Big]\Big\}, \end{aligned} \tag{19}$$

where the first line is the Lorentz invariant part and the second line is the Lorentz-violating (aether field) part. Using the commutation relation,

$$[\phi(x), \partial'^a\phi(x')] = i n_0^a \delta(\vec{x} - \vec{x'}), \tag{20}$$

where $n_0^a = (1, 0, 0, 0, 0)$ is a timelike vector and $\partial^a\theta(x_0 - x_0') = n_0^a\delta(x_0 - x_0')$. Then the energy-momentum tensor becomes

$$T^{cd}(x) = \lim_{x\to x'}\{\Gamma^{cd}(x,x')\tau[\phi(x)\phi(x')] + I^{cd}(x,x')\delta(x-x')\}, \tag{21}$$

with

$$\Gamma^{cd}(x,x') = -\partial^c\partial'^d + \frac{1}{2}g^{cd}\partial_a\partial'^a - \frac{1}{\mu_\phi^2}\left(u^a u^c\partial_a\partial'^d - \frac{1}{2}g^{cd}u^a u^b\partial_a\partial'_b\right), \tag{22}$$

$$I^{cd}(x,x') = -in_0^c n_0^d + \frac{1}{2}ig^{cd} - \frac{i}{\mu_\phi^2}\left(u^a u^c n_{0a} n_0^d - \frac{1}{2}g^{cd}u^a u^b n_{0a} n_{0b}\right). \tag{23}$$

The vacuum expectation value of the energy-momentum tensor is

$$\langle T^{cd}(x)\rangle = \lim_{x\to x'}\{i\Gamma^{cd}(x,x')G_0(x-x') + I^{cd}(x,x')\delta(x-x')\}, \tag{24}$$

where the propagator is

$$iG_0(x-x') = \langle 0|\tau[\phi(x)\phi(x')]|0\rangle. \tag{25}$$

Using the doublet notation, the physical energy-momentum tensor in terms of the α-dependent field is

$$\mathcal{T}^{cd(AB)}(x;\alpha) = i\lim_{x\to x'}\{\Gamma^{cd}(x,x')\bar{G}_0^{(AB)}(x-x';\alpha)\}, \tag{26}$$

where $\mathcal{T}^{cd(AB)}(x;\alpha) = \langle T^{cd(AB)}(x;\alpha)\rangle - \langle T^{cd(AB)}(x)\rangle$ and

$$\bar{G}_0^{(AB)}(x-x';\alpha) = G_0^{(AB)}(x-x';\alpha) - G_0^{(AB)}(x-x'). \quad (27)$$

A. Applications

1. Stefan-Boltzmann law: $\alpha = (\beta, 0, 0, 0, 0)$

The generalized Bogoliubov transformation, Eq. (16) for $d = 1$ and $s = 1$, is

$$w^2(\beta) = \sum_{l_0=1}^{\infty} e^{-\beta k^0 l_0}, \quad (28)$$

and the Green function becomes

$$\bar{G}_0(x-x';\beta) = 2\sum_{l_0=1}^{\infty} G_0(x-x'-i\beta l_0 n_0), \quad (29)$$

where $n_0 = (1,0,0,0,0)$. The energy-momentum tensor is given by

$$\mathcal{T}^{cd(11)}(x;\beta) = 2i\lim_{x\to x'}\left\{\Gamma^{cd}(x,x')\sum_{l_0=1}^{\infty} G_0(x-x'-i\beta l_0 n_0)\right\}. \quad (30)$$

Using $u^a = (0,0,0,0,v)$ its component $c = d = 0$, the Stefan-Boltzmann law, becomes

$$\begin{aligned}\mathcal{T}^{00(11)}(x;\beta) &= i\lim_{x\to x'}\left\{\sum_{l_0=1}^{\infty}\left[-\partial^0\partial'^0 + \frac{1}{2}g^{00}\partial_a\partial'^a \right.\right.\\ &\quad \left. - \frac{1}{\mu_\phi^2}\left(u^a u^0 \partial_a\partial'^0 - \frac{1}{2}g^{00}u^a u^b \partial_a \partial'_b\right)\right]\\ &\quad \left. \times G_0(x-x'-i\beta l_0 n_0)\right\}\\ &= \frac{\pi^2}{30}T^4\left[1 + \frac{1}{6}\left(1 - \frac{v^2}{\mu_\phi^2}\right)\right]. \end{aligned} \quad (31)$$

Therefore the aether field in the fifth dimension modifies the usual Stefan-Boltzmann law. Even if $v = 0$ the fifth dimension contributes $\sim 16\%$ to the law.

2. Casimir effect at zero temperature: $\alpha = (0, 0, 0, i2d, 0)$

The distance between two plates is d. Then

$$w^2(d) = \sum_{l_3=1}^{\infty} e^{-i2dk^3 l_3} \quad (32)$$

is the Bogoliubov transformation, Eq. (16) for $d = 1$ and $s = 1$, and

$$\bar{G}_0(x-x';d) = 2\sum_{l_3=1}^{\infty} G_0(x-x'-2dl_3 n_3) \quad (33)$$

is the Green function with $n_3 = (0,0,0,1,0)$. Then the Casimir energy and Casimir pressure, respectively, are given as

$$\mathcal{T}^{00(11)}(d) = -\frac{\pi^2}{1440d^4}\left[1 - \frac{1}{2}\left(1 - \frac{v^2}{\mu_\phi^2}\right)\right], \quad (34)$$

$$\mathcal{T}^{33(11)}(d) = -\frac{\pi^2}{480d^4}\left[1 - \frac{1}{6}\left(1 - \frac{v^2}{\mu_\phi^2}\right)\right]. \quad (35)$$

Both results are modified by the aether field. The Casimir energy and pressure are changed by 50% and 16%, respectively, with $v = 0$.

3. Casimir effect at finite temperature: $\alpha = (\beta, 0, 0, i2d, 0)$

The Casimir effect at finite temperature with the spatial compactification is calculated. The Bogoliubov transformation, Eq. (16) for $d = 2$ and $s = 2$, is

$$\begin{aligned} w^2(\beta, d) &= \sum_{l_0=1}^{\infty} e^{-\beta k^0 l_0} + \sum_{l_3=1}^{\infty} e^{-i2dk^3 l_3}\\ &\quad + 2\sum_{l_0,l_3=1}^{\infty} e^{-\beta k^0 l_0 - i2dk^3 l_3}. \end{aligned} \quad (36)$$

The first two terms are associated with the Stefan-Boltzmann law and the Casimir effect at zero temperature. The Green function for the third term is

$$\bar{G}_0(x-x';\beta,d) = 4\sum_{l_0,l_3=1}^{\infty} G_0(x-x'-i\beta l_0 n_0 - 2dl_3 n_3). \quad (37)$$

Then Casimir energy at finite temperature is

$$\mathcal{T}^{00(11)}(\beta,d) = \frac{\pi^2}{30}T^4\left[1+\frac{1}{6}\left(1-\frac{v^2}{\mu_\phi^2}\right)\right] - \frac{\pi^2}{1440d^4}\left[1-\frac{1}{2}\left(1-\frac{v^2}{\mu_\phi^2}\right)\right]$$
$$-\frac{2}{\pi^2}\sum_{l_0,l_3=1}^{\infty}\frac{(2dl_3)^2-3(\beta l_0)^2}{[(\beta l_0)^2+(2dl_3)^2]^3}\left[1-\frac{1}{2}\frac{(2dl_3)^2+(\beta l_0)^2}{[(2dl_3)^2-3(\beta l_0)^2]}\left(1-\frac{v^2}{\mu_\phi^2}\right)\right], \tag{38}$$

and the Casimir pressure at finite temperature is

$$\mathcal{T}^{33(11)}(\beta,d) = \frac{\pi^2}{90\beta^4}\left[1-\frac{1}{2}\left(1-\frac{v^2}{\mu_\phi^2}\right)\right] - \frac{\pi^2}{480d^4}\left[1-\frac{1}{6}\left(1-\frac{v^2}{\mu_\phi^2}\right)\right]$$
$$-\frac{2}{\pi^2}\sum_{l_0,l_3=1}^{\infty}\frac{3(2dl_3)^2-(\beta l_0)^2}{[(\beta l_0)^2+(2dl_3)^2]^3}\left[1+\frac{1}{2}\frac{(2dl_3)^2+(\beta l_0)^2}{[3(2dl_3)^2-(\beta l_0)^2]}\left(1-\frac{v^2}{\mu_\phi^2}\right)\right]. \tag{39}$$

These results display modifications due to the aether field and the extra dimension. If $v=0$, changes persist in the presence of the fifth dimension.

V. ELECTROMAGNETIC FIELD INTERACTING WITH AETHER FIELD

The Lagrangian for the electromagnetic field with the lowest-order coupling in u^a is

$$\mathcal{L}_A = \frac{1}{4}F_{ab}F^{ab} - \frac{1}{2\mu_A^2}u^a u^b g^{cd}F_{ac}F_{bd}. \tag{40}$$

The energy-momentum tensor becomes

$$\mathbf{t}^{cd} = -F^c{}_b\partial^d A^b + \frac{1}{4}g^{cd}F_{ab}F^{ab} - \frac{1}{\mu_A^2}(u^a u^c F_{ab} - u^a u_b F_a{}^c)\partial^d A^b + \frac{1}{2\mu_A^2}g^{cd}u^a u^b g^{lm}F_{al}F_{bm}, \tag{41}$$

which is

$$\mathbf{t}^{cd} = \mathbf{t}^{cd}_{\mathrm{EM}} + \mathbf{t}^{cd}_{\mathrm{LV}}, \tag{42}$$

where $\mathbf{t}^{cd}_{\mathrm{EM}}$ and $\mathbf{t}^{cd}_{\mathrm{LV}}$ are Lorentz invariant and Lorentz-violating (aether field) parts, respectively. The Belinfante method [24] is used to define the Lorentz invariant part. Then it leads to the symmetric energy-momentum tensor

$$T^{cd}_{\mathrm{EM}} = -F^c{}_b F^{db} + \frac{1}{4}g^{cd}F_{ab}F^{ab}. \tag{43}$$

The same method is not applicable for the Lorentz-violating part. Then the total energy-momentum tensor becomes

$$T^{cd} = -F^c{}_b F^{db} + \frac{1}{4}g^{cd}F_{ab}F^{ab} + \frac{1}{\mu_A^2}\left[\frac{1}{2}g^{cd}u^a u^b g^{ml}F_{am}F_{bl} - u^a u^c F_{ab}\partial^d A^b + u^a u_b F_a{}^c\partial^d A^b\right]. \tag{44}$$

This tensor is not completely symmetric. This is a feature of theories which exhibit Lorentz violation. To avoid divergences, the energy-momentum tensor at different space-time points is

$$T^{cd}(x) = T^{cd}_{\mathrm{EM}}(x) + T^{cd}_{\mathrm{LV}}(x), \tag{45}$$

where

$$T^{cd}_{\mathrm{EM}}(x) = \lim_{x\to x'}\tau\left[-F^{cb}(x)F^d{}_b(x') + \frac{1}{4}g^{cd}F_{ab}(x)F^{ab}(x')\right] \tag{46}$$

is the usual Lorentz invariant electromagnetic part in five dimensions and

$$T^{cd}_{\mathrm{LV}}(x) = \frac{1}{\mu_A^2}\lim_{x\to x'}\tau\left[\frac{1}{2}g^{cd}u^a u^b g^{ml}F_{am}(x)F_{bl}(x') - u^a u^c F_{ab}(x)\partial^d A^b(x') + u^a u_b F_a{}^c(x)\partial^d A^b(x')\right], \tag{47}$$

is the Lorentz-violating contribution due to the aether field.

Hereafter we choose the Coulomb gauge. The non-null commutation relations are defined by

$$[A_i(x),\pi_j(x')] = i\left[\delta_{ij} - \frac{1}{\nabla^2}\partial_i\partial_j\right]\delta(\vec{x}-\vec{x}'), \tag{48}$$

where $\pi_j(x) = \partial_0 A_j(x)$ is the momentum conjugate to A_j. Then the Lorentz invariant electromagnetic part becomes

$$T_{\mathrm{EM}}^{cd}(x) = \lim_{x\to x'}\left\{-\Delta^{cd,ij}(x,x')\tau[A_i(x)A_j(x')] + 2i\left(n_0^c n_0^d - \frac{1}{4}g^{cd}\right)\delta(x-x')\right\}, \tag{49}$$

with

$$\Delta^{cd,ij}(x,x') = \Gamma^{cm,d}{}_{m,}{}^{ij}(x,x') - \frac{1}{4}g^{cd}\Gamma^{ml,}{}_{ml,}{}^{ij}(x,x') \tag{50}$$

and

$$\Gamma^{cm,dl,ij}(x,x') = (g^{mi}\partial^c - g^{ci}\partial^m)(g^{lj}\partial'^d - g^{dj}\partial'^l). \tag{51}$$

The energy-momentum tensor associated with the Lorentz-violating part is

$$\begin{aligned} T_{\mathrm{LV}}^{cd}(x) = \frac{1}{\mu_A^2}\lim_{x\to x'}\Bigg\{&\frac{1}{2}g^{cd}u^a u^b g^{ml}[\Gamma_{am,bl,}{}^{ij}(x,x')\tau[A_i(x)A_j(x')] + I_{m,bl}(x,x')n_{0a}\delta(x-x') - I_{a,bl}(x,x')n_{0m}\delta(x-x')] \\ &- u^a u^c[\Gamma_{1a}{}^{d,ij}(x,x')\tau[A_i(x)A_j(x')] + I_{1a}{}^d(x,x')\delta(x-x')] \\ &+ u^a u_b[\Gamma_2{}^{cb,}{}_a{}^{d,ij}(x,x')\tau[A_i(x)A_j(x')] + I_2{}^d{}_{a,}{}^{cb}(x,x')\delta(x-x')]\Bigg\}, \end{aligned} \tag{52}$$

where

$$I_{m,bl}(x,x') = in_{0b}(g_{ml} - \nabla^{-2}\partial_m\partial_l)\delta(\vec{x}-\vec{x}') - in_{0l}(g_{mb} - \nabla^{-2}\partial_m\partial_b)\delta(\vec{x}-\vec{x}'), \tag{53}$$

$$\Gamma_{1a}{}^{d,ij}(x,x') = g^{ij}\partial_a\partial'^d - g_a^i\partial^j\partial'^d, \tag{54}$$

$$\Gamma_2{}^{cb,}{}_a{}^{d,ij}(x,x') = g^{ci}g^{bj}\partial_a\partial'^d - g_a^i g^{bj}\partial^c\partial'^d, \tag{55}$$

$$I_{1a}{}^d(x,x') = -in_0^d n_{0b}(\delta_a^b - \nabla^{-2}\partial_a\partial^b), \tag{56}$$

$$I_2{}^d{}_{a,}{}^{cb}(x,x') = in_0^d n_{0a}(\delta^{cb} - \nabla^{-2}\partial^c\partial^b) - in_0^d n_0^c(\delta_a^b - \nabla^{-2}\partial^b\partial_a). \tag{57}$$

Using the definition of the electromagnetic field propagator,

$$\langle 0|\tau[A_i(x)A_j(x')]|0\rangle = ig_{ij}G_0(x-x'), \tag{58}$$

with $G_0(x-x')$ being the massless scalar field propagator, the vacuum expectation value of the energy-momentum tensor is

$$\begin{aligned} \langle T^{cd}(x)\rangle = \lim_{x\to x'}\Bigg\{&-i\Gamma^{cd}(x,x')G_0(x-x') + 2i\left(n_0^c n_0^d - \frac{1}{4}g^{cd}\right)\delta(x-x') + \frac{1}{\mu_A^2}\Bigg[\frac{1}{2}g^{cd}u^a u^b g^{ml}(i\Gamma_{abml}(x,x')G_0(x-x') \\ &+ I_{m,bl}(x,x')n_{0a}\delta(x-x') - I_{a,bl}(x,x')n_{0m}\delta(x-x')) - u^a u^c[i\Gamma_{1a}{}^d(x,x')G_0(x-x') + I_{1a}{}^d(x,x')\delta(x-x')] \\ &+ u^a u_b[i\Gamma_2{}^{cb,}{}_a{}^d(x,x')G_0(x-x') + I_2{}^d{}_{a,}{}^{cb}(x,x')\delta(x-x')]\Bigg]\Bigg\}, \end{aligned} \tag{59}$$

where

$$\Gamma^{cd}(x,x') = 2\left(\partial^c\partial'^d - \frac{1}{4}g^{cd}\partial^b\partial'_b\right), \tag{60}$$

$$\Gamma_{ablm}(x,x') = g_{lm}\partial_a\partial'_b - g_{mb}\partial_a\partial'_l - g_{al}\partial_m\partial'_b + g_{ab}\partial_m\partial'_l, \tag{61}$$

$$\Gamma_{1a}{}^d(x,x') = 3\partial_a\partial'^d, \tag{62}$$

$$\Gamma_2{}^{cb,}{}_a{}^d(x,x') = g^{cb}\partial_a\partial'^d - g_a^b\partial^c\partial'^d. \tag{63}$$

The physical energy-momentum tensor is defined as

$$\mathcal{T}^{cd(AB)}(x;\alpha) = \langle T^{cd(AB)}(x;\alpha)\rangle - \langle T^{cd(AB)}(x)\rangle, \tag{64}$$

where the doublet notation is used to introduce the α-parameter. Then

$$\mathcal{T}^{cd(AB)}(x;\alpha) = -i\lim_{x\to x'}\left\{\Gamma^{cd}(x,x') - \frac{1}{\mu_A^2}\left[\frac{1}{2}g^{cd}u^a u^b g^{lm}\Gamma_{ablm}(x,x') - u^a u^c \Gamma_{1a}{}^{d}(x,x') + u^a u_b \Gamma_2{}^{cb,}{}_{a}{}^{d}(x,x')\right]\right\} \times \bar{G}_0^{(AB)}(x-x';\alpha), \tag{65}$$

where $\bar{G}_0^{(AB)}(x-x';\alpha)$ is given in Eq. (27).

Using Eq. (65) for different α-parameters, the Stefan-Boltzmann law and the Casimir effect at zero and finite temperature are calculated.

A. Stefan-Boltzmann law: $\boldsymbol{\alpha = (\beta, 0, 0, 0, 0)}$

The energy-momentum tensor is written as

$$\mathcal{T}^{cd(11)}(\beta) = -2i\lim_{x\to x'}\sum_{l_0=1}^{\infty}\left\{\Gamma^{cd}(x,x') - \frac{1}{\mu_A^2}\left[\frac{1}{2}g^{cd}u^a u^b g^{lm}\Gamma_{ablm}(x,x') - u^a u^c \Gamma_{1a}{}^{d}(x,x') + u^a u_b \Gamma_2{}^{cb,}{}_{a}{}^{d}(x,x')\right]\right\} \times G_0(x-x'-i\beta l_0 n_0), \tag{66}$$

where Eqs. (28) and (29) have been used. The component $c = d = 0$ with $u^a = (0,0,0,0,v)$ becomes

$$\mathcal{T}^{00(11)}(\beta) = \frac{\pi^2}{15}T^4\left[1 + \frac{1}{12}\left(1 - \frac{5v^2}{\mu_A^2}\right)\right]. \tag{67}$$

This is the Stefan-Boltzmann law modified by the aether field. If the aether field is zero, i.e., $v = 0$, the modification due to the fifth dimension is about 8%.

B. Casimir effect at zero temperature: $\boldsymbol{\alpha = (0, 0, 0, i2d, 0)}$

Using Eqs. (32) and (33), the energy-momentum tensor is

$$\mathcal{T}^{cd(11)}(d) = -2i\lim_{x\to x'}\sum_{l_3=1}^{\infty}\left\{\Gamma^{cd}(x,x') - \frac{1}{\mu_A^2}\left[\frac{1}{2}g^{cd}u^a u^b g^{lm}\Gamma_{ablm}(x,x') - u^a u^c \Gamma_{1a}{}^{d}(x,x') + u^a u_b \Gamma_2{}^{cb,}{}_{a}{}^{d}(x,x')\right]\right\} \times G_0(x-x'-2dl_3 n_3). \tag{68}$$

Then the Casimir energy, $\mathcal{T}^{00(11)}(d)$, and Casimir pressure, $\mathcal{T}^{33(11)}(d)$, are

$$\mathcal{T}^{00(11)}(d) = -\frac{\pi^2}{720d^4}\left[1 - \frac{1}{4}\left(1 - \frac{v^2}{\mu_A^2}\right)\right], \tag{69}$$

$$\mathcal{T}^{33(11)}(d) = -\frac{\pi^2}{240d^4}\left[1 + \frac{1}{12}\left(1 - \frac{9v^2}{\mu_A^2}\right)\right]. \tag{70}$$

The Casimir energy and Casimir pressure in four dimensions are obtained from this result when the aether field, v, vanishes. The extra dimension still contributes about 25% and 8% to energy and pressure, respectively.

C. Casimir effect at finite temperature: $\boldsymbol{\alpha = (\beta, 0, 0, i2d, 0)}$

The energy-momentum tensor associated with the third term of Eq. (36) is

$$\mathcal{T}^{cd(11)}(\beta,d) = -4i\lim_{x\to x'}\sum_{l_0,l_3=1}^{\infty}\left\{\Gamma^{cd}(x,x') - \frac{1}{\mu_A^2}\left[\frac{1}{2}g^{cd}u^a u^b g^{lm}\Gamma_{ablm}(x,x') - u^a u^c \Gamma_{1a}{}^{d}(x,x') + u^a u_b \Gamma_2{}^{cb,}{}_{a}{}^{d}(x,x')\right]\right\}G_0(x-x'-i\beta l_0 n_0 - 2dl_3 n_3), \tag{71}$$

where the Green function is given by Eq. (37). Then the Casimir energy, $\mathcal{T}^{00(11)}(\beta,d)$, at finite temperature is

$$\mathcal{T}^{00(11)}(\beta,d) = -\frac{4}{\pi^2}\sum_{l_0,l_3=1}^{\infty}\frac{(2dl_3)^2 - 3(\beta l_0)^2}{[(2dl_3)^2 + (\beta l_0)^2]^3}\left\{1 - \frac{(2dl_3)^2 + (\beta l_0)^2 - [(2dl_3)^2 + 9(\beta l_0)^2]\frac{v^2}{\mu_A^2}}{4[(2dl_3)^2 - 3(\beta l_0)^2]}\right\}. \tag{72}$$

The total Casimir energy, $E(\beta,d)$, at finite temperature, involving all terms in Eq. (36), is

$$E(\beta,d)=\frac{\pi^2}{15}T^4\left[1+\frac{1}{12}\left(1-\frac{5v^2}{\mu_A^2}\right)\right]-\frac{\pi^2}{720d^4}\left[1-\frac{1}{4}\left(1-\frac{v^2}{\mu_A^2}\right)\right]$$
$$-\frac{4}{\pi^2}\sum_{l_0,l_3=1}^{\infty}\frac{(2dl_3)^2-3(\beta l_0)^2}{[(2dl_3)^2+(\beta l_0)^2]^3}\left\{1-\frac{(2dl_3)^2+(\beta l_0)^2-[(2dl_3)^2+9(\beta l_0)^2]\frac{v^2}{\mu_A^2}}{4[(2dl_3)^2-3(\beta l_0)^2]}\right\}. \tag{73}$$

The Casimir pressure, $\mathcal{T}^{33(11)}(\beta,d)$, at finite temperature is

$$\mathcal{T}^{33(11)}(\beta,d)=-\frac{4}{\pi^2}\sum_{l_0,l_3=1}^{\infty}\frac{3(2dl_3)^2-(\beta l_0)^2}{[(2dl_3)^2+(\beta l_0)^2]^3}\left\{1+\frac{(2dl_3)^2+(\beta l_0)^2-[(\beta l_0)^2+9(2dl_3)^2]\frac{v^2}{\mu_A^2}}{4[3(2dl_3)^2-(\beta l_0)^2]}\right\}, \tag{74}$$

and the total Casimir pressure, $P(\beta,d)$, at finite temperature is

$$P(\beta,d)=\frac{\pi^2}{45\beta^4}\left[1-\frac{1}{4}\left(1-\frac{v^2}{\mu_A^2}\right)\right]-\frac{\pi^2}{240d^4}\left[1+\frac{1}{12}\left(1-\frac{9v^2}{\mu_A^2}\right)\right]$$
$$-\frac{4}{\pi^2}\sum_{l_0,l_3=1}^{\infty}\frac{3(2dl_3)^2-(\beta l_0)^2}{[(2dl_3)^2+(\beta l_0)^2]^3}\left\{1+\frac{(2dl_3)^2+(\beta l_0)^2-[(\beta l_0)^2+9(2dl_3)^2]\frac{v^2}{\mu_A^2}}{4[3(2dl_3)^2-(\beta l_0)^2]}\right\}. \tag{75}$$

The positive Stefan-Boltzmann law contributions in Eqs. (73) and (75) dominate in the high-temperature limit. The Casimir energy and the Casimir pressure are negative for low temperatures, since the Lorentz-violating contributions are small. The extra dimension contributes to the Casimir energy and pressure.

VI. FERMIONS FIELD INTERACTING WITH AETHER FIELD

The Lagrangian for the fermion-aether interaction is

$$\mathcal{L}_\psi=i\bar\psi\gamma^a\partial_a\psi-m\bar\psi\psi-\frac{i}{\mu_\psi^2}u^au^b\bar\psi\gamma_a\partial_b\psi, \tag{76}$$

which leads to the field equation

$$i\gamma^a\partial_a\psi-m\psi-\frac{i}{\mu_\psi^2}u^au^b\gamma_a\partial_b\psi=0. \tag{77}$$

The energy-momentum tensor is

$$T^{cd}=i\bar\psi\gamma^c\partial^d\psi-\frac{i}{\mu_\psi^2}u^au^c\bar\psi\gamma_a\partial^d\psi, \tag{78}$$

where Eq. (77) has been used. The first term is the Lorentz invariant part, and the second term is the Lorentz-violating (aether field) part. This result is rewritten at different points of space-time as

$$T^{cd}(x)=\lim_{x\to x'}\left[i\gamma^c\partial^d\tau[\bar\psi(x)\psi(x')]-\frac{i}{\mu_\psi^2}u^au^c\gamma_a\partial^d\tau[\bar\psi(x)\psi(x')]\right]. \tag{79}$$

To avoid divergences the point-splitting technique is used.

The vacuum expectation value of the energy-momentum tensor is

$$\langle T^{cd}(x)\rangle=-5i\lim_{x\to x'}\left[\left(\partial^c\partial^d-\frac{1}{\mu_\psi^2}u_au^c\partial^d\partial^a\right)G_0(x-x')\right], \tag{80}$$

where the fermion propagator definition

$$\langle 0|\tau[\bar\psi(x)\psi(x')]|0\rangle=iS(x-x'), \tag{81}$$

with $S(x-x')=(i\gamma\cdot\partial+m)G_0(x-x')$. For a massless fermion, $m=0$, is used.

Using the TFD formalism, the physical energy-momentum tensor is

$$\mathcal{T}^{cd(AB)}(x)=-5i\lim_{x\to x'}\{\Gamma^{cd}[G_{0F}^{(AB)}(x-x';\alpha)-G_{0F}^{(AB)}(x-x')]\}, \tag{82}$$

where

$$\Gamma^{cd} \equiv \partial^c \partial^d - \frac{1}{\mu_\psi^2} u_a u^c \partial^d \partial^a \tag{83}$$

and $G_{0F}^{(AB)}(x-x')$ is the Green function.

Applications of the energy-momentum tensor for different α-parameters will be used.

A. Stefan-Boltzmanna law: $\alpha = (\beta, 0, 0, 0, 0)$

The Bogoliubov transformation, Eq. (16) for $d = 1$ and $s = 1$, is

$$w^2(\beta) = \sum_{l_0=1}^{\infty} (-1)^{1+l_0} e^{-\beta k^0 l_0}. \tag{84}$$

This transformation leads to the thermal Green function,

$$\begin{aligned} G_{0F}^{(11)}(x-x';\beta) &= G_0(x-x') \\ &\quad + \sum_{l_0=1}^{\infty} (-1)^{1+l_0} [G_0^*(x'-x+i\beta l_0 n_0) \\ &\quad - G_0(x-x'-i\beta l_0 n_0)]. \end{aligned} \tag{85}$$

Then the energy-momentum tensor becomes

$$\begin{aligned} \mathcal{T}^{cd(11)}(\beta) &= 5i \lim_{x\to x'} \sum_{l_0=1}^{\infty} (-1)^{l_0} \{\Gamma^{cd}[G_0^*(x'-x+i\beta l_0 n_0) \\ &\quad - G_0(x-x'-i\beta l_0 n_0]\}. \end{aligned} \tag{86}$$

The component $c = d = 0$ is

$$\mathcal{T}^{00(11)}(\beta) = \frac{7\pi^2}{60} T^4 \left(1 + \frac{1}{4}\right). \tag{87}$$

This result is the modified Stefan-Boltzmann law. This modification is due to the fifth dimension. The aether field, v, does not contribute to this result. The extra dimension still contributes 25%.

B. Casimir effect at zero temperature: $\alpha = (0, 0, 0, i2d, 0)$

For parallel plates perpendicular to the z-direction and separated by a distance d, the Bogoliubov transformation, Eq. (16) for $d = 1$ and $s = 1$, is

$$w^2(d) = \sum_{l_3=1}^{\infty} (-1)^{1+l_3} e^{-i2dk^3 l_3}. \tag{88}$$

In order to calculate the Casimir effect at zero temperature, the energy-momentum tensor is

$$\begin{aligned} \mathcal{T}^{cd(11)}(d) &= 5i \lim_{x\to x'} \sum_{l_3=1}^{\infty} (-1)^{l_3} \{\Gamma^{cd}[G_0^*(x'-x+2dl_3 n_3) \\ &\quad - G_0(x-x'-2dl_3 n_3]\}. \end{aligned} \tag{89}$$

Then the Casimir energy and Casimir pressure are, respectively,

$$\mathcal{T}^{00(11)}(d) = -\frac{7\pi^2}{2880 a^4} \left(1 + \frac{1}{4}\right), \tag{90}$$

$$\mathcal{T}^{33(11)}(d) = -\frac{7\pi^2}{960 a^4} \left(1 + \frac{1}{4}\right). \tag{91}$$

It is important to note that the aether field, v, does not contribute to this result. The extra dimension still contributes 25% to the result.

C. Casimir effect at finite temperature: $\alpha = (\beta, 0, 0, i2d, 0)$

The Casimir effect at finite temperature and with spatial compactification is calculated. The Bogoliubov transformation, Eq. (16) for $d = 2$ and $s = 2$, is

$$\begin{aligned} w^2(\beta, d) &= \sum_{l_0=1}^{\infty} (-1)^{1+l_0} e^{-\beta k^0 l_0} + \sum_{l_3=1}^{\infty} (-1)^{1+l_3} e^{-i2dk^3 l_3} \\ &\quad + 2 \sum_{l_0,l_3=1}^{\infty} (-1)^{l_0+l_3} e^{-\beta k^0 l_0 - i2dk^3 l_3}. \end{aligned} \tag{92}$$

The first two terms correspond to the Stefan-Boltzmann law and the Casimir effect at zero temperature. The Casimir energy and Casimir pressure at finite temperature are calculated using the third term. The energy-momentum tensor is

$$\begin{aligned} \mathcal{T}^{cd(11)}(d) &= 10i \lim_{x\to x'} \sum_{l_0,l_3=1}^{\infty} (-1)^{l_0+l_3} \\ &\quad \times \{\Gamma^{cd}[G_0^*(x'-x+i\beta l_0 n_0 + 2dl_3 n_3) \\ &\quad - G_0(x-x'-i\beta l_0 n_0 - 2dl_3 n_3]\}. \end{aligned} \tag{93}$$

Then the Casimir energy, $\mathcal{T}^{00(11)}(\beta, d)$, and Casimir pressure, $\mathcal{T}^{33(11)}(\beta, d)$, at finite temperature are

$$\begin{aligned} \mathcal{T}^{00(11)}(\beta, d) &= -\frac{8}{\pi^2} \sum_{l_0,l_3=1}^{\infty} (-1)^{l_0+l_3} \\ &\quad \times \frac{(2dl_3)^2 - 3(\beta l_0)^2}{[(2dl_3)^2 + (\beta l_0)^2]^3} \left(1 + \frac{1}{4}\right) \end{aligned} \tag{94}$$

and

$$\mathcal{T}^{33(11)}(\beta,d) = -\frac{8}{\pi^2}\sum_{l_0,l_3=1}^{\infty}(-1)^{l_0+l_3} \times \frac{3(2dl_3)^2-(\beta l_0)^2}{[(2dl_3)^2+(\beta l_0)^2]^3}\left(1+\frac{1}{4}\right). \quad (95)$$

The total Casimir energy, $E(\beta,d)$, at finite temperature is

$$E(\beta,d) = \frac{7\pi^2}{60}T^4\left(1+\frac{1}{4}\right) - \frac{7\pi^2}{2880a^4}\left(1+\frac{1}{4}\right) - \frac{8}{\pi^2}\sum_{l_0,l_3=1}^{\infty}(-1)^{l_0+l_3}\frac{(2dl_3)^2-3(\beta l_0)^2}{[(2dl_3)^2+(\beta l_0)^2]^3}\left(1+\frac{1}{4}\right), \quad (96)$$

and the total Casimir pressure, $P(\beta,d)$, at finite temperature is

$$P(\beta,d) = \frac{7\pi^2}{180}T^4\left(1+\frac{1}{4}\right) - \frac{7\pi^2}{960a^4}\left(1+\frac{1}{4}\right) - \frac{8}{\pi^2}\sum_{l_0,l_3=1}^{\infty}(-1)^{l_0+l_3}\frac{3(2dl_3)^2-(\beta l_0)^2}{[(2dl_3)^2+(\beta l_0)^2]^3}\left(1+\frac{1}{4}\right). \quad (97)$$

The modifications in Eqs. (87), (90), (91), (96), (97) due the aether field are different from similar results for the scalar and electromagnetic fields. Explicit modifications involving the v parameter do not exist. However the extra dimension contributes 25% to the Casimir energy and pressure.

VII. CONCLUSIONS

The Standard Model is invariant under Lorentz and CPT symmetries, and experimentally no breakdown of these symmetries has been observed. At Planck scale (infinite energies), breakdown of these symmetries is postulated. In order to preserve the Lorentz invariance in four dimensions, a spacelike aether field with nonvanishing components along the fifth dimension, $u^a = (0,0,0,0,v)$, has been introduced. The aether field interacts with any matter field. Interactions with scalar, electromagnetic and fermions fields are considered. The aether field modifies the Stefan-Boltzmann law and the Casimir effect at zero and finite temperature. The TFD formalism has been used to introduce temperature effects. Since the v-parameter is small, these modifications may be considered as corrections. The temperature effect may be used to enforce constraint on the v-parameter. The modifications for scalar and electromagnetic fields consist of two parts: (i) contribution due to the fifth dimension and (ii) contribution of the aether field, v. For the fermions field, the modifications are only due to the fifth dimension, and there is no contribution by the aether field. The extra dimension contributions persist; i.e., an estimate of the change due to the fifth dimension to the Stefan-Boltzmann law and the Casimir effect are given.

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