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# Soft symmetry improvement of two particle irreducible effective actions

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Two particle irreducible effective actions (2PIEAs) are valuable nonperturbative techniques in quantum field theory; however, finite truncations of them violate the Ward identities (WIs) of theories with spontaneously broken symmetries. The symmetry improvement (SI) method of Pilaftsis and Teresi attempts to overcome this by imposing the WIs as constraints on the solution; however, the method suffers from the nonexistence of solutions in linear response theory and in certain truncations in equilibrium. Motivated by this, we introduce a new method called *soft-symmetry improvement* (SSI) which relaxes the constraint. Violations of WIs are allowed but punished in a least-squares implementation of the symmetry improvement idea. A new parameter $\xi$ controls the strength of the constraint. The method interpolates between the unimproved ($\xi \to \infty$) and SI ($\xi \to 0$) cases, and the hope is that practically useful solutions can be found for finite $\xi$. We study the SSI 2PIEA for a scalar $\mathrm{O}(N)$ model in the Hartree-Fock approximation. We find that the method is IR sensitive; the system must be formulated in finite volume $V$ and temperature $T = \beta^{-1}$, and the $V\beta \to \infty$ limit must be taken carefully. Three distinct limits exist. Two are equivalent to the unimproved 2PIEA and SI 2PIEA respectively, and the third is a new limit where the WI is satisfied but the phase transition is strongly first order and solutions can fail to exist depending on $\xi$. Further, these limits are disconnected from each other; there is no smooth way to interpolate from one to another. These results suggest that any potential advantages of SSI methods, and indeed any application of (S)SI methods out of equilibrium, must occur in finite volume.

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## I. INTRODUCTION

There is growing interest in techniques for nonperturbative and nonequilibrium quantum field theories. Potential applications for new methods range from cold atoms to cosmology (see, e.g., Ref. [1]). Recent progress on topics such as the dynamics of nonequilibrium critical points and phase transitions has come from the development of $n$ particle irreducible effective action ($n$PIEA; $n = 1, 2, 3, \ldots$) methods. These methods have a long history. The 1PIEA was introduced by Goldstone *et al.* [2] and Jona-Lasinio [3]. The 2PIEA was introduced independently by several authors [4–6] and finally received its modern formulation by Cornwall *et al.* [7]. This method has seen widespread use in both condensed matter and fundamental physics (see, e.g., Refs. [1,8] for fairly recent reviews). De Dominicis and Martin [9] then realized that these were special cases of a general formalism for arbitrary $n$. This work was then extended by others [8,10–12], but the practical use of effective actions for $n \geq 3$ remains minimal, largely due to difficulties with the renormalization of physically interesting theories.

$n$PIEAs can be thought of as generalizations of mean field theory which (a) are elegant, (b) are general, (c) are in principle exact, and (d) have been promoted for their applicability to nonequilibrium situations (see, e.g., Ref. [1] and references therein for extensive discussion of all these points). Nonperturbative methods are essential in nonequilibrium QFT because secular terms (i.e., terms which grow without bound over time) in the time evolution equations invalidate perturbation theory. $n$PIEAs with $n > 1$ achieve the required nonperturbative resummation in a manifestly self-consistent way which can be derived from first principles. "In principle exact" here means that the $n$PIEA equations of motion are exactly equivalent to the original nonperturbative definition of the quantum field theory. The only necessary approximation is in the numerical solution of these equations. The resulting equations of motion are also useful in equilibrium because many-body effects are included self-consistently. "General" means that the methods are applicable in principle to any quantum field theory whatsoever (although in a theory with many fields or with large $n$, the resulting $n$PIEA could be very bulky). Finally, "elegant" here means that few conceptually new elements are needed in the formulation of $n$PIEAs in addition to the usual terms of textbook quantum field theory. The complication is mainly of a technical, not conceptual, nature. To our knowledge, no other techniques satisfy all of these criteria.

$n$PIEA methods work by recasting perturbation theory as a variational method. Instead of working with standard Feynman diagrams built from bare propagators and vertices, one works with a reduced set of Feynman diagrams built from the *exact* mean field $\varphi$, propagators $\Delta$, and vertex functions $V^{(3)}, V^{(4)}, \ldots, V^{(n)}$. These quantities are

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determined by solving equations of motion $\delta\Gamma^{(n)}/\delta\varphi = \delta\Gamma^{(n)}/\delta\Delta = 0$, etc. The $\Gamma^{(n)}$ functionals are themselves built from $\varphi$, $\Delta$, $V^{(3)}$, and so on. The $\Gamma^{(n)}$ and accompanying equations of motion are exactly equivalent to the original quantum field theory but are sensitive to physical effects which are invisible to perturbation theory. Furthermore, this ability to capture nonperturbative physics is competitive with or exceeds other standard resummation methods such as Borel-Padé summation, at least in a toy model where exact solutions are available as a benchmark [13].

An unfortunate practical difficulty faced by would-be users of $n$PIEAs is that, once truncated to finite order, solutions of the equations of motion derived from $\Gamma^{(n>1)}$ no longer obey the expected symmetry properties [i.e., Ward identities (WIs)] which are obeyed by the exact solution, even if the truncation is manifestly invariant. This occurs simply because there is no guarantee that the pattern of partial resummations encoded in an approximation to $\Gamma^{(n)}$ will respect the order by order cancellations required to fully maintain the WIs. The most obvious effect of this is that Goldstone bosons are unphysically massive and the symmetry breaking phase transition is incorrectly predicted in models of spontaneous symmetry breaking treated within the Hartree-Fock approximation (see, e.g., Ref. [14] and references therein). The use of higher order truncations can cure this problem, but more subtle symmetry violating effects still occur. Similar remarks apply for gauge theories, where an unphysical gauge dependence remains in quantities that should be physical.

Several methods have been advocated in the literature to combat this problem, though none is without flaws. For example, the widely used external propagator method [15] is not fully self-consistent: *after* the variational solution is found, "external" correlation functions are constructed which *do* satisfy the WIs. However, the incorrect variational solutions are still the ones used in the self-consistent step. As a result, more subtle problems such as violations of unitarity persist. Ivanov *et al.* [16] developed a gapless version of the 2PIEA in the Hartree-Fock approximation which restores the second order phase transition and Goldstone theorem but requires the addition of an *ad hoc* correction term. There is not, as far as we know, any first principles motivation for the scheme or any systematic way of extending it. Leupold [17] discusses the use of nonlinear representations, which restores the symmetry at the expense of requiring nonpolynomial Lagrangians. Pilaftsis and Teresi [14] introduced a promising method called symmetry improvement (SI), which imposes the WIs directly as constraints on the solution through Lagrange multipliers. SI has been applied with some success with the SI 2PIEA [14,18–21] and extended to the SI 3PIEA [22]; however, the method is inconsistent out of equilibrium (at least at the linear response level) [23], and sometimes solutions fail to exist due to the constraint causing a renormalization group defying coupling between short and long distance physics [24]. Considering that the symptom in both cases is the nonexistence of solutions, and that the constraint in the SI method is singular and requires some careful treatment to begin with, it is reasonable to suspect that the culprit may be that the method is overconstraining. This motivates the investigation of whether it is possible to generalize the SI method and at the same time allow the solutions more freedom. That is what this paper does.

We introduce a new method which we call *soft-symmetry improvement* (SSI) which relaxes the constraint. Violations of WIs are allowed but punished in the solution of the SSI $n$PIEA. The method is essentially a least-squares implementation of the symmetry improvement idea. A new parameter, the stiffness $\xi$, controls the strength of the constraint. The method interpolates between the unimproved ($\xi \to \infty$) and SI ($\xi \to 0$) cases, and the hope is that practically useful solutions can be found for finite $\xi$. We study the SSI 2PIEA for a scalar $\mathrm{O}(N)$ model in the Hartree-Fock approximation. We find that the method is IR sensitive; the system must be formulated in finite volume $V$ and temperature $T = \beta^{-1}$, and the $V\beta \to \infty$ limit must be taken carefully. Three distinct limits exist. Two are equivalent to the unimproved 2PIEA and SI 2PIEA respectively, and the third is a new limit where the WI is satisfied but the phase transition is strongly first order and solutions can fail to exist depending on $\xi$. Further, these limits are disconnected from each other; there is no smooth interpolation from one limit to another. These results suggest that any potential advantages of SSI methods [and any consideration of (S)SI out of equilibrium] *must occur in finite volume*.

The structure of this paper is as follows. Following this Introduction, Sec. II introduces the SSI formalism. Then, in Sec. III, the SSI 2PIEA is renormalized in the Hartree-Fock approximation at finite $V\beta$. Solutions are then found in Sec. IV with careful consideration of the various $V\beta \to \infty$ limits. Finally, we discuss our results in Sec. V. The notation agrees with our previous papers [22,23] except where noted. In particular, the deWitt summation convention is used; i.e., sums over repeated indices imply integrations over corresponding spacetime arguments.

## II. SOFT SYMMETRY IMPROVEMENT OF 2PIEA

The soft-symmetry improved 2PIEA is a modification of the 2PIEA defined for theories with an internal symmetry. In order to have a concrete example, we use the $\mathrm{O}(N)$ symmetric scalar $(\phi^2)^2$ theory discussed in our previous papers [22,23]. We will focus on the spontaneous symmetry breaking regime where the field has a nonzero expectation value $\varphi_a = \langle\phi_a\rangle = (0, \ldots, 0, v)$, a "Higgs" boson with mass $m_H$ and $N-1$ massless Goldstone bosons. The definition of the SSI 2PIEA can be motivated by starting with the standard 2PIEA $\Gamma[\varphi, \Delta]$ (suppressing indices and spacetime arguments where these just clutter) and the trivial identity

$$\exp\left(\frac{i}{\hbar}\Gamma[\varphi,\Delta]\right)=\int\mathcal{D}\phi\delta(\phi-\varphi)\exp\left(\frac{i}{\hbar}\Gamma[\phi,\Delta]\right). \quad (1)$$

The usual symmetry improved action $\Gamma^{\mathrm{SI}}[\varphi,\Delta]$ is then obtained by inserting a delta function,

$$\exp\left(\frac{i}{\hbar}\Gamma^{\mathrm{SI}}[\varphi,\Delta]\right)=N\int\mathcal{D}\phi\delta(\phi-\varphi)\times\exp\left(\frac{i}{\hbar}\Gamma[\phi,\Delta]\right)\delta(\mathcal{W}[\phi,\Delta]), \quad (2)$$

where the Ward identity is [22]

$$0=\mathcal{W}_a^A[\phi,\Delta]\equiv\Delta_{ab}^{-1}T_{bc}^A\phi_c \quad (3)$$

and the normalization factor $N$ is chosen so that $\Gamma^{\mathrm{SI}}[\varphi,\Delta]$ numerically equals $\Gamma[\varphi,\Delta]$ when the arguments satisfy the Ward identity [25]. $T_{bc}^A$ is a generator of the $\mathrm{O}(N)$ symmetry where $A=1,\ldots,N(N-1)/2$ runs over the linearly independent generators. When an explicit basis of generators is required, we take $T_{ab}^{jk}=i(\delta_{ja}\delta_{kb}-\delta_{jb}\delta_{ka})$ where $A=(j,k)$ is thought of as an (antisymmetric) multi-index. Note that the implicit integration convention can be maintained if $T_{ab}^A(x,y)\propto\delta(x-y)$ contains a spacetime delta function, though in this notation one must remember that the upper indices do not have corresponding spacetime arguments since they merely label the particular generator. $\Gamma^{\mathrm{SI}}[\varphi,\Delta]$ is defined only for field configurations satisfying the Ward identity and equals the usual effective action on those configurations. Thus, $\Gamma^{\mathrm{SI}}[\varphi,\Delta]$ is nothing but the SI 2PIEA introduced by Pilaftsis and Teresi [14], arrived at in a new way.

Proceeding from the hypothesis that the problems with symmetry improvement are due to the strict imposition of the constraint, as embodied by the delta function above, we introduce a SSI effective action $\Gamma_\xi^{\mathrm{SSI}}[\varphi,\Delta]$ where the Ward identity is no longer strictly enforced. Small violations $\mathcal{W}\neq 0$ are allowed but punished in the functional integral. A new free parameter controls how strictly the constraint is enforced. The hope is that the added freedom allows consistent solutions with nontrivial dynamics (e.g., linear response to external sources), while the stiffness can be tuned to make violations of the Ward identity acceptably small in practice. To achieve this, we replace the delta function by a smoothed version $\delta(\mathcal{W})\to\delta_\xi(\mathcal{W})$ defined as follows:

$$\begin{aligned}\exp\left(\frac{i}{\hbar}\Gamma_\xi^{\mathrm{SSI}}[\varphi,\Delta]\right)&=N_0\int\mathcal{D}\phi\delta(\phi-\varphi)\exp\left(\frac{i}{\hbar}\Gamma[\phi,\Delta]\right)\delta_\xi(\mathcal{W}[\phi,\Delta])\\&=N_1\int\mathcal{D}[\phi,\lambda_\phi,\lambda_W]\exp\left(\frac{i}{\hbar}\left[\lambda_\phi(\phi-\varphi)+\Gamma[\phi,\Delta]+\lambda_W\mathcal{W}-\frac{1}{2}\xi\lambda_W^2\right]\right)\\&=N_2\int\mathcal{D}[\phi,\lambda_\phi]\exp\left(\frac{i}{\hbar}\left[\lambda_\phi(\phi-\varphi)+\Gamma[\phi,\Delta]+\frac{1}{2\xi}\mathcal{W}^2\right]\right)\\&=\exp\left(\frac{i}{\hbar}\left[\Gamma[\varphi,\Delta]+\frac{1}{2\xi}\mathcal{W}^2[\varphi,\Delta]\right]\right). \quad (4)\end{aligned}$$

The first line is a formal expression that is defined by the next line. The Fourier representation of the delta functions is used to replace $\delta(\phi-\varphi)\to\int\mathcal{D}\lambda_\phi\exp\frac{i}{\hbar}\lambda_\phi(\phi-\varphi)$, etc. The $\frac{1}{2}\xi\lambda_W^2$ term is responsible for smoothing the delta function, with the limit $\xi\to 0$ corresponding to a stiffening of the constraint. In the third line, the integral over $\lambda_W$, which is Gaussian, is performed. Finally, the integral over $\lambda_\phi$ yields a delta function which kills the $\phi$ integral, resulting in

$$\Gamma_\xi^{\mathrm{SSI}}[\varphi,\Delta]=\Gamma[\varphi,\Delta]+\frac{1}{2\xi}\mathcal{W}^2[\varphi,\Delta]. \quad (5)$$

The method can be generalized by using a weighted smoothing term $-\frac{1}{2}\xi\lambda_W R^{-1}\lambda_W$, where $R^{-1}$ is an arbitrary positive definite symmetric kernel which may depend on $\varphi$ and $\Delta$, which gives

$$\Gamma_{\xi R}^{\mathrm{SSI}}[\varphi,\Delta]=\Gamma[\varphi,\Delta]+\frac{1}{2\xi}\mathcal{W}R\mathcal{W}-\frac{i\hbar}{2}\mathrm{Tr}\,\ln R. \quad (6)$$

The simpler form $\Gamma_\xi^{\mathrm{SSI}}[\varphi,\Delta]$ corresponds to a trivial kernel (now with indices explicit),

$$R_{ab}^{AB}(x,y)=\delta^{AB}\delta_{ab}\delta(x-y), \quad (7)$$

which is used exclusively in the following, though one should note that the freedom to choose a nontrivial $R$ may be useful in certain circumstances. The end result is simply that $\mathcal{W}=0$ is enforced in the sense of (possibly weighted if $R$ is nontrivial) least-squared error, rather than as a strict constraint.

We define the SSI equations of motion as the result of the variational principle $\delta\Gamma_\xi^{\mathrm{SSI}}=0$, which gives

$$\frac{\delta\Gamma[\varphi,\Delta]}{\delta\varphi_a} = -\frac{1}{\xi}\mathcal{W}_c^A[\varphi,\Delta]\frac{\delta}{\delta\varphi_a}\mathcal{W}_c^A[\varphi,\Delta] = -\frac{1}{\xi}(\Delta_{cf}^{-1}T_{fg}^A\varphi_g)\Delta_{cd}^{-1}T_{da}^A, \tag{8}$$

$$\frac{\delta\Gamma[\varphi,\Delta]}{\delta\Delta_{ab}} = -\frac{1}{\xi}\mathcal{W}_c^A[\varphi,\Delta]\frac{\delta}{\delta\Delta_{ab}}\mathcal{W}_c^A[\varphi,\Delta] = \frac{1}{\xi}(\Delta_{cf}^{-1}T_{fg}^A\varphi_g)\Delta_{ca}^{-1}(\Delta_{bd}^{-1}T_{de}^A\varphi_e). \tag{9}$$

Now, the spontaneous symmetry breaking (SSB) ansatz

$$\varphi_a = v\delta_{aN}, \tag{10}$$

$$\Delta_{ab}^{-1} = \begin{cases} \Delta_G^{-1} & a = b \neq N, \\ \Delta_H^{-1} & a = b = N, \\ 0 & a \neq b \end{cases} \tag{11}$$

can be used, where $\Delta_{G/H}$ are the Goldstone/Higgs propagators respectively. This ansatz yields

$$\frac{\delta\Gamma[\varphi,\Delta]}{\delta\varphi_g(x)} = 0, \quad (g \neq N), \tag{12}$$

$$\frac{\delta\Gamma[\varphi,\Delta]}{\delta\varphi_N(x)} = \frac{1}{\xi}2(N-1)v\int_{yz}\Delta_G^{-1}(y,z)\Delta_G^{-1}(y,x) = \frac{1}{\xi}2(N-1)vm_G^4, \tag{13}$$

$$\frac{\delta\Gamma[\varphi,\Delta]}{\delta\Delta_G(x,y)} = -\frac{1}{\xi}2v^2\int_{wrz}\Delta_G^{-1}(w,r)\Delta_G^{-1}(w,x)\Delta_G^{-1}(y,z) = \frac{1}{\xi}2v^2m_G^6, \tag{14}$$

$$\frac{\delta\Gamma[\varphi,\Delta]}{\delta\Delta_H} = 0, \tag{15}$$

where $m_G$ is the Goldstone mass.

Note that if one takes $\xi \to 0$ proportionally to $vm_G^4$, one obtains for the nontrivial right-hand sides above $2(N-1)vm_G^4/\xi \to \text{constant}$ and $2v^2m_G^6/\xi \to (\text{const}) \times vm_G^2 \to 0$, and one recovers the usual SI 2PIEA scheme in the limit. In Sec. IV D, this is shown to hold with a careful treatment of the infinite volume limit. This confirms the intuition that $\xi \to 0$ approaches hard symmetry improvement and that $\Gamma_\xi^{\rm SSI}[\varphi,\Delta] \to \Gamma^{\rm SI}[\varphi,\Delta]$, which really is just the standard symmetry improved effective action. In the next sections, these equations of motion are renormalized and solved in the Hartree-Fock approximation.

## III. RENORMALIZATION OF THE HARTREE-FOCK TRUNCATION

There is a well-established renormalization theory for 2PIEAs (see, e.g., Refs. [15,26–28]). Our renormalization method is not particularly novel (we closely follow Refs. [14,22]), but it is important to carefully treat the behavior of the theory in the infrared which does lead to some new aspects. Therefore, we formulate the theory in Euclidean spacetime (i.e., the Matsubara formalism) in a box of volume $V = L^3$ with periodic boundary conditions of period $L$ in the space directions and $\beta$ in the time $\tau = it$ direction. It turns out that the SSI method is sensitive to the manner of taking the $V\beta \to \infty$ limit. The Euclidean continuation leads to $x = (t,\boldsymbol{x}) \to x_E \equiv (\tau,\boldsymbol{x})$, $\int_x \to -i\int_{x_E}$ and the conventions

$$f(x_E) = \frac{1}{V\beta}\sum_{n,\boldsymbol{k}} \mathrm{e}^{i(\omega_n\tau+\boldsymbol{k}\cdot\boldsymbol{x})}f(n,\boldsymbol{k}), \tag{16}$$

$$f(n,\boldsymbol{k}) = \int_{x_E}\mathrm{e}^{-i(\omega_n\tau+\boldsymbol{k}\cdot\boldsymbol{x})}f(x_E) \tag{17}$$

for Fourier transforms. The Matsubara frequencies are $\omega_n = 2\pi n/\beta$, and the wave vectors $\boldsymbol{k}$ are discretized on a lattice of spacing $2\pi/L$. The four-dimensional Euclidean shorthand $k_E = (\omega_n,\boldsymbol{k})$ is often useful.

We will work in the Hartree-Fock approximation, which normally leads a momentum independent self-energy and propagators of the form $\Delta_{G/H}^{-1}(n,\boldsymbol{k}) = k_E^2 + m_{G/H}^2$. However, it turns out that the SSI term leads to a momentum dependent Goldstone self-energy. The equations of motion can be solved by treating the Goldstone zero mode propagator $\Delta_G^{-1}(0,\boldsymbol{0})$ as a dynamical variable separate from the nonzero modes. We *define* $m_G$ to be the mass associated with the *nonzero* Goldstone modes, i.e., $\Delta_G^{-1}(n,\boldsymbol{k}) = k_E^2 + m_G^2$ for $n, \boldsymbol{k} \neq 0$. The zero mode propagator gains an independent scaling factor $\Delta_G^{-1}(0,\boldsymbol{0}) \equiv \epsilon m_G^2$. Note that the Goldstone theorem can be satisfied if $\epsilon = 0$ even if $m_G^2 \neq 0$. This is the case for the novel $\beta V \to \infty$ limit of the theory. The other limits are a reduction to the unimproved 2PIEA where $\epsilon = 1$ and $m_G^2 \neq 0$ and a reduction to the SI 2PIEA where $m_G^2 = 0$ and $\epsilon \neq 0$.

The 2PIEA is derived from the partition functional

$$Z[J,K] = \int\mathcal{D}[\phi]\exp\left(-S_E[\phi] - J_a\phi_a - \frac{1}{2}\phi_aK_{ab}\phi_b\right), \tag{18}$$

where

$$S_E[\phi] = \int_x \frac{1}{2}(\nabla\phi_a)^2 + \frac{1}{2}m^2\phi_a\phi_a + \frac{1}{4!}\lambda(\phi_a\phi_a)^2 \tag{19}$$

is the Euclidean action. Then, $W[J,K] = -\ln Z[J,K]$ is the connected generating functional and

$$\Gamma[\varphi,\Delta] = W - J\frac{\delta W}{\delta J} - K\frac{\delta W}{\delta K} \tag{20}$$

is the 2PIEA once $J$ and $K$ are eliminated in terms of $\varphi$ and $\Delta$ using

$$\frac{\delta W}{\delta J_a} = \langle\phi_a\rangle = \varphi_a, \tag{21}$$

$$\frac{\delta W}{\delta K_{ab}} = \frac{1}{2}\langle\phi_a\phi_b\rangle = \frac{1}{2}(\Delta_{ab} + \varphi_a\varphi_b). \tag{22}$$

The Legendre transform (20) can be evaluated by the saddle point method, which results in the standard expression [7]

$$\Gamma = S_E[\varphi] + \frac{1}{2}\mathrm{Tr}\,\ln(\Delta^{-1}) + \frac{1}{2}\mathrm{Tr}(\Delta_0^{-1}\Delta - 1) + \Gamma_2, \tag{23}$$

where $\Gamma_2$ is the set of two particle irreducible graphs and

$$\Delta_{0ab}^{-1} \equiv \frac{\delta^2 S_E}{\delta\phi_a\delta\phi_b}\bigg|_{\phi\to\varphi} = \left(-\nabla^2 + m^2 + \frac{1}{6}\lambda\varphi^2\right)\delta_{ab} + \frac{1}{3}\lambda\varphi_a\varphi_b \tag{24}$$

is the unperturbed propagator. To $\mathcal{O}(\lambda)$,

$$\Gamma_2 = \frac{1}{4!}\lambda\Delta_{aa}\Delta_{bb} + \frac{1}{12}\lambda\Delta_{ab}\Delta_{ab}. \tag{25}$$

To form $\Gamma_\xi^{\mathrm{SSI}}$, one adds the soft-symmetry improvement term $-\frac{1}{2\xi}\mathcal{W}^2$. Note that $\mathcal{W}$ is pure imaginary due to the $i$ in $T^A$, so $-\mathcal{W}^2$ is positive definite. After using the SSB ansatz, the condition $\mathcal{W}_a^A = 0$ becomes Goldstone's theorem $v\Delta_G^{-1}(0,\mathbf{0}) = 0$. We drop an irrelevant constant, use the SSB ansatz, and insert the renormalization constants $Z$, $Z_\Delta$, $\delta m_{0,1}^2$, $\delta\lambda_0$, and $\delta\lambda_{1,2}^{A,B}$ by making the replacements (cf. Ref. [22])

$$(\phi,\varphi,v) \to Z^{1/2}(\phi,\varphi,v), \tag{26}$$

$$m^2 \to Z^{-1}Z_\Delta^{-1}(m^2 + \delta m^2), \tag{27}$$

$$\lambda \to Z^{-2}(\lambda + \delta\lambda), \tag{28}$$

$$\Delta \to ZZ_\Delta\Delta. \tag{29}$$

Due to the presence of composite operators in the effective action, additional renormalization constants are required compared to the standard perturbative renormalization theory: $\delta m_0^2$ and $\delta\lambda_0$ for terms in the bare action, $\delta m_1^2$ for one-loop terms, $\delta\lambda_1^A$ for terms of the form $\phi_a\phi_a\Delta_{bb}$, $\delta\lambda_1^B$ for $\phi_a\phi_b\Delta_{ab}$ terms, $\delta\lambda_2^A$ for $\Delta_{aa}\Delta_{bb}$, and $\delta\lambda_2^B$ for $\Delta_{ab}\Delta_{ab}$. The fact that extra counterterms are required to renormalize the 2PIEA is not a problem so long as a sufficient number of renormalization conditions can be found. Altogether, there are nine renormalization constants which must be eliminated by imposing nine conditions. It turns out that $Z = Z_\Delta = 1$ in the Hartree-Fock approximation due to the momentum independence of the UV divergences in this approximation (this is well known in the 2PIEA literature [26,27], but it may not be immediately obvious that it continues to hold with the addition of the SSI terms; indeed, it does). One can introduce a renormalization constant $Z_\xi$ for $\xi$, but this turns out to be unnecessary. Thus, we arrive at the renormalized SSI effective action

$$\begin{aligned}\Gamma_\xi^{\mathrm{SSI}}[\varphi,\Delta] = &\int_x\left(\frac{m^2+\delta m_0^2}{2}v^2 + \frac{\lambda+\delta\lambda_0}{4!}v^4\right) + \frac{1}{2}(N-1)\mathrm{Tr}\,\ln(\Delta_G^{-1}) + \frac{1}{2}\mathrm{Tr}\,\ln(\Delta_H^{-1})\\ &+\frac{1}{2}(N-1)\mathrm{Tr}\left[\left(-\nabla^2 + m^2 + \delta m_1^2 + \frac{\lambda+\delta\lambda_1^A}{6}v^2\right)\Delta_G\right]\\ &+\frac{1}{2}\mathrm{Tr}\left[\left(-\nabla^2 + m^2 + \delta m_1^2 + \frac{3\lambda+\delta\lambda_1^A+2\delta\lambda_1^B}{6}v^2\right)\Delta_H\right] + \Gamma_2 + \frac{1}{\xi}(N-1)v^2\int_{xyz}\Delta_G^{-1}(x,y)\Delta_G^{-1}(x,z)\end{aligned} \tag{30}$$

with

$$\begin{aligned}\Gamma_2 = &\frac{1}{4!}(N-1)[(N+1)\lambda + (N-1)\delta\lambda_2^A + 2\delta\lambda_2^B]\Delta_G\Delta_G\\ &+\frac{1}{4!}(\lambda+\delta\lambda_2^A)2(N-1)\Delta_G\Delta_H\\ &+\frac{1}{4!}(3\lambda+\delta\lambda_2^A+2\delta\lambda_2^B)\Delta_H\Delta_H + \mathcal{O}(\lambda^2).\end{aligned} \tag{31}$$

$\Gamma_\xi^{\mathrm{SSI}}[\varphi,\Delta]$ can be simplified using the mode expansions

$$\Delta_{G/H}(x_E,y_E) = \frac{1}{V\beta}\sum_{n,\boldsymbol{k}} e^{ik_E\cdot(x_E-y_E)}\Delta_{G/H}(n,\boldsymbol{k}) \tag{32}$$

and doing the integrals, noting that the integrals in the SSI term give

$$\int_{xyz} \Delta_G^{-1}(x,y)\Delta_G^{-1}(x,z) = V\beta[\Delta_G^{-1}(0,\mathbf{0})]^2. \quad (33)$$

The result is

$$\begin{aligned}\Gamma_\xi^{\mathrm{SSI}}[\varphi,\Delta] = & V\beta\left(\frac{m^2+\delta m_0^2}{2}v^2+\frac{\lambda+\delta\lambda_0}{4!}v^4\right)\\ & +\frac{1}{2}(N-1)\sum_{n,\boldsymbol{k}}\ln\frac{1}{\Delta_G(n,\boldsymbol{k})}+\frac{1}{2}\sum_{n,\boldsymbol{k}}\ln\frac{1}{\Delta_H(n,\boldsymbol{k})}\\ & +\frac{1}{2}(N-1)\sum_{n,\boldsymbol{k}}\left(k_E^2+m^2+\delta m_1^2\right.\\ & \left.+\frac{\lambda+\delta\lambda_1^A}{6}v^2\right)\Delta_G(n,\boldsymbol{k})+\frac{1}{2}\sum_{n,\boldsymbol{k}}\left(k_E^2+m^2+\delta m_1^2\right.\\ & \left.+\frac{3\lambda+\delta\lambda_1^A+2\delta\lambda_1^B}{6}v^2\right)\Delta_H(n,\boldsymbol{k})+\Gamma_2\\ & +\frac{1}{\xi}(N-1)v^2V\beta[\Delta_G^{-1}(0,\mathbf{0})]^2. \end{aligned} \quad (34)$$

As a brief digression, a simple consistency check can be performed by examining the tree level equations of motion, which are (setting renormalization constants to their trivial values)

$$0 = v\left\{\left(m^2+\frac{\lambda}{6}v^2\right)+\frac{2(N-1)}{\xi}[\Delta_G^{-1}(0,\mathbf{0})]^2\right\}, \quad (35)$$

$$\Delta_G^{-1}(n,\boldsymbol{k}) = k_E^2+m^2+\frac{\lambda}{6}v^2, \quad n,\boldsymbol{k}\neq 0, \quad (36)$$

$$\Delta_G^{-1}(0,\mathbf{0}) = m^2+\frac{\lambda}{6}v^2-\frac{4V\beta}{\xi}v^2[\Delta_G^{-1}(0,\mathbf{0})]^3, \quad (37)$$

$$\Delta_H^{-1}(n,\boldsymbol{k}) = k_E^2+m^2+\frac{\lambda}{2}v^2. \quad (38)$$

Indeed, the classical solutions $v^2 = -6m^2/\lambda$, $\Delta_G^{-1}(n,\boldsymbol{k}) = k_E^2$ and $\Delta_H^{-1}(n,\boldsymbol{k}) = k_E^2+m_H^2 = k_E^2+\frac{\lambda}{3}v^2$ are consistent with these as expected. However, since these equations are self-consistent, spurious solutions are also possible. This can be investigated by solving (35) and (37) together on the assumption that $v^2\neq 0, -6m^2/\lambda$. Using (35) to reduce the degree of (37) to first order gives the potentially spurious solution

$$-\frac{\xi}{2(N-1)} = \left(m^2+\frac{\lambda}{6}v^2\right)\Big/\left[1-\frac{2V\beta}{N-1}v^2\left(m^2+\frac{\lambda}{6}v^2\right)\right]^2, \quad (39)$$

$$\Delta_G^{-1}(0,\mathbf{0}) = \left(m^2+\frac{\lambda}{6}v^2\right)\Big/\left[1-\frac{2V\beta}{N-1}v^2\left(m^2+\frac{\lambda}{6}v^2\right)\right]. \quad (40)$$

The condition that there are no tachyons requires $\Delta_G^{-1}(0,\mathbf{0})\geq 0$ which implies

$$0\leq m^2+\frac{\lambda}{6}v^2<\frac{N-1}{2V\beta v^2}. \quad (41)$$

This then implies that the right-hand side of (39) is positive, but then the left-hand side $\propto -\xi$ is negative, leading to a contradiction. Thus, the only spurious solutions are tachyonic and so are easily dismissable.

Returning to the main line of the argument, the rest of the paper restricts attention to the Hartree-Fock truncation where only the $\mathcal{O}(\lambda)$ terms in $\Gamma_2$ are kept. Thus,

$$\begin{aligned}\Gamma_2 = & \frac{1}{4!}(N-1)[(N+1)\lambda+(N-1)\delta\lambda_2^A+2\delta\lambda_2^B]\\ & \times\frac{1}{V\beta}\sum_{n,\boldsymbol{k}}\Delta_G(n,\boldsymbol{k})\sum_{j,\mathbf{q}}\Delta_G(j,\mathbf{q})\\ & +\frac{1}{4!}(\lambda+\delta\lambda_2^A)2(N-1)\frac{1}{V\beta}\sum_{n,\boldsymbol{k}}\Delta_G(n,\boldsymbol{k})\sum_{j,\mathbf{q}}\Delta_H(j,\mathbf{q})\\ & +\frac{1}{4!}(3\lambda+\delta\lambda_2^A+2\delta\lambda_2^B)\frac{1}{V\beta}\sum_{n,\boldsymbol{k}}\Delta_H(n,\boldsymbol{k})\sum_{j,\mathbf{q}}\Delta_H(j,\mathbf{q}). \end{aligned} \quad (42)$$

The resulting equations of motion are the vacuum expectation value (vev) equation

$$\begin{aligned}0 = & V\beta\left(\frac{m^2+\delta m_0^2}{2}2v+\frac{\lambda+\delta\lambda_0}{4!}4v^3\right)\\ & +(N-1)\frac{\lambda+\delta\lambda_1^A}{6}v\sum_{n,\boldsymbol{k}}\Delta_G(n,\boldsymbol{k})\\ & +\frac{3\lambda+\delta\lambda_1^A+2\delta\lambda_1^B}{6}v\sum_{n,\boldsymbol{k}}\Delta_H(n,\boldsymbol{k})\\ & +\frac{1}{\xi}(N-1)2vV\beta[\Delta_G^{-1}(0,\mathbf{0})]^2, \end{aligned} \quad (43)$$

the Goldstone propagator equation

$$\begin{aligned}\frac{1}{\Delta_G(n,\boldsymbol{k})} = & k_E^2+m^2+\delta m_1^2+\frac{\lambda+\delta\lambda_1^A}{6}v^2\\ & +\frac{1}{6}[(N+1)\lambda+(N-1)\delta\lambda_2^A+2\delta\lambda_2^B]\\ & \times\frac{1}{V\beta}\sum_{j,\mathbf{q}}\Delta_G(j,\mathbf{q})\\ & +\frac{1}{3!}(\lambda+\delta\lambda_2^A)\frac{1}{V\beta}\sum_{j,\mathbf{q}}\Delta_H(j,\mathbf{q})\\ & -\delta_{n0}\delta_{\boldsymbol{k}\mathbf{0}}\frac{4v^2}{\xi}V\beta[\Delta_G^{-1}(0,\mathbf{0})]^3, \end{aligned} \quad (44)$$

and the Higgs propagator equation

$$\frac{1}{\Delta_H(n,\boldsymbol{k})} = k_E^2 + m^2 + \delta m_1^2 + \frac{3\lambda + \delta\lambda_1^A + 2\delta\lambda_1^B}{6} v^2 + \frac{1}{3!}(\lambda + \delta\lambda_2^A)(N-1)\frac{1}{V\beta}\sum_{j,\mathbf{q}}\Delta_G(j,\mathbf{q}) + \frac{1}{3!}\frac{3\lambda + \delta\lambda_2^A + 2\delta\lambda_2^B}{V\beta}\sum_{j,\mathbf{q}}\Delta_H(j,\mathbf{q}). \tag{45}$$

As previously mentioned, the self-energies are momentum independent except for the $\delta_{n0}\delta_{\boldsymbol{k}\mathbf{0}}$ term in $\Delta_G$. Therefore, we write the propagators as

$$\Delta_G(n,\boldsymbol{k}) = \begin{cases} \Delta_G(0,\mathbf{0}) & n = \boldsymbol{k} = 0 \\ \frac{1}{k_E^2 + m_G^2} & n, \boldsymbol{k} \neq 0 \end{cases}, \tag{46}$$

$$\Delta_H(n,\boldsymbol{k}) = \frac{1}{k_E^2 + m_H^2} \tag{47}$$

and define $\Delta_G^{-1}(0,\mathbf{0}) \equiv \epsilon m_G^2$ which is now independent of the nonzero modes. The zero mode obeys the equation

$$\Delta_G^{-1}(0,\mathbf{0}) = m_G^2 - 4\frac{1}{\xi}v^2 V\beta[\Delta_G^{-1}(0,\mathbf{0})]^3. \tag{48}$$

Now, there are two cases which must be distinguished. In the first, $m_G^2 = 0$, and the zero mode equation has the solutions

$$\Delta_G^{-1}(0,\mathbf{0}) = \begin{cases} 0, \\ \pm i\sqrt{\frac{\xi}{4v^2 V\beta}}, \end{cases} \tag{49}$$

the latter two of which are clearly unphysical. However, the first solution is just what one would have if $\Delta_G(n,\boldsymbol{k}) = k_E^{-2}$ as usual for a massless particle (i.e., the zero mode no longer needs to be treated separately). Then, $\sum_{j,\mathbf{q}}\Delta_G(j,\mathbf{q})$ and $\sum_{j,\mathbf{q}}\Delta_H(j,\mathbf{q})$ are just the familiar Hartree-Fock tadpole sums, which in the infinite volume limit are

$$\sum_{j,\boldsymbol{q}}\Delta_G(j,\boldsymbol{q}) = \beta V(\mathcal{T}_G^\infty + \mathcal{T}_G^{\text{fin}} + \mathcal{T}_G^{\text{th}}), \tag{50}$$

$$\sum_{j,\boldsymbol{q}}\Delta_H(j,\boldsymbol{q}) = \beta V(\mathcal{T}_H^\infty + \mathcal{T}_H^{\text{fin}} + \mathcal{T}_H^{\text{th}}), \tag{51}$$

where

$$\mathcal{T}_{G/H}^\infty = -\frac{m_{G/H}^2}{16\pi^2}\left[\frac{1}{\eta} - \gamma + 1 + \ln(4\pi)\right], \tag{52}$$

$$\mathcal{T}_{G/H}^{\text{fin}} = \frac{m_{G/H}^2}{16\pi^2}\ln\frac{m_{G/H}^2}{\mu^2} \tag{53}$$

are the vacuum contributions in dimensional regularization in $4 - 2\eta$ dimensions with $\overline{\text{MS}}$ subtraction at the scale $\mu$ ($\gamma \approx 0.577$ is the Euler gamma constant) and $\mathcal{T}_{G/H}^{\text{th}}$ are the Bose-Einstein integrals

$$\mathcal{T}_{G/H}^{\text{th}} = \int_{\boldsymbol{k}} \frac{1}{\omega_{\boldsymbol{k}}}\frac{1}{e^{\beta\omega_{\boldsymbol{k}}} - 1}, \qquad \omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + m_{G/H}^2}. \tag{54}$$

If, on the other hand, $m_G^2 \neq 0$, then $\Delta_G$ no longer has the usual form, and the Goldstone tadpole must be handled differently. In this case, it can be rewritten as

$$\sum_{j,\mathbf{q}}\Delta_G(j,\mathbf{q}) = \sum_{j,\mathbf{q}\neq 0}\Delta_G(j,\mathbf{q}) + \Delta_G(0,\mathbf{0}) = \sum_{j,\mathbf{q}\neq 0}\Delta_G(j,\mathbf{q}) + \frac{1}{m_G^2} + \Delta_G(0,\mathbf{0}) - \frac{1}{m_G^2} = \sum_{j,\mathbf{q}}\tilde{\Delta}_G(j,\mathbf{q}) + \Delta_G(0,\mathbf{0}) - \frac{1}{m_G^2}, \tag{55}$$

where $\tilde{\Delta}_G$ is an auxiliary propagator defined to have the usual form

$$\tilde{\Delta}_G(n,\boldsymbol{k}) = \frac{1}{k_E^2 + m_G^2}. \tag{56}$$

Then, $\sum_{j,\mathbf{q}}\tilde{\Delta}_G(j,\mathbf{q})$ is just the familiar Hartree-Fock tadpole sum for a particle of mass $m_G$. The terms $\Delta_G(0,\mathbf{0}) - \frac{1}{m_G^2}$ in the Goldstone tadpole account for the shift in the zero mode propagator from its usual value. The zero mode equation can be rewritten as

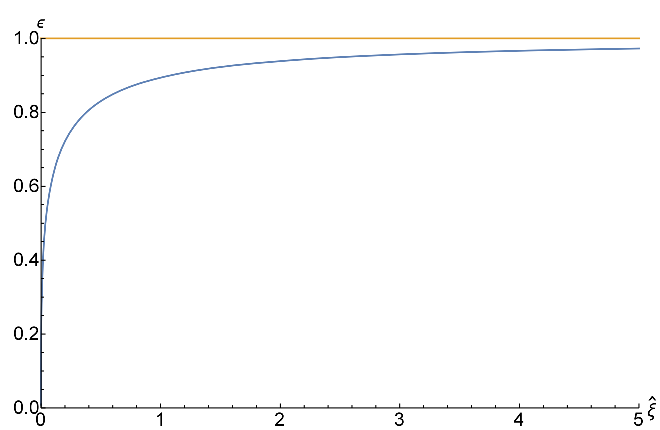


FIG. 1. Plot of $\epsilon$ (blue) vs $\hat{\xi}$ solving the Goldstone boson zero mode equation (59). The line at $\epsilon = 1$ is to guide the eye.

$$\epsilon = 1 - \frac{4v^2 V\beta m_G^4}{\xi}\epsilon^3 = 1 - \frac{4\epsilon^3}{27\hat{\xi}}, \tag{57}$$

where

$$\hat{\xi} = \frac{\xi}{27v^2 V\beta m_G^4} \tag{58}$$

is dimensionless and the numeric factor has been chosen for later convenience. The real solution of this cubic equation is

$$\epsilon = \frac{3}{2}\sqrt[3]{\hat{\xi}(\sqrt{\hat{\xi}+1}-1)}\left(\sqrt[3]{\frac{\hat{\xi}}{(\sqrt{\hat{\xi}+1}-1)^2}}-1\right), \tag{59}$$

which is monotonically increasing from 0 to 1 as $\hat{\xi}$ goes from 0 to $+\infty$ and behaves asymptotically as

$$\epsilon \sim \begin{cases} \frac{3\hat{\xi}^{1/3}}{2^{2/3}} + \mathcal{O}(\hat{\xi}^{2/3}), & \hat{\xi}\to 0, \\ 1 - \frac{4}{27\hat{\xi}} + \mathcal{O}(\hat{\xi}^{-2}), & \hat{\xi}\to\infty. \end{cases} \tag{60}$$

The behavior of $\epsilon$ is shown in Fig. 1.

The remaining equations are renormalized by demanding that kinematically distinct divergences vanish, essentially duplicating the renormalization method of Refs. [14,22]. This is done by substituting the tadpoles (50)–(51) in the equations of motion, rearranging to obtain expressions for $v$, $m_G^2$, and $m_H^2$, then demanding that the divergences proportional to $v$, $\mathcal{T}_G^{\rm fin}$, and $\mathcal{T}_H^{\rm fin}$ independently vanish. This leads to 11 equations which are nontrivially consistent and determine the nine counterterms. No new difficulties are found here compared to the standard treatment, and the details are left in a *Mathematica* notebook in the Supplemental Material [29]. The resulting finite equations of motion are the vev equation

$$v = 0, \tag{61}$$

or

$$\begin{aligned} 0 = m^2 &+ \frac{\lambda}{6}v^2 + (N-1)\frac{\lambda}{6} \\ &\times \left[\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\rm th} + \frac{1}{V\beta}\frac{1}{m_G^2}\left(\frac{1}{\epsilon}-1\right)\right] \\ &+ \frac{\lambda}{2}\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\rm th}\right) \\ &+ \frac{(N-1)2}{\xi}(m_G^2\epsilon)^2, \end{aligned} \tag{62}$$

the Goldstone gap equation

$$\begin{aligned} m_G^2 = m^2 &+ \frac{\lambda}{6}v^2 + (N+1)\frac{\lambda}{6} \\ &\times \left[\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\rm th} + \frac{1}{V\beta}\frac{1}{m_G^2}\left(\frac{1}{\epsilon}-1\right)\right] \\ &+ \frac{\lambda}{6}\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\rm th}\right), \end{aligned} \tag{63}$$

and the Higgs gap equation

$$\begin{aligned} m_H^2 = m^2 &+ \frac{\lambda}{2}v^2 + (N-1)\frac{\lambda}{6} \\ &\times \left[\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\rm th} + \frac{1}{V\beta}\frac{1}{m_G^2}\left(\frac{1}{\epsilon}-1\right)\right] \\ &+ \frac{\lambda}{2}\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\rm th}\right). \end{aligned} \tag{64}$$

## IV. SOLUTION IN THE INFINITE VOLUME/LOW TEMPERATURE LIMIT

### A. Scaling of the solutions

We desire solutions of (62)–(64) in the $V\beta\to\infty$ limit. It is possible in general to look for solutions in the symmetric and broken phases with scalings $\xi\sim(V\beta)^\alpha$ and $m_G^2\sim(V\beta)^{-\gamma}$ for $\gamma\geq 0$. In Sec. IV B, we examine the symmetric phase ($v=0$, $m_G\neq 0$) and show that it is unaffected by SSI as expected. In Sec. IV C, we examine the broken phase $v\neq 0$ with $m_G\neq 0$. Ordinarily, Goldstone's theorem is broken in this regime; however, with the additional freedom afforded by SSI, we find a scaling for $\xi$ such that $\epsilon\to 0$ and Goldstone's theorem is nevertheless satisfied. Finally, in Sec. IV D, we examine the broken phase $v\neq 0$ with massless Goldstones $m_G=0$. We expect SSI in this regime to be close to the old symmetry improvement method, and in fact it turns out to be exactly equivalent. The apparent extra freedom of the SSI method (the choice of $\xi$) is equivalent the freedom to choose the Lagrange multiplier of the SI method, which we demonstrate by deriving the explicit connection between them. This gives a new insight into why the SI equations of motion do not depend on the Lagrange multiplier, which previously appeared as a remarkable coincidence but now can be understood as a consequence of the $V\beta\to\infty$ limit.

The effects of SSI enter into (62)–(64) through two terms, for which we introduce the shorthands

$$\mathcal{S}_1 = \frac{1}{V\beta}\frac{1}{m_G^2}\left(\frac{1}{\epsilon}-1\right), \tag{65}$$

$$\mathcal{S}_2 = \frac{2(N-1)}{\xi}(m_G^2\epsilon)^2. \tag{66}$$

In the following sections, we consider all possible scalings of $\xi$ and $m_G$ and their effect on these terms, ruling out most possibilities. For reference purposes, we collect here the scalings that work in each section. In Sec. IV B, we find that the symmetric phase exists independently of the $V\beta \to \infty$ limit. In Sec. IV C, it is necessary to let $\xi$ scale as $\xi = (V\beta)^{-2}\zeta$ where $\zeta$ is a constant (with mass dimension $[\zeta] = -6$), for which

$$\epsilon \to \frac{1}{V\beta}\left(\frac{\zeta}{4v^2 m_G^4}\right)^{1/3} \to 0, \tag{67}$$

$$\mathcal{S}_1 \to \left(\frac{4v^2}{\zeta m_G^2}\right)^{1/3}, \tag{68}$$

$$\mathcal{S}_2 \to \frac{1}{\zeta}(N-1)\left(\frac{\zeta m_G^2}{\sqrt{2}v^2}\right)^{2/3}. \tag{69}$$

The equations of motion are then nondimensionalized and studied using three methods: perturbation theory in $\zeta^{-1}$, at leading order in $1/N$, and through exact numerical solutions. Finally, in Sec. IV D, both $\xi$ and $m_G$ must be scaled as

$$\xi = (V\beta)^{\alpha}\mu^{2+4\alpha}\hat{\zeta}, \tag{70}$$

$$m_G^2 = (V\beta)^{-\gamma}\mu^{2-4\gamma}y, \tag{71}$$

where $\gamma > 0$, $\alpha + 2\gamma + 2 = 0$, and $\hat{\zeta}$ and $y$ are dimensionless. Any scaling satisfying these conditions leads to identical equations of motion and solutions. Then,

$$\epsilon \to \frac{1}{V\beta\mu^4}\left(\frac{\hat{\zeta}}{4xy^2}\right)^{1/3}, \tag{72}$$

$$\mathcal{S}_1 \to 0, \tag{73}$$

$$\mathcal{S}_2 \to \mu^2(N-1)\left(\frac{y^2}{2\hat{\zeta}x^2}\right)^{1/3}, \tag{74}$$

where $x = v^2/\mu^2$ is the dimensionless vev.

### B. Symmetric phase

At high temperatures, there should be a symmetric phase solution to the equations of motion. We therefore examine the $v \to 0$ limit of the equations of motion. As $v \to 0$ at fixed $\xi$, $\epsilon \to 1 - \frac{4v^2 V\beta m_G^4}{\xi}$ provided $m_G$ does not go to infinity faster than $1/\sqrt{v}$. Then,

$$\mathcal{S}_1 \to \frac{4v^2 m_G^2}{\xi} \to 0, \tag{75}$$

and the equations of motion (62)–(64) reduce to

$$m_G^2 = m_H^2 = m^2 + \frac{1}{6}(N+2)\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\text{th}}\right), \tag{76}$$

which is a symmetric (i.e., equal mass) phase as expected. Indeed, the gap equation is unmodified by SSI in the symmetric phase because there $\mathcal{W} = 0$ trivially. This phase terminates at the critical point $m_G^2 = m_H^2 = 0$ which gives the critical temperature

$$T_\star = \sqrt{\frac{12\bar{v}^2}{N+2}}, \tag{77}$$

where $m^2 = -\lambda\bar{v}^2/6$ and the overbar denotes the zero temperature value of a quantity. That $T_\star$ is independent of $\xi$ is consistent with the previously known result that the same critical point is found in all symmetry improvement schemes [22]. There is no subtlety involved in the $V\beta \to \infty$ limit in this case.

### C. Broken phase with $m_G^2 \neq 0$

Attempting to describe the broken phase with the SSI equations of motion is rather more complicated than the symmetric phase. Decreasing temperature at *fixed* $\xi$ gives $\hat{\xi} \to 0$ and

$$\mathcal{S}_1 \to \left(\frac{4v^2}{\xi m_G^2(V\beta)^2}\right)^{1/3}, \tag{78}$$

so the equations of motion (62)–(64) become for the vev

$$\begin{aligned} 0 = m^2 &+ \frac{\lambda}{6}v^2 + (N-1)\frac{\lambda}{6} \\ &\times\left[\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\text{th}} + \left(\frac{4v^2}{\xi m_G^2(V\beta)^2}\right)^{1/3}\right] \\ &+\frac{\lambda}{2}\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\text{th}}\right) \\ &+\frac{1}{\xi}(N-1)\left(\frac{\xi m_G^2}{\sqrt{2}v^2 V\beta}\right)^{2/3}, \end{aligned} \tag{79}$$

and for the Goldstone

$$\begin{aligned} m_G^2 = m^2 + \frac{\lambda}{6}v^2 + (N+1)\frac{\lambda}{6}&\left[\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\text{th}}\right. \\ &\left.+\left(\frac{4v^2}{\xi m_G^2(V\beta)^2}\right)^{1/3}\right] + \frac{\lambda}{6}\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\text{th}}\right), \end{aligned} \tag{80}$$

and for the Higgs

$$m_H^2 = m^2 + \frac{\lambda}{2}v^2 + (N-1)\frac{\lambda}{6}\left[\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\rm th} + \left(\frac{4v^2}{\xi m_G^2 (V\beta)^2}\right)^{1/3}\right] + \frac{\lambda}{2}\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\rm th}\right). \tag{81}$$

Note that all of the soft-symmetry improvement terms vanish in the limit $V\beta \to \infty$. Thus, the SSI 2PIEA reduces to the unimproved 2PIEA if $V\beta \to \infty$ at fixed $\xi$. It is necessary to allow $\xi$ to vary as the $V\beta \to \infty$ limit is taken to obtain a nontrivial correction to the unimproved 2PIEA. This is the first sign that the limit is nontrivial.

This section examines the simplest scheme to find a nontrivial limit. This turns out to be the novel limit mentioned in the Introduction. It is shown in Sec. IV D that the only other nontrivial limit is equivalent to the old SI 2PIEA. We proceed by letting $\xi$ vary with $V\beta$ as $\xi = (V\beta)^\alpha \zeta$ where $\zeta$ is a constant (with mass dimension $[\zeta] = 2 + 4\alpha$). If $\alpha \geq 1$, the SSI terms vanish in the limit. If $\alpha < 1$,

$$\epsilon \to \left(\frac{(V\beta)^\alpha \zeta}{4v^2 V\beta m_G^4}\right)^{1/3}, \tag{82}$$

and the symmetry improvement terms are

$$\mathcal{S}_1 \to \frac{1}{m_G^2}\left[\left(\frac{4v^2 m_G^4}{\zeta}\right)^{1/3}(V\beta)^{(-\alpha-2)/3} - \frac{1}{V\beta}\right], \tag{83}$$

$$\mathcal{S}_2 \to \frac{1}{\zeta}(N-1)\left(\frac{\zeta m_G^2}{\sqrt{2}v^2}\right)^{2/3}(V\beta)^{(-2-\alpha)/3}. \tag{84}$$

The only nontrivial possibility is $\alpha = -2$, for which

$$\epsilon \to \frac{1}{V\beta}\left(\frac{\zeta}{4v^2 m_G^4}\right)^{1/3} \to 0, \tag{85}$$

$$\mathcal{S}_1 \to \left(\frac{4v^2}{\zeta m_G^2}\right)^{1/3}, \tag{86}$$

$$\mathcal{S}_2 \to \frac{1}{\zeta}(N-1)\left(\frac{\zeta m_G^2}{\sqrt{2}v^2}\right)^{2/3}. \tag{87}$$

The equations of motion are for the vev

$$0 = m^2 + \frac{\lambda}{6}v^2 + (N-1)\frac{\lambda}{6}\left[\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\rm th} + \left(\frac{4v^2}{\zeta m_G^2}\right)^{1/3}\right] + \frac{\lambda}{2}\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\rm th}\right) + (N-1)\left(\frac{m_G^2}{\sqrt{2\zeta}v^2}\right)^{2/3}, \tag{88}$$

for the Goldstone

$$m_G^2 = m^2 + \frac{\lambda}{6}v^2 + (N+1)\frac{\lambda}{6}\left[\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\rm th} + \left(\frac{4v^2}{\zeta m_G^2}\right)^{1/3}\right] + \frac{\lambda}{6}\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\rm th}\right), \tag{89}$$

and for the Higgs

$$m_H^2 = m^2 + \frac{\lambda}{2}v^2 + (N-1)\frac{\lambda}{6}\left[\frac{m_G^2}{16\pi^2}\ln\frac{m_G^2}{\mu^2} + \mathcal{T}_G^{\rm th} + \left(\frac{4v^2}{\zeta m_G^2}\right)^{1/3}\right] + \frac{\lambda}{2}\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\rm th}\right). \tag{90}$$

Importantly, note that the mass appearing in Goldstone's theorem is $\epsilon m_G^2$ [from the definition of $\epsilon$: $\Delta_G^{-1}(0,\mathbf{0}) = \epsilon m_G^2$], which obeys $\epsilon m_G^2 \to 0$ as $V\beta \to \infty$ thanks to the scaling chosen for $\xi$. Thus, this scheme satisfies Goldstone's theorem even if $m_G^2 \neq 0$. What $m_G^2 \neq 0$ indicates here is not actually a violation of Goldstone's theorem but a *noncommunication* of the masslessness of the Goldstone zero mode to the other modes.

Defining the dimensionless variables

$$x = v^2/\mu^2, \tag{91}$$

$$y = m_G^2/\mu^2, \tag{92}$$

$$z = m_H^2/\mu^2, \tag{93}$$

$$\bar{x} = \bar{v}^2/\mu^2, \tag{94}$$

$$\bar{X} = -6m^2/\lambda\mu^2, \tag{95}$$

$$\hat{\zeta} = \zeta\mu^6, \tag{96}$$

$$T_{G/H} = \mu^{-2}\mathcal{T}_{G/H}^{\rm th} \tag{97}$$

(note the distinction between the Lagrangian parameter $\bar{X}$ and the zero temperature value of the mean field $\bar{x}$, which happen to be equal at tree level and in the usual renormalization scheme for the Hartree-Fock approximation), this system becomes

$$0=\frac{\lambda}{6}(x-\bar{X})+(N-1)\frac{\lambda}{6}\left[\frac{1}{16\pi^2}y\ln y+T_G+\left(\frac{4x}{\hat{\zeta}y}\right)^{1/3}\right]+\frac{\lambda}{2}\left(\frac{1}{16\pi^2}z\ln z+T_H\right)+(N-1)\left(\frac{y}{\sqrt{2\hat{\zeta}x}}\right)^{2/3}, \tag{98}$$

$$y=\frac{\lambda}{6}(x-\bar{X})+(N+1)\frac{\lambda}{6}\left[\frac{1}{16\pi^2}y\ln y+T_G+\left(\frac{4x}{\hat{\zeta}y}\right)^{1/3}\right]+\frac{\lambda}{6}\left(\frac{1}{16\pi^2}z\ln z+T_H\right), \tag{99}$$

$$z=\frac{\lambda}{3}x-(N-1)\left(\frac{y}{\sqrt{2\hat{\zeta}x}}\right)^{2/3}. \tag{100}$$

Looking for a zero temperature solution gives the system

$$0=\frac{\lambda}{6}(\bar{x}-\bar{X})+(N-1)\frac{\lambda}{6}\left[\frac{1}{16\pi^2}\bar{y}\ln\bar{y}+\left(\frac{4\bar{x}}{\hat{\zeta}\,\bar{y}}\right)^{1/3}\right]+\frac{\lambda}{2}\frac{1}{16\pi^2}\bar{z}\ln\bar{z}+(N-1)\left(\frac{\bar{y}^2}{2\hat{\zeta}\bar{x}^2}\right)^{1/3}, \tag{101}$$

$$\bar{y}=\frac{\lambda}{6}(\bar{x}-\bar{X})+(N+1)\frac{\lambda}{6}\left[\frac{1}{16\pi^2}\bar{y}\ln\bar{y}+\left(\frac{4\bar{x}}{\hat{\zeta}\,\bar{y}}\right)^{1/3}\right]+\frac{\lambda}{6}\frac{1}{16\pi^2}\bar{z}\ln\bar{z}, \tag{102}$$

$$\bar{z}=\frac{\lambda\bar{x}}{3}-(N-1)\left(\frac{\bar{y}^2}{2\hat{\zeta}\bar{x}^2}\right)^{1/3}. \tag{103}$$

First, ignoring the SSI terms, one finds the usual unimproved 2PI solution $\bar{x}=\bar{X}$, $\bar{y}=0$, $\bar{z}=\lambda\bar{x}/3=1$. Now, examine the large $N$ limit of these equations, taking as the scaling limit $g=\lambda N=\text{constant}$ and $\bar{x}$, $\bar{X}\sim N^1$, $\bar{y}$, $\bar{z}\sim N^0$ and $\hat{\zeta}\sim N^a$ with $a$ to be determined. To leading order,

$$0=\frac{g}{6N}(\bar{x}-\bar{X})+\frac{g}{6}\left\{\frac{1}{16\pi^2}\bar{y}\ln\bar{y}+N^{(1-a)/3}\left[\frac{4\bar{x}/N}{(\hat{\zeta}/N^a)\bar{y}}\right]^{1/3}\right\}+N^{(1-a)/3}\left[\frac{\bar{y}^2}{2(\hat{\zeta}/N^a)(\bar{x}/N)^2}\right]^{1/3}, \tag{104}$$

$$\bar{y}=\frac{g}{6N}(\bar{x}-\bar{X})+\left\{\frac{g}{6}\frac{1}{16\pi^2}\bar{y}\ln\bar{y}+N^{(1-a)/3}\left[\frac{4\bar{x}/N}{(\hat{\zeta}/N^a)\bar{y}}\right]^{1/3}\right\}, \tag{105}$$

$$\bar{z}=\frac{g\bar{x}}{3N}-N^{(1-a)/3}\left[\frac{\bar{y}^2}{2(\hat{\zeta}/N^a)(\bar{x}/N)^2}\right]^{1/3}. \tag{106}$$

Note that the $z$ dependence of the first two equations is higher order in $1/N$. Scaling limits exist if $a\geq 1$. Note that the SSI term in $\Gamma_\xi^{\rm SSI}$ goes as $\xi^{-1}Nv^2\sim N^{3-a}$ so that one needs $a\geq 2$ for a scaling limit for $\Gamma_\xi^{\rm SSI}$ to exist. $a=1$ can also be ruled out by considering the equations of motion, for in this case the leading approximation is

$$0=\frac{g}{6N}(\bar{x}-\bar{X})+\frac{g}{6}\left\{\frac{1}{16\pi^2}\bar{y}\ln\bar{y}+\left[\frac{4\bar{x}/N}{(\hat{\zeta}/N)\bar{y}}\right]^{1/3}\right\}+\left[\frac{\bar{y}^2}{2(\hat{\zeta}/N)(\bar{x}/N)^2}\right]^{1/3}, \tag{107}$$

$$\bar{y}=\frac{g}{6N}(\bar{x}-\bar{X})+\frac{g}{6}\left\{\frac{1}{16\pi^2}\bar{y}\ln\bar{y}+\left[\frac{4\bar{x}/N}{(\hat{\zeta}/N)\bar{y}}\right]^{1/3}\right\}, \tag{108}$$

$$\bar{z}=\frac{g\bar{x}}{3N}-\left[\frac{\bar{y}^2}{2(\hat{\zeta}/N)(\bar{x}/N)^2}\right]^{1/3}. \tag{109}$$

Using (107) to simplify (108),

$$\bar{y}=-\left(\frac{\bar{y}^2}{2(\hat{\zeta}/N)(\bar{x}/N)^2}\right)^{1/3}, \tag{110}$$

which has the solution

$$\bar{y}=-\frac{1}{2(\hat{\zeta}/N)(\bar{x}/N)^2}<0, \tag{111}$$

with an unphysical tachyonic Goldstone $m_G^2<0$. This is not *necessarily* a problem because the zero mode $\Delta_G(0,\mathbf{0})$ is always positive and, in finite volume with $\beta$ and $L$ sufficiently small (i.e., $\beta$, $L<2\pi/|m_G|$), each mode $\Delta_G(n,\boldsymbol{k})=1/(\omega_n^2+\boldsymbol{k}^2+m_G^2)$ with $n$, $\boldsymbol{k}\neq\mathbf{0}$ is still positive. Physically, confinement energy is stabilizing the tachyon. However, a second condition is that the imaginary part of the first equation of motion vanishes, yielding

$$0=-\frac{1}{16\pi^2}|\bar{y}|(2k+1)\pi-\sin\left[\frac{(2k+1)\pi}{3}\right]\left[\frac{4\bar{x}/N}{(\hat{\zeta}/N)|\bar{y}|}\right]^{1/3}, \tag{112}$$

where the branch chosen is $\bar{y}=|\bar{y}|\exp(i\pi(2k+1))$ where $k$ is an integer. Using the solution for $\bar{y}$, this becomes

$$k = -\frac{1}{2} - 32\pi(\hat{\zeta}/N)(\bar{x}/N)^3 \sin\left(\frac{(2k+1)\pi}{3}\right), \qquad (113)$$

which only has solutions of the form $k = 3j$ if

$$j = -\frac{1}{6} - \frac{16\pi}{\sqrt{3}}(\hat{\zeta}/N)(\bar{x}/N)^3 \qquad (114)$$

is an integer. The existence of solutions only for certain discrete values of $\hat{\zeta}\bar{x}^3$ is troubling and highly counterintuitive (note especially that the relationship between $\hat{\zeta}$ and $\bar{x}$ for a given $j$ is independent of $g$, so that no matter how $g$ is varied at fixed $m^2$ and $\hat{\zeta}$, $\bar{x}$ is fixed even though one expects $\bar{x} \sim N/g$).

If $a > 1$, the SSI terms in the equation of motion are of higher order, and the leading large $N$ approximation is just the standard one, i.e.,

$$0 = \frac{g}{6N}(\bar{x} - \bar{X}) + \frac{g}{6}\frac{1}{16\pi^2}\bar{y}\ln\bar{y}, \qquad (115)$$

$$\bar{y} = \frac{g}{6N}(\bar{x} - \bar{X}) + \frac{g}{6}\frac{1}{16\pi^2}\bar{y}\ln\bar{y}, \qquad (116)$$

$$\bar{z} = \frac{g\bar{x}}{3N}, \qquad (117)$$

which has the solution $\bar{x} = \bar{X}$, $\bar{y} = 0$ and $\bar{z} = \lambda\bar{x}/3$ as expected. Now, note that if $1 < a < 4$ the SSI terms go as a fractional power of $N$ between $N^0$ and $N^{-1}$ which cannot balance any of the terms coming from diagrams, which all go as integer powers of $N^{-1}$. This implies that, if $a > 1$, it must be of the form $a = 4 + 3k$ where $k = 0, 1, 2, \ldots$. The SSI terms then scale as $N^{-(1+k)}$ in the equation of motion and $N^{-1-3k}$ in $\Gamma_{\xi}^{\rm SSI}$. Thus, the SSI equations of motion possess a satisfactory leading large $N$ limit, but only if the scaling is such that the SSI terms are of higher order. This is the first sign that the SSI terms are problematic.

Now, consider the case where symmetry improvement is only weakly imposed, i.e., the SSI terms are a small perturbation. Intuitively, this can be achieved by taking $\hat{\zeta}$ sufficiently large. It is thus natural to solve the equations of motion (101)–(103) perturbatively in powers of $\hat{\zeta}^{-1/3}$ as $\hat{\zeta} \to \infty$. Writing $\bar{x} = \bar{x}_0 + \hat{\zeta}^{-1/3}\bar{x}_1 + \hat{\zeta}^{-2/3}\bar{x}_2 + \cdots$ and so on, the leading equations of motion are just the unimproved two particle irreducible (2PI) ones:

$$0 = \frac{\lambda}{6}(\bar{x}_0 - \bar{X}) + (N-1)\frac{\lambda}{6}\frac{1}{16\pi^2}\bar{y}_0\ln\bar{y}_0 + \frac{\lambda}{2}\frac{1}{16\pi^2}\bar{z}_0\ln\bar{z}_0, \qquad (118)$$

$$\bar{y}_0 = \frac{\lambda}{6}(\bar{x}_0 - \bar{X}) + (N+1)\frac{\lambda}{6}\frac{1}{16\pi^2}\bar{y}_0\ln\bar{y}_0 + \frac{\lambda}{6}\frac{1}{16\pi^2}\bar{z}_0\ln\bar{z}_0, \qquad (119)$$

$$\bar{z}_0 = \frac{\lambda\bar{x}_0}{3}. \qquad (120)$$

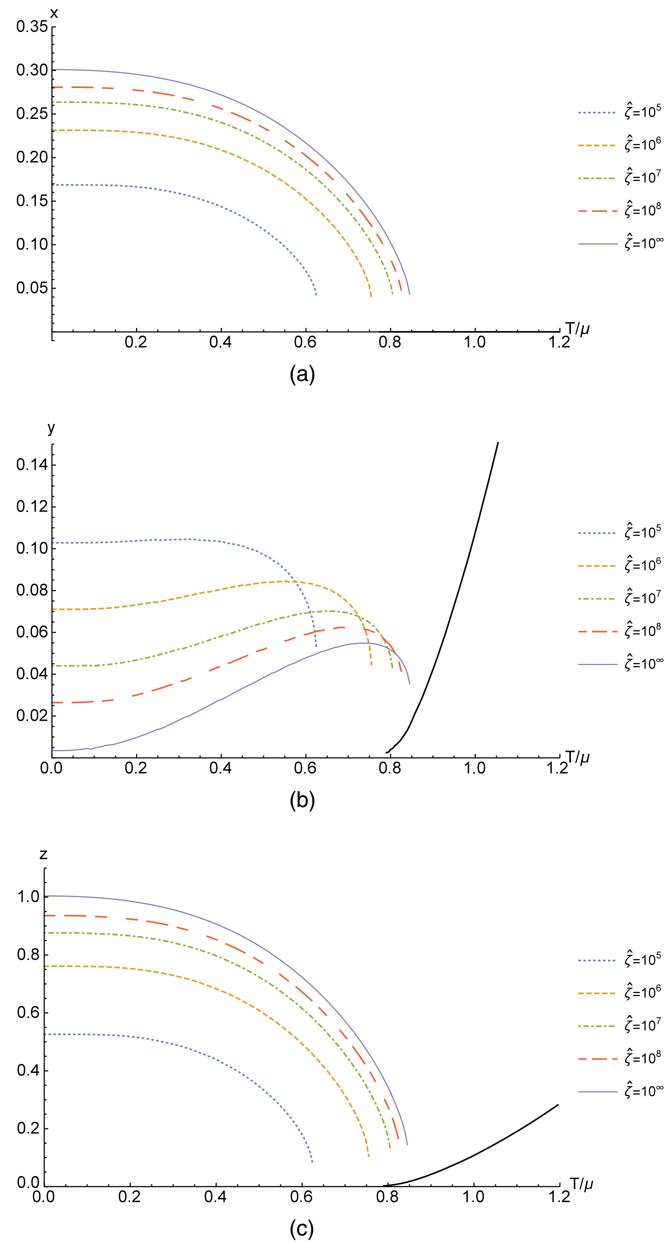


FIG. 2. Solutions of (98)–(100), the SSI equations of motion in the Hartree-Fock approximation. The colored curves are broken phase $x$ (Fig. 2(a)), $y$ (Fig. 2(b)), and $z$ (Fig. 2(c)) and symmetric phase (solid black) solutions vs temperature for $\lambda = 10$, $N = 4$, $\bar{X} = 0.3$. and several values of $\hat{\zeta}$ from $10^5$ to $\infty$ (unimproved). The critical temperature is $T_\star/\mu \approx 0.775$.

The first order perturbation obeys a system of equations which can be arranged as the matrix equation

$$\begin{pmatrix} \frac{\lambda}{6} & \frac{(N-1)\lambda}{96\pi^2}(1+\ln\bar{y}_0) & \frac{\lambda}{32\pi^2}(1+\ln\bar{z}_0) \\ \frac{\lambda}{6} & \frac{(N+1)\lambda}{96\pi^2}(1+\ln\bar{y}_0)-1 & \frac{\lambda}{96\pi^2}(1+\ln\bar{z}_0) \\ \frac{\lambda}{3} & 0 & -1 \end{pmatrix}\begin{pmatrix}\bar{x}_1\\ \bar{y}_1\\ \bar{z}_1\end{pmatrix} = (N-1)\left(\frac{\bar{y}_0^2}{2\bar{x}_0^2}\right)^{1/3}\begin{pmatrix} -\frac{\lambda\bar{x}_0}{3\bar{y}_0}-1 \\ -\frac{N+1}{N-1}\frac{\lambda\bar{x}_0}{3\bar{y}_0} \\ 1\end{pmatrix}. \tag{121}$$

Note that this equation is singular in the limit $\bar{y}_0 \to 0$. The solution for $\bar{y}_1$ in this limit is

$$\bar{y}_1 \to -\frac{32\pi^2}{\ln\bar{y}_0}\left(\frac{\bar{x}_0}{2\bar{y}_0}\right)^{1/3} \to \infty. \tag{122}$$

There is no sense in which the SSI terms are a small perturbation, no matter the value of $\hat{\zeta}$. This can also be seen from a direct examination of the full equations of motion. In the limit $\bar{y} \to 0$, the $(\frac{4\bar{x}}{\hat{\zeta}\,\bar{y}})^{1/3}$ terms always dominate for any finite value of $\hat{\zeta}$. The result is that the SSI solution must always have $\bar{y} \neq 0$, even at zero temperature. For the same reason, a perturbation analysis near the critical temperature also fails, and, in fact, real valued solutions do not exist in a ($\hat{\zeta}$ dependent) range of temperatures beneath the critical temperature. Further, $m_G^2$ appears to increase as the SSI terms are more strongly imposed. Physically, the unimproved 2PI equations of motion "would like to have" a nonzero Goldstone mass. When the mass of the zero mode is forced to vanish, the SSI 2PIEA adjusts by *increasing* the mass of the other modes. This can be verified by examining numerical solutions.

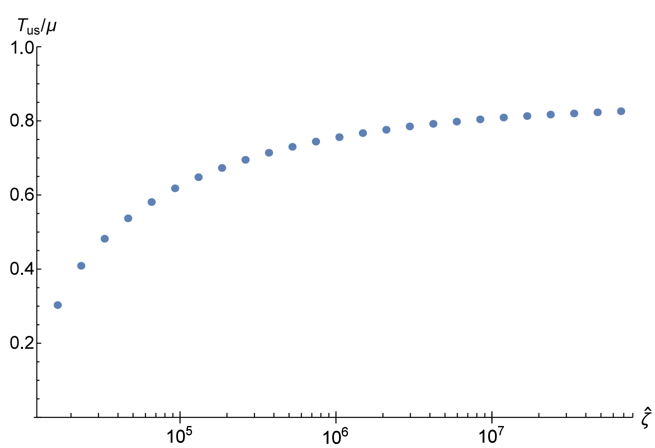


FIG. 3. The upper spinodal temperature $T_{\mathrm{us}}$ vs $\hat{\zeta}$ for broken phase solutions of the SSI equations of motion in the Hartree-Fock approximation for $\lambda = 10$, $N = 4$, and $\bar{X} = 0.3$. The critical temperature is $T_\star/\mu \approx 0.775$.

Numerical solutions of (98)–(100) are shown in Fig. 2 for $\lambda = 10$, $N = 4$, $\bar{X} = 0.3$, and several values of $\hat{\zeta}$ from $10^4$ to $\infty$. The critical temperature for these values is $T_\star/\mu \approx 0.775$. These parameter values are chosen for illustrative, not physical, purposes. For very large $\hat{\zeta}$, the solution is near the unimproved solution. However, as $\hat{\zeta}$ is decreased, $x$ and $z$ decrease, and $y$ increases (this is consistent with the perturbation $y_1$ being positive). Broken phase solutions cease to exist above the upper spinodal temperature $T_{\mathrm{us}}(\hat{\zeta})$ which depends on $\hat{\zeta}$. Note that $T_{\mathrm{us}}(\hat{\zeta})$ drops below $T_\star$ for all $\hat{\zeta} < \hat{\zeta}_c$ where $\hat{\zeta}_c$ is somewhere between $10^6$ and $10^7$. This means that for $\hat{\zeta} < \hat{\zeta}_c$ there is *no solution* between $T_{\mathrm{us}}(\hat{\zeta})$ and $T_\star$. Further, as $\hat{\zeta} \to 0$, $T_{\mathrm{us}}(\hat{\zeta}) \to 0$. This behavior can be seen in Fig. 3. At a critical value $\hat{\zeta} = \hat{\zeta}_\star$, one has $T_{\mathrm{us}}(\hat{\zeta}_\star) = 0$, and real solutions cease to exist for all $\hat{\zeta} \leq \hat{\zeta}_\star$.

The mathematical origin of this loss of solution can be understood by considering the zero temperature equations of motion (101)–(103). We now use (103) to eliminate $\bar{z}$ in the (101) and (102) and consider the real and imaginary parts of the right-hand sides of these equations as functions of $\bar{x}$ and $\bar{y}$. The relevant regions of the $\bar{x}-\bar{y}$ plane are shown in Fig. 4. The blue vertically meshed regions satisfy $\Re((101)) > 0$, the yellow horizontally meshed regions satisfy $\Im((101)) \neq 0$, and the green diagonally meshed regions satisfy $\Re((102)) > \bar{y}$. Note that, for $\bar{x}$, $\bar{y} > 0$, $\Im((101)) \neq 0$ is equivalent to $\bar{z} > 0$, as is $\Im((102)) \neq 0$ which does not give anything new.

Valid solutions of the equations of motion are on the boundary of the blue and green regions simultaneously *and* outside of the yellow horizontally meshed region. As $\hat{\zeta}$ is decreased, it can be seen that the blue vertically meshed region "closes in" toward the origin, the green diagonally meshed region grows upward, and the yellow horizontally meshed region grows to the right. Solutions cease to exist for $\hat{\zeta} = \hat{\zeta}_\star \approx 12200$ where all three regions intersect at a common point. For all $\hat{\zeta} < \hat{\zeta}_\star$, there are no solutions (intersection points between the blue and green curves) which are also real (outside the yellow horizontally meshed region). If the temperature is nonzero, the thermal contributions increase the real parts of the right-hand sides which, in comparison with Fig. 4, hastens the onset of the loss of solutions, which is therefore achieved at a greater value $\hat{\zeta}$. Conversely, the temperature at which solutions are lost for a given $\hat{\zeta}$ increases as a function of $\hat{\zeta}$, which, of course, matches the behavior seen in Figs. 2 and 3.

### D. Broken phase with $m_G^2 \to 0$

In order to find a broken phase solution without the pathological properties of the previous section, one can try to find solutions with $m_G^2 \to 0$ in the $V\beta \to \infty$ limit. To achieve this, take the scalings

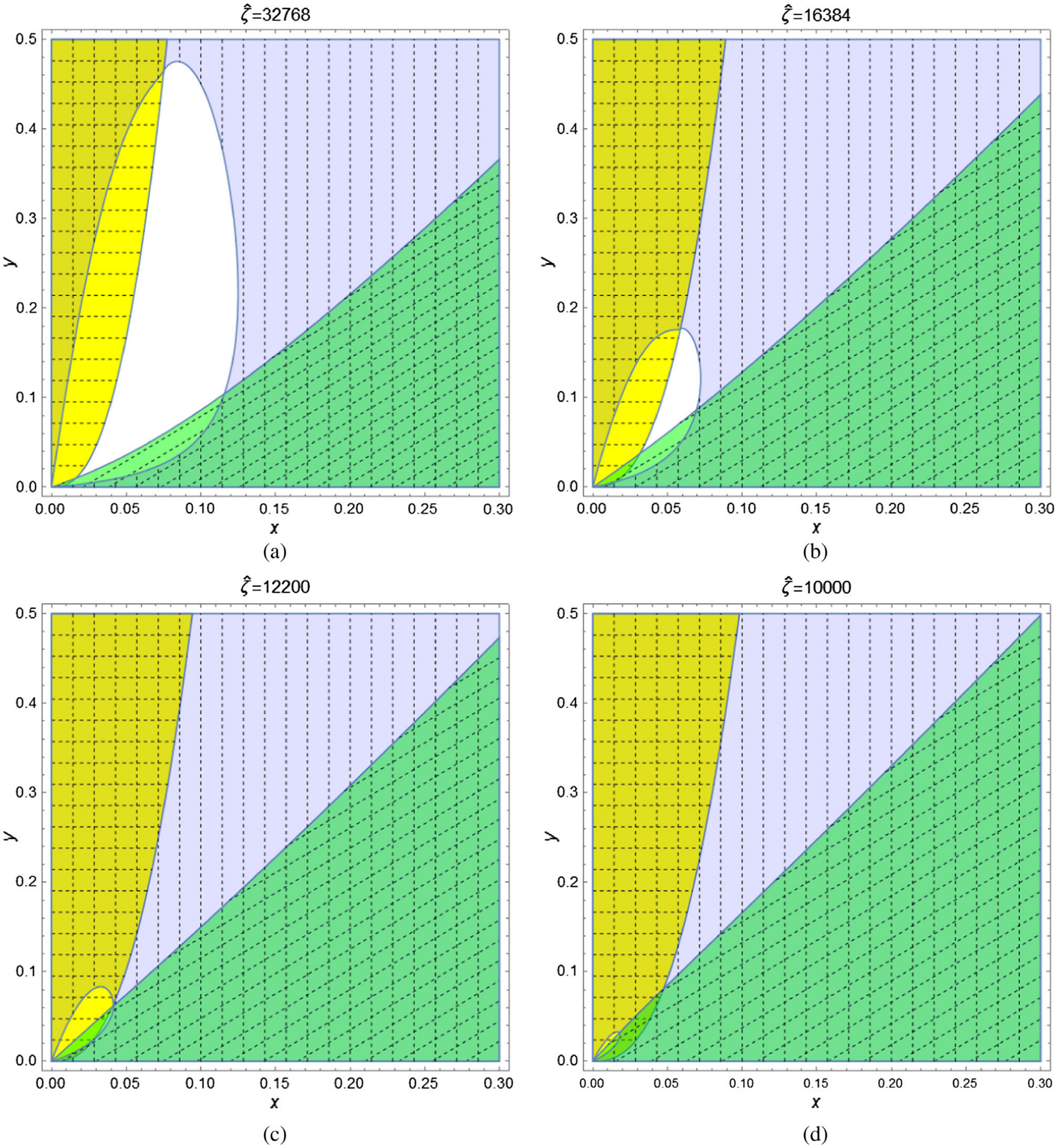


FIG. 4. Real and imaginary parts of the right-hand sides of (101) and (102) as functions of $\bar{x}$ and $\bar{y}$ for $\lambda = 10$, $N = 4$, $\bar{X} = 0.3$, and $\hat{\zeta} = 2^{15}$ (upper left), $2^{14}$ (upper right), 12200 (lower left), and $10^4$ (lower right). The blue vertically meshed regions satisfy $\Re((101)) > 0$, the yellow horizontally meshed regions satisfy $\Im((101)) \neq 0$, and the green diagonally meshed regions satisfy $\Re((102)) > \bar{y}$.

$$\xi = (V\beta)^\alpha \zeta = (V\beta)^\alpha \mu^{2+4\alpha}\hat{\zeta}, \tag{123}$$

$$m_G^2 = (V\beta)^{-\gamma}\mu^{2-4\gamma} y, \tag{124}$$

where $\gamma > 0$. The definitions of the other dimensionless variables ($x$, $z$, etc.) are as before. Then,

$$\epsilon \sim \begin{cases} \left(\frac{(V\beta)^{\alpha+2\gamma-1}}{(\mu^{-4})^{\alpha+2\gamma-1}}\right)^{1/3}\left(\frac{\hat{\zeta}}{4xy^2}\right)^{1/3}, & \alpha+2\gamma-1<0, \\ 1-\frac{(\mu^{-4})^{\alpha+2\gamma-1}}{(V\beta)^{\alpha+2\gamma-1}}\frac{4xy^2}{\hat{\zeta}}, & \alpha+2\gamma-1>0. \end{cases} \tag{125}$$

One can take the equations of motion (62)–(64) with the prescription $\mathcal{S}_1 \to 0$ because, as discussed in Sec. III, the

Goldstone tadpole reduces to the unmodified form in the massless case. The result is the equation of motion for the vev,

$$0 = m^2 + \frac{\lambda}{6}v^2 + (N-1)\frac{\lambda}{6}\frac{T^2}{12} + \frac{\lambda}{2}\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\rm th}\right) + \mathcal{S}_2, \tag{126}$$

for the Goldstone mass

$$0 = m^2 + \frac{\lambda}{6}v^2 + (N+1)\frac{\lambda}{6}\frac{T^2}{12} + \frac{\lambda}{6}\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\rm th}\right), \tag{127}$$

and for the Higgs mass

$$m_H^2 = m^2 + \frac{\lambda}{2}v^2 + (N-1)\frac{\lambda}{6}\frac{T^2}{12} + \frac{\lambda}{2}\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\rm th}\right), \tag{128}$$

having used that $\mathcal{T}_G^{\rm fin} = T^2/12$ for $m_G^2 = 0$. Note that, remarkably, Eqs. (127) and (128) are nothing but the SI 2PIEA equations of motion (cf. Ref. [22]). The only thing new is the modification of the vev equation by the term $\mathcal{S}_2$. To examine this further, one must consider the three cases $\alpha + 2\gamma - 1 \gtreqless 0$ which govern the possible scaling behaviors of this term.

In the $\alpha + 2\gamma - 1 > 0$ case, $\epsilon \to 1$, and

$$\mathcal{S}_2 \to \mu^2\frac{(\mu^{-4})^{\alpha+2\gamma}}{(V\beta)^{\alpha+2\gamma}}\frac{(N-1)2y^2}{\hat{\zeta}} \to 0. \tag{129}$$

If, on the other hand, $\alpha + 2\gamma - 1 = 0$, $\epsilon$ is a constant as $V\beta \to \infty$, and

$$\mathcal{S}_2 \to \mu^2\frac{(\mu^{-4})}{(V\beta)}\frac{1}{\hat{\zeta}}(N-1)2y^2\epsilon^2 \to 0. \tag{130}$$

In both of these cases, Eq. (126) is unmodified by SSI and cannot hold at the same time as the other two equations of motion. To see this, solve the SI 2PI equations to get

$$m_H^2 = -2m^2 - \frac{1}{3}\lambda(N+2)\frac{T^2}{12}, \tag{131}$$

and

$$\frac{\lambda}{6}v^2 = -m^2 - \frac{1}{6}(N+1)\lambda\frac{T^2}{12} - \frac{1}{6}\lambda\left(\frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\rm th}\right). \tag{132}$$

Now, use these in (126) to get

$$\frac{T^2}{12} = \frac{m_H^2}{16\pi^2}\ln\frac{m_H^2}{\mu^2} + \mathcal{T}_H^{\rm th}, \tag{133}$$

which only holds at $T = 0$ (for $\mu = \bar{m}_H$) and $T = T_\star$. There is no solution at any other temperature.

The remaining case is $\alpha + 2\gamma - 1 < 0$. For this case, the SSI term becomes

$$\mathcal{S}_2 \to \mu^2\left[\frac{(\mu^{-4})^{\alpha+2\gamma+2}}{(V\beta)^{\alpha+2\gamma+2}}\right]^{1/3}(N-1)\left(\frac{y^2}{2\hat{\zeta}x^2}\right)^{1/3}. \tag{134}$$

Looking for asymptotic balance, the only solution is $\alpha + 2\gamma + 2 = 0$ (which automatically satisfies the condition $\alpha + 2\gamma - 1 < 0$). In this case, Eq. (126) reduces to (in terms of dimensionless variables now)

$$0 = -\frac{\lambda}{6}\bar{X} + \frac{\lambda}{6}x + (N-1)\frac{\lambda}{6}\frac{T^2/\mu^2}{12} + \frac{\lambda}{2}\left(\frac{z}{16\pi^2}\ln z + T_H\right) + (N-1)\left(\frac{y^2}{2\hat{\zeta}x^2}\right)^{1/3}. \tag{135}$$

Subtracting (128) from this gives

$$(N-1)\left(\frac{y^2}{2\hat{\zeta}x^2}\right)^{1/3} = \frac{\lambda}{3}x - z, \tag{136}$$

which can be easily solved for $y$, giving

$$y^2 = 2\hat{\zeta}x^2\left(\frac{\frac{\lambda}{3}x - z}{N-1}\right)^3. \tag{137}$$

Note that $m_G^2 = 0$ regardless of the value of $y$. The only constraint is that $0 \le y < \infty$ which requires $\lambda x/3 \ge z$. This can be verified using the solution of the SI 2PI equations of motion

$$z = 1 - \frac{T^2}{T_\star^2}, \tag{138}$$

$$x = \bar{X} - \left[(N+1)\frac{T^2/\mu^2}{12} + T_H\right] \tag{139}$$

[recalling $\bar{z} = 1$ and $\bar{X} = 3/\lambda$ are the zero temperature solutions and $T_\star^2 = 12\bar{X}\mu^2/(N+2)$] so that

$$\frac{\lambda}{3}x - z = \frac{1}{\bar{X}}\left(\frac{T^2/\mu^2}{12} - T_H\right) \ge 0, \tag{140}$$

since the thermal integral $T_H$ is maximized for massless particles. Thus, $0 \le y^2 < \infty$, and $y$ can always be chosen in $0 \le y < \infty$. Thus, all of the limits $\xi \sim (V\beta)^{-2\gamma-2}$ and $m_G^2 \sim (V\beta)^{-\gamma}$ with $\gamma > 0$ are equivalent and are identified as the unique limiting procedure that gives back the old SI 2PIEA from the SSI 2PIEA.

One can also see that this procedure is the unique way of connecting the SI and SSI methods by directly matching the SSI term in $\Gamma_{\xi}^{\rm SSI}$ with the Lagrange multiplier term in $\Gamma^{\rm SI}$. To do this, one must recall the original formulation of the symmetry improvement method. The constraint term in the SI 2PIEA is (cf. "the simple constraint" discussed in Ref. [23])

$$\mathcal{C} = \frac{i}{2}\ell_A^a \mathcal{W}_a^A. \tag{141}$$

The constraint is singular, meaning one must proceed by violating the constraint by an amount $\sim\eta$ then taking a limit $\eta \to 0$ such that $\ell\eta$ is a constant. In the previous literature [14,22,23], this procedure was carried out at the level of the equations of motion. Now, it is convenient to implement this at the level of the action by shifting the constraint term to

$$\frac{i}{2}\ell_A^a(\mathcal{W}_a^A - i\mathcal{F}_a^A), \tag{142}$$

where $\mathcal{F}_a^A \sim \eta$ is the regulator written in O($N$)-covariant form. Setting the SI constraint term equal to the SSI term gives

$$\frac{i}{2}\ell_A^a(\mathcal{W}_a^A - i\mathcal{F}_a^A) = -\frac{1}{2\xi}\mathcal{W}_a^A\mathcal{W}_a^A. \tag{143}$$

This can be simplified by recalling that $\mathcal{W}_a^A = \Delta_{ab}^{-1}T_{bc}^A\varphi_c$, going to an antisymmetric multi-index $A \to jk$ for the Lie algebra indices, and using $T_{bc}^{jk} = i(\delta_{jb}\delta_{kc} - \delta_{jc}\delta_{kb})$ and $\varphi_c = v\delta_{cN}$. This gives

$$\begin{aligned} 2\frac{i}{2}V\beta\ell_{cN}^a(iP_{ca}^{\perp}[\Delta_G^{-1}(0,\mathbf{0})]v - i\mathcal{F}_a^{cN}) \\ = -\frac{1}{2\xi}(-2(N-1)v^2V\beta[\Delta_G^{-1}(0,\mathbf{0})]^2), \end{aligned} \tag{144}$$

having used $\int_y \Delta_G^{-1}(x,y) = \Delta_G^{-1}(0,\mathbf{0})$ and introduced the transverse projector $P_{ca}^{\perp} = \delta_{ca} - \varphi_c\varphi_a/\varphi^2$. Without loss of generality, one can set

$$\ell_{cN}^a = P_{ac}^{\perp}\left(\frac{1}{N-1}\ell_{dN}^d\right), \tag{145}$$

$$\mathcal{F}_a^{cN} = P_{ac}^{\perp}\mathcal{F} \tag{146}$$

and find

$$-\ell_{cN}^c(\Delta_G^{-1}(0,\mathbf{0})v - \mathcal{F}) = \frac{1}{\xi}(N-1)v^2[\Delta_G^{-1}(0,\mathbf{0})]^2. \tag{147}$$

Now, recall that the usual form of the SI regulator is $\Delta_G^{-1}(0,\mathbf{0})v = m_G^2 v = \eta m^3$ where $m$ is some arbitrary mass scale (it is convenient to take $m = \mu$). This identifies $\mathcal{F} = \eta m^3$. The $\eta \to 0$ limit is taken so that $\eta\ell_{cN}^c = \ell_0 v$ is a constant. Using this and $\Delta_G^{-1}(0,\mathbf{0}) = \epsilon m_G^2$ gives

$$-\frac{\ell_0 v}{\eta}(\epsilon m_G^2 v - \eta\mu^3) = \frac{1}{\xi}(N-1)v^2[\epsilon m_G^2]^2. \tag{148}$$

It is now convenient to take $\eta = (V\beta)^{-\delta}\mu^{-4\delta}$ with $\delta > 0$. Taking also the usual scalings for $\xi$ and $m_G^2$, one finds

$$-(V\beta)^{\delta-\gamma}\mu^{4(\delta-\gamma)}\epsilon\ell_0 xy + \ell_0\sqrt{x} = \frac{\mu^{-4\alpha-8\gamma}}{(V\beta)^{\alpha+2\gamma}}\epsilon^2(N-1)\frac{xy^2}{\hat{\zeta}}. \tag{149}$$

If $\alpha + 2\gamma - 1 > 0$, asymptotic balance is impossible (dominant terms can be matched, but not subdominant terms). Likewise, balance cannot be achieved for $\alpha + 2\gamma - 1 = 0$. If $\alpha + 2\gamma - 1 < 0$; however,

$$\begin{aligned} &-(V\beta)^{\delta-\gamma}\mu^{4(\delta-\gamma)}\left[\frac{(V\beta)^{\alpha+2\gamma-1}}{(\mu^{-4})^{\alpha+2\gamma-1}}\right]^{1/3}\left(\frac{\hat{\zeta}}{4xy^2}\right)^{1/3}\ell_0 xy + \ell_0\sqrt{x} \\ &= \frac{\mu^{-4\alpha-8\gamma}}{(V\beta)^{\alpha+2\gamma}}\left[\frac{(V\beta)^{\alpha+2\gamma-1}}{(\mu^{-4})^{\alpha+2\gamma-1}}\right]^{2/3}\left(\frac{\hat{\zeta}}{4xy^2}\right)^{2/3}(N-1)\frac{xy^2}{\hat{\zeta}}. \end{aligned} \tag{150}$$

Matching powers of $(V\beta)$ on both sides gives

$$0 = 3\delta - 1 + \alpha - \gamma, \tag{151}$$

$$0 = \alpha + 2\gamma + 2, \tag{152}$$

which of course duplicates the previous result. These equations have the solutions

$$\alpha = -2\delta, \tag{153}$$

$$\gamma = \delta - 1. \tag{154}$$

$\gamma > 0$ requires $\delta > 1$. Substituting this into (150) gives

$$-\left(\frac{\hat{\zeta}}{4xy^2}\right)^{1/3}\ell_0 xy + \ell_0\sqrt{x} = \left(\frac{\hat{\zeta}}{4xy^2}\right)^{2/3}(N-1)\frac{xy^2}{\hat{\zeta}}, \tag{155}$$

which can be solved for $\hat{\zeta}$, giving

$$\hat{\zeta}^{1/3} = \left(\frac{1}{2\sqrt{xy}}\right)^{1/3}\left(1 \pm \sqrt{1-(N-1)\frac{y}{\ell_0}}\right). \tag{156}$$

This is the desired connection between the SSI stiffness parameter $\hat{\zeta}$ and the SI Lagrange multiplier $\ell_0$.

## V. DISCUSSION

In this paper, we have introduced a new method of soft-symmetry improvement which relaxes the constraint of the symmetry improvement method. Violations of Ward identities are allowed but punished in the solution of the SSI effective action. The method is essentially a least-squares implementation of the symmetry improvement idea. A new parameter, the stiffness $\xi$, controls the strength of the constraint. We studied the SSI 2PIEA for a scalar O($N$) model in the Hartree-Fock approximation and found that the method is IR sensitive. The system must be formulated in finite volume $V$ and temperature $T = \beta^{-1}$, and the $V\beta \to \infty$ limit must be taken carefully.

We found three distinct limits in Sec. IV. In all cases, the symmetric phase is the same and is unmodified from either the unimproved 2PIEA or SI 2PIEA methods. Only the broken phase is affected by SSI. Two of the limits are equivalent to the unimproved 2PIEA and SI 2PIEA respectively. The third is a new limit where $\hat{\zeta} = (V\beta)^2\xi\mu^6$ is taken to be fixed and finite as $V\beta \to \infty$. In this limit, the WI is satisfied, but the phase transition is strongly first order and strongly dependent on the scaled stiffness $\hat{\zeta}$. Also, the upper spinodal temperature decreases as $\hat{\zeta}$ decreases, and, for $\hat{\zeta} < \hat{\zeta}_c$, solutions fail to exist between the upper spinodal temperature and the critical temperature. For $\hat{\zeta} = \hat{\zeta}_\star$, the upper spinodal temperature is equal to zero, and broken phase solutions cease to exist entirely. The limit was studied in both the leading large $N$ limit and in perturbation theory in $\hat{\zeta}^{-1/3}$. The large $N$ limit is trivial to leading order, and the perturbation theory does not exist since the SSI term is singular at the unimproved solution. These results all suggest that the new limit is pathological.

The results of this paper are primarily restricted by the use of the Hartree-Fock approximation. Investigations of higher order approximations are motivated but would be far more involved, numerically, than anything attempted here. It is possible that a higher order truncation could ameliorate some or all of the problems with SSI found here. However, assuming the Hartree-Fock results hold true, we can summarize the findings as follows: we have found a method which subsumes both the unimproved 2PIEA and SI 2PIEA and contains a new dynamical limit as $V\beta \to \infty$. However, these limits are disconnected from each other; there is no smooth way to interpolate from one to another. Further, each limit is in one way or other pathological. These results suggest that any potential advantages of SSI methods [and likely any consideration of (S)SI out of equilibrium] must occur in finite volume. Whether this is possible or not depends on the particular system being studied. Thus, ultimately, symmetry improvement methods cannot be trusted as a "black box"; their validity must be decided on a case by case basis.

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[29] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.95.025018 for *Mathematica* notebooks for the renormalization and solution of the SSI 2PIEA in the Hartree-Fock approximation.