a-function in three dimensions: Beyond the leading order

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Recently, evidence was provided for the existence of an *a*-function for renormalizable quantum field theories in three dimensions. An explicit expression was given at lowest order for general theories involving scalars and fermions and shown to be related to the *β*-functions by a gradient flow equation with positive-definite metric as in four dimensions. Here, we extend this lowest-order calculation to a general Abelian Chern-Simons gauge theory coupled to fermions and scalars and derive a prediction for part of the four-loop Yukawa *β*-function. We also compute the complete four-loop Yukawa *β*-function for the scalar-fermion theory and show that it is entirely consistent with the gradient flow equations at next-to-leading order.

I. INTRODUCTION

Following Cardy's suggestion [1] that Zamolodchikov's two-dimensional *c* theorem [2] might have an analogue in four dimensions, considerable progress has been made in proving the so-called *a* theorem in even dimensions [3–9]. In a recent paper [10] we provided evidence that for a wide range of renormalizable quantum field theories in three dimensions we can similarly define a function *A* which satisfies the equation

$$\partial_I A = T_{IJ}\beta^J, \qquad (1.1)$$

for a function T_{IJ}; we denote the function by *A* since the notation *a* is often used in four dimensions for the Euler density coefficient in the Weyl anomaly. A crucial consequence of Eq. (1.1) is that we then have

$$\mu \frac{d}{d\mu} A = \beta^I \frac{\partial}{\partial g^I} A = G_{IJ}\beta^I\beta^J, \qquad (1.2)$$

where $G_{IJ} = T_{(IJ)}$, thus demonstrating a function with monotonic behavior under renormalization group (RG) flow and providing a three-dimensional version of the strong *a* theorem so long as G_{IJ} is positive definite. This is remarkable, since attempts [11] to extend the methods [6] used to prove the strong *a* theorem to three dimensions did not appear to lead to a relation of the desired form. In Ref. [10] we firstly used the leading-order (two-loop) *β*-functions computed in Refs. [12,13] to construct a solution of Eq. (1.1) for Abelian and non-Abelian [for the case $SU(n)$] Chern-Simons theories at leading order. Our method was essentially that employed in four dimensions in the classic work of Ref. [14]. The "metric" G_{IJ} was indeed found to be positive definite at

this order, at least perturbatively. The Yukawa and scalar couplings in these theories were of a restricted form. However, by considering completely general scalar and fermion theories (but without gauge interactions) we were able to argue that the existence of the *a*-function was somewhat trivial for these theories at leading order, but that predictions for the scalar-coupling-dependent contributions to the next-to-leading-order (four-loop) Yukawa *β*-function emerged and could be verified by an explicit computation.

In this paper our purpose is first of all to extend the general leading-order calculation to the gauged case (we present results for the Abelian case, but the extension to the non-Abelian case is straightforward) and secondly to complete the four-loop computation[1] of the Yukawa *β*-function [10] for a general scalar or fermion theory and show that we can extend the definition of the *a*-function in Eq. (1.1) to this order. It turns out that in the gauged case the existence of the *a*-function is nontrivial even at leading order; it imposes constraints on the *β*-function coefficients which we will show are satisfied. A by-product of our extended leading-order computation is a prediction for the scalar-coupling-dependent contribution to the four-loop Yukawa *β*-function for a completely general (i.e. gauged) renormalizable theory in three dimensions.

It has already been proposed that the free energy *F* in three dimensions may have similar properties to the four-dimensional *a*-function, leading to a conjectured "*F* theorem" [15–18]. It has been shown that for certain theories in three dimensions the free energy does indeed decrease monotonically along RG trajectories. It has also been shown that *F* obeys a gradient flow equation at leading order for theories which may be regarded as a

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[1]Modulo the anomalous dimensions for which we do not have a fully independent computation, as we shall explain later.

perturbation around a conformal field theory. This covers theories with a scalar potential, which may be regarded a perturbation around a free field theory. However the only nontrivial example at leading order, namely a gauged scalar-fermion theory, does not fall into this class. This is the reason why we have pursued the computation beyond leading order despite its complexities. On the other hand, our method does not provide any general insight as to the origin of the gradient flow, so it would be interesting to investigate the relation between the "F-function" and our a-function.

The structure of the paper is as follows. In Sec. II we discuss the construction of the a-function at leading order, corresponding to the two-loop Yukawa β-function. Here we consider a completely general Chern-Simons gauge theory coupled to fermions and scalars, and we show that in this gauged case Eq. (1.1) imposes nontrivial constraints on the β-function coefficients which are indeed satisfied. In Sec. III we proceed to the next-to-leading order, but for a general ungauged scalar or fermion theory. Here the a-function is determined by the two-loop scalar β-function and the four-loop Yukawa β-function. We show that Eq. (1.1) imposes a plethora of constraints upon the four-loop Yukawa β-function coefficients; and we compute the four-loop β-function to demonstrate that these are all satisfied. Various remarks are offered in a conclusion. Finally, a number of technical details are postponed to Appendixes: namely, an explicit list of the tensor structures in which we expressed the β-function results, together with an explanation of our choice of these structures, and the full set of consistency conditions and four-loop Yukawa β-function results at next-to-leading order. We also discuss there the scheme dependence of our results. We present in a final Appendix our prediction for the scalar-coupling-dependent contribution to the general four-loop Yukawa β-function, as mentioned above.

II. LEADING-ORDER RESULTS

In this section we define the general three-dimensional Abelian Chern-Simons theory, present its β-functions at lowest order (two loops) and construct the leading term in the a-function. The Lagrangian is given by

$$L = \frac{1}{2}[\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + (D_\mu\phi_i)^2 + i\bar{\psi}_a D\psi_a]$$
$$+ \frac{1}{4}Y_{abij}\psi_a\psi_b\phi_i\phi_j - \frac{1}{6!}h_{ijklmn}\phi_i\phi_j\phi_k\phi_l\phi_m\phi_n, \quad (2.1)$$

where we employ a real basis for both scalar and fermion fields and $D_\mu = \partial_\mu - iEA_\mu$, where E is a charge matrix (E^ϕ and E^ψ for scalar and fermion fields, respectively). Recall that in $d = 3$, $\bar{\psi} = \psi^{*T}$, and there is no obstacle to decomposing ψ into real Majorana fields. Gauge invariance entails the identities

$$E^\psi_{ac}Y_{cbij} + E^\psi_{bc}Y_{acij} + E^\phi_{im}Y_{abmj} + E^\phi_{jm}Y_{abim} = 0,$$
$$E^\phi_{ip}h_{pjklmn} + \text{perms} = 0. \quad (2.2)$$

The L-loop Yukawa and scalar β-functions take the respective forms

$$(\beta_Y^{(L)})_{abij} = \sum_{\alpha=1}^{n_L} c_\alpha^{(L)}(U_\alpha^{(L)})_{abij},$$
$$(\beta_h^{(L)})_{ijklmn} = \sum_{\alpha=1}^{m_L} d_\alpha^{(L)}(V_\alpha^{(L)})_{ijklmn}, \quad (2.3)$$

where $U_\alpha^{(L)}$ and $V_\alpha^{(L)}$ denote L-loop tensor structures. In the interest of brevity, in the main body of the text we shall simply give a diagrammatic representation of the various tensor structures appearing here; to avoid any ambiguity the full expressions will be given in Appendix A. In these diagrams the Yukawa and scalar couplings will be represented by vertices, with the fermion and scalar legs indicated thus, with lines indicating contracted indices:
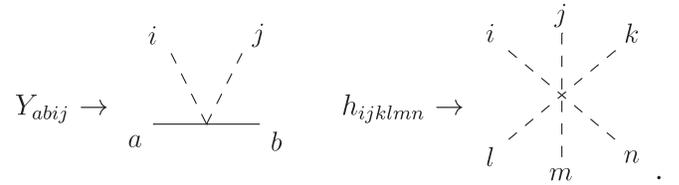
At two loops, the number of tensor structures appearing in the Yukawa β-function in Eq. (2.3) is given by $n_2 = 29$, and the two-loop tensor structures are displayed in Fig. 1 and written explicitly in Eqs. (A1)–(A3). A small circle represents a single gauge matrix E^ϕ or E^ψ, and a square represents a product of two E^ϕ or E^ψ. Each tensor structure is defined so as to have a "weight" of one, as explained in Appendix A, where we also explain our choice for these structures, which is not unique since tensor structures containing gauge matrices may be related through the gauge-invariance identity Eq. (2.2). We note here that $U_{22}^{(2)}$–$U_{25}^{(2)}$ correspond to anomalous dimension contributions and consequently we may simply read off the corresponding values of $c_{22}^{(2)}$–$c_{25}^{(2)}$ from the results of Ref. [12] with no further calculation. In the case of $U_{24}^{(2)}$ and $U_{25}^{(2)}$, there is also a graph with a fermion loop which is not depicted but whose contribution may be seen in Eq. (A3). We have assumed that the contributions from single fermion loops and single scalar loops are equal; this is consistent with our other explicit calculations but in any case does not affect any of our conclusions.

The computation of the two-loop Yukawa β-function is very straightforward and we simply quote the result, namely
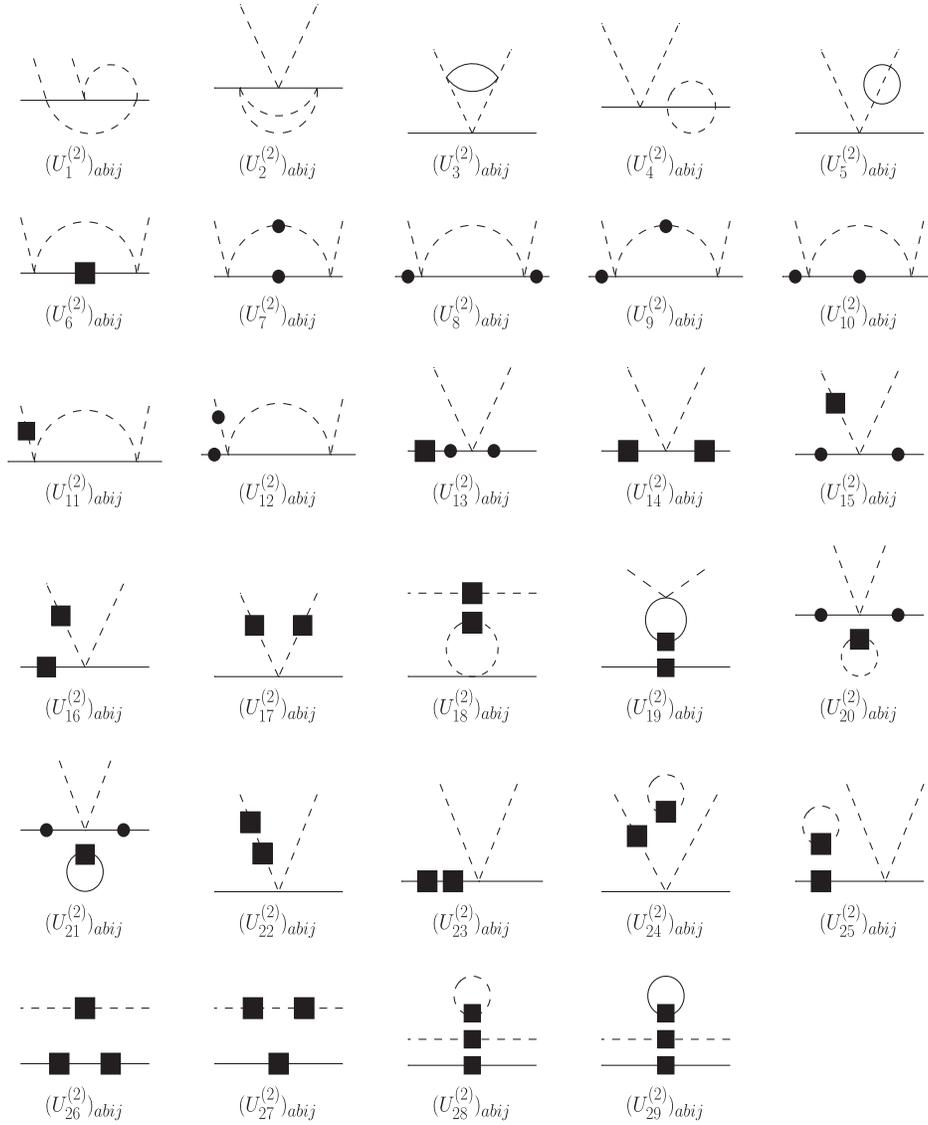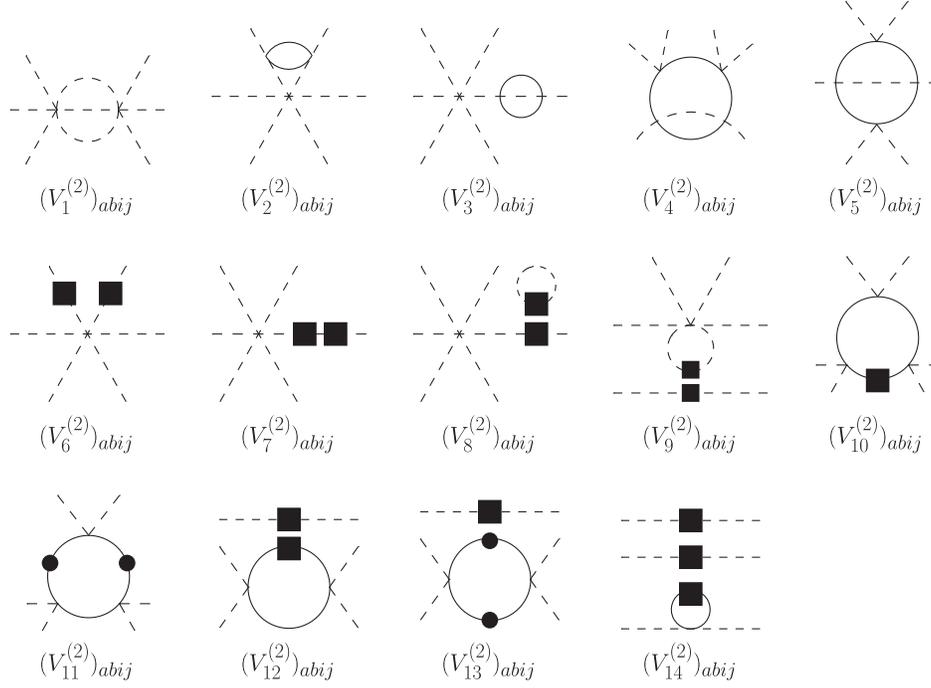
FIG. 1.   Two-loop tensor structures appearing in $\beta_Y^{(2)}$.

$$\beta_Y^{(2)} = 8U_1^{(2)} + 2(U_2^{(2)} + U_3^{(2)}) + \frac{2}{3}(U_4^{(2)} + U_5^{(2)}) + 8(U_6^{(2)} + U_7^{(2)} + U_8^{(2)})$$
$$- 16U_9^{(2)} + 24U_{13}^{(2)} + 4U_{14}^{(2)} + 24U_{15}^{(2)} - 16U_{16}^{(2)} - 8(U_{17}^{(2)} + U_{18}^{(2)} + U_{19}^{(2)})$$
$$+ 2(U_{20}^{(2)} + U_{21}^{(2)}) - \frac{40}{3}U_{22}^{(2)} - \frac{4}{3}U_{23}^{(2)} - \frac{8}{3}U_{24}^{(2)} - \frac{2}{3}U_{25}^{(2)}$$
$$- 32(U_{26}^{(2)} + U_{27}^{(2)}) - 8(U_{28}^{(2)} + U_{29}^{(2)}). \tag{2.4}$$

Here and elsewhere we suppress a factor of $(8\pi)^{-1}$ for each loop order. The individual coefficients $c_\alpha^{(2)}$ defined in Eq. (2.3) may then easily be read off, e.g. $c_1^{(2)} = 8$, etc. The coefficients $c_1^{(2)}$, $c_4^{(2)}$, and $c_5^{(2)}$ differ by factors from the corresponding results in Ref. [10] due to our slightly different choice of basis tensors here, as described above. Note further that with our choice of basis for $U_6^{(2)}$–$U_{12}^{(2)}$, as explained in Appendix A, each nonvanishing term here corresponds to a single Feynman diagram.

There are $m_2 = 14$ tensor structures in the two-loop scalar β-function defined in Eq. (2.3), and they are of the form depicted in Fig. 2 and defined precisely in Eq. (A4). Once again, in the case of $V_8^{(2)}$ there is also a graph with a fermion loop

FIG. 2. Two-loop tensor structures appearing in $\beta_h^{(2)}$.

which is not depicted but whose contribution may be seen in Eq. (A4). The computation of the two-loop scalar β-function is again straightforward and it is given by

$$\beta_h^{(2)} = \frac{20}{3}V_1^{(2)} + 30V_2^{(2)} + 4V_3^{(2)} - 360(V_4^{(2)} + V_5^{(2)}) - 120V_6^{(2)}$$
$$- 40V_7^{(2)} - 8V_8^{(2)} - 120V_9^{(2)} + 360(V_{10}^{(2)} - V_{11}^{(2)}) + 720(V_{13}^{(2)} + 2V_{14}^{(2)}). \tag{2.5}$$

Once again, the individual coefficients $d_\alpha^{(2)}$ defined in Eq. (2.3) may easily be read off when required later.

We now turn to the construction of the a-function at lowest order. As discussed in Ref. [10], we impose Eq. (1.1) in the form

$$\frac{\partial A^{(5)}}{\partial Y_{abij}} = \beta_{abij}^{(2)}, \tag{2.6}$$

where we define

$$\frac{\partial}{\partial Y_{abij}} Y_{a'b'i'j'} = \frac{1}{4}(\delta_{aa'}\delta_{bb'} + \delta_{ab'}\delta_{ba'})(\delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{ji'}). \tag{2.7}$$

The corresponding (lowest-order) contribution to the metric T_{IJ} is therefore effectively chosen to be the unit matrix in coupling space. The most general lowest-order a-function which may satisfy Eq. (2.6) is given by

$$A^{(5)} = \sum_{\alpha=1}^{9} a_\alpha^{(5)}A_\alpha^{(5)} + \sum_{\alpha=13}^{29} a_\alpha^{(5)}A_\alpha^{(5)}, \tag{2.8}$$

where the $A_\alpha^{(5)}$, $\alpha = 1\ldots9$, are given in Eq. (A5) and depicted in Fig. 3.

The terms $A_{13}^{(5)}$–$A_{29}^{(5)}$ are defined by

$$A_\alpha^{(5)} = (U_\alpha^{(2)})_{abij}Y_{abij}, \tag{2.9}$$

with $U_\alpha^{(2)}$ as defined in Eq. (A3); we choose not to display the corresponding diagrams which are of a very simple symmetric form. The reader will note that we have not defined structures corresponding to $A_{10}^{(5)}$–$A_{12}^{(5)}$. This is purely for notational convenience, in order to ensure that $A_\alpha^{(5)}$ corresponds with $U_\alpha^{(2)}$ in Eq. (2.9). Differentiating the terms in $A^{(5)}$ with respect to Y_{abij} corresponds to removing each vertex in turn, leaving a structure which may be expressed in terms of one or possibly several of the $U_\alpha^{(2)}$. For instance,

$$Y_{acij}Y_{cdkl}Y_{dbkl} \in \beta_Y^{(2)}.$$
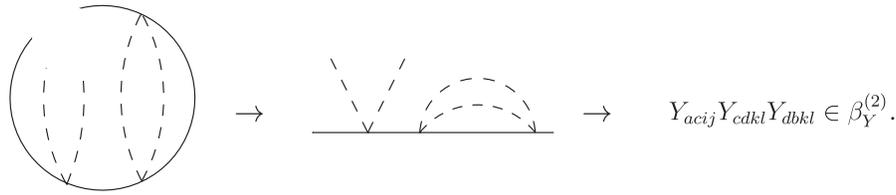
We then find immediately upon comparison of the coefficients of $U_\alpha^{(2)}$ in Eq. (2.6) that

$$a_\alpha^{(5)} = \frac{1}{4}c_\alpha^{(2)}, \quad \alpha = 1,\dots 5. \qquad (2.10)$$

There is no constraint on the two-loop coefficients $c_1^{(2)}$–$c_5^{(2)}$ in Eq. (2.4). This is because the symmetries of the corresponding tensor structures appearing in Eq. (A5) imply a one-to-one relation between a-function contributions and Yukawa β-function contributions. This may be seen in Fig. 3 where in each of the diagrams $A_1^{(5)}$–$A_5^{(5)}$ the removal of any vertex leads to the same β-function contribution. These a-function coefficients can thus be tailored term by term to match any values for the β-function coefficients. The same is true of $A_{13}^{(5)}$–$A_{29}^{(5)}$, and again we immediately find

$$a_\alpha^{(5)} = \frac{1}{2}c_\alpha^{(2)}, \quad \alpha = 13,\dots 25,$$
$$a_\alpha^{(5)} = c_\alpha^{(2)}, \quad \alpha = 26,\dots 29. \qquad (2.11)$$

However, the situation is different for $A_6^{(5)}$–$A_9^{(5)}$. There are four of these independent a-function structures, fewer than the seven β-function structures in our basis. Differentiation of each of these structures with respect to Y_{abij} leads to two distinct β-function structures, which in turn should be written in terms of our basis. Consequently we find a nontrivial set of equations relating $a_6^{(5)}$–$a_9^{(5)}$ with $c_6^{(2)}$–$c_{10}^{(2)}$ whose solution is

$$a_6^{(5)} = a_8^{(5)} = c_6^{(2)}, \qquad a_7^{(5)} = 0, \qquad a_9^{(5)} = c_8^{(2)}, \quad (2.12)$$

subject to the three consistency conditions

$$c_6^{(2)} = c_7^{(2)} = -\frac{1}{2}c_9^{(2)}, \qquad c_8^{(2)} = c_7^{(2)} + \frac{1}{2}c_{10}^{(2)}, \quad (2.13)$$

which are indeed satisfied by the coefficients in Eq. (2.4). In obtaining these equations, we imposed the vanishing of two of the seven potential coefficients, namely $c_{11}^{(2)}$ and $c_{12}^{(2)}$, which are manifestly zero for reasons explained in Appendix A. These consistency conditions correspond in the obvious way to relations among the simple poles in the Feynman diagrams corresponding to $U_6^{(2)}$–$U_{10}^{(2)}$. This is because, with our basis choice, the nonbasis structures which could potentially have contributed to $c_6^{(2)}$–$c_{10}^{(2)}$ when written in terms of the basis, in fact, correspond to vanishing Feynman diagram contributions.

Combining Eqs. (2.10)–(2.12) with coefficients $c_\alpha^{(2)}$ read off from Eq. (2.4), we find that the lowest-order contribution to the a-function is
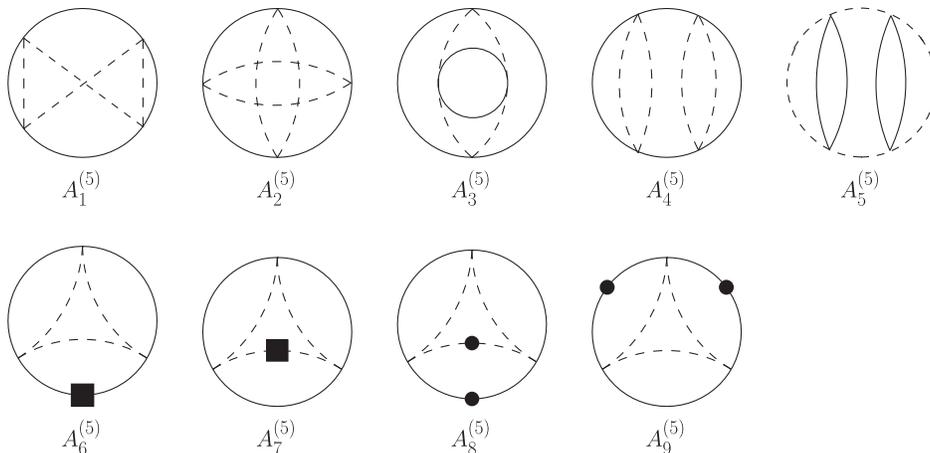

FIG. 3. Contributions to $A^{(5)}$ from Yukawa couplings.

$$A^{(5)} = 2A_1^{(5)} + \frac{1}{2}(A_2^{(5)} + A_3^{(5)}) + \frac{1}{6}(A_4^{(5)} + A_5^{(5)}) + 8(A_6^{(5)} + A_8^{(5)} + A_9^{(5)})$$
$$+ 12A_{13}^{(5)} + 2A_{14}^{(5)} + 12A_{15}^{(5)} - 8A_{16}^{(5)} - 4(A_{17}^{(5)} + A_{18}^{(5)} + A_{19}^{(5)})$$
$$+ A_{20}^{(5)} + A_{21}^{(5)} - \frac{20}{3}A_{22}^{(5)} - \frac{2}{3}A_{23}^{(5)} - \frac{4}{3}A_{24}^{(5)} - \frac{1}{3}A_{25}^{(5)}$$
$$- 32(A_{26}^{(5)} + A_{27}^{(5)}) - 8(A_{28}^{(5)} + A_{29}^{(5)}). \tag{2.14}$$

We cannot extend this leading-order (i.e. five-loop) a-function in order to generate the two-loop scalar β-function in a similar way via Eq. (1.1), as was pointed out in Ref. [10]. It is clear, on the analogy of the Yukawa β-function, that in order to satisfy Eq. (1.1) with regard to the two-loop scalar β-function the a-function must contain terms such as $(V_\alpha^{(4)})_{ijklmn} h_{ijklmn}$ [with $V_\alpha^{(4)}$ as defined in Eq. (2.3)] which correspond to seven-loop diagrams. We therefore postpone this discussion to the next section where we provide a full discussion of the seven-loop a-function.

III. HIGHER-ORDER RESULTS

In this section we consider the seven-loop a-function, which will be determined via Eq. (1.1) by the two-loop scalar β-function and the four-loop Yukawa β-function. This hierarchy of loop orders for different couplings was first noticed in the four-dimensional context [19]. At this order we need to consider also next-to-leading-order contributions on the right-hand side of Eq. (1.1). Including all relevant terms, we obtain (using a somewhat schematic notation)

$$d_h A^{(7)} = dh T_{hh}^{(5)} \beta_h^{(2)}, \tag{3.1}$$

$$d_Y A^{(7)} = dY T_{YY}^{(5)} \beta_Y^{(2)} + dY T_{YY}^{(3)} \beta_Y^{(4)}. \tag{3.2}$$

Here $T_{YY}^{(3)}$ is the leading-order contribution to T_{IJ} which as mentioned before is effectively a unit tensor, and $T_{YY}^{(5)}$ is a potential higher-order contribution described in detail later; we can easily see that no $T_{Yh}^{(5)}$ or $T_{hY}^{(5)}$ contributions are possible. We write

$$A^{(7)} = A_h^{(7)} + A_{hY}^{(7)} + A_Y^{(7)} + a(\beta_Y^{(2)})_{abij}(\beta_Y^{(2)})_{abij}. \tag{3.3}$$

Here $A_h^{(7)}$, $A_{hY}^{(7)}$ and $A_Y^{(7)}$ are the pure scalar, mixed scalar and Yukawa and pure Yukawa contributions to $A^{(7)}$, respectively, while the last term represents the usual arbitrariness [7] in the definition of A satisfying Eq. (1.1). We remark that since the β-functions are renormalization scheme dependent beyond one loop, the A-function we construct will also be scheme dependent. We analyze the scheme dependence in more detail at the end of Appendix B. Here we simply note that in any particular

scheme, at a critical point where the β-functions vanish, the dependence on a in Eq. (3.3) disappears and the A-function is universal at the critical point (up to a numerical factor which we have fixed by requiring that the metric is the unit matrix at leading order).

We can see how $\beta_h^{(2)}$ will determine $A_h^{(7)}$ and $A_{hY}^{(7)}$ through Eq. (3.1) while $\beta_Y^{(4)}$ will determine $A_Y^{(7)}$ through Eq. (3.2). Equation (3.2) will also provide consistency checks on the mixed a-function terms $A_{hY}^{(7)}$. Starting with Eq. (3.1), then, we expand

$$A_h^{(7)} = a_{h_1}^{(7)} A_{h_1}^{(7)}, \qquad A_{hY}^{(7)} = \sum_{\alpha=2}^{14} a_{h_\alpha}^{(7)} A_{h_\alpha}^{(7)}, \tag{3.4}$$

as depicted in Fig. 4, while $A_Y^{(7)}$ will be defined later.

Notice that, as suggested at the end of Sec. II, we have

$$A_{h_\alpha}^{(7)} = (V_\alpha^{(4)})_{ijklmn} h_{ijklmn}, \tag{3.5}$$

with $V_\alpha^{(4)}$ as defined in Eq. (2.3). The significance of the labels H_1–H_{13} on the diagrams will be explained shortly. Equations (2.3), (3.1), and (3.4) then imply

$$a_{h_1}^{(7)} = \frac{1}{3}\lambda d_1^{(2)},$$

$$a_{h_\alpha}^{(7)} = \frac{1}{2}\lambda d_\alpha^{(2)}, \quad \alpha = 2, 3 \quad \text{and} \quad \alpha = 6, \dots 9,$$

$$a_{h_\alpha}^{(7)} = \lambda d_\alpha^{(2)}, \quad \alpha = 4, 5 \quad \text{and} \quad \alpha = 10\dots14, \tag{3.6}$$

so that $T_{hh}^{(5)}$ is the unit tensor up to a factor of λ which will be determined shortly by the four-loop calculation. Reading off the coefficients $d_\alpha^{(2)}$ from Eq. (2.5), and substituting Eq. (3.6) into Eq. (3.4), we find $A_h^{(7)}$ and $A_{hY}^{(7)}$ in Eq. (3.3) are given by

$$A_h = \frac{20}{9}\lambda A_{h_1}^{(7)},$$

$$A_{hY} = \lambda[15A_{h_2}^{(7)} + 2A_{h_3}^{(7)} - 360(A_{h_4}^{(7)} + A_{h_5}^{(7)}) - 60A_{h_6}^{(7)}$$
$$- 20A_{h_7}^{(7)} - 4A_{h_8}^{(7)} - 60A_{h_9}^{(7)} + 360(A_{h_{10}}^{(7)} - A_{h_{11}}^{(7)})$$
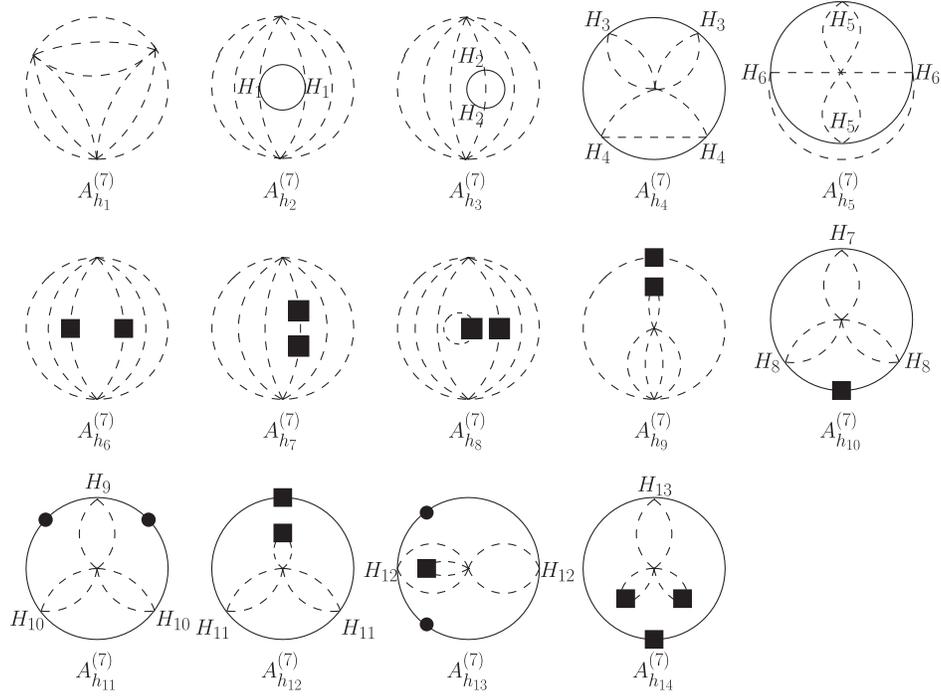$$+ 720(A_{h_{13}}^{(7)} + 2A_{h_{14}}^{(7)})]. \tag{3.7}$$

FIG. 4.   Contributions to $A_h^{(7)}$ and $A_{hY}^{(7)}$.

In the main text we shall henceforth consider the ungauged case and omit $A_{h_6}^{(7)}$–$A_{h_{14}}^{(7)}$, in order to focus on our main purpose of providing evidence for the *a* theorem in three dimensions. One piece of this evidence, presented already in Ref. [10], is the consistency check mentioned earlier, arising from the fact that $A_{hY}^{(7)}$ is determined by both $\beta_h^{(2)}$ and $\beta_Y^{(4)}$. However, if we regard the three-dimensional *a* theorem as sufficiently established, we can use Eq. (3.7) in conjunction with Eq. (3.2) to obtain a prediction for a hitherto-unknown part of the four-loop Yukawa β-function in the general gauged case. We shall postpone this to Appendix C.

Turning now to Eq. (3.2) in the ungauged case, we need to consider in Eq. (3.3) $A_{h_1}^{(7)}$–$A_{h_{14}}^{(7)}$ in $A_{hY}^{(7)}$, together with the pure Yukawa contributions, $A_Y^{(7)}$. We expand $A_Y^{(7)}$ as

$$A_Y^{(7)} = \sum_{\alpha=1}^{52} a_\alpha^{(7)} A_\alpha^{(7)}, \tag{3.8}$$

where the tensor structures $A_\alpha^{(7)}$ are depicted in Figs. 5 and 6. At this order we shall not give explicit expressions for the terms in the *a*-function since these are often quite unwieldy and may easily be reconstructed from the diagrams. In order to avoid having to specify all four-loop Yukawa β-function tensor structures explicitly, we have simply labeled every vertex in every *a*-function diagram in Figs. 4–6. We now label each Yukawa β-function term according to the vertex which, when differentiated, yields

that structure, denoting mixed gauge-Yukawa contributions from Fig. 4 by $U_{H_\alpha}^{(4)}$ and pure Yukawa contributions from Figs. 5 and 6 as $V_\alpha^{(4)}$. For instance, we have

$$\frac{\partial}{\partial Y_{abij}} A_{h_2}^{(7)} = 2(U_{H_1}^{(4)})_{abij}, \qquad \frac{\partial}{\partial Y_{abij}} A_3^{(7)} = 6(V_3^{(4)})_{abij}, \tag{3.9}$$

where

$$(U_{H_1}^{(4)})_{abij} = h_{ikmnpq}h_{jlmnpq}Y_{abkl},$$
$$(V_3^{(4)})_{abij} = Y_{ackl}Y_{cdmn}Y_{deij}Y_{efkl}Y_{fbmn}. \tag{3.10}$$

An X in Figs. 5 and 6 corresponds to a structure which cannot occur in the β-function (for instance by virtue of being one-particle reducible). The full four-loop Yukawa β-function accordingly takes the form

$$\beta_Y^{(4)} = \sum_{\alpha=1}^{6} c_{H_\alpha} U_{H_\alpha}^{(4)} + \sum_{\alpha=1}^{105} c_\alpha V_\alpha^{(4)}, \tag{3.11}$$

writing separately the mixed scalar-Yukawa and pure Yukawa terms. Similarly, the associated next-to-leading-order metric takes the form
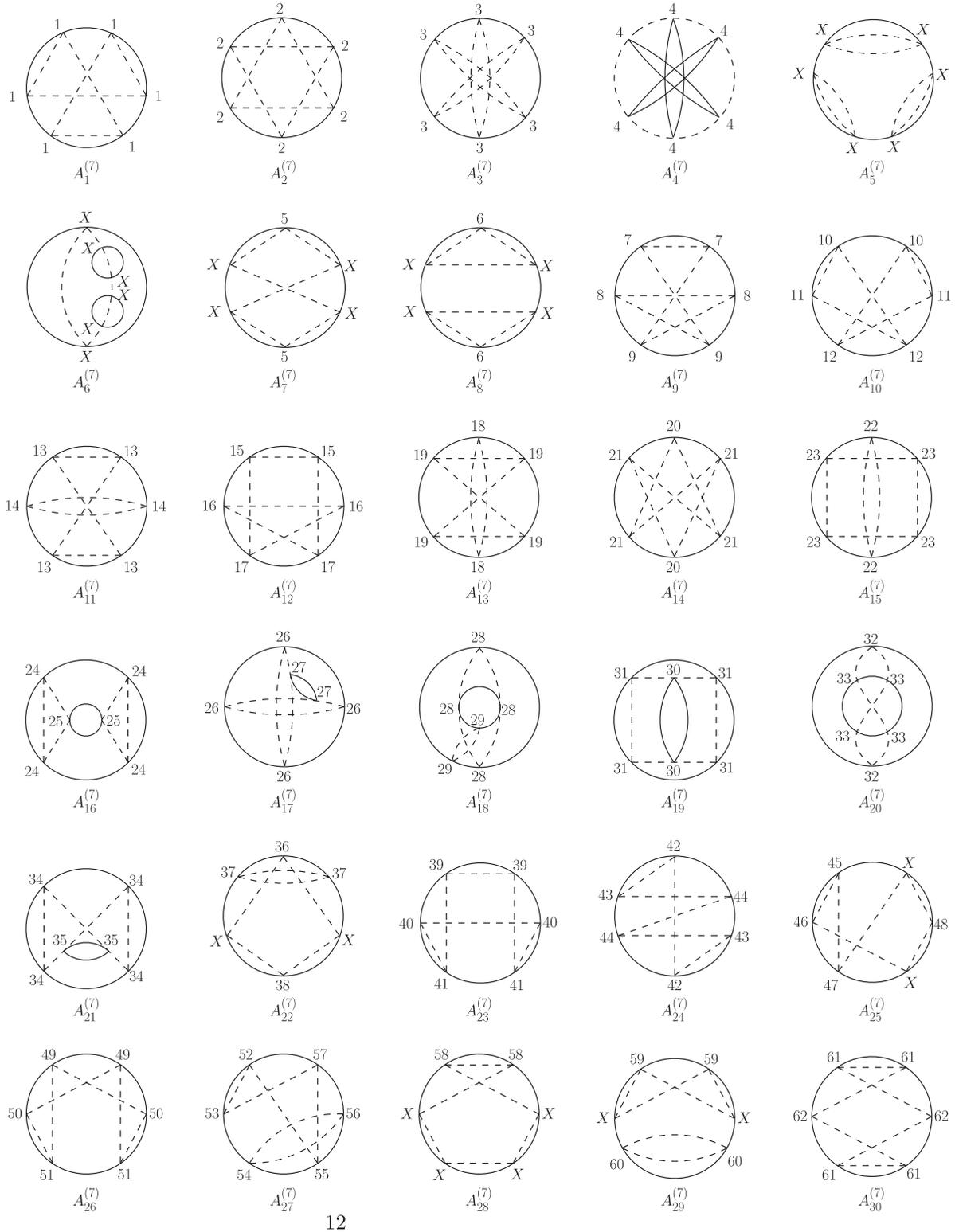
$$T_{YY}^{(5)} = \sum_{\alpha=1}^{18} t_\alpha^{(5)} (T_\alpha^{(5)})_{YY}, \tag{3.12}$$

FIG. 5. Contributions to $A^{(7)}$, terms 1–30.

where the tensor structures $T_\alpha^{(5)}$ are depicted in Fig. 7 (which for convenience show $T_\alpha^{(5)}$ contracted with dY and β_Y, denoted by a cross and a lozenge, respectively). At this order T_{IJ} is manifestly symmetric, as can easily be seen from the

diagrams in Fig. 7; this was also the case at next-to-leading order in both four [7,20] and six [21] dimensions.

At two loops, in the nongauge case, Eq. (1.1) imposed no constraints on the pure Yukawa contributions. By
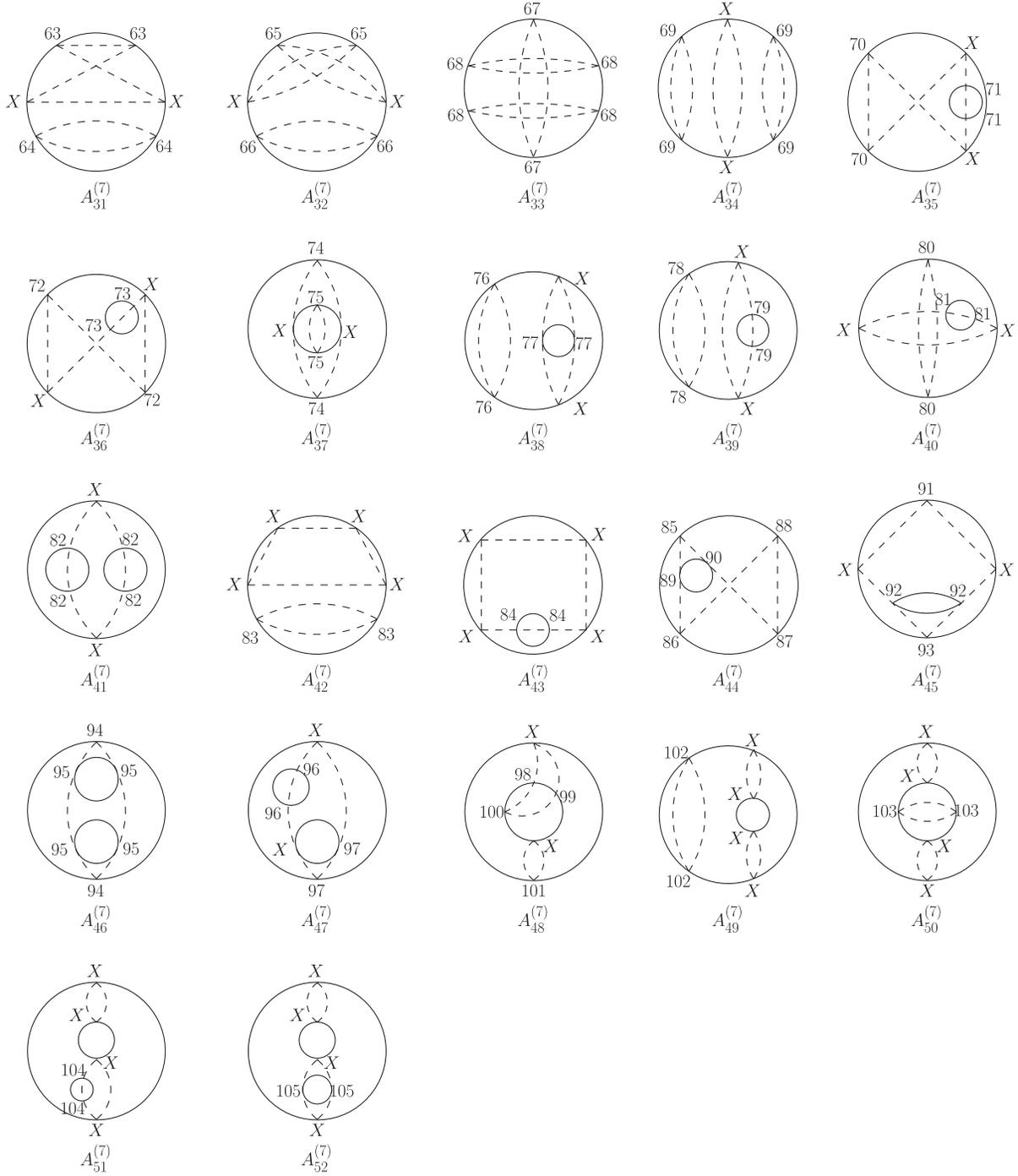
FIG. 6. Contributions to $A^{(7)}$, terms 31–52.

contrast, we shall find that Eq. (3.2) imposes a large set of consistency conditions on the four-loop β-function coefficients, which we shall confirm by direct computation. The full system of equations derived using (3.2) is highly nontrivial. Firstly we obtain immediately from inserting $A_{hY}^{(7)}$ into Eq. (3.2) and comparing with Eq. (3.11)

$$c_{H_1}^{(4)} = \lambda d_2^{(2)}, \qquad c_{H_2}^{(4)} = \lambda d_3^{(2)}, \qquad c_{H_3}^{(4)} = 2\lambda d_8^{(2)},$$
$$c_{H_4}^{(4)} = 2\lambda d_8^{(2)}, \qquad c_{H_5}^{(4)} = 2\lambda d_9^{(2)}, \qquad c_{H_6}^{(4)} = 2\lambda d_9^{(2)}, \quad (3.13)$$

relating four-loop Yukawa β-function coefficients to two-loop scalar β-function coefficients. These relations were checked already in Ref. [10]; they are satisfied provided $\lambda = \frac{1}{90}$.
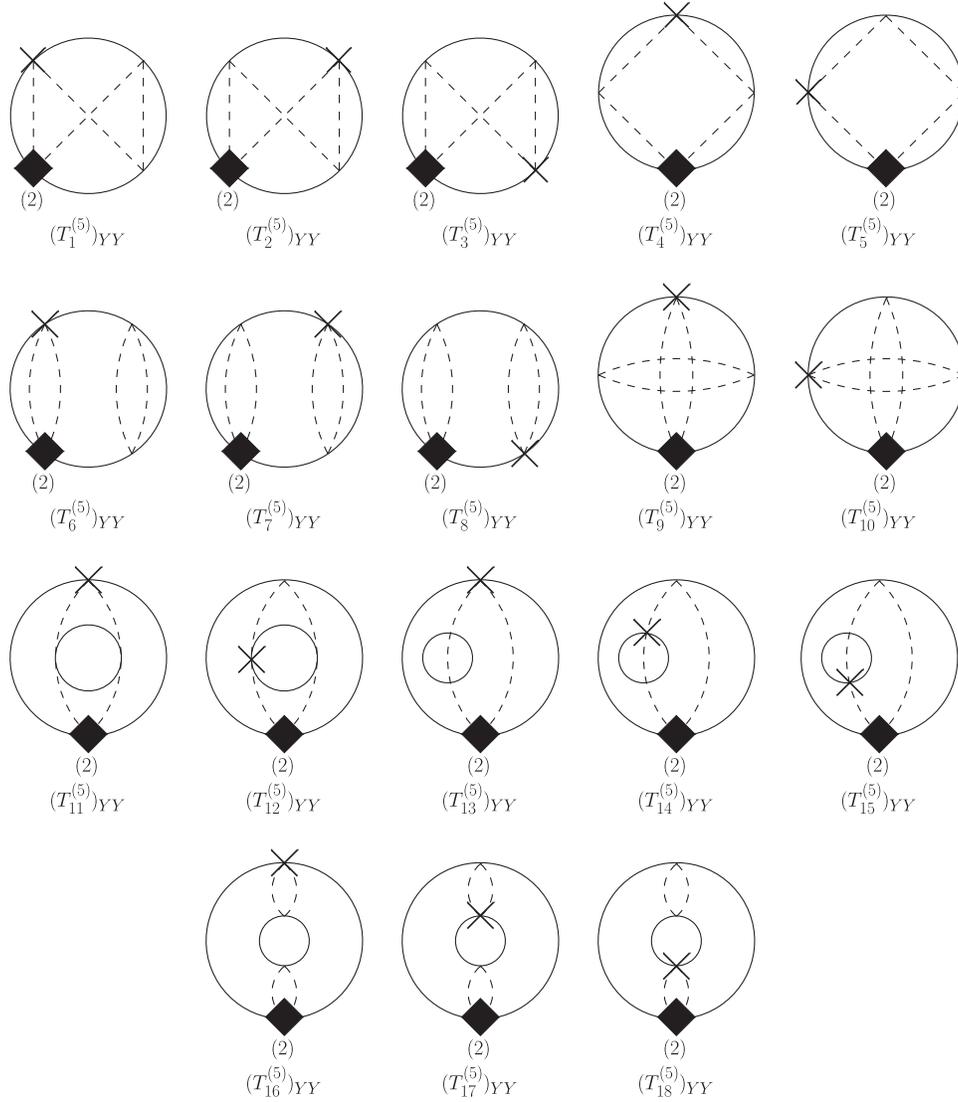
FIG. 7. Contributions to $T_{YY}^{(5)}$.

The terms in $A_Y^{(7)}$ have been arranged such that substituting into (3.2) produces conditions in the following order.

(i) Diagrams 1–6 simply relate the a-function coefficients $a_{1-6}^{(7)}$ to the $\beta_Y^{(4)}$ coefficients and give no consistency conditions.

(ii) Diagrams 7 and 8 relate tensor structures that appear in $\beta_Y^{(4)}$ to tensor structures that do not appear in $\beta_Y^{(4)}$, hence setting the corresponding β-function coefficients to zero.

(iii) Diagrams 9–21 relate tensor structures that appear in $\beta_Y^{(4)}$ but not in any metric contributions, giving simple consistency conditions.

(iv) Diagrams 22–47 relate tensor structures which appear both in $\beta_Y^{(4)}$ and in metric contributions, giving nontrivial consistency conditions.

(v) Diagrams 48–52 along with metric terms $t_{16-18}^{(5)}$ form a closed system of equations independent of the rest of the system.

With our choice of leading-order metric $T_{IJ}^{(3)} = \delta_{IJ}$, examples from the first four categories are the following.

(i) Substituting $A_1^{(7)}$ gives $6a_1^{(7)} = 4c_1^{(4)}$, so that $a_1^{(7)} = \frac{2}{3}c_1^{(4)}$ in a similar manner to the lowest-order calculation.

(ii) Substituting $A_7^{(7)}$ gives $2a_7^{(7)} = 2c_5^{(4)}$ and $4a_7^{(7)} = 0$; hence, $c_5^{(4)} = 0$.

(iii) Substituting $A_9^{(7)}$ gives $2a_7^{(7)} = 4c_7^{(4)}$, $2a_7^{(7)} = 4c_8^{(4)}$ and $2a_7^{(7)} = 4c_9^{(4)}$; hence, $c_7^{(4)} = c_8^{(4)} = c_9^{(4)}$.

(iv) Substituting $A_{25}^{(7)}$ and $A_{28}^{(7)}$ gives a set of nine equations:

$$a_{25}^{(7)} = 4c_{45}^{(4)}, \qquad a_{25}^{(7)} = 4c_{46}^{(4)}, \qquad a_{25}^{(7)} = 4c_{47}^{(4)},$$

$$a_{25}^{(7)} = 8t_5^{(5)}, \qquad a_{25}^{(7)} = 16t_4^{(5)} + 4c_{48}^{(4)}, \qquad a_{25}^{(7)} = 8t_5^{(5)},$$

$$2a_{28}^{(7)} = 4c_{58}^{(4)}, \qquad 2a_{28}^{(7)} = 8t_5^{(5)}, \qquad 2a_{28}^{(7)} = 16t_4^{(5)} + 8t_5^{(5)}, \qquad (3.14)$$

leading to the consistency conditions $t_4^{(5)} = 0$ and $c_{45}^{(4)} = c_{46}^{(4)} = c_{47}^{(4)} = c_{48}^{(4)} = c_{58}^{(4)}$.

Deriving the full system of equations and eliminating the $a_\alpha^{(7)}$ and $t_\alpha^{(5)}$ coefficients therefore leaves a large set of consistency conditions on the $\beta_Y^{(4)}$ coefficients, which are given in full in Appendix B. We have computed these four-loop coefficients and checked that they satisfy all these conditions; again, the details are given in Appendix B. Finally, using the $\overline{\text{MS}}$ values in Eqs. (B6) and (B7), the coefficients $a_i^{(7)}$ defined in Eq. (3.8) may all be computed. Combining with the nongauge parts of Eq. (3.7) and substituting into Eq. (3.3), our final result for the a-function at next-to-leading order in the general scalar-fermion theory is

$$A^{(7)} = \frac{20}{9} A_{h_1}^{(7)} + 15 A_{h_2}^{(7)} + 2 A_{h_3}^{(7)} - 360(A_{h_8}^{(7)} + A_{h_9}^{(7)}) - \frac{4}{3} A_1^{(7)} + \frac{16}{3} A_2^{(7)} - \frac{2}{3} A_3^{(7)} - \frac{1}{3} A_4^{(7)} - \frac{7}{162} A_5^{(7)} + \frac{11}{162} A_6^{(7)}$$

$$+ (2\pi^2 - 16) A_9^{(7)} + 8 A_{10}^{(7)} - 2 A_{11}^{(7)} + 2\pi^2 A_{12}^{(7)} + (\pi^2 - 8) A_{13}^{(7)} + \left(\frac{8}{3} - 16\pi^2\right) A_{14}^{(7)} + \frac{\pi^2}{2} A_{15}^{(7)}$$

$$+ 4 A_{16}^{(7)} + 8 A_{17}^{(7)} + \frac{\pi^2}{2} A_{19}^{(7)} + \pi^2 A_{20}^{(7)} + 2 A_{21}^{(7)} + \pi^2 A_{22}^{(7)} - \frac{16}{3} A_{23}^{(7)} - \frac{28}{3} A_{24}^{(7)} + 2\pi^2 A_{25}^{(7)} - \frac{4}{3} A_{26}^{(7)} - \frac{56}{3} A_{27}^{(7)} + \pi^2 A_{28}^{(7)}$$

$$- \frac{10}{9} A_{29}^{(7)} - \frac{28}{3} A_{30}^{(7)} - \frac{16}{9} A_{31}^{(7)} - \frac{8}{9} A_{32}^{(7)} - \frac{7}{3} A_{33}^{(7)} - \frac{1}{54} A_{34}^{(7)} + \frac{8}{9} A_{35}^{(7)} + \frac{2}{9} A_{36}^{(7)} - \frac{14}{3} A_{37}^{(7)} - \frac{2}{9} A_{38}^{(7)} + \frac{10}{27} A_{39}^{(7)} - \frac{2}{9} A_{40}^{(7)}$$

$$+ \frac{11}{54} A_{41}^{(7)} + \frac{\pi^2}{6}(A_{42}^{(7)} + A_{43}^{(7)}) - \frac{8}{3} A_{44}^{(7)} + \pi^2 A_{45}^{(7)} - \frac{4}{3} A_{46}^{(7)} + \frac{1}{9} A_{47}^{(7)} + 2\pi^2 A_{48}^{(7)} + \frac{\pi^2}{12} A_{49}^{(7)} + \frac{\pi^2}{4} A_{50}^{(7)} + \frac{\pi^2}{12} A_{51}^{(7)} + \frac{\pi^2}{4} A_{52}^{(7)}.$$

$$(3.15)$$

The next-to-leading-order metric coefficients in Eq. (3.12) are likewise given by

$$t_1^{(5)} = -\frac{28}{3} + 3a, \qquad t_2^{(5)} = -\frac{40}{3} + 3a, \qquad t_3^{(5)} = -\frac{16}{3} + 3a,$$

$$t_4^{(5)} = 0, \qquad t_5^{(5)} = \pi^2, \qquad t_6^{(5)} = -\frac{1}{9} + \frac{1}{4}a,$$

$$t_7^{(5)} + t_8^{(5)} = -\frac{2}{3} + \frac{1}{2}a, \qquad t_9^{(5)} = -\frac{13}{3} + \frac{3}{4}a, \qquad t_{10}^{(5)} = -\frac{14}{3} + \frac{3}{2}a,$$

$$t_{11}^{(5)} = -\frac{1}{3} + \frac{3}{4}a, \qquad t_{12}^{(5)} = -\frac{20}{3} + \frac{1}{2}a, \qquad t_{13}^{(5)} = \frac{11}{9} + \frac{3}{4}a,$$

$$t_{14}^{(5)} + t_{15}^{(5)} = \frac{1}{2}a, \qquad t_{16}^{(5)} = \frac{\pi^2}{4}, \qquad t_{17}^{(5)} = 0, \qquad t_{18}^{(5)} = \frac{\pi^2}{4}. \qquad (3.16)$$

As in previous calculations in four [7,20] and six [21] dimensions, we find that at each order the a-function is determined up to the expected freedom parametrized by a in Eq. (3.3). However, in the case of the metric, only the sums of $t_7^{(5)}, t_8^{(5)}$, and $t_{14}^{(5)}, t_{15}^{(5)}$ are determined, leading to an additional arbitrariness. $t_4^{(5)}$ vanishes identically, whereas $t_{17}^{(5)}$ is proportional to $c_{103}^{(4)} - c_{105}^{(4)}$ and hence vanishes by Eq. (B4).

IV. CONCLUSIONS

We have shown that Eq. (1.1) is valid at next-to-leading order for a general scalar-fermion theory. It seems possible that effectively we are constructing the F-function, defined in Refs. [15–18] as the free energy for the theory defined on S^3. A proof was given in Ref. [16] that the F-function satisfies Eq. (1.1) in the neighborhood of a conformal field theory at leading order; but our explicit results here provide further evidence beyond leading order [where, except as we have shown here for a gauge theory, the existence of an a-function with the properties of Eq. (1.1) is in any case trivial]. We have also shown that the two-loop scalar coupling β-function determines part of the next-to-leading-order a-function which in turn gives a prediction for the scalar-coupling-dependent sector of the four-loop Yukawa β-function, as displayed in Eq. (C3). We already checked
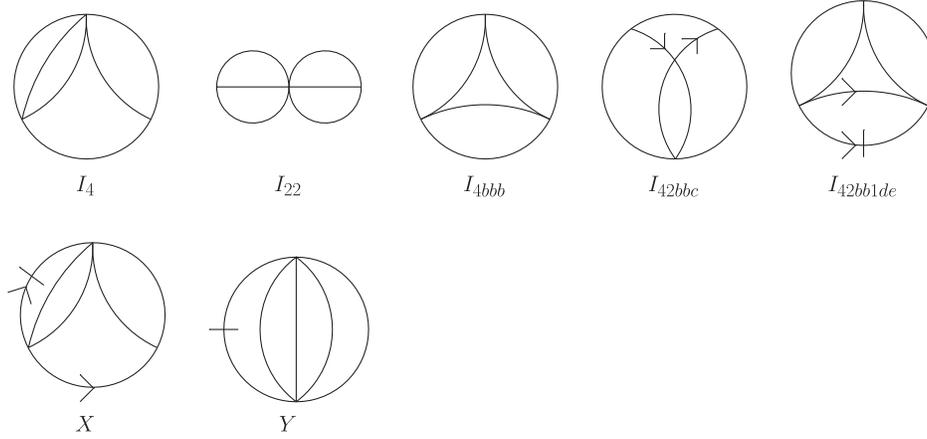
FIG. 8.   Frequently occurring Feynman integrals.

the nongauge part of this prediction in Ref. [10], but we have now extended this calculation to include the gauge terms.

We also note that there has been considerable recent attention devoted to three-dimensional $U(N)$ Chern-Simons theories in the large-N 't Hooft limit where there appear to be interesting duality properties [22–24]; see also Ref. [25] and further references therein. The A-function which we have constructed might have a role in elucidating the phase structure of these theories. The free energy also plays an important role here, but for the theory defined on $R^2 \times S^1$ corresponding to finite temperature.

A natural next step would be to investigate the case of $\mathcal{N} = 2$ supersymmetry, where partial results are available for the four-loop Yukawa β-function [26]. This would potentially extend the verification of Eq. (1.1) at next-to-leading order beyond the simple scalar-fermion case; and then assuming the equation applied more generally, it might be possible to deduce or at least strongly constrain the four-loop β-functions for a general gauge theory. One could further look for evidence for an all-orders form for the a-function in the $\mathcal{N} = 2$ supersymmetric case, such as was found in the four-dimensional case in Refs. [7,20,27–30]. The discussion in Ref. [31] may provide some pointers in this direction.

An interesting corollary of our calculations is that the consistency conditions yield a set of relationships among the divergent contributions from three-dimensional Feynman integrals. As we have seen, for $A_9^{(7)}$–$A_{21}^{(7)}$ these conditions relate the simple poles in the diagrams obtained by deleting one of the vertices in a single a-function contribution. It was shown in Ref. [32] that the Feynman diagrams obtained in the $\mathcal{N} = 2$ supersymmetric calculation could be expressed in terms of a relatively small set of integrals; and in fact this set can be further reduced using simple integration by parts. On the other hand in the current nonsupersymmetric case additional basic integrals are required with double propagators. In Fig. 8 we display the minimal required set of integrals from Ref. [32], namely

I_4, I_{22}, I_{4bbb}, I_{42bbc}, $I_{42bb1de}$ together with the most frequently occurring additional integrals, denoted X and Y. Our consistency conditions from $A_9^{(7)}$–$A_{21}^{(7)}$ imply the extra relations

$$I_{42bbc} = I_4 - \frac{1}{2}I_{22} = -2X = -2Y, \quad I_{42bb1de} = -\frac{1}{2}I_{4bbb},$$
(4.1)

so that everything can be expressed in terms of I_4, I_{22} and I_{4bbb}. These relations cannot be obtained by any simple process of integration by parts and appear to be entirely new. They are also very easy to phrase diagrammatically at least for $A_9^{(7)}$–$A_{21}^{(7)}$. The diagrams of Fig. 8 are in any case straightforward to compute using standard methods. However, it seems likely that the calculation for a general gauge theory would impose further relations among diagrams, possibly including those which are otherwise difficult to evaluate. It would be interesting to investigate whether there is any underlying topological explanation for these relations.

ACKNOWLEDGMENTS

APPENDIX A: TENSOR STRUCTURES

In this appendix we give the exact expression for the various tensor structures which were described pictorially in the main text and also explain in detail our choice of these tensor structures in view of relations among them resulting from gauge invariance as in Eq. (2.2). The contributions to the two-loop Yukawa β-function defined in Eq. (2.3) were depicted in Fig. 1. The explicit

expressions for pure Yukawa terms are given by

$$(U_1^{(2)})_{abij} = \frac{1}{4}[Y_{acil}Y_{cdjm}Y_{dblm} + Y_{aclm}Y_{cdjm}Y_{dbil} + Y_{acjl}Y_{cdim}Y_{dblm} + Y_{aclm}Y_{cdim}Y_{dbjl}],$$

$$(U_2^{(2)})_{abij} = Y_{aclm}Y_{cdij}Y_{dblm},$$

$$(U_3^{(2)})_{abij} = Y_{cdik}Y_{abkl}Y_{cdlj},$$

$$(U_4^{(2)})_{abij} = \frac{1}{2}[Y_{acij}Y_{cdlm}Y_{dblm} + Y_{adlm}Y_{dclm}Y_{cbij}],$$

$$(U_5^{(2)})_{abij} = \frac{1}{2}[Y_{abik}Y_{cdkl}Y_{dclj} + Y_{cdil}Y_{dclk}Y_{abkj}], \tag{A1}$$

those with two gauge insertions are given by

$$(U_6^{(2)})_{abij} = \frac{1}{4}[Y_{acil}(E^{\psi 2})_{cd}Y_{dblj} + (a \leftrightarrow b, \quad i \leftrightarrow j)],$$

$$(U_7^{(2)})_{abij} = \frac{1}{4}[Y_{acil}E^{\psi}_{cd}E^{\phi}_{lm}Y_{dbmj} + (a \leftrightarrow b, \quad i \leftrightarrow j)],$$

$$(U_8^{(2)})_{abij} = \frac{1}{4}[E^{\psi}_{ac}Y_{cdil}Y_{delj}E^{\psi}_{be} + (a \leftrightarrow b, \quad i \leftrightarrow j)],$$

$$(U_9^{(2)})_{abij} = \frac{1}{4}[E^{\psi}_{ac}Y_{cdil}E^{\phi}_{ml}Y_{dbmj} + (a \leftrightarrow b, \quad i \leftrightarrow j)],$$

$$(U_{10}^{(2)})_{abij} = \frac{1}{4}[E^{\psi}_{ac}Y_{cdil}E^{\psi}_{de}Y_{eblj} + (a \leftrightarrow b, \quad i \leftrightarrow j)],$$

$$(U_{11}^{(2)})_{abij} = \frac{1}{4}[(E^{\phi 2})_{im}Y_{acml}Y_{cblj} + (a \leftrightarrow b, \quad i \leftrightarrow j)],$$

$$(U_{12}^{(2)})_{abij} = \frac{1}{4}[E^{\psi}_{ac}E^{\phi}_{im}Y_{ceml}Y_{eblj} + (a \leftrightarrow b, \quad i \leftrightarrow j)], \tag{A2}$$

and those with four or six are given by

$$(U_{13}^{(2)})_{abij} = \frac{1}{2}[(E^{\psi 3})_{ac}Y_{cdij}E^{\psi}_{bd} + (a \leftrightarrow b)], \qquad (U_{14}^{(2)})_{abij} = \frac{1}{2}[(E^{\psi 2})_{ac}Y_{cdij}(E^{\psi 2})_{db} + (a \leftrightarrow b)],$$

$$(U_{15}^{(2)})_{abij} = \frac{1}{4}[(E^{\phi 2})_{ik}E^{\psi}_{ac}Y_{cdkj}E^{\psi}_{bd} + (a \leftrightarrow b, \quad i \leftrightarrow j)],$$

$$(U_{16}^{(2)})_{abij} = \frac{1}{4}[(E^{\phi 2})_{ik}(E^{\psi 2})_{ac}Y_{cbkj} + (a \leftrightarrow b, \quad i \leftrightarrow j)],$$

$$(U_{17}^{(2)})_{abij} = (E^{\phi 2})_{ik}(E^{\phi 2})_{jl}Y_{abkl}, \qquad (U_{18}^{(2)})_{abij} = (E^{\phi 2})_{ij}(E^{\phi 2})_{kl}Y_{abkl},$$

$$(U_{19}^{(2)})_{abij} = (E^{\psi 2})_{ab}(E^{\psi 2})_{cd}Y_{cdij}, \qquad (U_{20}^{(2)})_{abij} = \text{tr}(E^{\phi 2})E^{\psi}_{ac}Y_{cdij}E^{\psi}_{bd},$$

$$(U_{21}^{(2)})_{abij} = \text{tr}(E^{\psi 2})E^{\psi}_{ac}Y_{cdij}E^{\psi}_{bd}, \qquad (U_{22}^{(2)})_{abij} = \frac{1}{2}[(E^{\phi 4})_{ik}Y_{abkj} + (i \leftrightarrow j)],$$

$$(U_{23}^{(2)})_{abij} = \frac{1}{2}[(E^{\psi 4})_{ac}Y_{cbij} + (a \leftrightarrow b)],$$

$$(U_{24}^{(2)})_{abij} = \frac{1}{2}[\text{tr}(E^{\phi 2}) + \text{tr}(E^{\psi 2})][(E^{\phi 2})_{ik}Y_{abkj} + (i \leftrightarrow j)],$$

$$(U_{25}^{(2)})_{abij} = \frac{1}{2}[\text{tr}(E^{\phi 2}) + \text{tr}(E^{\psi 2})][(E^{\psi 2})_{ac}Y_{cbij} + (a \leftrightarrow b)],$$

$$(U_{26}^{(2)})_{abij} = (E^{\phi 2})_{ij}(E^{\psi 4})_{ab}, \qquad (U_{27}^{(2)})_{abij} = (E^{\phi 4})_{ij}(E^{\psi 2})_{ab},$$

$$(U_{28}^{(2)})_{abij} = \text{tr}(E^{\phi 2})(E^{\phi 2})_{ij}(E^{\psi 2})_{ab}, \qquad (U_{29}^{(2)})_{abij} = \text{tr}(E^{\psi 2})(E^{\phi 2})_{ij}(E^{\psi 2})_{ab}. \tag{A3}$$
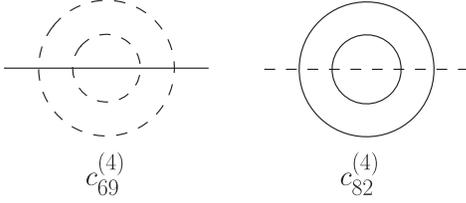
$$c_{69}^{(4)} \qquad c_{82}^{(4)}$$

FIG. 9. Feynman integrals for terms undetermined by (3.2).

Note that each tensor structure is symmetrized on a, b and i, j and has a "weight" of one. This is a different choice of basis convention from that adopted in Ref. [10]; but it means that differentiating a vertex in an a-function term gives a Yukawa β-function term with a factor of 1. A word is in order regarding the choice of tensor structures in Eq. (A2). There are in fact 13 distinct tensor structures with two insertions of E; however there are six identities among them resulting from application of Eq. (2.2), so that we can select seven of the structures as a basis. This will facilitate the construction of the a-function as we shall see shortly. Each of the 13 structures corresponds to a two-loop Feynman diagram. However, eight of these manifestly give no β-function contribution, for the following reasons. Firstly, several of them are one-particle reducible. For the remainder, the corresponding Feynman diagrams are logarithmically divergent so we can set the external momenta to zero. Structures corresponding to a Feynman diagram with only a single γ matrix then manifestly give no contribution to the β-function by Lorentz invariance; and diagrams with an E^ϕ (and therefore a gauge vertex) on an external scalar line give no contribution due to antisymmetry of the gauge propagator (resulting in a $\epsilon^{\mu\nu\rho}k_\mu k_\nu$ contribution).

We have therefore selected the seven structures $U_6^{(2)}$–$U_{12}^{(2)}$ as a convenient basis since it includes all the diagrams (namely those corresponding to $U_6^{(2)}$–$U_{10}^{(2)}$) with potentially nonvanishing β-function contributions. Note that in the Abelian case which we are currently considering, some of them correspond to more than one Feynman diagram, with different orderings of the gauge matrices.

In the case of contributions with four and six insertions of $E^{\phi,\psi}$, we have not constructed a basis, since in this case there is a one-to-one equivalence between β-function and a-function contributions, and consequently the construction of the a-function is trivial. We have simply listed in Eq. (A3) all the structures corresponding to nonvanishing contributions from Feynman diagrams.

The contributions to the two-loop scalar β function, as defined in Eq. (2.3), were depicted in Fig. 2 and are given explicitly by

$$(V_1^{(2)})_{ijklmn} = \frac{1}{6!}[h_{ijkpqr}h_{lmnpqr} + \text{perms}],$$

$$(V_2^{(2)})_{ijklmn} = \frac{1}{6!}[h_{ijklpq}Y_{abmp}Y_{abnq} + \text{perms}],$$

$$(V_3^{(2)})_{ijklmn} = \frac{1}{6!}[h_{ijklmp}Y_{abpq}Y_{abnq} + \text{perms}],$$

$$(V_4^{(2)})_{ijklmn} = \frac{1}{6!}[Y_{abij}Y_{bckl}Y_{cdmp}Y_{dapn} + \text{perms}],$$

$$(V_5^{(2)})_{ijklmn} = \frac{1}{6!}[Y_{abij}Y_{bcmp}Y_{cdkl}Y_{dapn} + \text{perms}],$$

$$(V_6^{(2)})_{ijklmn} = \frac{1}{6!}[h_{ijklpq}(E^{\phi 2})_{pm}(E^{\phi 2})_{qn} + \text{perms}],$$

$$(V_7^{(2)})_{ijklmn} = \frac{1}{6!}[h_{ijklmp}(E^{\phi 4})_{pm} + \text{perms}],$$

$$(V_8^{(2)})_{ijklmn} = \frac{1}{6!}[h_{ijklmp}(E^{\phi 2})_{pm}\{\text{tr}(E^{\phi 2}) + \text{tr}(E^{\psi 2})\}$$
$$+ \text{perms}],$$

$$(V_9^{(2)})_{ijklmn} = \frac{1}{6!}[h_{ijklpq}(E^{\phi 2})_{qp}(E^{\phi 2})_{mn} + \text{perms}],$$

$$(V_{10}^{(2)})_{ijklmn} = \frac{1}{6!}[Y_{abij}E_{bc}^{\psi 2}Y_{cdkl}Y_{damn} + \text{perms}],$$

$$(V_{11}^{(2)})_{ijklmn} = \frac{1}{6!}[Y_{abij}E_{bc}^{\psi}Y_{cdkl}E_{de}^{\psi}Y_{eamn} + \text{perms}],$$

$$(V_{12}^{(2)})_{ijklmn} = \frac{1}{6!}[E_{ij}^{\phi 2}Y_{abkl}E_{bc}^{\psi 2}Y_{camn} + \text{perms}],$$

$$(V_{13}^{(2)})_{ijklmn} = \frac{1}{6!}[E_{ij}^{\phi 2}Y_{abkl}E_{bc}^{\psi}Y_{cdmn}E_{da}^{\psi} + \text{perms}],$$

$$(V_{14}^{(2)})_{ijklmn} = \frac{1}{6!}[E_{ij}^{\phi 2}E_{kl}^{\phi 2}Y_{abmn}E_{ba}^{\psi 2} + \text{perms}], \quad (A4)$$

where "+ perms" completes the 6! permutations of the indices $\{ijklmn\}$. The choice of convention for the factors is similar to the Yukawa β-function terms. Once again, and for similar reasons as in the case of Eq. (A3), we have not constructed a complete basis of independent tensor structures but have simply listed in Eq. (A4) all the structures corresponding to nonvanishing contributions from Feynman diagrams.

The lowest-order (five-loop) a-function structures were depicted in Fig. 3 and given explicitly by

$$A_1^{(5)} = Y_{abij}Y_{bckl}Y_{cdik}Y_{dajl}, \qquad A_2^{(5)} = Y_{abij}Y_{bckl}Y_{cdij}Y_{dakl},$$

$$A_3^{(5)} = Y_{abij}Y_{cdjk}Y_{abkl}Y_{cdli}, \qquad A_4^{(5)} = Y_{acij}Y_{cbij}Y_{bdlm}Y_{dalm},$$

$$A_5^{(5)} = Y_{abik}Y_{bakj}Y_{cdil}Y_{dclj}, \qquad A_6^{(5)} = Y_{abij}(E^{\psi 2})_{bc}Y_{cdjk}Y_{daki},$$

$$A_7^{(5)} = Y_{abij}(E^{\phi 2})_{jk}Y_{bckl}Y_{cali}, \qquad A_8^{(5)} = Y_{abij}E_{bc}^{\psi}E_{jk}^{\phi}Y_{cdkl}Y_{dali},$$

$$A_9^{(5)} = Y_{abij}E_{bc}^{\psi}Y_{cdjk}E_{de}^{\psi}Y_{eaki}. \tag{A5}$$

Note that we have for the nongauge terms

$$A_\alpha^{(5)} = (U_\alpha^{(4)})_{abij} Y_{abij}, \quad \alpha = 1, \ldots 5. \tag{A6}$$

APPENDIX B: DETAILS OF FOUR-LOOP RESULTS

In this Appendix we list the full set of consistency conditions obtained by imposing Eq. (1.1) at next-to-leading order, together with the detailed four-loop results for the Yukawa β-function coefficients. The simple consistency conditions obtained from $A_9^{(7)}$–$A_{21}^{(7)}$ are

$$
\begin{aligned}
c_5^{(4)} &= 0, \qquad c_6^{(4)} = 0, \qquad c_7^{(4)} = c_8^{(4)} = c_9^{(4)}, \\
c_{10}^{(4)} &= c_{11}^{(4)} = c_{12}^{(4)}, \qquad c_{13}^{(4)} = 2c_{14}^{(4)}, \qquad c_{15}^{(4)} = c_{16}^{(4)} = c_{17}^{(4)}, \\
2c_{18}^{(4)} &= c_{19}^{(4)}, \qquad 2c_{20}^{(4)} = c_{21}^{(4)}, \qquad 2c_{22}^{(4)} = c_{23}^{(4)}, \\
c_{24}^{(4)} &= 2c_{25}^{(4)}, \qquad 2c_{26}^{(4)} = c_{27}^{(4)}, \qquad c_{28}^{(4)} = c_{29}^{(4)}, \\
2c_{30}^{(4)} &= c_{31}^{(4)}, \qquad 2c_{32}^{(4)} = c_{33}^{(4)}, \qquad c_{34}^{(4)} = 2c_{35}^{(4)}, \\
c_{54}^{(4)} &= c_{56}^{(4)}, \qquad c_{89}^{(4)} = c_{90}^{(4)},
\end{aligned} \tag{B1}
$$

and those resulting from $A_{22}^{(7)}$–$A_{47}^{(7)}$ are

$$
\begin{aligned}
c_{40}^{(4)} - c_{39}^{(4)} &= c_{42}^{(4)} - c_{44}^{(4)} = c_{50}^{(4)} - c_{49}^{(4)} \\
&= c_{52}^{(4)} - c_{57}^{(4)} = 3(c_{70}^{(4)} - c_{72}^{(4)}) = c_{87}^{(4)} - c_{86}^{(4)}, \\
c_{40}^{(4)} - c_{41}^{(4)} &= c_{42}^{(4)} - c_{43}^{(4)} = c_{50}^{(4)} - c_{51}^{(4)} = c_{52}^{(4)} - c_{53}^{(4)} \\
&= 6(c_{70}^{(4)} - c_{72}^{(4)}) - c_{61}^{(4)} + c_{62}^{(4)} = c_{87}^{(4)} - c_{88}^{(4)}, \\
c_{55}^{(4)} - 4c_{67}^{(4)} &= \frac{1}{2}(c_{56}^{(4)} - 2c_{68}^{(4)}), \qquad 3(c_{70}^{(4)} + c_{72}^{(4)}) + c_{85}^{(4)} = c_{88}^{(4)} + 12c_{97}^{(4)}, \\
c_{75}^{(4)} &= 12(c_{80}^{(4)} - c_{97}^{(4)}), \qquad c_{85}^{(4)} - c_{89}^{(4)} = 2(2c_{94}^{(4)} - c_{95}^{(4)}),
\end{aligned} \tag{B2}
$$

and

$$
\begin{aligned}
c_{52}^{(4)} - c_{55}^{(4)} - 6c_{63}^{(4)} + 12c_{65}^{(4)} + 6c_{70}^{(4)} + c_{85}^{(4)} - c_{87}^{(4)} - 12c_{97}^{(4)} &= 0, \\
3c_{59}^{(4)} + 3c_{70}^{(4)} - 6c_{77}^{(4)} - c_{87}^{(4)} + c_{89}^{(4)} - 6c_{97}^{(4)} &= 0, \\
6c_{63}^{(4)} - 3c_{70}^{(4)} + 3c_{72}^{(4)} - 12c_{77}^{(4)} - c_{85}^{(4)} - c_{88}^{(4)} + 2c_{89}^{(4)} &= 0, \\
3c_{65}^{(4)} + c_{74}^{(4)} - 3c_{77}^{(4)} &= 0, \\
4c_{67}^{(4)} - c_{68}^{(4)} + 2c_{74}^{(4)} + 6c_{80}^{(4)} + 4c_{94}^{(4)} - 2c_{95}^{(4)} - 6c_{97}^{(4)} &= 0, \\
2c_{36}^{(4)} = c_{37}^{(4)} = 2c_{38}^{(4)} = c_{45}^{(4)} = c_{46}^{(4)} = c_{47}^{(4)} = c_{48}^{(4)} = c_{58}^{(4)} = 2c_{91}^{(4)} = c_{92}^{(4)} = 2c_{93}^{(4)}. &
\end{aligned} \tag{B3}
$$

The condition resulting from $A_{48}^{(7)}$–$A_{52}^{(7)}$ is

$$c_{98}^{(4)} = c_{99}^{(4)} = c_{100}^{(4)} = c_{101}^{(4)} = 6c_{102}^{(4)} = 4c_{103}^{(4)} = 6c_{104}^{(4)} = 4c_{105}^{(4)}. \tag{B4}$$

All anomalous dimension terms (except $c_{102}^{(4)}$ and $c_{104}^{(4)}$) have been eliminated from the above consistency conditions; the consistency conditions which do involve anomalous dimension coefficients may be expressed in the following form:

$$c_{60}^{(4)} = c_{59}^{(4)} - 2c_{63}^{(4)} + 12c_{69}^{(4)} + 2c_{70}^{(4)} - \frac{1}{6}c_{86}^{(4)} - \frac{1}{6}c_{87}^{(4)} + \frac{1}{3}c_{90}^{(4)} + \frac{4}{3}c_{94}^{(4)} - \frac{2}{3}c_{95}^{(4)} - 4c_{97}^{(4)},$$

$$c_{64}^{(4)} = -c_{63}^{(4)} + 12c_{69}^{(4)} + c_{70}^{(4)} - \frac{1}{6}c_{86}^{(4)} + \frac{1}{6}c_{90}^{(4)} + \frac{2}{3}c_{94}^{(4)} - \frac{1}{3}c_{95}^{(4)} - 2c_{97}^{(4)},$$

$$c_{66}^{(4)} = -\frac{1}{2}c_{63}^{(4)} + 6c_{69}^{(4)} + \frac{1}{2}c_{70}^{(4)} + \frac{1}{6}c_{74}^{(4)} - \frac{1}{2}c_{80}^{(4)} + \frac{1}{3}c_{94}^{(4)} - \frac{1}{6}c_{95}^{(4)} - \frac{1}{2}c_{97}^{(4)},$$

$$c_{71}^{(4)} = 12c_{82}^{(4)} - \frac{1}{6}c_{86}^{(4)} + \frac{1}{6}c_{90}^{(4)} + \frac{2}{3}c_{94}^{(4)} - \frac{1}{3}c_{95}^{(4)} - 2c_{97}^{(4)},$$

$$c_{73}^{(4)} = 12c_{82}^{(4)} - \frac{1}{6}c_{87}^{(4)} + \frac{1}{6}c_{90}^{(4)} + \frac{2}{3}c_{94}^{(4)} - \frac{1}{3}c_{95}^{(4)} - 2c_{97}^{(4)},$$

$$c_{76}^{(4)} = -c_{63}^{(4)} + 6c_{69}^{(4)} + c_{70}^{(4)} + c_{77}^{(4)} + \frac{2}{3}c_{94}^{(4)} - \frac{1}{3}c_{95}^{(4)} - 2c_{97}^{(4)},$$

$$c_{78}^{(4)} = -\frac{1}{6}c_{63}^{(4)} + 2c_{69}^{(4)} + \frac{1}{6}c_{70}^{(4)},$$

$$c_{79}^{(4)} = \frac{1}{6}c_{63}^{(4)} - \frac{1}{6}c_{70}^{(4)} + 2c_{82}^{(4)},$$

$$c_{81}^{(4)} = \frac{1}{3}c_{74}^{(4)} + 6c_{82}^{(4)} + \frac{2}{3}c_{94}^{(4)} - \frac{1}{3}c_{95}^{(4)} - c_{97}^{(4)},$$

$$c_{83}^{(4)} = \frac{1}{6}c_{92}^{(4)},$$

$$c_{84}^{(4)} = \frac{1}{6}c_{92}^{(4)},$$

$$c_{96}^{(4)} = 6c_{82}^{(4)} + \frac{1}{3}c_{94}^{(4)} - \frac{1}{6}c_{95}^{(4)} - c_{97}^{(4)}. \tag{B5}$$

We see that in this form these conditions have the effect of predicting the values of all the anomalous dimension coefficients except two, namely $c_{69}^{(4)}$ and $c_{82}^{(4)}$. We need only calculate the two explicit Feynman integrals depicted in Fig. 9 corresponding to $c_{69}^{(4)}$ and $c_{82}^{(4)}$ in order to obtain all 14 possible anomalous dimension coefficients. This is useful since the Feynman diagrams corresponding to anomalous dimensions are typically harder to evaluate than the others, being linearly or quadratically divergent.

The nonanomalous dimension terms can all be calculated in $\overline{\text{MS}}$ via integration by parts, using master integrals computed in Ref. [32]. The coefficients are

$$c_1^{(4)} = -8, \qquad c_2^{(4)} = 32, \qquad c_3^{(4)} = -4, \qquad c_4^{(4)} = -2, \qquad c_5^{(4)} = 0,$$

$$c_6^{(4)} = 0, \qquad c_7^{(4)} = 4(\pi^2 - 8), \qquad c_8^{(4)} = 4(\pi^2 - 8), \qquad c_9^{(4)} = 4(\pi^2 - 8), \qquad c_{10}^{(4)} = 16,$$

$$c_{11}^{(4)} = 16, \qquad c_{12}^{(4)} = 16, \qquad c_{13}^{(4)} = -8, \qquad c_{14}^{(4)} = -4, \qquad c_{15}^{(4)} = 4\pi^2,$$

$$c_{16}^{(4)} = 4\pi^2, \qquad c_{17}^{(4)} = 4\pi^2, \qquad c_{18}^{(4)} = 2(\pi^2 - 8), \qquad c_{19}^{(4)} = 4(\pi^2 - 8), \qquad c_{20}^{(4)} = 16\left(\frac{\pi^2}{3} - 2\right),$$

$$c_{21}^{(4)} = 32\left(\frac{\pi^2}{3} - 2\right), \qquad c_{22}^{(4)} = \pi^2, \qquad c_{23}^{(4)} = 2\pi^2, \qquad c_{24}^{(4)} = 16, \qquad c_{25}^{(4)} = 8,$$

$$c_{26}^{(4)} = 32, \qquad c_{27}^{(4)} = 64, \qquad c_{28}^{(4)} = 0, \qquad c_{29}^{(4)} = 0, \qquad c_{30}^{(4)} = \pi^2,$$

$$c_{31}^{(4)} = 2\pi^2, \qquad c_{32}^{(4)} = 2\pi^2, \qquad c_{33}^{(4)} = 4\pi^2, \qquad c_{34}^{(4)} = 8, \qquad c_{35}^{(4)} = 4,$$

$$c_{36}^{(4)} = \pi^2, \qquad c_{37}^{(4)} = 2\pi^2, \qquad c_{38}^{(4)} = \pi^2, \qquad c_{39}^{(4)} = 8, \qquad c_{40}^{(4)} = 16,$$

$$c_{41}^{(4)} = 0, \qquad c_{42}^{(4)} = 8, \qquad c_{43}^{(4)} = -8, \qquad c_{44}^{(4)} = 0, \qquad c_{45}^{(4)} = 2\pi^2,$$

$$c_{46}^{(4)} = 2\pi^2, \qquad c_{47}^{(4)} = 2\pi^2, \qquad c_{48}^{(4)} = 2\pi^2, \qquad c_{49}^{(4)} = 16, \qquad c_{50}^{(4)} = 24$$

$$c_{51}^{(4)} = 8, \qquad c_{52}^{(4)} = 8, \qquad c_{53}^{(4)} = -8, \qquad c_{54}^{(4)} = 0, \qquad c_{55}^{(4)} = 16,$$

$$c_{56}^{(4)} = 0, \qquad c_{57}^{(4)} = 0, \qquad c_{58}^{(4)} = 2\pi^2, \qquad c_{59}^{(4)} = \frac{16}{3}, \qquad c_{61}^{(4)} = 0,$$

$$c_{62}^{(4)} = 0, \qquad c_{63}^{(4)} = \frac{8}{3}, \qquad c_{65}^{(4)} = \frac{8}{3}, \qquad c_{67}^{(4)} = 4, \qquad c_{68}^{(4)} = 0,$$

$$c_{70}^{(4)} = 8, \qquad c_{72}^{(4)} = \frac{16}{3}, \qquad c_{74}^{(4)} = 4, \qquad c_{75}^{(4)} = 0, \qquad c_{77}^{(4)} = 4,$$

$$c_{80}^{(4)} = \frac{8}{3}, \qquad c_{85}^{(4)} = 0, \qquad c_{86}^{(4)} = 16, \qquad c_{87}^{(4)} = 24, \qquad c_{88}^{(4)} = 8,$$

$$c_{89}^{(4)} = 24, \qquad c_{90}^{(4)} = 24, \qquad c_{91}^{(4)} = \pi^2, \qquad c_{92}^{(4)} = 2\pi^2, \qquad c_{93}^{(4)} = \pi^2,$$

$$c_{94}^{(4)} = -2, \qquad c_{95}^{(4)} = 8, \qquad c_{97}^{(4)} = \frac{8}{3}, \qquad c_{98}^{(4)} = 2\pi^2, \qquad c_{99}^{(4)} = 2\pi^2,$$

$$c_{100}^{(4)} = 2\pi^2, \qquad c_{101}^{(4)} = 2\pi^2, \qquad c_{103}^{(4)} = \frac{\pi^2}{2}, \qquad c_{105}^{(4)} = \frac{\pi^2}{2}. \tag{B6}$$

One can therefore see that these coefficients satisfy every equation in Eqs. (B1)–(B4). Evaluating the integrals in Fig. 9 and subtracting the central two-loop subdivergences, we find that

$$c_{69}^{(4)} = \frac{4}{27}, \qquad c_{82}^{(4)} = \frac{22}{27}, \tag{B7}$$

and hence using (B5) the other anomalous dimension coefficients are predicted to be

$$c_{60}^{(4)} = \frac{4}{9}, \qquad c_{64}^{(4)} = -\frac{8}{9}, \qquad c_{66}^{(4)} = -\frac{4}{9}, \qquad c_{71}^{(4)} = \frac{16}{9}$$

$$c_{73}^{(4)} = \frac{4}{9}, \qquad c_{76}^{(4)} = \frac{8}{9}, \qquad c_{78}^{(4)} = \frac{32}{27}, \qquad c_{79}^{(4)} = \frac{20}{27},$$

$$c_{81}^{(4)} = -\frac{4}{9}, \qquad c_{83}^{(4)} = \frac{\pi^2}{3}, \qquad c_{84}^{(4)} = \frac{\pi^2}{3}, \qquad c_{96}^{(4)} = \frac{2}{9},$$

$$c_{102}^{(4)} = \frac{\pi^2}{3}, \qquad c_{104}^{(4)} = \frac{\pi^2}{3}. \tag{B8}$$

We have checked most, though not all, of these predictions by explicit computation.

Finally, a word on scheme dependence. A change in scheme can be effected by a redefinition of the couplings. In our case we may consider

$$\delta Y^{(2)} = \sum_{\alpha=1}^{5} \delta_\alpha U_\alpha^{(2)}, \tag{B9}$$

where the $U_\alpha^{(2)}$ are defined in Eq. (A1) (we continue to focus on the scalar-fermion case and omit potential gauge contributions). The resulting changes in $\beta_Y^{(4)}$ and $A^{(7)}$ are given by

$$\delta A^{(7)} = -\delta Y^{(2)} \cdot \frac{\partial}{\partial Y} A^{(5)},$$

$$\delta \beta_Y^{(4)} = \beta_Y^{(2)} \cdot \frac{\partial}{\partial Y} \delta Y^{(2)} - \delta Y^{(2)} \cdot \frac{\partial}{\partial Y} \beta_Y^{(2)}. \tag{B10}$$

Using the two-loop β-function (for the nongauged theory) as given by Eqs. (2.3), (A1), and (2.4), the induced changes in the *a*-function coefficients as defined in Eq. (3.8) are given by

$$\delta a_5^{(7)} = -\frac{4}{3}\delta_5, \qquad \delta a_6^{(7)} = -\frac{4}{3}\delta_4, \qquad \delta a_{23}^{(7)} = -8\delta_1,$$

$$\delta a_{24}^{(7)} = -8\delta_1, \qquad \delta a_{26}^{(7)} = -8\delta_1, \qquad \delta a_{27}^{(7)} = -8(4\delta_1 + \delta_2),$$

$$\delta a_{29}^{(7)} = -\frac{4}{3}\delta_1 + 16\delta_5, \qquad \delta a_{30}^{(7)} = -8\delta_1, \qquad \delta a_{31}^{(7)} = -\frac{4}{3}\delta_1 + 16\delta_5,$$

$$\delta a_{32}^{(7)} = -\frac{8}{3}\delta_2 + 8\delta_5, \qquad \delta a_{33}^{(7)} = -8\delta_2, \qquad \delta a_{34}^{(7)} = -\frac{4}{3}\delta_5,$$

$$\delta a_{35}^{(7)} = -\frac{4}{3}\delta_1 + 16\delta_4, \qquad \delta a_{36}^{(7)} = -\frac{4}{3}\delta_1 + 16\delta_4, \qquad \delta a_{37}^{(7)} = -8(\delta_2 + \delta_3),$$

$$\delta a_{38}^{(7)} = -\frac{8}{3}\delta_3 + 8\delta_5, \qquad \delta a_{39}^{(7)} = -\frac{8}{3}(\delta_4 + \delta_5), \qquad \delta a_{40}^{(7)} = -\frac{8}{3}\delta_2 + 8\delta_4,$$

$$\delta a_{41}^{(7)} = -\frac{4}{3}\delta_4, \qquad \delta a_{44}^{(7)} = -8(4\delta_1 + \delta_3), \qquad \delta a_{46}^{(7)} = -8\delta_3,$$

$$\delta a_{47}^{(7)} = -\frac{8}{3}\delta_3 + 8\delta_4, \tag{B11}$$

and the corresponding changes in the $\beta_Y^{(4)}$ coefficients defined are found to be

$$\delta c_{52}^{(4)} = 2(\delta_1 - 4\delta_2), \qquad \delta c_{53}^{(4)} = 2(\delta_1 - 4\delta_2), \qquad \delta c_{54}^{(4)} = 4(4\delta_2 - \delta_1), \qquad \delta c_{55}^{(4)} = 2(4\delta_2 - \delta_1),$$

$$\delta c_{56}^{(4)} = 4(4\delta_2 - \delta_1), \qquad \delta c_{57}^{(4)} = 2(\delta_1 - 4\delta_2), \qquad \delta c_{59}^{(4)} = \frac{2}{3}\delta_1 - 8\delta_4, \qquad \delta c_{60}^{(4)} = 8\delta_4 - \frac{2}{3}\delta_1,$$

$$\delta c_{63}^{(4)} = \frac{2}{3}\delta_1 - 8\delta_4, \qquad \delta c_{64}^{(4)} = 8\delta_4 - \frac{2}{3}\delta_1, \qquad \delta c_{65}^{(4)} = \frac{4}{3}\delta_2 - 4\delta_4, \qquad \delta c_{66}^{(4)} = 4\delta_4 - \frac{4}{3}\delta_2,$$

$$\delta c_{70}^{(4)} = \frac{2}{3}\delta_1 - 8\delta_5, \qquad \delta c_{71}^{(4)} = 8\delta_5 - \frac{2}{3}\delta_1, \qquad \delta c_{72}^{(4)} = \frac{2}{3}\delta_1 - 8\delta_5, \qquad \delta c_{73}^{(4)} = 8\delta_5 - \frac{2}{3}\delta_1,$$

$$\delta c_{74}^{(4)} = 4(\delta_3 - \delta_2), \qquad \delta c_{75}^{(4)} = 16(\delta_2 - \delta_3), \qquad \delta c_{76}^{(4)} = 4\delta_4 - \frac{4}{3}\delta_3, \qquad \delta c_{77}^{(4)} = \frac{4}{3}\delta_3 - 4\delta_4,$$

$$\delta c_{78}^{(4)} = \frac{4}{3}(\delta_4 - \delta_5), \qquad \delta c_{79}^{(4)} = \frac{4}{3}(\delta_5 - \delta_4), \qquad \delta c_{80}^{(4)} = \frac{4}{3}\delta_2 - 4\delta_5, \qquad \delta c_{81}^{(4)} = 4\delta_5 - \frac{4}{3}\delta_2,$$

$$\delta c_{85}^{(4)} = 2(4\delta_3 - \delta_1), \qquad \delta c_{86}^{(4)} = 2(\delta_1 - 4\delta_3), \qquad \delta c_{87}^{(4)} = 2(\delta_1 - 4\delta_3), \qquad \delta c_{88}^{(4)} = 2(\delta_1 - 4\delta_3),$$

$$\delta c_{89}^{(4)} = 2(4\delta_3 - \delta_1), \qquad \delta c_{90}^{(4)} = 2(4\delta_3 - \delta_1), \qquad \delta c_{96}^{(4)} = 4\delta_5 - \frac{4}{3}\delta_3, \qquad \delta c_{97}^{(4)} = \frac{4}{3}\delta_3 - 4\delta_5, \tag{B12}$$

all the other coefficients remaining unchanged. Given the method of derivation, it is expected that the consistency conditions will be scheme independent. It is indeed easy to verify that all consistency conditions, including the expressions for the anomalous dimension coefficients, are invariant under the changes in Eq. (B12) and hence hold in an arbitrary renormalization scheme. This constitutes an additional check on the validity of these consistency conditions. We finally remark that at this order and in the ungauged case, we have from Eqs. (2.8) and (2.10) that $A^{(5)} = \frac{1}{4}Y_{abij}(\beta_Y^{(2)})_{abij}$ and then the freedom in $A^{(7)}$ corresponds simply to taking $\delta Y^{(2)} = -\frac{1}{4}a\beta_Y^{(2)}$. It is in fact easy to check from Eqs. (B10) and (B11) that this reproduces at lowest order the freedom expressed by the a term in Eq. (3.3) but leaves $\beta_Y^{(4)}$ unchanged. This redefinition of course vanishes

at the fixed point and so the fixed-point coupling is unchanged, as is the fixed-point value of the A-function. More general coupling redefinitions will correspond to a change in renormalization scheme with an attendant change in the β-functions, and furthermore the fixed-point coupling value and fixed-point A-function value will correspondingly be redefined.

APPENDIX C: PREDICTION FOR GENERAL YUKAWA-β-FUNCTION

As we explained in Sec. III, if we assume that the a theorem in three dimensions does indeed hold, then we may use it to derive a prediction for the part of the four-loop Yukawa β-function involving factors of the scalar coupling h, in the full gauged case. We now write

$$\beta_Y^{(4)} = \sum_{\alpha=1}^{13} c_{H_\alpha} U_{H_\alpha}^{(4)} + ..., \tag{C1}$$

extending the mixed scalar-Yukawa terms in Eq. (3.11) to the gauged case. As explained earlier, the tensor structures $U_{h_\alpha}^{(4)}$, $\alpha = 1 \ldots 13$, may be read off from the vertices in Fig. 4; the ellipsis in Eq. (C1) subsumes all the contributions with no factors of h. We computed the pure Yukawa contributions in Sec. III, but of course there will also be mixed Yukawa-gauge contributions in this general case. The coefficients $c_{H_1}^{(4)}$–$c_{H_6}^{(4)}$ were already given in Eq. (3.13). We now examine the consequences of Eq. (3.2). It is easy to see that there is no mixed scalar and Yukawa contribution from the $T_{YY}^{(5)}$ terms, since there is no possible contribution to $T_{YY}^{(5)}$ itself containing h and of course $\beta_Y^{(2)}$ does not contain h either. Equations (3.4) and (C1) now imply

$$c_{H_7}^{(4)} = \lambda d_{10}^{(2)}, \qquad c_{H_8}^{(4)} = 2\lambda d_{10}^{(2)}, \qquad c_{H_9}^{(4)} = \lambda d_{11}^{(2)},$$
$$c_{H_{10}}^{(4)} = 2\lambda d_{11}^{(2)}, \qquad c_{H_{11}}^{(4)} = 2\lambda d_{12}^{(2)}, \qquad c_{H_{12}}^{(4)} = 2\lambda d_{13}^{(2)},$$
$$c_{H_{13}}^{(4)} = \lambda d_{14}^{(2)}, \tag{C2}$$

with no contribution to $T_{YY}^{(5)}$ in Eq. (3.2). These coefficients form a prediction for the part of the four-loop Yukawa β-function involving scalar-coupling contributions; namely, combining Eqs. (2.5), (3.13), and (C2) and taking $\lambda = \frac{1}{90}$,

$$\beta_Y^{(4)} = \frac{1}{3} U_{H_1}^{(4)} + \frac{2}{45} U_{H_2}^{(4)} + 4(-2U_{H_3}^{(4)} - 2U_{H_4}^{(4)} + 2U_{H_5}^{(4)}$$
$$+ 2U_{H_6}^{(4)} - U_{H_7}^{(4)} - 2U_{H_8}^{(4)} + U_{H_9}^{(4)} + 2U_{H_{10}}^{(4)}$$
$$+ 4U_{H_{13}}^{(4)} + 4U_{H_{14}}^{(4)}) + ..., \tag{C3}$$

where we have included the nongauge terms $U_{H_1}^{(4)}$–$U_{H_6}^{(4)}$. This subsumes all the contributions to $\beta_Y^{(4)}$ for a general Abelian Chern-Simons theory involving a factor of h; as we mentioned, the purely Y-dependent terms were obtained in Sec. III. Of course the computation of the remaining mixed Yukawa-gauge terms would still require considerable labor, even after exploiting any additional consistency conditions which might arise.

Since there are many potential four-loop Yukawa β-function structures involving h which are not included in Eq. (C1), this prediction might appear to give a great deal of additional information in the form of requirements for vanishing coefficients; but we should also consider the relations among these coefficients following from Eq. (2.2). In the case of contributions with two gauge matrices, the four structures $U_{H_7}^{(4)}$–$U_{H_{10}}^{(4)}$ may be extended to a basis by adding just one more structure, whose coefficient in the β-function is easily seen to be zero without any calculation. Therefore Eq. (1.1) effectively yields only the four constraints expressed by the coefficient predictions in Eq. (C2). Similar remarks would be expected to apply to the contributions with four and six gauge matrices.

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