a-function for $\mathcal{N} = 2$ supersymmetric gauge theories in three dimensions

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Recently, the existence of a candidate a-function for renormalizable theories in three dimensions was demonstrated for a general theory at leading order and for a scalar-fermion theory at next-to-leading order. Here we extend this work by constructing the a-function at next-to-leading order for an $\mathcal{N} = 2$ supersymmetric Chern-Simons theory. This increase in precision for the a-function necessitated the evaluation of the underlying renormalization-group functions at *four* loops.

I. INTRODUCTION

Following Cardy's suggestion [1] that Zamolodchikov's two-dimensional c-theorem [2] might have an analogue in four dimensions, considerable progress has been made in proving the so-called a-theorem in even dimensions [3–15] (for a review see Ref. [16]). The Weyl anomaly played a central role in the derivation of the c-theorem and the a-theorem in even dimensions, and therefore it seems unlikely that the a-theorem could be extended to odd dimensions where there is no Weyl anomaly. An alternative candidate for a function which evolves monotonically along renormalization group (RG) flows in odd dimensions is the so-called F-function [17–20]. This is the Euclidean path integral of the theory (or "free energy") conformally mapped (in the case of three dimensions) to S^3. It has been shown to increase between UV and IR fixed points for a variety of theories. However, an additional important feature of the a-function in even dimensions is the gradient flow property; for theories with couplings g^I and corresponding RG β functions β^I, it satisfies

$$\partial_I a \equiv \frac{\partial}{\partial g^I} a = T_{IJ} \beta^J \qquad (1.1)$$

for a function T_{IJ}. A crucial consequence of Eq. (1.1) is that we then have

$$\mu \frac{d}{d\mu} a = \beta^I \frac{\partial}{\partial g^I} a = G_{IJ} \beta^I \beta^J \qquad (1.2)$$

where $G_{IJ} = T_{(IJ)}$, (IJ) denoting symmetrization. The a-theorem then follows immediately if G_{IJ} is positive definite. There is, however, so far no evidence that F possesses the gradient flow property except in simple

cases at leading order, where its existence is trivial in the sense that no conditions are imposed on the β-function coefficients.

Accordingly a different approach has recently been taken [21,22] in which a function with the gradient flow property (1.1) has been constructed order by order, using as a starting point the β functions for a range of three-dimensional theories. The method was essentially that employed in four dimensions in the classic work of Ref. [23]. Initially [21] the leading-order (two-loop) β functions computed by Avdeev *et al.* in Refs. [24,25] were used to construct a solution of Eq. (1.1) for Abelian and non-Abelian [for the case $SU(n)$] Chern-Simons theories at leading order. Moreover the "metric" G_{IJ} was indeed positive definite at this order. The Yukawa and scalar couplings in these theories were of a restricted form. However, it was then shown [21,22] that the results extended at leading order to completely general Abelian Chern-Simons theories (and there was no reason to doubt that the extension to the non-Abelian case would be fairly immediate). The extension to next-to-leading order involves the four-loop β functions (recall that there are no divergences at odd loop orders in odd dimensions). The four-loop Yukawa β function for a general (ungauged) fermion/scalar theory was therefore computed [22] and it was shown that the definition of the a-function in Eq. (1.1) could be extended to this order as well. In the general gauged case at leading order, and in the ungauged case at next-to-leading order, Eq. (1.1) imposes stringent conditions on the β-function coefficients. Clearly (in the absence of a general proof of the gradient flow property) the next natural step would be to extend the calculation for a general gauge theory to next-to-leading order. The β-function computation for a general theory at four loops would be very involved; however, the supersymmetric case is much more tractable and consequently we consider here the case of $\mathcal{N} = 2$ supersymmetry. Here we can avail ourselves of the superspace formalism to simplify the calculations and

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furthermore as a consequence of the nonrenormalization theorem [26], the Yukawa β function is determined by the chiral field anomalous dimension. Another motivation for consideration of the supersymmetric case is that the F-theorem has mostly been studied in this context and therefore a comparison might be facilitated. In order to check the validity of Eq. (1.1), it is sufficient to compute only the contributions to the anomalous dimension which contain Yukawa couplings. This is because Eq. (1.1) places constraints on the Yukawa-dependent contributions (which we shall show are satisfied) but not on the Yukawa-independent terms. This is a fortunate situation since it saves us a great deal of arduous computation.

We find that indeed we can construct the a-function satisfying Eq. (1.1) at next-to-leading order in the case of a general $\mathcal{N} = 2$ supersymmetric Chern-Simons theory. On the one hand there are far fewer constraints (only two, in fact) on the RG coefficients than we found even in the ungauged nonsupersymmetric case at next-to-leading order, so that the imposition of supersymmetry itself must guarantee that most of the original constraints are satisfied. On the other hand, one of the remaining constraints is highly nontrivial since it involves a constraint on a Feynman diagram which had not hitherto been computed and which required advanced techniques to evaluate.

The structure of the paper is as follows. In Sec. II we describe the $\mathcal{N} = 2$ Chern-Simons theory and its quantization, recall the lowest-order (two-loop) result for the anomalous dimension and use it to construct the leading-order terms in the a-function. In Sec. III we consider the a-function at next-to-leading order, and show how its existence imposes consistency conditions on some of the coefficients in the next-to-leading-order (four-loop) anomalous dimension. In Sec. IV we present our calculation of the Yukawa-dependent terms in the four-loop anomalous dimension (as explained earlier, these are all we need), in particular checking that the consistency conditions are satisfied. We explain in some detail the computation of the particular diagram mentioned above. Finally, some closing remarks are offered in Sec. V. Details of the superspace conventions and some explicit results for momentum integrals are deferred to appendixes.

II. GENERAL PROCEDURE AND LOWEST-ORDER RESULTS

In this section we describe the $\mathcal{N} = 2$ Chern-Simons theory and the general framework for our calculations. We also review the two-loop anomalous dimension calculation and construct the corresponding a-function. The action for the theory can be written as

$$S = S_{\text{SUSY}} + S_{\text{GF}} \qquad (2.1)$$

where S_{SUSY} is the usual supersymmetric action [27]

$$S_{\text{SUSY}} = \int d^3x \int d^4\theta \left(k \int_0^1 dt \text{Tr}[\bar{D}^\alpha(e^{-tV}D_\alpha e^{tV})] \right.$$
$$\left. + \Phi^j (e^{V_A R_A})^i{}_j \Phi_i \right)$$
$$+ \left(\int d^3x \int d^2\theta W(\Phi) + \text{H.c.} \right). \qquad (2.2)$$

Here V is the vector superfield, Φ is the chiral matter superfield and $W(\Phi)$ is the superpotential (quartic for renormalizability in three dimensions; see Appendix A for our $\mathcal{N} = 2$ superspace conventions. We take $W(\Phi)$ to be given by

$$W(\Phi) = \frac{1}{4!} Y^{ijkl} \Phi_i \Phi_j \Phi_k \Phi_l. \qquad (2.3)$$

[We use the convention that $\Phi^i = (\Phi_i)^*$, and also denote $\bar{Y}_{ijkl} = (Y^{ijkl})^*$.] We assume a simple gauge group, though we comment later on the extension to nonsimple groups. Gauge invariance requires the gauge coupling k to be quantized, so that $2\pi k$ is an integer. The vector superfield V is in the adjoint representation, $V = V_A T_A$ where T_A are the generators of the fundamental representation, satisfying

$$[T_A, T_B] = if_{ABC}T_C,$$
$$\text{Tr}(T_A T_B) = \delta_{AB}. \qquad (2.4)$$

The chiral superfield can be in a general representation, with gauge matrices denoted R_A satisfying

$$[R_A, R_B] = if_{ABC}R_C,$$
$$\text{Tr}(R_A R_B) = T_R \delta_{AB}. \qquad (2.5)$$

In three dimensions the Yukawa couplings Y^{ijkl} are dimensionless and (as mentioned earlier) the theory is renormalizable. In Eq. (2.1) the gauge-fixing term S_{GF} is given by [28]

$$S_{\text{GF}} = -\frac{k}{2\xi} \int d^3x d^2\theta \text{tr}[f\bar{f}] - \frac{k}{2\xi} \int d^3x d^2\bar{\theta}\text{tr}[f\bar{f}], \qquad (2.6)$$

where ξ is a gauge-fixing parameter, and we introduce into the functional integral a corresponding ghost term

$$\int \mathcal{D}f \mathcal{D}\bar{f} \Delta(V) \Delta^{-1} V \qquad (2.7)$$

with

$$\Delta(V) = \int d\Lambda d\bar{\Lambda} \delta(F(V, \Lambda, \bar{\Lambda}) - f)\delta(\bar{F}(V, \Lambda, \bar{\Lambda}) - \bar{f}),$$
$$\qquad (2.8)$$

and $\bar{F} = D^2 V$, $F = \bar{D}^2 V$. With the gauge-fixing parameter $\xi = 0$ this results in a gauge propagator

$$\langle V^A(1)V^B(2)\rangle = -\frac{1}{k}\frac{1}{\partial^2}\bar{D}^\alpha D_\alpha \delta^4(\theta_1 - \theta_2)\delta^{AB}. \qquad (2.9)$$

The gauge vertices are obtained by expanding $S_{\text{SUSY}} + S_{\text{GF}}$ as given by Eqs. (2.2) and (2.6):

$$S_{\text{SUSY}} + S_{\text{GF}} \to -\frac{i}{6} f^{ABC} \int d^3x d^4\theta \bar{D}^\alpha V^A D_\alpha V^B V^C$$

$$-\frac{1}{24} f^{ABE} f^{CDE} \int d^3x d^4\theta \bar{D}^\alpha V^A V^B D_\alpha V^C V^D + \cdots .$$

$$(2.10)$$

The ghost action resulting from Eq. (2.8) has the same form as in the four-dimensional $\mathcal{N} = 1$ case [26,29]; we refrain from quoting it explicitly since we do not need to consider diagrams with ghost propagators. Finally the chiral propagator and chiral-gauge vertices are readily obtained by expanding Eq. (2.2); the chiral propagator is given by

$$\langle \Phi^i(1) \Phi_j(2) \rangle = -\frac{1}{\partial^2} \delta^4(\theta_1 - \theta_2) \delta^i{}_j. \quad (2.11)$$

The regularization of the theory is effected by replacing V, Φ, Y by corresponding bare quantities V_B, Φ_B, Y_B, with the bare and renormalized fields related by

$$V_B = Z_V^{\frac{1}{2}} V, \qquad \Phi_B = Z_\Phi^{\frac{1}{2}} \Phi. \quad (2.12)$$

Since the Chern-Simons level k is expected to be unrenormalized for a generic Chern-Simons theory due to the topological nature of the theory (so that $k_B = k$), the only β functions are those for the superpotential coupling and its conjugate. These will be given according to the nonrenormalization theorem [26] by

$$\beta_Y^{ijkl} = (\gamma_\Phi)_m{}^{(i} Y^{jkl)m}, \qquad \beta_{\bar{Y}ijkl} = \bar{Y}_{m(ijk}(\gamma_\Phi)_{l)}{}^m \quad (2.13)$$

where the anomalous dimension γ_Φ is defined by

$$\gamma_\Phi = \frac{1}{2} \mu \frac{d}{d\mu} \ln Z_\Phi. \quad (2.14)$$

Using dimensional regularization with $d = 3 - \epsilon$ dimensions, we have

$$Z_\Phi = \sum_{L \text{ even}, m=1\ldots\frac{L}{2}} \frac{Z_\Phi^{(L,m)}}{\epsilon^m} \quad (2.15)$$

where L is the loop order. γ_Φ is determined by the simple poles in Z_Φ according to

$$\gamma_\Phi^{(L)} = -\frac{L}{2} Z_\Phi^{(L,1)}. \quad (2.16)$$

The anomalous dimension of the chiral superfield is given at two loops by [28,30]

$$(8\pi)^2 \gamma_\Phi^{(2)} = \frac{1}{3} Y_2 - \frac{2}{k^2} C_R C_R - \frac{1}{k^2} \tilde{T} C_R \quad (2.17)$$

where

$$(Y_2)_i{}^j = \bar{Y}_{iklm} Y^{jklm},$$

$$C_R = R_A R_A,$$

$$C_G \delta_{AB} = f_{ACD} f_{BCD},$$

$$\tilde{T} = T_R - C_G, \quad (2.18)$$

and T_R is defined in Eq. (2.5). This result may readily be obtained by $\mathcal{N} = 2$ superfield methods [24,28,30,31]: see Appendix A for our $\mathcal{N} = 2$ superfield conventions. Henceforth we set $k = 1$ for simplicity, and also neglect factors of $(8\pi)^2$ (one for each loop order); these factors of k and $(8\pi)^2$ may easily be restored if desired. The two-loop results for general Chern-Simons theories obtained in Ref. [25] are not directly comparable since they were computed in the $\mathcal{N} = 1$ framework.

The β functions β_Y and $\beta_{\bar{Y}}$ are given at lowest order by inserting Eq. (2.17) into Eq. (2.13). It is then clear that Eq. (1.1) is satisfied to this order in the form[1]

$$\frac{\partial}{\partial Y^{ijkl}} A = \beta_{\bar{Y}ijkl}, \qquad \frac{\partial}{\partial \bar{Y}_{ijkl}} A = \beta_Y^{ijkl}, \quad (2.19)$$

[hence effectively with a unit T_{IJ} on the right-hand side of Eq. (1.1)] by taking

$$A^{(5)} = a_1^{(5)} A_1^{(5)} + a_2^{(5)} A_2^{(5)} + a_3^{(5)} A_3^{(5)} \quad (2.20)$$

where

$$A_1^{(5)} = (Y_2)_i{}^j (Y_2)_j{}^i,$$

$$A_2^{(5)} = (Y_2)_i{}^j (C_R^2)_j{}^i,$$

$$A_3^{(5)} = C_G (Y_2)_i{}^j (C_R)_j{}^i, \quad (2.21)$$

with Y_2 given by Eq. (2.18), and

$$a_1^{(5)} = \frac{2}{3}, \qquad a_2^{(5)} = -8, \qquad a_3^{(5)} = -4\tilde{T}. \quad (2.22)$$

Notice that at this order the construction of the a-function imposes no constraints on the anomalous dimension coefficients, since there is a one-to-one correspondence between a-function structures and anomalous dimension structures. For later convenience it will be useful to rewrite Eq. (2.19) in the form

$$d_Y A \equiv dY \circ \partial_Y A = dY \circ \beta_{\bar{Y}},$$

$$d_{\bar{Y}} A \equiv d\bar{Y} \circ \partial_{\bar{Y}} A = d\bar{Y} \circ \beta_Y, \quad (2.23)$$

where $\circ$ is a scalar product on Yukawa couplings so that for instance $Y \circ \bar{Y} \equiv Y^{ijkl} \bar{Y}_{ijkl}$.

III. CONSISTENCY CONDITIONS FOR FOUR-LOOP ANOMALOUS DIMENSION

In this section we derive the consistency conditions on the four-loop anomalous dimension coefficients required for Eq. (1.1) to be satisfied at next-to-leading order. We assume an a-function at this order of the general form

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[1]We prefer to use the notation A (rather than a) in three dimensions in the absence of any connection to a conformal anomaly coefficient.

$$A^{(7)} = \sum_{i=1}^{14} a_i^{(7)} A_i^{(7)} + \tilde{a}(\beta_Y^{(2)}) \circ (\beta_{\bar{Y}}^{(2)}) \qquad (3.1)$$

where the structures $A_i^{(7)}$ are depicted in Fig. 1, except for $A_{14}^{(7)}$ which it is more convenient simply to define explicitly, viz.

$$A_{14}^{(7)} = \frac{1}{4} \mathrm{tr}(Y_2\{R_A, R_B\})\mathrm{tr}(Y_2\{R_A, R_B\}), \qquad (3.2)$$

and the final term represents the usual arbitrariness [10] in defining the a-function. Our convention for the chiral lines is that the arrows point from a Y vertex to a $\bar{Y}$ vertex; furthermore, a box represents an insertion of C_R, and an A or B represents an insertion of a gauge generator R_A or R_B respectively. At this order we expect

$$d_Y A^{(7)} = dY \circ T^{(3)} \circ \beta_{\bar{Y}}^{(4)} + dY \circ T^{(5)} \circ \beta_{\bar{Y}}^{(2)}$$
$$+ dY \circ K^{(5)} \circ \beta_Y^{(2)} \qquad (3.3)$$

where as we saw in the previous section, $T^{(3)}$ is effectively the unit matrix and we write

$$T^{(5)} = \sum_{i=1}^{7} t_i^{(5)} T_i^{(5)}, \qquad K^{(5)} = \sum_{i=1}^{2} k_i^{(5)} K_i^{(5)}, \qquad (3.4)$$

where the individual contributions $dY \circ T_i^{(5)} \circ d\bar{Y}$, $dY \circ K_i^{(5)} \circ dY$ are depicted in Figs. 2 and 3, with a cross denoting dY and a diamond denoting $d\bar{Y}$. The

corresponding expression for $d_{\bar{Y}}$ may be obtained by conjugation and is not given explicitly. Finally the four-loop anomalous dimension is expected to take the form

$$(8\pi)^4 \gamma_\Phi^{(4)} = \sum_{i=1}^{12} g_i \gamma_i^{(4)} + \cdots \qquad (3.5)$$

where the invariants involving Yukawa couplings are given by

$$(\gamma_1^{(4)})_i{}^j = \bar{Y}_{ilmn}(Y_2)_k{}^l Y^{kmnj},$$
$$(\gamma_2^{(4)})_i{}^j = \bar{Y}_{ipqr} Y^{pqmn} \bar{Y}_{mnkl} Y^{klrj},$$
$$\gamma_3^{(4)} = Y_2 C_R C_R,$$
$$(\gamma_4^{(4)})_i{}^j = \bar{Y}_{ikln}(C_R C_R)^n{}_m Y^{klmj}$$
$$(\gamma_5^{(4)})_i{}^j = \bar{Y}_{iklm}(C_R)_n{}^m Y^{pkln}(C_R)_p{}^j,$$
$$(\gamma_6^{(4)})_i{}^j = \bar{Y}_{iklm}(C_R)_n{}^l(C_R)_p{}^m Y^{knpj},$$
$$(\gamma_7^{(4)})_i{}^j = \bar{Y}_{iklm}(R_A R_B)_n{}^m Y^{pkln}(R_B R_A)_p{}^j,$$
$$(\gamma_8^{(4)})_i{}^j = \bar{Y}_{iklm}(R_A R_B)_n{}^l(R_A R_B)_p{}^m Y^{knpj},$$
$$\gamma_9^{(4)} = Y_2 C_R,$$
$$(\gamma_{10}^{(4)})_i{}^j = \bar{Y}_{iklm}(C_R)_n{}^m Y^{klnj},$$
$$(\gamma_{11}^{(4)})_i{}^j = \frac{1}{2}\mathrm{tr}(Y_2 R_A R_B)(\{R_A, R_B\})_i{}^j,$$
$$(\gamma_{12}^{(4)})_i{}^j = \bar{Y}_{iklm}(R_A C_R)_n{}^l(R_A)_p{}^m Y^{knpj}. \qquad (3.6)$$
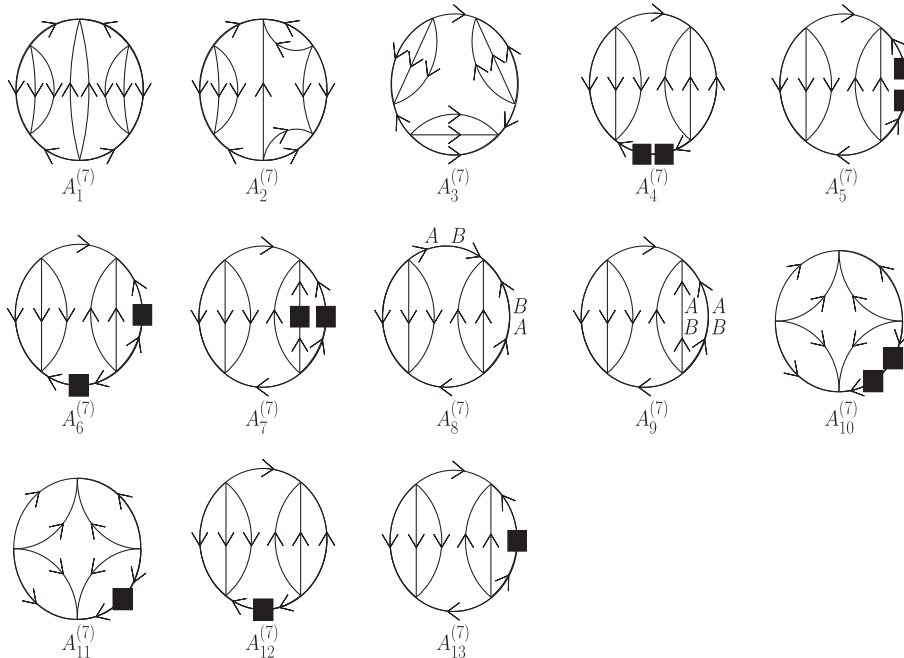

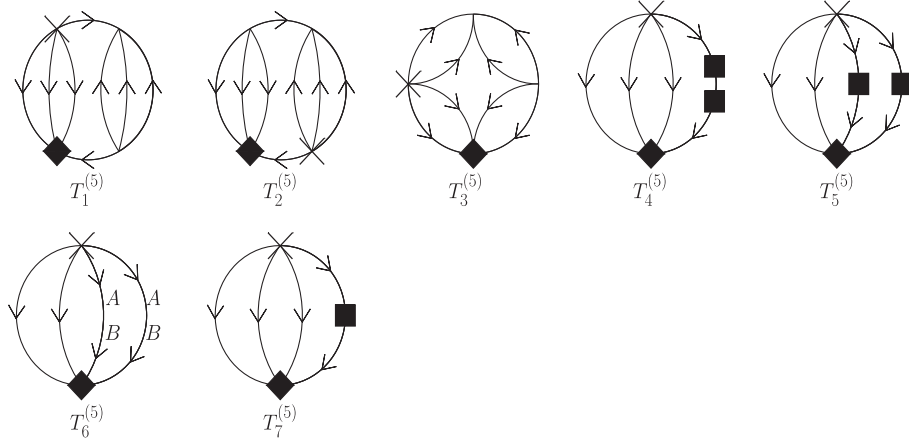FIG. 1.　Contributions to $A^{(7)}$ (for notation, see Sec. III).

FIG. 2. Contributions to $T^{(5)}$.

The structures $\gamma^{(4)}_{1-11}$ form a basis for 2nd rank tensors with four gauge matrices; $\gamma^{(4)}_{12}$ is not independent but has been retained since it appears naturally in diagrammatic calculations (and in fact ultimately cancels). The ellipsis in Eq. (3.5) indicates Yukawa-independent terms which we have not computed. We then find that Eq. (1.1) entails

$$a_1^{(7)} + \frac{1}{3}\tilde{a} = t_1^{(5)} = 2g_1^{(4)} + \frac{1}{2}(t_2^{(5)} + k_1^{(5)}),$$

$$a_2^{(7)} = 4g_2^{(4)} = \frac{2}{3}t_3^{(5)} = \frac{2}{3}t_3^{(5)} + \frac{4}{3}k_2^{(5)},$$

$$a_3^{(7)} + \frac{1}{9}\tilde{a} = \frac{1}{9}(t_1^{(5)} + t_2^{(5)} + k_1^{(5)}),$$

$$a_4^{(7)} - \frac{4}{3}\tilde{a} = 2g_3^{(4)} - t_1^{(5)} - t_2^{(5)} - k_1^{(5)} + \frac{1}{6}t_4^{(5)},$$

$$a_5^{(7)} - 4\tilde{a} = 4g_4^{(4)} - 6t_2^{(5)} - 6k_1^{(5)} = -6t_1^{(5)} + t_4^{(5)},$$

$$a_6^{(7)} = 4g_5^{(4)} = \frac{2}{3}t_5^{(5)},$$

$$a_7^{(7)} = 4g_6^{(4)} = \frac{2}{3}t_5^{(5)},$$

$$a_8^{(7)} = 4g_7^{(4)} = \frac{2}{3}t_6^{(5)},$$

$$a_9^{(7)} = 4g_8^{(4)} = \frac{2}{3}t_6^{(5)},$$

$$a_{10}^{(7)} = -4t_3^{(5)} = -4t_3^{(5)} - 8k_2^{(5)},$$

$$a_{11}^{(7)} = -2\tilde{T}t_3^{(5)} = -2\tilde{T}t_3^{(5)} - 4\tilde{T}k_2^{(5)},$$

$$a_{12}^{(7)} - \frac{2}{3}\tilde{T}\tilde{a} = 2g_9^{(4)} - \frac{1}{2}\tilde{T}(t_1^{(5)} + t_2^{(5)} + k_1^{(5)}) + \frac{1}{6}t_7^{(5)},$$

$$a_{13}^{(7)} - 2\tilde{T}\tilde{a} = 4g_{10}^{(4)} - 3\tilde{T}(t_2^{(5)} + k_1^{(5)}) = -3\tilde{T}t_1^{(5)} + t_7^{(5)},$$

$$a_{14}^{(7)} = 2g_{11}^{(4)}, \qquad (3.7)$$

(where $\tilde{a}$ is the parameter introduced in Eq. (3.1)). The a-function coefficients $a_2^{(7)}$, $a_{6-9}^{(7)}$, $a_{14}^{(7)}$, are given directly in terms of the anomalous dimension coefficients in Eq. (3.7); while the remaining ones are given by

$$a_1^{(7)} = \frac{4}{3}g_1^{(4)},$$

$$a_3^{(7)} = 0,$$

$$a_4^{(7)} = \frac{8}{3}g_1^{(4)} + 2g_3^{(4)} + \frac{2}{3}g_4^{(4)},$$

$$a_5^{(7)} = 8g_1^{(4)} + 4g_4^{(4)},$$

$$a_{10}^{(7)} = -24g_2^{(4)},$$

$$a_{11}^{(7)} = -12\tilde{T}g_2^{(4)},$$

$$a_{12}^{(7)} = \frac{4}{3}\tilde{T}g_1^{(4)} + 2g_9^{(4)} + \frac{2}{3}g_{10}^{(4)},$$

$$a_{13}^{(7)} = 4\tilde{T}g_1^{(4)} + 4g_{10}^{(4)}, \qquad (3.8)$$

together with

$$t_1^{(5)} = \frac{1}{3}(4g_1^{(4)} + \tilde{a}),$$

$$t_2^{(5)} + k_1^{(5)} = \frac{2}{3}(-2g_1^{(4)} + \tilde{a}),$$

$$t_3^{(6)} = 6g_2^{(4)},$$

$$t_4^{(5)} = 16g_1^{(4)} + 4g_4^{(4)} - 2\tilde{a},$$

$$t_5^{(5)} = 6g_5^{(4)},$$

$$t_6^{(5)} = 6g_7^{(4)},$$

$$t_7^{(5)} = 8\tilde{T}g_1^{(4)} + 4g_{10}^{(4)} - \tilde{T}\tilde{a},$$
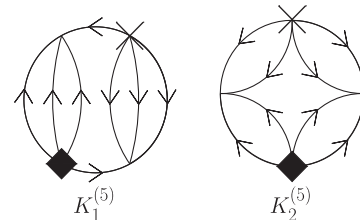
$$k_2^{(5)} = 0, \qquad (3.9)$$

FIG. 3. Contributions to $K^{(5)}$.

subject to the consistency conditions

$$g_5^{(4)} = g_6^{(4)}, \qquad g_7^{(4)} = g_8^{(4)}. \qquad (3.10)$$

Turning to the Yukawa-independent terms, it is clear that we may satisfy Eq. (1.1) if for each Yukawa-independent term $X^i{}_j$, we add to $A^{(7)}$ a term $Y^{ijkl}X^m{}_{(i}Y_{jkl)m}$, and therefore there will be no further constraints.

As we have observed already in four dimensions [10,32] and six dimensions [33], and indeed at lower orders in three dimensions [21,22], the a-function coefficients are determined completely (within a given renormalization scheme) up to the arbitrariness parametrized by the coefficient $\tilde{a}$.

IV. FOUR-LOOP CALCULATION

In this section we describe the diagrammatic computation of the four-loop anomalous dimension, again focusing on contributions containing Yukawa couplings. We are therefore concerned with the calculation of four-loop two-point diagrams. Two large classes of diagrams may be immediately discarded as giving no contribution to the anomalous dimension [31]. The first consists of those diagrams in which the first (last) vertex encountered along the incoming (outgoing) chiral line has a single gauge line. In this case after performing the superspace D-algebra[2] one is left with a diagram which is finite by power counting. These diagrams are shown schematically in Fig. 4(a). The second class consists of those diagrams which contain a one-loop subdiagram with one gauge and one chiral line, depicted in Fig. 4(b); in this case one finds that one is left with fewer than two D's and two $\bar{D}$'s on the loop shown, hence giving a vanishing contribution.

The diagrams which do potentially give nontrivial contributions to the four-loop anomalous dimension are depicted in Fig. 5. With the exception of Fig. 5(h) (which will be discussed in more detail shortly), the momentum integrals obtained after performing the superspace D-algebra in these diagrams may be expressed using integration by parts in terms of a relatively small basis of momentum integrals [34,35] which are depicted in Fig. 6, and whose divergences are also listed in Appendix B. (The graph labeled X in Fig. 6 will also be discussed in more detail shortly.) The massless four-loop two-point functions depicted in Fig. 6 are assumed to have their UV subdivergences subtracted; the "dot" on the propagator in $\tilde{Y}$ in Fig. 6 represents a double propagator. The results given later, and also most of these conventions for labeling the diagrams, are taken from Refs. [34,35], except for $\tilde{Y}$ which was defined (without the tilde, used here to avoid confusion with the Yukawa coupling) in Ref. [22]. Our results for the diagrams of Fig. 5 are listed in Table I.

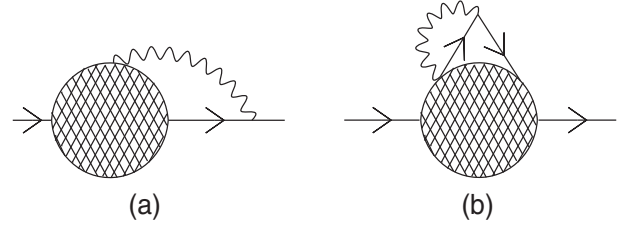[2]See Appendix A for conventions and definitions.

FIG. 4.   Classes of diagrams that do not contribute.

The central columns of Table I show the divergent contribution from each diagram [again, except Fig. 5(h)] expressed in terms of this basis. These momentum integrals multiply a variety of group structures, as defined in Eqs. (2.18) and (3.6), which are tabulated in the final column of Table I. Finally the first column of Table I contains an overall symmetry factor. The resulting contribution to the two-point function for each diagram is therefore obtained by adding the momentum integrals with the coefficients listed in the appropriate row and multiplying the resulting sum by the corresponding symmetry factor and group structure. For instance, row (j) of Table I denotes a contribution

$$(+1)(-2I_4 + I_{4bbb})\left(\gamma_7^{(4)} - \frac{1}{12}C_G\gamma_9^{(4)}\right). \qquad (4.1)$$

We note here the cancellation of $\gamma_{12}^{(4)}$ between rows (e) and (i), as mentioned in the previous section.

We now return to Fig. 5(h). After performing the superspace D-algebra, this results in the momentum integral labeled X in Fig. 6. In it there is an implicit spinor trace over the momenta of the rim propagators, $\not{k}$, where we use three-dimensional γ matrices with $\text{Tr}I = 2$. This particular integral provided us with a technical challenge compared to the other graphs we had to evaluate. Accordingly we describe our method for evaluating it in more detail than usual. First, by power counting it is straightforward to see that the graph is primitively divergent, which provides a shortcut to finding the divergence. Either we can reroute the external momentum through the diagram in such a way that it becomes simpler to compute, or we can use a vacuum bubble expansion, such as that discussed in Refs. [36,37]. In the former case one has to be careful that infrared divergences are not introduced. However, we have chosen to follow the latter course as it is more systematic and accessible to recently developed integration-by-parts algorithms. In converting the massless four-loop two-point function to vacuum bubbles we recursively apply the identity

$$\frac{1}{(k-p)^2} = \frac{1}{[k^2+m^2]} + \frac{2kp - p^2 + m^2}{(k-p)^2[k^2+m^2]} \qquad (4.2)$$

to all propagators, where k can be regarded as a loop momentum. The recursion terminates when all resulting
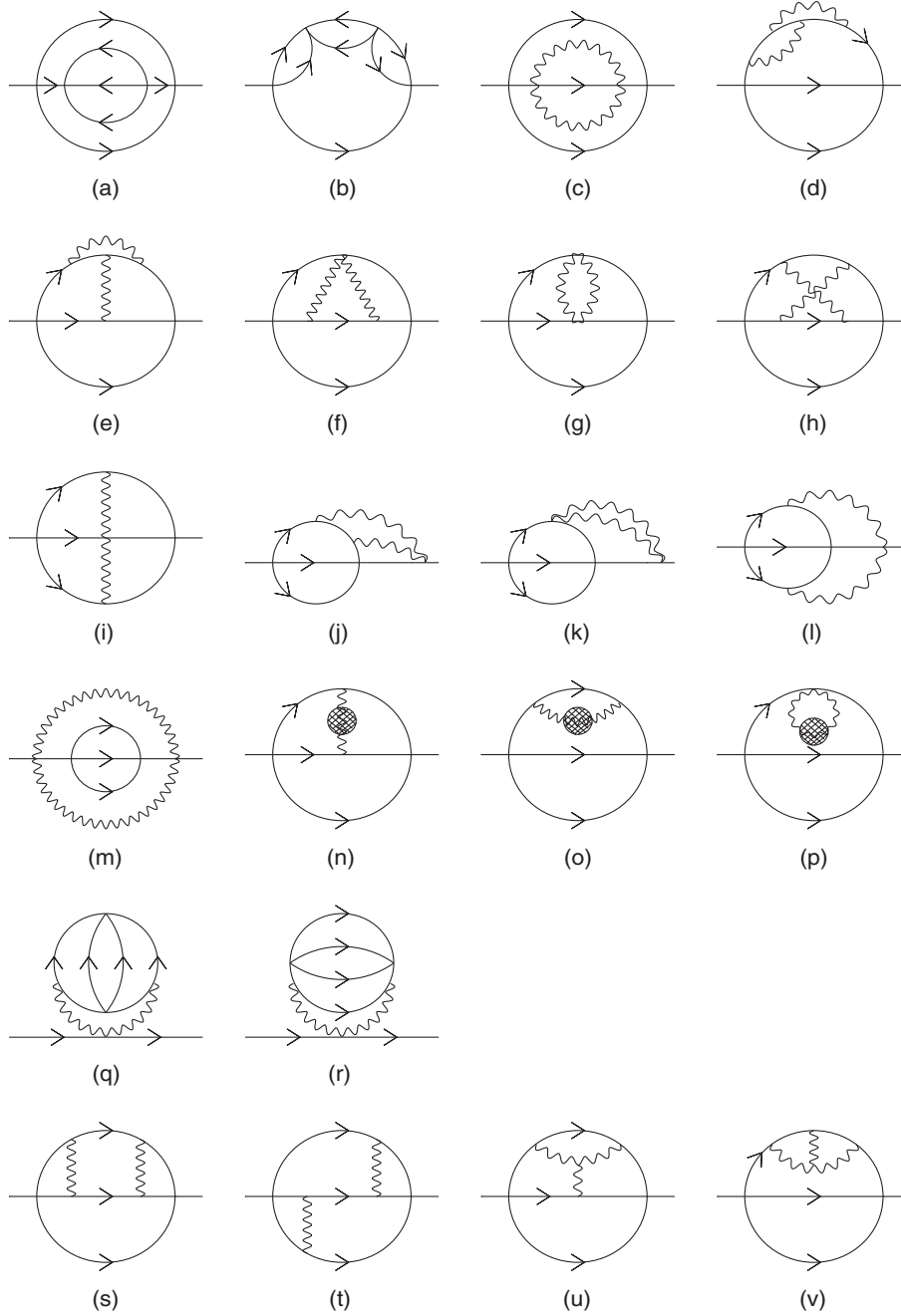
FIG. 5.   Four-loop diagrams contributing to the Yukawa-dependent part of the anomalous dimension, with pole terms listed in Table I.
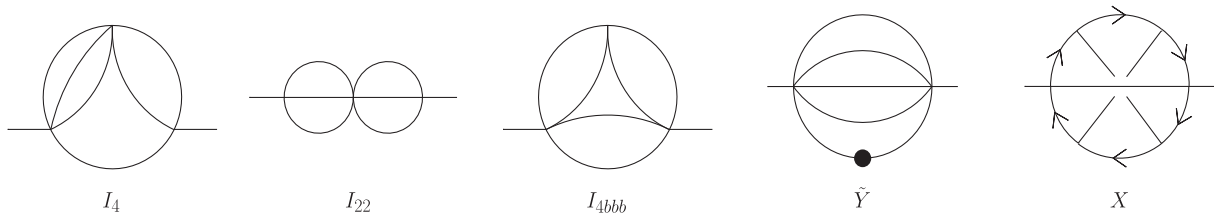

FIG. 6.   The basis of massless two-point momentum integrals. For definitions and results, see Appendix B as well as the main text. The dotted propagator is squared, and the arrows on the last integral denote numerator factors of k.

TABLE I. Results for diagrams listed in Fig. 5 in terms of master integrals (see Fig. 6) and invariants involving Yukawa couplings of Eq. (3.6)

	symm	I_4	I_{22}	I_{4bbb}	$\tilde{Y}$	overall group factor
(a)	$-\frac{1}{12}$	1	0	0	0	$\gamma_1^{(4)}$
(b)	$-\frac{1}{8}$	0	0	1	0	$\gamma_2^{(4)}$
(c)	$-\frac{1}{4}$	-2	0	0	0	$\gamma_4^{(4)} - \frac{1}{4}C_G\gamma_{10}^{(4)}$
(d)	$-\frac{1}{2}$	0	0	1	0	$\gamma_4^{(4)} - \frac{1}{2}C_G\gamma_{10}^{(4)}$
(e)	-1	0	0	$\frac{2}{3}$	0	$-\frac{1}{2}\gamma_4^{(4)} - \frac{1}{2}\gamma_{12}^{(4)} - \frac{1}{12}C_G\gamma_9^{(4)} + \frac{1}{4}C_G\gamma_{10}^{(4)}$
(f)	1	0	0	$-\frac{2}{3}$	0	$\gamma_8^{(4)} + \frac{1}{12}C_G\gamma_9^{(4)} - \frac{1}{4}C_G\gamma_{10}^{(4)}$
(g)	$-\frac{1}{4}$	0	0	-2	0	$\gamma_8^{(4)} + \frac{1}{12}C_G\gamma_9^{(4)} - \frac{1}{4}C_G\gamma_{10}^{(4)}$
(i)	$\frac{1}{2}$	0	0	$-\frac{2}{3}$	0	$\frac{1}{2}\gamma_4^{(4)} + \frac{1}{2}\gamma_7^{(4)} - \gamma_8^{(4)} + \gamma_{12}^{(4)}$
(j)	1	-2	0	1	0	$\gamma_7^{(4)} - \frac{1}{12}C_G\gamma_9^{(4)}$
(k)	$-\frac{1}{2}$	-2	0	0	0	$\gamma_7^{(4)} - \frac{1}{12}C_G\gamma_9^{(4)}$
(l)	1	-2	0	$\frac{4}{3}$	0	$\frac{1}{6}\gamma_3^{(4)} - \frac{1}{2}\gamma_7^{(4)}$
(m)	$-\frac{1}{12}$	-2	0	0	0	$\gamma_3^{(4)} - \frac{1}{4}C_G\gamma_9^{(4)}$
(n)	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$(\tilde{T} + \frac{1}{2}C_G)(\frac{1}{6}\gamma_9^{(4)} - \frac{1}{2}\gamma_{10}^{(4)})$
(o)	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$(\tilde{T} + \frac{1}{2}C_G)\gamma_{10}^{(4)}$
(p)	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$(\tilde{T} + \frac{1}{2}C_G)\gamma_{10}^{(4)}$
(q)	$-\frac{1}{12}$	0	1	0	-2	$\gamma_{11}^{(4)}$
(r)	$\frac{1}{6}$	1	0	0	-1	$\gamma_{11}^{(4)}$

integrals involving the external momentum are finite by Weinberg's theorem. In our case one in effect replaces the scalar propagators of the graph by the first term of the identity. What remains is an integral with products of scalar products of internal and external momenta after the trace has been taken. To proceed we have applied the Laporta algorithm [38]. This constructs identities based on integration by parts for all such scalar product integrals and then reduces these to a base set of what is termed master integrals. Specifically these are four-loop massive vacuum diagrams. In particular they have been evaluated numerically to very high precision in three dimensions in Ref. [39] using the approach of Ref. [40], running parallel to the same calculation in four dimensions [41]. Therefore for our particular graph we have constructed a database of relations between all possible integrals within that of graph X of Fig. 6 using the REDUZE formulation [42] of the Laporta algorithm. We have used FORM [43,44] to handle the resulting algebra. One aspect of the integration-by-parts approach is that one has to substitute terms from the masters beyond the poles in ϵ. This is because factors of $1/(d-3)$ will appear as coefficients of the master integrals

in the decomposition of the original integral. In addition, as several of the masters have double poles in ϵ then in order to have an answer which corresponds to a primitive the poles higher than the simple one have to cancel. It is reassuring to find that this is indeed the case when we perform the actual computation, giving us a strong check. Interestingly it transpires that integral X of Fig. 6 is in fact finite.

The simple pole contributions in Table I may now be summed using Eq. (B3) and we obtain the Yukawa-dependent contribution to the four-loop anomalous dimension using Eq. (2.16). Our final result is

$$
\begin{aligned}
(8\pi)^4\gamma_\Phi^{(4)} = {} & \frac{2}{3}\gamma_1^{(4)} + \frac{\pi^2}{4}\gamma_2^{(4)} + \frac{4}{3}\left(1 - \frac{1}{3}\pi^2\right)\gamma_3^{(4)} \\
& - \left(4 - \frac{2}{3}\pi^2\right)\gamma_4^{(4)} - \frac{1}{3}\pi^2(\gamma_7^{(4)} + \gamma_8^{(4)}) \\
& + \left[2\left(1 - \frac{1}{8}\pi^2\right)\tilde{T} + \frac{1}{8}\pi^2 C_G\right]\left(\frac{1}{3}\gamma_9^{(4)} - \gamma_{10}^{(4)}\right) \\
& - \frac{4}{3}\gamma_{11}^{(4)} + \cdots,
\end{aligned}
\tag{4.3}
$$

where as before the ellipsis indicates Yukawa-independent terms. We may now read off the coefficients $g_i^{(4)}$ as defined in Eq. (3.5), and in particular we see that $g_5^{(4)} = g_6^{(4)} = 0$, $g_7^{(4)} = g_8^{(4)} = -\frac{1}{3}\pi^2$, in accord with Eq. (3.10).

V. CONCLUSIONS

We have demonstrated the existence of an a-function having the gradient flow properties of Eq. (1.1) at next-to-leading order, for a general three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory in this paper, and for a completely general (nonsupersymmetric) ungauged three-dimensional theory in Ref. [22]. It is worth emphasizing that in our a-function construction we have had to compute a new class of Feynman diagram at four loops to ensure full consistency. For instance, we found by computation that graph X of Fig. 6 was finite. Alternatively we could have used properties of our a-function construction to have predicted this a priori. That the two approaches tally is indicative that our a-function and the field theories we have examined are fully informed of each other. It seems highly likely that the gradient flow property will extend to a completely general three-dimensional Chern-Simons theory coupled to scalars and fermions at next-to-leading order and probably beyond; and again, based on this one might obtain predictions for further Feynman diagrams which would otherwise require advanced techniques to evaluate.

It would be very desirable to find a general all-orders proof, or to make contact with the F-function described in Refs. [17–19] which has been argued to have similar properties at least at leading order. In this connection it

might be interesting to compute the *a*-function as in Eqs. (3.1), (3.7), and (3.8) for the particular theories considered in Refs. [17,18] in order to make a direct comparison. It would be all the more desirable to relate our *a*-function with the *F*-function since as we mentioned earlier, it has been shown that the latter increases as expected between IR and UV fixed points; whereas although we have demonstrated monotonic behavior of our *a*-function perturbatively, i.e. for weak couplings (since our metric is the unit matrix at leading order), we currently have no way to prove this in general.

ACKNOWLEDGMENTS

APPENDIX A: CONVENTIONS

In this appendix we list our superspace and supersymmetry conventions. We use a metric signature $(+--)$ so that a possible choice of γ matrices is $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_3$, $\gamma^2 = i\sigma_1$ with

$$(\gamma^0)_\alpha{}^\beta = (\sigma_2)_\alpha{}^\beta, \tag{A1}$$

etc. We then have

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu} - i\epsilon^{\mu\nu\rho}\gamma_\rho. \tag{A2}$$

We have [28] a complex two-spinor θ^α (with conjugate denoted $\bar\theta^\alpha$) with indices raised and lowered according to

$$\theta^\alpha = C^{\alpha\beta}\theta_\beta, \qquad \theta_\alpha = \theta^\beta C_{\beta\alpha}, \tag{A3}$$

with $C^{12} = -C_{12} = i$. We then have

$$\theta_\alpha\theta_\beta = C_{\beta\alpha}\theta^2, \qquad \theta^\alpha\theta^\beta = C^{\beta\alpha}\theta^2, \tag{A4}$$

where

$$\theta^2 = \frac{1}{2}\theta^\alpha\theta_\alpha. \tag{A5}$$

The supercovariant derivatives are defined by

$$D_\alpha = \partial_\alpha + \frac{i}{2}\bar\theta^\beta\partial_{\alpha\beta}, \tag{A6}$$

$$\bar D_\alpha = \bar\partial_\alpha + \frac{i}{2}\theta^\beta\partial_{\alpha\beta}, \tag{A7}$$

where

$$\partial_{\alpha\beta} = \partial_\mu(\gamma^\mu)_{\alpha\beta}, \tag{A8}$$

satisfying

$$\{D_\alpha, \bar D_\beta\} = i\partial_{\alpha\beta}. \tag{A9}$$

We also define

$$d^2\theta = \frac{1}{2}d\theta^\alpha d\theta_\alpha$$

$$d^2\bar\theta = \frac{1}{2}d\bar\theta^\alpha d\bar\theta_\alpha,$$

$$d^4\theta = d^2\theta d^2\bar\theta, \tag{A10}$$

so that

$$\int d^2\theta\,\theta^2 = \int d^2\bar\theta\,\bar\theta^2 = -1. \tag{A11}$$

The vector superfield $V(x, \theta, \bar\theta)$ is expanded in Wess-Zumino gauge as

$$V = i\theta^\alpha\bar\theta_\alpha\sigma + \theta^\alpha\bar\theta^\beta A_{\alpha\beta} - \theta^2\bar\theta^\alpha\bar\lambda_\alpha$$
$$- \bar\theta^2\theta^\alpha\lambda_\alpha + \theta^2\bar\theta^2 D, \tag{A12}$$

and the chiral field is expanded as

$$\Phi = \phi(y) + \theta^\alpha\psi_\alpha(y) - \theta^2 F(y), \tag{A13}$$

where

$$y^\mu = x^\mu + i\theta\gamma^\mu\bar\theta. \tag{A14}$$

APPENDIX B: INTEGRALS

Here, we list the UV divergences of our basis of momentum integrals. As in Refs. [34,35], these are subdivergence-subtracted massless two-point functions, depicted schematically in Fig. 6. The basic massless one-loop integral (defined to be dimensionless here) is given by[3]

$$G(a, b) = \int \frac{d^d k}{(2\pi)^d} \frac{p^{2(a+b-d/2)}}{k^{2a}(k-p)^{2b}}$$
$$= \frac{\Gamma(a+b-\frac{d}{2})\Gamma(\frac{d}{2}-a)\Gamma(\frac{d}{2}-b)}{(4\pi)^{d/2}\Gamma(d-a-b)\Gamma(a)\Gamma(b)}. \tag{B1}$$

A standard method is then to iteratively integrate out massless subgraphs in higher-loop integrals in terms of the function G, and this is indeed sufficient to evaluate the first four of the integrals of Fig. 6. Due to dimensional reasons, in odd dimensions the first logarithmic UV divergence can only occur at even loop orders, which here is parametrized by the two-loop massless sunset-type integral

---

[3]The *G*-function notation was introduced in Ref. [45].

$$I_2 = \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \frac{p^{2(3-d)}}{k^2 q^2 (k+q-p)^2}$$
$$= G(1,1) G\left(2 - \frac{d}{2}, 1\right). \qquad \text{(B2)}$$

We are now ready to define our basis of momentum integrals, using an operator $\hat{K}[f(\epsilon)]$ that extracts the pole parts of the function $f(\epsilon)$. Recalling that we work in $d = 3 - \epsilon$ dimensions, we obtain

$$I_4 = \hat{K}\left[I_2\left(G(1,1)G\left(5 - \frac{3d}{2}, 1\right) - \hat{K}[I_2]\right)\right] = \frac{1}{(8\pi)^4}\left(-\frac{2}{\epsilon^2} + \frac{4}{\epsilon}\right),$$

$$I_{22} = \hat{K}[I_2(I_2 - 2\hat{K}[I_2])] = \frac{1}{(8\pi)^4}\left(-\frac{4}{\epsilon^2} + \frac{0}{\epsilon}\right),$$

$$I_{4bbb} = \hat{K}\left[G^3(1,1)G\left(4 - d, 2 - \frac{d}{2}\right)\right] = \frac{1}{(8\pi)^4}\left(\frac{\pi^2}{\epsilon}\right),$$

$$\tilde{Y} = \hat{K}\left[G^2(1,1)G\left(2, 2 - \frac{d}{2}\right)G\left(2 - \frac{d}{2}, 4 - d\right)\right] = \frac{1}{(8\pi)^4}\left(-\frac{2}{\epsilon}\right), \qquad \text{(B3)}$$

the first three of which agree with Refs. [34,35] after adjusting for our definition of ϵ which differs by a factor of 2. Furthermore, the integrals satisfy the consistency condition $4\tilde{Y} = I_{22} - 2I_4$ given in Eq. (4.1) of Ref. [22]. We note that in fact the result for $\tilde{Y}$ is not required since it cancels between rows (q) and (r) of Table I.

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