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$1/c$ expansion of nonminimally coupled curvature-matter gravity models and constraints from planetary precession

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The effects of a nonminimally coupled curvature-matter model of gravity on a perturbed Minkowski metric are presented. The action functional of the model involves two functions $f^1(R)$ and $f^2(R)$ of the Ricci scalar curvature R. This work expands upon previous results, extending the framework developed there to compute corrections up to order $O(1/c^4)$ of the 00 component of the metric tensor. It is shown that additional contributions arise due to both the nonlinear form $f^1(R)$ and the nonminimal coupling $f^2(R)$, including exponential contributions that cannot be expressed as an expansion in powers of $1/r$. Some possible experimental implications are assessed with application to perihelion precession.

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I. INTRODUCTION

Dark matter and dark energy are key contemporary concepts used to account, for instance, for the astrophysical problem of the flattening of galactic rotation curves and the cosmological issue of the accelerated expansion of the Universe, respectively. Dark energy accounts for 68% of the energy budget of the Universe [1]; among several other proposals, it has been the object of several so-called "quintessence" models [2], which posit the existence of scalar fields with negative pressure, as an alternative to a suitably adjusted cosmological constant, which presents the eponymous problem of reconciling the large order of magnitude difference between its observed and predicted values [3]. Dark matter searches focus on the characterization of additional matter species arising from extensions to the Standard Model of particles, collectively dubbed as weak-interacting massive particles (WIMPS) such as, for instance, neutralinos or axions [4]. As an alternative, some proposals assume that both dark components may be described in a unified fashion [5,6].

Other models assume that, instead of additional matter species, the fundamental laws of general relativity (GR) may be incomplete, prompting, e.g., corrections and alternatives to the Einstein-Hilbert action. Among such theories, those involving a nonlinear correction to the geometric part of the action via the scalar curvature, aptly called $f(R)$ theories, have gained much attention (see Ref. [7] for a thorough discussion). These can be extended also to include a nonminimal coupling (NMC) between the scalar curvature and the matter Lagrangian density, leading to an even richer phenomenology and implying that the energy-momentum tensor may not be (covariantly) conserved [8] (see also Ref. [9] for a more general model).

Considering that $f(R)$ are phenomenological models that should be derived from the low-energy regime of a more complete theory [7,10], strong fundamental motivations for the presence of a nonminimal coupling (NMC) arise from, for instance, one-loop vacuum-polarization effects in the formulation of quantum electrodynamics in a curved spacetime [11], as well as in the context of multi-scalar-tensor theories, when considering matter scalar fields [12] (as explicitly shown in Ref. [13]). Furthermore, a NMC was put forward in the context of Riemann-Cartan geometry [14], with another study showing that it has clear implications in the characterization of the ground state [15].

NMC models have yielded several interesting results, including the impact on stellar observables [16], energy conditions [17], equivalence with multi-scalar-tensor theories [18], possibility to account for galactic [19] and cluster [20] dark matter, cosmological perturbations [21], a mechanism for mimicking a cosmological constant at

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astrophysical scales [22], post-inflationary reheating [23], dark energy [24–26], dynamical impact of the choice of the Lagrangian density of matter [27,28], gravitational collapse [29], and black hole solutions [30], its Newtonian limit [31], the existence of closed timelike curves [32] and the modified Layzer-Irvine equation [33] (see Ref. [34] for a review and Refs. [35] for other NMC gravity theories and their potential applications).

Recently, the impact of NMC gravity on the spacetime metric surrounding a spherical central body was considered in Ref. [36], where the additional degree of freedom arising from a nontrivial $f(R)$ function is light, thus yielding a long-range additional force which requires considering the background cosmological setting; following the procedure set out in Ref. [37] for $f(R)$ gravity, the parametrized post-Newtonian (PPN) parameter γ was computed, provided that a set of requirements for $f(R)$ and the NMC function are obeyed. Then the compatibility has been assessed between a NMC model which accounts for the observed accelerated expansion of the Universe and Solar System experiments.

Conversely, the case where the former is short-ranged enables one to neglect the background cosmological setting and derive the ensuing corrections to the gravitational potential [38], which are shown to be of the Yukawa-type—as previously reported in Ref. [39] for $f(R)$ gravity. In particular, it is found that the range of this Yukawa potential is given solely by $f(R)$, with the NMC affecting only its strength: this is a natural result, since the effect of the latter vanishes in vacuum, but affects the gravitational source.

The purpose of this work is thus to further examine those findings, extending the formalism used in Ref. [38] to include terms up to order $O(1/c^4)$ in the 00 component of the metric tensor. The nonlinear correction to the geometry part of the action is represented by a function $f^1(R)$, and the NMC is represented by a function $f^2(R)$ which multiplies the matter Lagrangian density. Both functions are assumed analytic at $R = 0$ and the coefficients of the Taylor expansions around $R = 0$ are considered as the parameters of the model.

This work is organized as follows: In Sec. II, the NMC model is presented and in Sec. III its nonrelativistic limit is derived. Section IV computes the post-Newtonian and Yukawa corrections to the metric tensor by considering matter as a perfect fluid (without assumptions of symmetry). In Sec. V, the metric around a static, spherically symmetric body is computed. Section VI addresses the ensuing Solar System constraints, namely through perturbations to perihelion precession. Recent observations of Mercury, including data from the NASA orbiter MESSENGER (Mercury Surface, Space Environment, Geochemistry and Ranging) spacecraft, are used to constrain the parameters of the model. Finally, conclusions are drawn in Sec. VII.

II. NONMINIMALLY COUPLED GRAVITY

The action functional of NMC gravity is of the form [8]

$$S = \int \left[\frac{1}{2}f^1(R) + [1 + f^2(R)]\mathcal{L}\right]\sqrt{-g}d^4x, \qquad (1)$$

where $f^i(R)$ (with $i = 1, 2$) are functions of the Ricci scalar curvature R, $\mathcal{L}$ is the Lagrangian density of matter, and g is the metric determinant.

The Einstein-Hilbert action is recovered by choosing:

$$f^1(R) = 2\kappa(R - 2\Lambda), \qquad f^2(R) = 0, \qquad (2)$$

where $\kappa \equiv c^4/16\pi G$, G is Newton's gravitational constant and Λ the cosmological constant.

The variation of the action functional with respect to the metric $g_{\mu\nu}$ yields the field equations

$$(f^1_R + 2f^2_R\mathcal{L})R_{\mu\nu} - \frac{1}{2}f^1 g_{\mu\nu} = (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)(f^1_R + 2f^2_R\mathcal{L}) + (1 + f^2)T_{\mu\nu}, \qquad (3)$$

where $f^i_R \equiv df^i/dR$. The trace of the field equations is given by

$$(f^1_R + 2f^2_R\mathcal{L})R + 3\Box(f^1_R + 2f^2_R\mathcal{L}) - 2f^1 = (1 + f^2)T, \qquad (4)$$

where T is the trace of the energy-momentum tensor $T_{\mu\nu}$.

A rather striking feature of NMC gravity is that the energy-momentum tensor of matter is not covariantly conserved: indeed, applying the Bianchi identities to Eq. (3), one finds that

$$\nabla_\mu T^{\mu\nu} = \frac{f^2_R}{1 + f_2}(g^{\mu\nu}\mathcal{L} - T^{\mu\nu})\nabla_\mu R, \qquad (5)$$

a result that, as discussed thoroughly in Refs. [18,40], cannot be "gauged away" by a convenient conformal transformation, but is instead a distinctive feature of the model under scrutiny.

A. Assumptions on the metric

We assume that the metric can be written as a small perturbation around flat spacetime,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \text{with} \quad |h_{\mu\nu}| \ll 1, \qquad (6)$$

where $\eta_{\mu\nu}$ is the Minkowski metric with signature $(-,+,+,+)$. In the following, Greek letters denote space-time indices ranging from 0 to 3, whereas Latin letters denote spatial indices ranging from 1 to 3.

In analogy with the post-Newtonian approximation of general relativity, we expand the metric tensor in powers of $1/c$:

$$g_{00} = -1 + h_{00}^{(2)} + h_{00}^{(4)} + O\left(\frac{1}{c^6}\right),$$
$$g_{0i} = h_{0i}^{(3)} + O\left(\frac{1}{c^5}\right),$$
$$g_{ij} = \delta_{ij} + h_{ij}^{(2)} + O\left(\frac{1}{c^4}\right), \tag{7}$$

where

$$h_{\mu\nu}^{(n)} = O\left(\frac{1}{c^n}\right), \quad \text{for } n = 2, 3, 4. \tag{8}$$

The above choice of metric is an apt description for an approximately flat spacetime, and asymptotically approaches a Minkowsi form if the higher-order contributions $h_{\mu\nu}^{(n)}$ vanish as $r \to \infty$. The order of the expansion for each metric component is chosen so as to ensure that all the physics is derived to post-Newtonian order (see Ref. [41] for a thorough discussion).

We impose the following gauge conditions [41],

$$h_{i0,i}^{(3)} = \frac{1}{2c} h_{ii,0}^{(2)} + O\left(\frac{1}{c^5}\right),$$
$$h_{ij,j}^{(2)} = \frac{1}{2} h_{jj,i}^{(2)} - \frac{1}{2} h_{00,i}^{(2)} + O\left(\frac{1}{c^4}\right), \tag{9}$$

so that the Ricci tensor $R_{\mu\nu}$ is expanded as

$$R_{00} = -\frac{1}{2}\nabla^2 h_{00}^{(2)} - \frac{1}{2}\nabla^2 h_{00}^{(4)} - \frac{1}{2}|\nabla h_{00}^{(2)}|^2 + \frac{1}{2} h_{ij}^{(2)} h_{00,ij}^{(2)} + O\left(\frac{1}{c^6}\right), \tag{10}$$

$$R_{0i} = -\frac{1}{2}\nabla^2 h_{0i}^{(3)} - \frac{1}{4c} h_{00,i0}^{(2)} + O\left(\frac{1}{c^5}\right), \tag{11}$$

$$R_{ij} = -\frac{1}{2}\nabla^2 h_{ij}^{(2)} + O\left(\frac{1}{c^4}\right), \tag{12}$$

where ∇^2 denotes the usual Laplacian operator in three-dimensional Euclidean space.

We also expand the Ricci scalar as follows:

$$R = R^{(2)} + R^{(4)} + O\left(\frac{1}{c^6}\right), \tag{13}$$

where $R^{(n)} = O(1/c^n)$, for $n = 2$, 4.

B. Energy-momentum tensor

As in the PPN framework, the components of the energy-momentum tensor, $T_{\mu\nu}$, to the relevant order, are [41]

$$T_{00} = \rho c^2 \left(1 + \frac{v^2}{c^2} + \frac{\Pi}{c^2} - h_{00}^{(2)}\right) + O\left(\frac{1}{c^2}\right), \tag{14}$$

$$T_{0i} = -\rho c v_i + O\left(\frac{1}{c}\right), \tag{15}$$

$$T_{ij} = \rho v_i v_j + p\delta_{ij} + O\left(\frac{1}{c^2}\right), \tag{16}$$

where matter is considered as a perfect fluid with matter density ρ, velocity field v_i, pressure p, and specific energy density Π (ratio of energy density to rest-mass density). The trace of the energy-momentum tensor is given by

$$T = -\rho c^2 \left(1 + \frac{\Pi}{c^2}\right) + 3p + O\left(\frac{1}{c^2}\right). \tag{17}$$

If Ω denotes the portion of three-dimensional space occupied by a body with mass density ρ, and $\rho = 0$ outside of the body, in order for the field Eqs. (3) to be well defined, we require that both the function $\rho = \rho(t, x)$ and its spatial derivatives are continuous across the surface of the body:

$$\rho(t, x) = 0, \qquad \nabla\rho(t, x) = 0, \qquad x \in \partial\Omega, \tag{18}$$

where the operator ∇ denotes the three-dimensional gradient.

In what follows, we use $\mathcal{L} = -\rho(c^2 + \Pi)$ for the Lagrangian density of matter (see Ref. [27] for a discussion).

C. Assumptions on $f^1(R)$ and $f^2(R)$

We assume the functions $f^1(R)$ and $f^2(R)$ to be analytic at $R = 0$. Hence, the function f^1 admits the following Taylor expansion around $R = 0$,

$$f^1(R) = 2\kappa \sum_{i=1}^{\infty} a_i R^i, \qquad a_1 = 1, \tag{19}$$

where the condition $a_1 = 1$ allows for recovering GR when the function f^1 is linear and $f^2 = 0$.

Analogously, the function f^2 admits the following Taylor expansion,

$$f^2(R) = \sum_{j=1}^{\infty} q_j R^j. \tag{20}$$

The $1/c$ expansion of the metric, which is the subject of the present paper, will show how the coefficients a_i, q_j affect the weak-field limit of NMC gravity, in such a way that some of these coefficients can be constrained by means of experiments in gravitational physics.

One should notice that the above Taylor expansion of $f^i(R)$ around a flat spacetime is in direct contradiction with the assumption of a nonvanishing scalar curvature $R \neq 0$ in a cosmological context, as required by the current accelerated expansion of the Universe: indeed, in order to account for dark matter [19,20] and dark energy [24,26] one must resort to inverse power-law functions $f^i(R) \sim 1/R^n$ (with $n > 0$); furthermore, no known, universal exponent n can account for both the dark matter and dark energy problems.

To account for this, one usually assumes that both functions $f^i(R)$ can be, in general, written as a Laurent series of the form

$$f^i(R) = \sum_{j=-\infty}^{+\infty} \left(\frac{R}{R_{ij}}\right)^j, \tag{21}$$

which is clearly nonanalytical: as discussed in Ref. [34], the assumption of a simpler power-law form for $f^i(R)$ in a particular cosmological context assumes that, in the vicinity of the values for the scalar curvature $R \sim R_0$ relevant for that scenario, one of the terms of the Laurent series above is dominant, $f^i(R \sim R_0) \sim (R/R_{\rm in})^n$.

Given the above, the proposed Taylor expansion restates the tension existing between the assumption of an asymptotically flat metric and the knowledge that, at cosmological scales, the metric is dynamical. In this work and in Ref. [38] it is assumed that the relevant interaction has a range smaller than the scale of the Solar System: as such, one needs not consider the matching of the assumed metric with a cosmological Friedmann-Robertson-Walker spacetime, and no contribution from a cosmological constant is included, as this would contradict the assumption of an asymptotically flat spacetime. Conversely, if the interaction is long-ranged, the contribution from the cosmological value of the scalar curvature must be taken into account and the dynamical impact are quite different, as reported in Ref. [36].

Finally, it should be noted that, from a mathematical standpoint, the lack of analyticity of an inverse-power law form for $f^i(R)$ may be overcome by resorting to a regularization procedure, whereas a vanishingly small curvature scale r is added to the model

$$f^i(R) = \sum_{j=-\infty}^{+\infty} \left(\frac{R+r}{R_{ij}}\right)^j \tag{22}$$

so that, while its dynamical behavior at cosmological scales is unperturbed if $R \sim R_{\rm cosmo} \gg r$, it can be Taylor expanded around $R = 0$.

III. NONRELATIVISTIC LIMIT

In this section, we compute the quantity $h_{00}^{(2)}$, which yields the nonrelativistic limit of NMC gravity. First, we compute the trace of the field Eqs. (4) at order $O(1/c^2)$, obtaining

$$\nabla^2 R^{(2)} - \frac{R^{(2)}}{6a_2} = -\frac{4\pi G}{3c^2 a_2}(\rho - 6q_1 \nabla^2 \rho). \tag{23}$$

In the following, we assume that $a_2 > 0$ and set $m^2 = 1/(6a_2)$.

The above admits a Yukawa-type solution,

$$R^{(2)} = \frac{G}{3c^2 a_2} \times \int d^3y \frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}[\rho(t,\mathbf{y}) - 6q_1 \nabla^2 \rho(t,\mathbf{y})]. \tag{24}$$

We now introduce the Green function

$$G(\mathbf{x}-\mathbf{y}) = -\frac{1}{4\pi}\frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}, \tag{25}$$

which satisfies the following equation in the sense of a distribution,

$$(\nabla^2 - m^2)G(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}), \tag{26}$$

where $\delta(\mathbf{x}-\mathbf{y})$ is the Dirac distribution.

Hence, if the mass density ρ is zero outside of a body which occupies a region Ω of three-dimensional space, using Green's identity and the boundary conditions Eq. (18), we have

$$\int \nabla^2 \rho(t,\mathbf{y}) \frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} d^3y = -4\pi\rho + m^2 \int \rho(t,\mathbf{y}) \frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} d^3y. \tag{27}$$

Collecting the above results we find for the Ricci scalar R at order $O(1/c^2)$:

$$R^{(2)} = \frac{8\pi G}{c^2}\frac{q_1}{a_2}\rho + \frac{G}{3c^2 a_2}\left(1 - \frac{q_1}{a_2}\right) \int \rho(t,\mathbf{y}) \frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} d^3y. \tag{28}$$

Note that, if $a_2 < 0$, then the solution for $R^{(2)}$ would be oscillatory, which would lead to an unphysical behavior at asymptotically large distances.

The $0-0$ component of the field Eqs. (3), written at order $O(1/c^2)$, is

$$\nabla^2 \left(h_{00}^{(2)} + 4a_2 R^{(2)} - \frac{2q_1}{\kappa}\rho c^2 \right) = R^{(2)} - \frac{1}{\kappa}\rho c^2, \tag{29}$$

where the $O(1/c^2)$ contributions to R_{00} and T_{00} have been taken into account using Eqs. (10) and (14), respectively.

Combining Eq. (29) with the trace Eq. (23) yields the modified Poisson equation

$$\nabla^2\left(h_{00}^{(2)} - 2a_2R^{(2)} + \frac{16\pi G}{c^2}q_1\rho\right) = -\frac{8\pi G}{c^2}\rho, \qquad (30)$$

which admits the solution

$$h_{00}^{(2)} = 2\left(\frac{U}{c^2} + a_2R^{(2)} - \frac{8\pi G}{c^2}q_1\rho\right), \qquad (31)$$

where U is the usual Newtonian potential

$$U = G\int\frac{\rho(t,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d^3y. \qquad (32)$$

In the particular case of a body with a static and spherically symmetric distribution of mass, the solution Eq. (31) coincides, outside of the body, with the metric found in Ref. [38]; in the case of pure $f(R)$ gravity, i.e., $q_1 = 0$, it reduces to the solution for $h_{00}^{(2)}$ found in Ref. [42].

Eventually, the solution for $h_{00}^{(2)}$ shows that the nonrelativistic limit of NMC gravity, outside of a massive body, is constituted by the sum of the Newtonian potential plus a Yukawa potential proportional to $R^{(2)}$. The characteristic length of the Yukawa potential is given by $\lambda \equiv 1/m$, as in $f(R)$ gravity, whereas the strength of such a potential depends on both a_2 and the NMC parameter q_1.

The gravitational effects of this Yukawa potential and consequent experimental constraints on the parameters a_2 and q_1 have been discussed in detail in Ref. [38].

IV. POST-NEWTONIAN + YUKAWA APPROXIMATION OF NMC GRAVITY

In this section, we compute a parametrized post-Newtonian plus Yukawa (PPNY) approximation of NMC gravity (see also Ref. [39] for $f(R)$ gravity): this reflects the impossibility of expanding a Yukawa perturbation $\sim(1/r)\exp(-r/\lambda)$ in powers of $1/r$, so that both contributions must be considered. More precisely, in the following subsections, we compute the metric contributions $h_{ij}^{(2)}$, $h_{0i}^{(3)}$ and $h_{00}^{(4)}$, by solving the field equations of NMC gravity.

A. Solution for h_{ij} at second order

The $i-j$ components of the field Eqs. (3), written at order $O(1/c^2)$, are

$$\nabla^2\left(\frac{1}{2}h_{ij}^{(2)} - 2a_2\delta_{ij}R^{(2)} + \frac{16\pi G}{c^2}q_1\rho\delta_{ij}\right) + \frac{1}{2}\delta_{ij}R^{(2)} + 2a_2R_{,ij}^{(2)} = \frac{c^2}{\kappa}q_1\rho_{,ij}, \qquad (33)$$

where the $O(1/c^2)$ contributions to R_{ij} and $T_{ij}/(2\kappa)$ have been taken into account using Eqs. (12) and (14), respectively.

In order to rewrite Eq. (33) in the form of a Poisson equation, we observe that, using Eqs. (10) and (12) at order $O(1/c^2)$, we have

$$R^{(2)} = \frac{1}{2}(\nabla^2h_{00}^{(2)} - \nabla^2h_{ii}^{(2)}). \qquad (34)$$

Using this result and the $0-0$ component of the field Eqs. (29), the trace Eq. (23) can be rewritten as

$$\nabla^2(h_{ii}^{(2)} + 5h_{00}^{(2)}) = -\frac{64\pi G}{c^2}\rho. \qquad (35)$$

Moreover, using the Poisson equation for the Newtonian potential, $\nabla^2U = -4\pi G\rho$, we have

$$\rho_{,ij} = -\frac{1}{4\pi G}\nabla^2U_{,ij}, \qquad (36)$$

while the solution (31) for $h_{00}^{(2)}$ and Eqs. (34)–(36) enable to write

$$R_{,ij}^{(2)} = \nabla^2\left(6a_2R_{,ij}^{(2)} - \frac{2}{c^2}U_{,ij} - \frac{48\pi G}{c^2}q_1\rho_{,ij}\right). \qquad (37)$$

Now, substituting Eqs. (36) and (37) into the $i-j$ components of the field Eqs. (33), and using again Eq. (34) of $R^{(2)}$, we obtain the following,

$$\nabla^2\left[\frac{1}{2}h_{ij}^{(2)} + a_2\delta_{ij}R^{(2)} + 12a_2^2R_{,ij}^{(2)} - \frac{4}{c^2}(a_2-q_1)U_{,ij} - \frac{8\pi G}{c^2}q_1(\rho\delta_{ij} + 12a_2\rho_{,ij})\right] = -\frac{4\pi G}{c^2}\rho\delta_{ij}. \qquad (38)$$

This is a system of decoupled Poisson equations with solution

$$h_{ij}^{(2)} = 2\left[\frac{U}{c^2}\delta_{ij} - a_2\delta_{ij}R^{(2)} - 12a_2^2R_{,ij}^{(2)} + \frac{4}{c^2}(a_2-q_1)U_{,ij} + \frac{8\pi G}{c^2}q_1(\rho\delta_{ij} + 12a_2\rho_{,ij})\right]. \qquad (39)$$

In the case of pure $f(R)$ gravity, i.e., if $q_1 = 0$, the above reduces to the solution for $h_{ij}^{(2)}$ found in Ref. [42].

Notice that the obtained solution is not diagonal, and hence it is not in the standard post-Newtonian gauge. In a subsequent section, it will be written as a diagonal spatial metric by means of a suitable gauge transformation.

B. Solution for h_{0i} at third order

The $0-i$ components of the field Eqs. (3), written at order $O(1/c^3)$, are

$$\nabla^2 h^{(3)}_{0i} + \frac{1}{2c} h^{(2)}_{00,0i} + \frac{4a_2}{c} R^{(2)}_{,0i} - \frac{2c}{\kappa} q_1 \rho_{,0i} = \frac{c}{\kappa} \rho v_i, \quad (40)$$

where the $O(1/c^3)$ contributions to R_{0i} and T_{0i} have been taken into account using Eqs. (11) and (15), respectively.

In order to solve Eqs. (40) we use the following set of PPN potentials [41],

$$V_i = G \int \frac{\rho(t,\mathbf{y}) v_i(t,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d^3y,$$
$$W_i = G \int \frac{\rho(t,\mathbf{y})[\mathbf{v}(t,\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})](x-y)_i}{|\mathbf{x}-\mathbf{y}|^3} d^3y. \quad (41)$$

Using the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{v}) = 0, \quad (42)$$

one can show that (cf. Ref. [41])

$$\nabla^2 (W_i - V_i) = 2U_{,0i}. \quad (43)$$

Then, arguing as in the previous subsection, we have

$$R^{(2)}_{,0i} = \nabla^2 \left(6a_2 R^{(2)}_{,0i} - \frac{2}{c^2} U_{,0i} - \frac{48\pi G}{c^2} q_1 \rho_{,0i} \right),$$
$$\rho_{,0i} = -\frac{1}{4\pi G} \nabla^2 U_{,0i}. \quad (44)$$

Inserting Eqs. (43) and (44) into the $0-i$ components of the field Eqs. (40) and using the solution (31) for $h^{(2)}_{00}$, we obtain

$$\nabla^2 \left[h^{(3)}_{0i} + 30\frac{a_2^2}{c} R^{(2)}_{,0i} - \frac{10}{c^3}(a_2 - q_1) U_{,0i} - \frac{1}{2c^3} V_i + \frac{1}{2c^3} W_i - \frac{240\pi G}{c^3} a_2 q_1 \rho_{,0i} \right] = \frac{16\pi G}{c^3} \rho v_i. \quad (45)$$

This is a system of decoupled Poisson equations with solution

$$h^{(3)}_{0i} = -\frac{7}{2c^3} V_i - \frac{1}{2c^3} W_i + \frac{10}{c^3}(a_2 - q_1) U_{,0i} - 30\frac{a_2^2}{c} R^{(2)}_{,0i} + \frac{240\pi G}{c^3} a_2 q_1 \rho_{,0i}. \quad (46)$$

Again, in the case of pure $f(R)$ gravity the above reduces to the solution for $h^{(3)}_{0i}$ found in Ref. [42].

C. Solution for h_{00} at fourth order

The solution of the $0-0$ component of the field Eqs. (3) at order $O(1/c^4)$ is more involved and its computation is deferred to Appendix A, leading to the lengthy expression shown below,

$$\begin{aligned} h^{(4)}_{00} =& -\frac{2}{c^4}U^2 - 2a_2^2 R^2 - 4\frac{a_2}{c^2} UR + \frac{32\pi G}{c^4} q_1 \rho U + \frac{32\pi G}{c^2} a_2 q_1 \rho R - \frac{128\pi^2 G^2}{c^4} q_1^2 \rho^2 - 36\frac{a_2^2}{c^2} R_{,00} + \frac{12}{c^4}(a_2 - q_1) U_{,00} \\ &+ \frac{288\pi G}{c^4} a_2 q_1 \rho_{,00} + \frac{8}{c^4}(a_2 - q_1)|\nabla U|^2 - 24 a_2^3 |\nabla R|^2 - \frac{1536\pi^2 G^2}{c^4} a_2 q_1^2 |\nabla\rho|^2 - 8\frac{a_2}{c^2}(2a_2 + q_1)\nabla U\cdot\nabla R \\ &+ \frac{64\pi G}{c^4} q_1 (2a_2 + q_1)\nabla\rho\cdot\nabla U + \frac{384\pi G}{c^2} a_2^2 q_1 \nabla\rho\cdot\nabla R - \frac{a_2}{3\pi}\mathcal{V}(R^2) + \frac{4G}{c^4}\mathcal{V}(\rho U) - \frac{2G}{c^2}\left(\frac{8}{3}a_2 - 5q_1\right)\mathcal{V}(\rho R) \\ &+ \frac{64\pi G^2}{c^4} q_1 \mathcal{V}(\rho^2) - \frac{8G}{c^4}(a_2 - q_1)\mathcal{V}(\nabla\rho\cdot\nabla U) + 24\frac{G}{c^2} a_2^2 \mathcal{V}(\nabla\rho\cdot\nabla R) - \frac{192\pi G^2}{c^4} a_2 q_1 \mathcal{V}(|\nabla\rho|^2) + \frac{2G}{c^4}\mathcal{V}(\rho\Pi) \\ &+ \frac{4G}{c^4}\mathcal{V}(\rho v^2) + \frac{6G}{c^4}\mathcal{V}(p) - \frac{1}{6\pi c^2} X(UR) + \frac{1}{4\pi}\left(a_2 + \frac{a_3}{2a_2}\right) X(R^2) + \frac{4G}{3c^4} X(\rho U) \\ &- \frac{G}{6c^2}\left(16a_2 + 20q_1 + 8\frac{q_2}{a_2}\right) X(\rho R) + \frac{16\pi G^2}{3c^4} q_1 \left(4 - \frac{q_1}{a_2}\right) X(\rho^2) - \frac{8G}{3c^4}\left[a_2 - q_1\left(2 - \frac{q_1}{a_2}\right)\right] X(\nabla\rho\cdot\nabla U) \\ &+ \frac{8G}{c^2} a_2 (a_2 - q_1) X(\nabla\rho\cdot\nabla R) - \frac{64\pi G^2}{c^4} q_1 (a_2 - q_1) X(|\nabla\rho|^2) - \frac{2G}{c^4} X(p) \\ &+ \frac{2G}{3c^4}\left(1 - \frac{q_1}{a_2}\right) X(\rho\Pi) - \frac{1}{c^4}\sqrt{\frac{2}{3}a_2}\left(1 - \frac{q_1}{a_2}\right)\hat{\chi}_{,00}, \end{aligned} \quad (47)$$

where, for brevity, R denotes $R^{(2)}$, and the Poisson and Yukawa potentials $\mathcal{V}$ and X, respectively, are defined by

$$\mathcal{V}(Q) = \int \frac{Q(t,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d^3y, \qquad X(Q) = \int Q(t,\mathbf{y}) \frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} d^3y, \quad (48)$$

while the potential $\hat{\chi}$ is given by

$$\hat{\chi} = G \int \rho(t, \mathbf{y}) e^{-m|\mathbf{x}-\mathbf{y}|} d^3 y. \tag{49}$$

In the case of pure $f(R)$ gravity Eq. (47) differs from the solution for $h_{00}^{(4)}$ found in Ref. [42] for some coefficients of order of unity. The reason of such a difference is explained in Appendix A.

The expression for $h_{00}^{(4)}$ is not in the usual PPN form, since it contains both the time derivatives $U_{,00}$, $R_{,00}$, $\rho_{,00}$ and $\hat{\chi}_{,00}$, and terms depending on the gradients ∇U, ∇R and $\nabla \rho$. In the following section, these will be eliminated by means of suitable gauge transformations, thus yielding a more adequate PPNY form for the metric.

D. PPNY metric

So far, the metric has been computed in the gauge specified by conditions Eqs. (9), which are convenient in the PPN framework [41]. However, the solution Eq. (39) for the metric perturbation h_{ij} at second order is not diagonal, hence it is not in the standard post-Newtonian gauge. Moreover, we recall that the metric perturbation h_{00} at fourth order also contains terms that do not appear in the standard post-Newtonian approximation.

To correct this, we follow Ref. [42] and make a further gauge transformation

$$x^\mu \to x^\mu + \xi^\mu, \tag{50}$$

so that the metric perturbation transforms as

$$h_{\mu\nu} \to h_{\mu\nu} - \nabla_\nu \xi_\mu - \nabla_\mu \xi_\nu + O(\xi^2). \tag{51}$$

Following the calculations depicted in Appendix B, which work out the gauge transformation above and also the transformation to a form with no gradient terms, we finally arrive at the desired metric coefficients.

In order to write the metric we need to define the following potentials. We denote by $\mathcal{Y}$ the Yukawa potential generated by a distribution of masses with density ρ:

$$\mathcal{Y} = G \int \rho(t, \mathbf{y}) \frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} d^3 y, \tag{52}$$

so that Eq. (28) can be written as

$$R^{(2)} = \frac{1-\theta}{3c^2 a_2} \mathcal{Y} + \frac{8\pi G}{c^2} \theta \rho, \tag{53}$$

where we define the dimensionless parameter $\theta = q_1/a_2$. Next we introduce the standard PPN potentials [41], constructed with the Poisson kernel:

$$\Phi_1 = G\mathcal{V}(\rho v^2), \qquad \Phi_2 = G\mathcal{V}(\rho U),$$
$$\Phi_3 = G\mathcal{V}(\rho \Pi), \qquad \Phi_4 = G\mathcal{V}(p), \tag{54}$$

and the analogous potentials, constructed with the Yukawa kernel, which are characteristic of the NMC gravity model:

$$\Sigma_2 = GX(\rho U), \quad \Sigma_3 = GX(\rho\Pi), \quad \Sigma_4 = GX(p). \tag{55}$$

Moreover, we introduce the following new potentials

$$\Phi_5 = G^2 \mathcal{V}(\rho^2), \quad \Sigma_5 = G^2 X(\rho^2), \quad \Theta_i = GX(\rho v_i), \tag{56}$$

and

$$Y_i = G \int \frac{\rho(t,\mathbf{y})[\mathbf{v}(t,\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y})](x-y)_i}{|\mathbf{x}-\mathbf{y}|^3} e^{-m|\mathbf{x}-\mathbf{y}|} d^3 y,$$
$$Z_i = G \int \frac{\rho(t,\mathbf{y})[\mathbf{v}(t,\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y})](x-y)_i}{|\mathbf{x}-\mathbf{y}|^2} e^{-m|\mathbf{x}-\mathbf{y}|} d^3 y. \tag{57}$$

The final expression of the metric tensor is the following,

$$\begin{aligned}
g_{00} = {}& -1 + 2\frac{U}{c^2} + (1-\theta)\frac{2}{3c^2}\mathcal{Y} - \frac{2}{c^4}U^2 + \frac{2}{c^4}(2\Phi_1 + 2\Phi_2 + \Phi_3 + 3\Phi_4) + \frac{2}{c^4}\left[\frac{2}{3}(1-\theta)\Sigma_2 + \frac{1}{3}(1-\theta)\Sigma_3 - \Sigma_4\right] \\
& + \frac{16\pi}{3c^4}\theta(4a_2 + 11q_1)\Phi_5 + \frac{8\pi}{c^4}\theta\left(-2q_1 + \frac{a_3 q_1}{a_2^2} - \frac{4}{3}\frac{q_2}{a_2}\right)\Sigma_5 + \frac{(1-\theta)}{c^4}\left\{-\frac{2}{9}(1-\theta)\mathcal{Y}^2 - \frac{4}{3}U\mathcal{Y} - \frac{1}{18\pi a_2}X(U\mathcal{Y})\right. \\
& - \frac{1}{9\pi}\frac{(1-\theta)}{a_2}\left[\frac{1}{3}\mathcal{V}(\mathcal{Y}^2) - \frac{1}{4}\left(1 + \frac{a_3}{2a_2^2}\right)X(\mathcal{Y}^2)\right] - \frac{14}{9}(2-\theta)\psi_0 - 8a_2\psi_1 + 2\sqrt{\frac{a_2}{3}}\psi_2 + 8a_2\psi_3 \\
& \left. - \frac{4}{3}(1-\theta)a_2\left[\sqrt{\frac{2}{3a_2}}\psi_4 + 2\psi_5 - 2\psi_6 - \sqrt{\frac{2}{3a_2}}(\psi_7 + \psi_8) - \frac{1}{3a_2}\psi_9\right] + \frac{2}{3a_2}\left(-2a_2 + q_1 + \frac{a_3 q_1}{a_2^2} - \frac{2}{3}\frac{q_2}{a_2}\right)\psi_{10}\right\}, \\
g_{0i} = {}& -\frac{7}{2c^3}V_i - \frac{1}{2c^3}W_i + \frac{1}{6c^3}(1-\theta)\left(\Theta_i - Y_i - \frac{1}{\sqrt{6a_2}}Z_i\right), \\
g_{ij} = {}& \left[1 + 2\frac{U}{c^2} - (1-\theta)\frac{2}{3c^2}\mathcal{Y}\right]\delta_{ij},
\end{aligned} \tag{58}$$

where the potentials ψ_i $(i = 0, \ldots, 10)$ are given in Appendix B.

V. STATIC, SPHERICALLY SYMMETRIC METRIC AROUND A BODY WITH UNIFORM DENSITY

In this section, we give the expression for the PPNY metric in vacuum, around a spherical body of radius R_S and with a static, uniform mass density: hence, we assume $\rho(t, x) = \text{const}$ inside the body and $v = 0$. This is a simple model which allows us to achieve an explicit expression for the metric amenable for computation of orbits around a body (either the Sun or a planet) in the Solar System.

Note that such a mass density does not satisfy the boundary conditions Eq. (18) at the surface of the body. Nevertheless, in order to satisfy such boundary conditions, we may model the mass density of the body with a constant value in an interior region and a sharp transition in a thin layer close to the surface. When the thickness of the layer tends to zero, the various potentials appearing in the PPNY metric converge to the potentials corresponding to a uniform density model, since such potentials depend only on the density ρ and not on spatial derivatives of ρ. Hence, the uniform density model is an approximation (limit case) of a density model with a thin layer. In what follows, we set the origin of the spatial coordinates at the center of the spherical body and set $r = |x|$.

A. Effective mass

In order to find the expression for the metric, we first observe that all the potentials in the g_{00} coefficient of the PPNY metric which involve the Poisson integral—i.e., the potentials of the type $\mathcal{V}(Q)$ with the exception of $\mathcal{V}(\mathcal{Y}^2)$, under our assumptions on the density ρ—are proportional to $1/r$ outside of the body, whenever $r > R_S$. The potential $\mathcal{V}(\mathcal{Y}^2)$ has to be decomposed into the sum of two potentials $\mathcal{V}_1(\mathcal{Y}^2) + \mathcal{V}_2(\mathcal{Y}^2)$, where $\mathcal{V}_1$ is proportional to $1/r$ in vacuum, while $\mathcal{V}_2$ contains other functions of r (see Appendix C).

Hence, we can take into account the potentials proportional to $1/r$, for $r > R_S$, absorbing such contributions in the effective mass M_S of the body, defined as follows:

$$\begin{aligned}\frac{GM_S}{r} &= U + \frac{1}{c^2}(2\Phi_1 + 2\Phi_2 + \Phi_3 + 3\Phi_4) \\ &\quad + \frac{8}{3}\frac{\pi}{c^2}\theta(4a_2 + 11q_1)\Phi_5 \\ &\quad - \frac{(1-\theta)}{2c^2}\left[\frac{1}{27\pi}\frac{(1-\theta)}{a_2}\mathcal{V}_1(\mathcal{Y}^2) + \frac{14}{9}(2-\theta)\psi_0\right],\end{aligned} \tag{59}$$

where $\psi_0 = G\mathcal{V}(\rho\mathcal{Y})$. In order to compute the effective mass, the potentials in Eq. (59) are evaluated under the assumption $R_S \ll \lambda = 1/m$ (this assumption will be justified in Sec. V B). Then, keeping powers $(R_S/\lambda)^n$ with $n \leq 2$, the following expression of the effective mass follows:

$$M_S = M_N\left[1 + \frac{GM_N}{R_S c^2}\sum_{n=-2}^{2} B_n\left(\frac{R_S}{\lambda}\right)^n\right], \tag{60}$$

where $M_N = \int \rho(\mathbf{x}) d^3x$ is the Newtonian mass, and the coefficients B_n are given by

$$\begin{aligned} B_{-2} &= \frac{\theta}{3}(4 + 11\theta), \\ B_{-1} &= 0, \\ B_0 &= 3 - \frac{14}{15}(1-\theta)(2-\theta), \\ B_1 &= \frac{1}{9}(1-\theta)(12-5\theta), \\ B_2 &= \frac{2}{35}(1-\theta)(-10+3\theta). \end{aligned} \tag{61}$$

In all the potentials of g_{00} which are divided by c^4 (i.e., all the potentials with the exception of the Yukawa potential $\mathcal{Y}$), we can replace the Newtonian mass with the effective mass keeping the accuracy of the $O(1/c^4)$ approximation.

B. Yukawa potential

In the case of pure $f(R)$ gravity, i.e., $q_1 = q_2 = 0$, it turns out that most of the terms in g_{00} are negligible because of exponential suppression [42]: this reflects the requirement for a short ranged Yukawa interaction, so as to make it compatible with observations [37]. Conversely, in NMC gravity the Yukawa interaction can be long ranged, as it has been shown in Ref. [38], so that terms which are not exponentially suppressed arise in g_{00}, which in principle allow to constrain the theory by means of Solar System experiments.

Notice that the above is strictly valid only in the case where the density vanishes outside the central body, as assumed throughout this study. However, if one considers a nonvanishing density $\rho \neq 0$, a chameleon effect may arise where the dynamical impact of a nonlinear $f(R)$ (with a range up to the Mpc scale) is hidden from local tests of gravity, due to the reduction of the related Compton wavelength in regions of deep gravitational potential wells [43,44]. The possibility of achieving a chameleon effect in the context of NMC theories is alluring, given the existing coupling between geometry and matter, and shall be considered in a future work.

We now observe that, assuming a constant density ρ, all the potentials in the g_{00} coefficient of the PPNY metric which involve the Yukawa integral, hence the potentials of the type $X(Q)$ with the exception of $X(U\mathcal{Y})$ and $X(\mathcal{Y}^2)$, are proportional to $\exp(-r/\lambda)/r$ outside of the body, $r > R_S$

(where $\lambda = 1/m = \sqrt{6a_2}$). The potentials $X(U\mathcal{Y})$ and $X(\mathcal{Y}^2)$, evaluated in vacuum, contain both terms proportional to $\exp(-r/\lambda)/r$ and other functions of r (see Appendix C).

We can take into account the potentials proportional to $\exp(-r/\lambda)/r$, for $r > R_S$, absorbing such contributions in the effective strength α of a Yukawa potential, which yields the following contribution to g_{00}:

$$\frac{2}{c^2} G M_S \alpha \frac{e^{-r/\lambda}}{r}. \tag{62}$$

The expressions of all the potentials of the type $X(Q)$ appearing in g_{00} are listed in Appendix C. Assembling such expressions, it turns out that the effective strength α is a function of the following four dimensionless quantities built with the parameters of the considered NMC model:

$$\theta = \frac{q_1}{a_2}, \qquad \mu = \frac{a_3}{a_2^2}, \qquad \nu = \frac{q_2}{a_2^2}, \qquad \frac{R_S}{\lambda}. \tag{63}$$

We assume that the range of the Yukawa potential satisfies the condition $\lambda \gg R_S$, and we expand the potentials in power series of R_S/λ. Again, we remark that in the case of pure $f(R)$ gravity, if R_S is the radius either of the Sun or of the Earth, then the condition $\lambda \gg R_S$ is not compatible with Solar System observations [37,39].

Expanding in power series of R_S/λ, and using the results given in Appendix C, it follows that α can be decomposed into a zeroth-order and a first-order contribution on $1/c^2$:

$$\begin{aligned}
\alpha &= \alpha_0 + \frac{G M_S}{c^2 R_S} \alpha_1, \\
\alpha_0 &= \frac{1}{3}(1-\theta)\left[1 + \frac{1}{10}\left(\frac{R_S}{\lambda}\right)^2 + \frac{1}{280}\left(\frac{R_S}{\lambda}\right)^4\right], \\
\alpha_1 &= \sum_{n=-2}^{2} A_n \left(\frac{R_S}{\lambda}\right)^n,
\end{aligned} \tag{64}$$

with the coefficients A_n given by

$$\begin{aligned}
A_{-2} &= -\frac{4}{9}\theta\left[1 + \frac{3}{2}\nu + \theta\left(4 - \frac{9}{8}\mu\right) - \frac{11}{4}\theta^2\right], \\
A_{-1} &= 0, \\
A_0 &= -\frac{1}{45}\left[25 + 12\nu + \theta(11 - 18\mu - 9\nu) - \theta^2\left(31 - \frac{63}{4}\mu\right) + \frac{17}{2}\theta^3\right], \\
A_1 &= -\frac{1}{108}(1-\theta)\left\{-20 - 36\theta + 20\theta^2 + 36\theta\mu - 24\nu + 36\mathrm{Ei}\left(-2\frac{R_S}{\lambda}\right) + 9(1-\theta)(2+\mu)\left[\mathrm{Ei}\left(-\frac{R_S}{\lambda}\right) - \mathrm{Ei}\left(-3\frac{R_S}{\lambda}\right)\right]\right\}, \\
A_2 &= -\frac{1}{3150}\left[1121 - 255\mu + 440\nu - \frac{1}{2}\theta(228 + 300\mu + 865\nu) - \theta^2\left(1215 - \frac{3195}{8}\mu\right) + \frac{1057}{4}\theta^3\right],
\end{aligned} \tag{65}$$

where $\mathrm{Ei}(x)$ denotes the exponential integral function:

$$\mathrm{Ei}(x) = -\int_{-x}^{+\infty} \frac{e^{-t}}{t} dt. \tag{66}$$

In Sec. VI, perihelion precession will be computed under the assumption $\lambda \gg L$, where L is the characteristic distance from the planet to the Sun, and by Taylor expanding the involved quantities to second order in L/λ. For the purpose of achieving such an order of approximation, in Eq. (64) we keep powers $(R_S/\lambda)^n$ with $n \leq 4$ for α_0 (since α_0 will be multiplied by λ^2 in some terms of the equations of motion) and $n \leq 2$ for α_1.

We conclude this section by observing that, if the condition $\lambda \gg R_S$ is not satisfied, then most of the terms in g_{00} are exponentially suppressed if $r > R_S$, so that they become quickly negligible by increasing r outside of the body. The only potentials which are not present in GR and are not exponentially suppressed (see also Ref. [42]) are ψ_1 (if either $\lambda \ll R_S$ or $\lambda \approx R_S$) and ψ_2, ψ_3 (if $\lambda \approx R_S$). However, for a static, spherically symmetric (not necessarily uniform) mass density $\rho = \rho(r)$, we find that such potentials vanish identically for $r > R_S$.

C. Further potentials

Using the expression Eq. (58) for the PPNY metric and the results given in Appendix C, it turns out that the coefficient g_{00} contains the following combination of functions of r:

$$\begin{aligned}
&-\frac{2}{c^4}\left(\frac{G M_S}{r}\right)^2 (1 + \beta_1 e^{-r/\lambda} + \beta_2 e^{-2r/\lambda}) \\
&-\frac{2}{c^2}\frac{G M_S}{r}\left(\frac{G M_S}{c^2 R_S}\right)\sum_{i=1}^{3} \zeta_i F_i(r),
\end{aligned} \tag{67}$$

with the coefficients

$$\beta_1 = \frac{2}{3}(1-\theta)\left[1+\frac{1}{10}\left(\frac{R_S}{\lambda}\right)^2\right],$$
$$\beta_2 = \frac{1}{9}(1-\theta)^2\left[1+\frac{1}{5}\left(\frac{R_S}{\lambda}\right)^2\right],$$
$$\zeta_1 = \frac{1}{3}(1-\theta)\frac{R_S}{\lambda},$$
$$\zeta_2 = \frac{2}{9}(1-\theta)^2\frac{R_S}{\lambda},$$
$$\zeta_3 = -\frac{1}{6}(1-\theta)^2\left(1+\frac{\mu}{2}\right)\frac{R_S}{\lambda}, \qquad (68)$$

and the three functions $F_i(r)$ given by

$$F_1(r) = e^{-r/\lambda}\ln\left(\frac{r}{R_S}\right) - e^{r/\lambda}\mathrm{Ei}\left(-\frac{2r}{\lambda}\right),$$
$$F_2(r) = e^{-2r/\lambda} + 2\frac{r}{\lambda}\mathrm{Ei}\left(-\frac{2r}{\lambda}\right),$$
$$F_3(r) = e^{-r/\lambda}\mathrm{Ei}\left(-\frac{r}{\lambda}\right) - e^{r/\lambda}\mathrm{Ei}\left(-\frac{3r}{\lambda}\right). \qquad (69)$$

D. PPNY metric around the spherical body

Collecting the results of the previous sections, we find the expression for the metric tensor,

$$g_{00} = -1 + 2\frac{GM_S}{rc^2}(1+\alpha e^{-r/\lambda}) - \frac{2}{r}\left(\frac{GM_S}{c^2}\right)^2\left(\frac{1}{R_S}[\zeta_1 F_1(r) + \zeta_2 F_2(r) + \zeta_3 F_3(r)] + \frac{1}{r}[1+\beta_1 e^{-r/\lambda} + \beta_2 e^{-2r/\lambda}]\right),$$
$$g_{0i} = 0,$$
$$g_{ij} = \left[1 + 2\frac{GM_S}{rc^2}(1-\alpha e^{-r/\lambda})\right]\delta_{ij}. \qquad (70)$$

VI. PERIHELION PRECESSION

In this section, we use the previously obtained expression for the PPNY metric, Eq. (70), to assess the impact of the NMC gravity model on the precession of the perihelion of closed orbits.

Alternatively, a coordinate transformation to the usual nonisotropic Schwarzschild frame could be performed (see Refs. [45–47]). As shown in Ref. [48] for the case of general relativity, both approaches naturally lead to the same result, highlighting the general covariance of the theory, maintained by the NMC model here considered.

The action for a point particle with mass m is given by

$$S = mc\int d\tau[1+f^2(R)]\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}, \qquad (71)$$

where τ is an affine parameter (which, for the case of timelike geodesics, can be identified with the proper time). This is invariant for reparametrizations of the form $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu$, so that variations with respect to δx^μ yield the equations of motion [31],

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \frac{f_R^2(R)}{1+f^2(R)}g^{\alpha\beta}R_{,\beta}, \qquad (72)$$

clearly showing that the NMC gravity model under scrutiny leads to a deviation from geodesic motion [8,40].

Naturally, we are considering the test body to travel outside the central mass. However, this does not imply that the scalar curvature vanishes, as Eq. (53) shows to order $O(1/c^2)$. Furthermore, one must consider the contribution to this term of both the Yukawa potential given in Eq. (52) generated by both the central body as well as the test body itself, $\mathcal{Y} = \mathcal{Y}_S + \mathcal{Y}_B$, thus giving rise to the possibility of a self-acceleration.

If the test body has inner structure [e.g., a density $\rho_B(t,x)$], this will further complicate the computation of the additional force arising from the nonconservation of the energy-momentum tensor depicted on the rhs of the above. As such, we consider that the test body is homogeneous and static, $\rho_B(t,x) = \mathrm{const}$, consistent with the approximation considered in the previous section for the central body itself (for a thorough discussion of the effect of the inner structure on the nongeodesic motion induced by a NMC model, see Ref. [35]).

We must still consider the effect of the Yukawa potentials arising both from the central body as well as the test body. For this, we resort to Eq. (C1) of Appendix C, where this quantity is computed assuming a homogeneous density ρ,

$$\mathcal{Y} = GM_S\frac{e^{-r/\lambda}}{r}\left[1+\frac{1}{10}\left(\frac{R_S}{\lambda}\right)^2\right]. \qquad (73)$$

Anticipating the comparison with the observed precession of the perihelion of Mercury, we may compute the proportion between both contributions,

$$\frac{\mathcal{Y}_B}{\mathcal{Y}_S} \approx \frac{M_B}{M_S}\frac{L}{r}, \qquad (74)$$

where condition $\lambda \gg R_S$ was considered, $L \sim 55 \times 10^9$ m is the characteristic distance from Mercury to the Sun and r is the distance to the centre of the planet. Since $M_B \sim 1.7 \times 10^{-7} M_S$, we find that the Yukawa potential created by Mercury itself is only dominant up to a distance to its centre $r \lesssim 10$ km $\approx 0.3\%$ of its radius. Thus, we conclude that we may safely disregard the self-acceleration of Mercury due to the Yukawa potential it generates, and focus solely on the contribution of the Sun, $\mathcal{Y} \sim \mathcal{Y}_S$.

In order to compute perturbations to the Newtonian orbit, it is useful to write the equations of motion in the form

$$\frac{d^2x^i}{dt^2} = -\left(\Gamma^i_{\alpha\beta} - \Gamma^0_{\alpha\beta}\frac{\dot{x}^i}{c}\right)\dot{x}^\alpha\dot{x}^\beta + \frac{x^i}{r}N(r), \tag{75}$$

where dot denotes time derivative, $N(r)$ is the additional potential due to the nonconservation of the energy-momentum tensor,

$$N(r) = c^2\frac{g_{00} + v^2/c^2}{g_{jj}}\frac{q_1 + 2q_2R}{1 + q_1R + q_2R^2}R'(r), \tag{76}$$

and the factor g_{00} is due to the transformation $ds \to dt$. To the desired order $O(1/c^4)$ on the metric $g_{\mu\nu}$ and the scalar curvature R, we have

$$N(r) = -\frac{c^2}{6}\theta\lambda^2\left[R^{(4)\prime}(r) + \left(1 - \frac{v^2}{c^2} - h^{(2)}_{00} - h^{(2)}_{jj} + \frac{\lambda^2}{6}R^{(2)}\left[2\frac{\nu}{\theta} - \theta\right]\right)R^{(2)\prime}(r)\right]. \tag{77}$$

Here and in the sequel the prime denotes derivative with respect to r. Using Eq. (53), the potentials defined in Appendix C and the definitions Eq. (64), we can write the scalar curvature to the required order,

$$R^{(2)} = 2\frac{1-\theta}{c^2\lambda^2}\mathcal{Y} = \frac{6\alpha_0}{\lambda^2}\frac{GM_S}{rc^2}e^{-r/\lambda} + O\left(\frac{1}{c^4}\right),$$

$$R^{(2)} + R^{(4)} = \frac{4}{c^4}Y_0'[(1-\theta)U' - 3Y_0'] + \frac{6}{c^4\lambda^2}\left[c^2Y + \left(1 - \frac{3}{2}\mu\right)Y_0^2 - \frac{(GM_S)^2}{R_Sr}(\zeta_1F_1 + \zeta_3F_3)\right], \tag{78}$$

where we define the Yukawa contributions

$$Y_0(r) = \alpha_0\frac{GM_S}{r}e^{-r/\lambda},$$

$$Y_1(r) = \alpha_1\frac{(GM_S)^2}{R_Src^2}e^{-r/\lambda},$$

$$Y(r) = Y_0(r) + Y_1(r) = \alpha\frac{GM_S}{r}e^{-r/\lambda}. \tag{79}$$

We thus obtain the expression below,

$$N(r) = \frac{\theta}{c^2}\left[\left[4U + v^2 - c^2 - \left(2 - \theta - 3\mu + 2\frac{\nu}{\theta}\right)Y_0\right]Y_0' - c^2Y_1' + 4\lambda^2Y_0'Y_0'' - \frac{2}{3}\lambda^2(1-\theta)(U''Y_0' + U'Y_0'') + \frac{(GM_S)^2}{R_Sr}\left[\zeta_1\left(F_1' - \frac{F_1}{r}\right) + \zeta_3\left(F_3' - \frac{F_3}{r}\right)\right]\right], \tag{80}$$

valid to order $O(1/c^2)$.

In the following, we set

$$F(r) = -\frac{(GM_S)^2}{r}\left[\frac{1}{r}(1 + \beta_1e^{-r/\lambda} + \beta_2e^{-2r/\lambda}) + \frac{1}{R_S}\sum_{i=1}^{3}\zeta_iF_i(r)\right]. \tag{81}$$

Using the metric Eq. (70), the equations of motion yield

$$\frac{d\mathbf{v}}{dt} = -\frac{GM_S\mathbf{r}}{r^3} + \mathbf{\Delta}, \tag{82}$$

with the perturbative force

$$\mathbf{\Delta} = -4\frac{\dot{r}}{c^2}U'\mathbf{v} + \left[Y' - \frac{2}{c^2}(U-Y)(U'+Y') + \frac{v^2}{c^2}(U'-Y') + \frac{F'}{c^2} + N\right]\frac{\mathbf{r}}{r} \equiv \Delta_v\frac{\mathbf{v}}{v} + \Delta_r\frac{\mathbf{r}}{r}, \tag{83}$$

where $v = |\mathbf{v}|$, and Δ_r, Δ_v are defined implicitly.

To compute the precession of the perihelion, we follow Refs. [48,49] and begin by recalling that, in Newtonian Mechanics, orbits are ellipses (with perihelion at an angle $\phi = \phi_P$), described by

$$r(\phi) = \frac{L}{1 + e\cos(\phi - \phi_P)}, \tag{84}$$

where e is the orbit's eccentricity, L is the previously mentioned *semilatus rectum*,

$$\frac{1}{L} = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right), \tag{85}$$

and r_+ and r_- are the apoapsis and periapsis, i.e., the distances to the central body at aphelion and perihelion, respectively. The following relations are also valid,

$$\frac{d\phi}{dt} = \frac{\sqrt{GM_SL}}{r^2},$$

$$\frac{dr}{dt} = e\sqrt{\frac{GM_S}{L}}\sin(\phi - \phi_P),$$

$$\mathbf{r}\cdot\mathbf{v} = r\frac{dr}{dt} = \frac{e|\mathbf{h}|\sin(\phi - \phi_P)}{1 + e\cos(\phi - \phi_P)},$$

$$v^2 = \frac{GM_S}{L}[1 + e^2 + 2e\cos(\phi - \phi_P)],$$

$$|\mathbf{h}| = \sqrt{GM_SL}, \qquad |\mathbf{A}| = eGM_S. \tag{86}$$

The constants of motion of closed Newtonian orbits are not only the total energy and angular momentum (per mass), $\mathbf{h} = \mathbf{r}\times\mathbf{v}$, but also the Runge-Lenz vector,

$$\mathbf{A} = -\frac{GM_S\mathbf{r}}{r} + \mathbf{v} \times \mathbf{h}, \tag{87}$$

which points towards the perihelion.

Thus, in order to compute the precession of the latter due to a small perturbing force, it suffices to obtain the (small) variation of the Runge-Lenz vector along the line perpendicular to both **A** and the angular momentum,

$$\frac{d\phi_P}{dt} = (\mathbf{h} \times \mathbf{A}) \cdot \frac{\frac{d\mathbf{A}}{dt}}{|\mathbf{h}|\mathbf{A}^2}, \tag{88}$$

using

$$\frac{d\mathbf{A}}{dt} = \mathbf{\Delta} \times \mathbf{h} + \mathbf{v} \times (\mathbf{r} \times \mathbf{\Delta}). \tag{89}$$

Integrating, we can finally get

$$\delta\phi_P = \int_0^{2\pi} \frac{d\phi_P}{dt}\frac{dt}{d\phi} d\phi = \int_0^{2\pi} \frac{d\phi_P}{dt} \frac{L^2}{|\mathbf{h}|[1 + e\cos(\phi - \phi_P)]^2} d\phi. \tag{90}$$

In the case under scrutiny, inserting Eq. (83) into Eq. (89) yields

$$\frac{d\mathbf{A}}{dt} = \frac{\Delta_r}{r}\mathbf{r} \times \mathbf{h} + \frac{2\Delta_v}{v}\mathbf{v} \times \mathbf{h}, \tag{91}$$

so that Eq. (88) becomes

$$\frac{d\phi_P}{dt} = \frac{1}{e^2}\sqrt{\frac{L}{GM_S}}\left[\Delta_r\left(1 - \frac{L}{r}\right) + 2\Delta_v \frac{\mathbf{r}\cdot\mathbf{v}}{rv}\right] = \frac{1}{e}\sqrt{\frac{L}{GM_S}} \times \left[\frac{2\Delta_v \sin(\phi - \phi_P)}{\sqrt{1 + e^2 + 2e\cos(\phi - \phi_P)}} - \Delta_r \cos(\phi - \phi_P)\right]. \tag{92}$$

Notice that $\Delta_v \equiv -4U'\dot{r}v/c^2$ has no dependence on the additional parameters of the model under scrutiny.

In the following we consider the regime $\lambda \gg L \sim r$ and we Taylor expand the involved quantities to second order in r/λ (except in the nonrelativistic terms); as such, using Eqs. (83) and (90), we may write

$$\delta\phi_P = \frac{6\pi GM_S}{Lc^2} + \frac{1-\theta}{3e}\int_0^{2\pi} I(\phi)\cos(\phi - \phi_P)d\phi, \tag{93}$$

with

$$\begin{aligned} I(\phi) = {} & (1-\theta)\left[1 + \frac{1}{10}\left(\frac{R_S}{\lambda}\right)^2\right]\left(1 + \frac{r}{\lambda}\right)\exp\left(-\frac{r}{\lambda}\right) + \frac{GM_S}{R_S c^2}\left\{\frac{\theta}{3}\left[-11\theta^2 + \theta\left(16 - \frac{9}{2}\mu\right) + 4 + 6\nu\right]\left[\frac{1}{2}\left(\frac{r}{R_S}\right)^2 - \left(\frac{\lambda}{R_S}\right)^2\right]\right. \\ & + \left(\frac{R_S}{\lambda}\right)^2\left[\frac{19}{525}(1-\theta)\theta\left(\frac{R_S}{r}\right)^3 + \frac{1}{15}(1-\theta)[\theta(\theta + 3\mu - 4) - 2\nu - 9]\frac{R_S}{r} + \frac{1}{10}(1-e^2)(1-\theta)\frac{R_S}{L}\right. \\ & - \frac{151}{600}\theta^3 + \frac{88 - 493\mu}{560}\theta^2 + \frac{228 + 2400\mu + 865\nu}{2100}\theta - \frac{71 + 440\nu + 270\mu}{1050} + (1-\theta)\left(\frac{10}{3} + \frac{\mu}{2}(1-\theta) - \theta\right)\frac{r}{R_S} \\ & - \frac{1}{2}(1-e^2)(1-\theta)\frac{r}{L}\frac{r}{R_S} + \frac{\theta^3 34 - \theta^2(124 - 63\mu) + 4\theta(11 - 18\mu - 9\nu) + 4(25 + 12\nu)}{120}\left(\frac{r}{R_S}\right)^2 \\ & \left. + \frac{1}{24}\theta\left[-11\theta^2 + \theta\left(16 - \frac{9}{2}\mu\right) + 4 + 6\nu\right]\left(\frac{r}{R_S}\right)^4\right] - \frac{R_S}{3\lambda}\left[\frac{(1-\theta)}{3}[8\theta^2 - \theta(17 - 18\mu) + 1 - 12\nu]\right. \\ & \left. + \frac{\theta}{3}\left[-11\theta^2 + \theta\left(16 - \frac{9}{2}\mu\right) + 4 + 6\nu\right]\left(\frac{r}{R_S}\right)^3\right] \\ & + (1-e^2)(1-\theta)\frac{R_S}{L} - \frac{\theta^3 34 - \theta^2(124 - 63\mu) + 4\theta(11 - 18\mu - 9\nu) + 4(25 + 12\nu)}{60} \\ & \left. + (1-\theta)[\theta(\theta + 3\mu - 4) - 2(9 + \nu)]\frac{R_S}{3r} + \frac{4}{15}\theta(1-\theta)\left(\frac{R_S}{r}\right)^3\right\}, \end{aligned} \tag{94}$$

so that the familiar result from GR is recovered by setting $\theta = 1$, as expected (except in the case of a perfectly circular orbit, $e = 0$, when the perihelion is ill defined).

In the above, the exponential contribution may be first expanded to third order in the eccentricity e

$$\left(1+\frac{r}{\lambda}\right)\exp\left(-\frac{r}{\lambda}\right) \approx \exp\left(-\frac{L}{\lambda}\right) \times \left[1+\frac{L}{\lambda}+e\left(\frac{L}{\lambda}\right)^2\cos(\phi-\phi_P)+\frac{e^2}{2}\left(\frac{L}{\lambda}\right)^2\left(\frac{L}{\lambda}-3\right)\cos^2(\phi-\phi_P) + \frac{e^3}{6}\left(\frac{L}{\lambda}\right)^2\left(\left[\frac{L}{\lambda}\right]^2-8\frac{L}{\lambda}+12\right)\cos^3(\phi-\phi_P)\right], \quad (95)$$

so that the third power leads to a contribution of second order in e to Eq. (93). The remaining terms in Eq. (94) may be directly integrated, using

$$\int_0^{2\pi}\frac{\cos x}{(1+e\cos x)^n}dx = \begin{cases} 3\pi e\left(1+\frac{1}{4}e^2\right) & n=-3 \\ \pi e & n=-1 \\ 0 & n=0 \\ \frac{2\pi}{e}\left(1-\frac{1}{\sqrt{1-e^2}}\right) \approx -\pi e\left(1+\frac{3}{4}e^2\right) & n=1 \\ -\frac{2\pi e}{(1-e^2)^{3/2}} \approx -2\pi e\left(1+\frac{3}{2}e^2\right) & n=2 \\ -\frac{3\pi e}{(1-e^2)^{5/2}} \approx -3\pi e\left(1+\frac{5}{2}e^2\right) & n=3 \\ -\frac{\pi e(4+e^2)}{(1-e^2)^{7/2}} \approx -4\pi e\left(1+\frac{15}{4}e^2\right) & n=4 \end{cases}. \quad (96)$$

However, the ensuing expressions are too cumbersome, so we choose to instead also expand the ensuing integral to second order in e: the overall result is then given by

$$\begin{aligned} \delta\phi_P = {} & \frac{6\pi GM_S}{Lc^2} + (1-\theta)^2\frac{\pi}{3}\left\{1+e^2\left[\frac{3}{2}-\frac{L}{\lambda}+\frac{1}{8}\left(\frac{L}{\lambda}\right)^2\right]\right\}\left[1+\frac{1}{10}\left(\frac{R_S}{\lambda}\right)^2\right]\left(\frac{L}{\lambda}\right)^2\exp\left(-\frac{L}{\lambda}\right) \\ & + (1-\theta)\frac{\pi GM_S}{12Lc^2}\left\{\left[\frac{\theta}{3}\left[-11\theta^2+\theta\left(16-\frac{9}{2}\mu\right)+4+6\nu\right]\left[-2(2+3e^2)+(4+10e^2)\frac{L}{\lambda}-\left(2+\frac{15}{2}e^2\right)\left(\frac{L}{\lambda}\right)^2\right]\right.\right. \\ & \left. -\frac{1}{30}(2+3e^2)(\theta^3 34-\theta^2[124-63\mu]+4\theta[11-18\mu-9\nu]+4[25+12\nu])\left(\frac{R_S}{\lambda}\right)^2\right]\left(\frac{L}{R_S}\right)^3 \\ & -(1-\theta)\left(\frac{28}{3}+2(1-\theta)\mu-4\theta+e^2\left[8+\frac{3}{2}(\mu[1-\theta]-2\theta)\right]\right)\left(\frac{L}{\lambda}\right)^2+\frac{4}{5}(1-\theta)\theta(4+e^2)\left[1+\frac{19}{140}\left(\frac{R_S}{\lambda}\right)^2\right]\left(\frac{R_S}{L}\right)^2 \\ & \left. +\frac{4}{3}(1-\theta)\left([\theta(\theta-4+3\mu)-2\nu-9]\left[1+\frac{1}{5}\left(\frac{R_S}{\lambda}\right)^2\right]-9\right)\right\}. \end{aligned} \quad (97)$$

Notice that the above collapses to $\delta\phi_P = 4\pi GM_S/Lc^2$ when the model parameters θ, μ and ν vanish and $\lambda \to \infty$: this falls short of the GR prediction of $\delta\phi_P^{(\mathrm{GR})} = 6\pi GM_S/Lc^2 = 42.98''$ by a factor 2/3.

The prediction for the precession of the perihelion assuming a PPN metric [41] together with the Newtonian effect of a quadrupole moment $J_2 = (2.2 \pm 0.1) \times 10^{-7}$ [50] of the Sun is given by

$$\delta\phi_P = \left[\frac{2(1+\gamma)-\beta}{3}+3\times 10^3 J_2\right]\frac{6\pi GM_S}{Lc^2}, \quad (98)$$

with the most stringent bounds on the PPN parameters β [51] and γ [52] given by

$$\begin{aligned} \beta - 1 &= (-4.1 \pm 7.8)\times 10^{-5}, \\ \gamma - 1 &= (2.1 \pm 2.3)\times 10^{-5}. \end{aligned} \quad (99)$$

The bound on β results from recent observations of Mercury, including data from the MESSENGER spacecraft.

The result $\delta\phi_P = 4\pi GM_S/Lc^2$ is equivalent to having $\beta = 2\gamma$ [41]. In particular, this is precisely what stems from the extraneous comparison of $f(R)$ models with a

Brans-Dicke theory with parameter $\omega = 0$, which incorrectly leads to $\gamma = 1/2$ and $\beta = 1$.

Conversely, inspection shows that setting $\theta = 1$ immediately yields the GR prediction for the precession of the perihelion, independently of the remaining model parameters: this reflects the dependence of the model parameters α_0, β_i, $\zeta_i \sim 1 - \theta$, and confirms the previous findings of Ref. [38], where it was noted that the vanishing of the zeroth-order coupling $\alpha_0 = 0$ when $\theta = 1$ evades the stringent constraints of Yukawa forces existing for characteristic length scales 1 mm $< \lambda <$ 1000 AU [53].

Using the experimental bounds for the PPN parameters β and γ given in Eq. (99), we find that the additional perihelion precession due to the model under scrutiny is bounded by

$$\varepsilon_1 < \frac{\delta\phi_P - \delta\phi_P^{(\mathrm{GR})}}{\delta\phi_P^{(\mathrm{GR})}} < \varepsilon_2,$$
$$\varepsilon_1 = -1.367 \times 10^{-5}, \qquad \varepsilon_2 = 6.9 \times 10^{-5}. \tag{100}$$

Hence, Eq. (97) for $\delta\phi_P$ permits us to obtain the admissible region in the four-dimensional parameter space with dimensionless coordinates $\theta = q_1/a_2$, $\mu = a_3/a_2^2$, $\nu = q_2/a_2^2$ and $R_S/\lambda = R_S/\sqrt{6a_2} \ll 1$.

For the computation of exclusion plots, we use the values for the mass of the Sun, $M_\odot = 1.989 \times 10^{30}$ kg, the radius of the Sun, $R_\odot = 6.957 \times 10^8$ m, and the *semilatus rectum* of Mercury, $L = 5.546 \times 10^{10}$ m.

Using Eq. (97), the bounds Eq. (100) can be written in the following form which involves a linear combination of parameters μ and ν,

$$\varepsilon_1 < \frac{L}{R_g} F_\lambda(\theta) e^{-L/\lambda} + P_\lambda(\theta)\mu + Q_\lambda(\theta)\nu + S_\lambda(\theta) < \varepsilon_2, \tag{101}$$

where $R_g = 2GM_\odot/c^2$ is the Schwarzschild radius of the Sun, and $P_\lambda(\theta)$, $Q_\lambda(\theta)$, $S_\lambda(\theta)$ and $F_\lambda(\theta)$ are polynomials in θ which depend on powers of the dimensionless quantities L/λ and $R_\odot/\lambda$, and do not depend on c^2.

For fixed values of L/λ and θ, the admissible region described by bounds Eq. (101) is a strip in the $\mu - \nu$ plane bounded by two parallel straight lines. The bounds Eq. (101) take a simple form in the following particular case. Assume $|1-\theta| \ll 1$, which implies that the strength of the Yukawa potential is small in the nonrelativistic limit, and also assume $(L/\lambda)|1-\theta| \ll 10^{-6}$. Then, neglecting $(1-\theta)^2$ with respect to $|1-\theta|$, and neglecting $(L/\lambda)^2$, e^2 with respect to unity, the bounds Eq. (101) can be approximated as follows:

$$\varepsilon_1 < \frac{\theta}{3}(1-\theta)\left(1 - \frac{L}{\lambda}\right)\left(\frac{L}{R_\odot}\right)^3 \left(\frac{1}{4}\theta\mu - \frac{\nu}{3} - \frac{\theta}{2}\right) < \varepsilon_2. \tag{102}$$

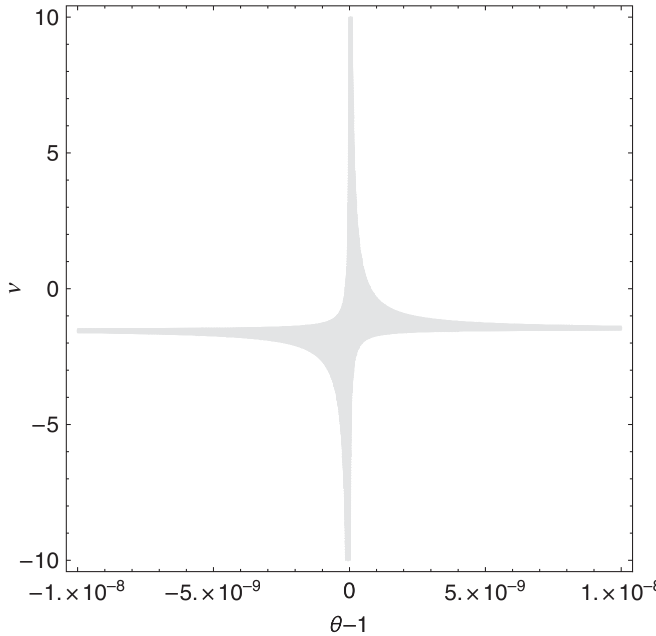

FIG. 1. Exclusion plot for the model parameters (θ, ν), for $\lambda = 50L$ and $\mu = 0$.

In this case, the slope m of the parallel lines and the width Δ of the strip in the ν direction are given by

$$m = \frac{3}{4}\theta, \qquad \Delta = 9\lambda \left(\frac{R_\odot}{L}\right)^3 \frac{\varepsilon_2 - \varepsilon_1}{\theta|1-\theta|(\lambda - L)}. \tag{103}$$

If the above assumptions are not satisfied, then the general expression Eq. (101) of the bounds has to be used. Using

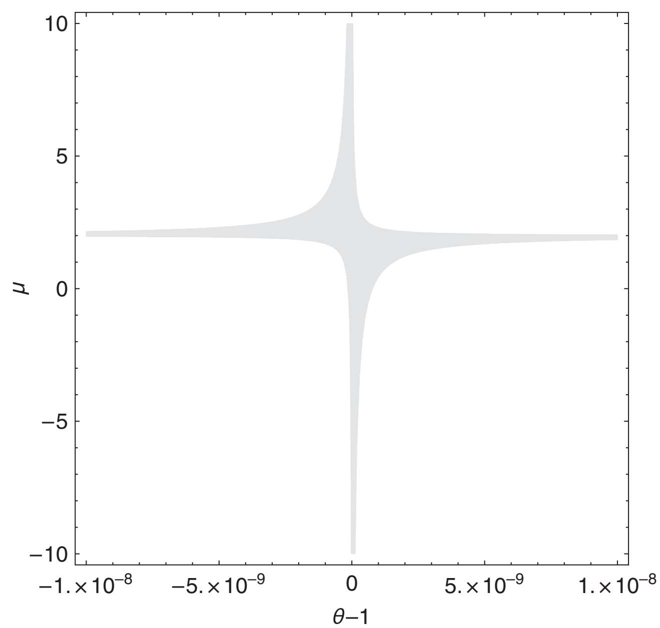

FIG. 2. Exclusion plot for the model parameters (θ, μ), for $\lambda = 50L$ and $\nu = 0$.

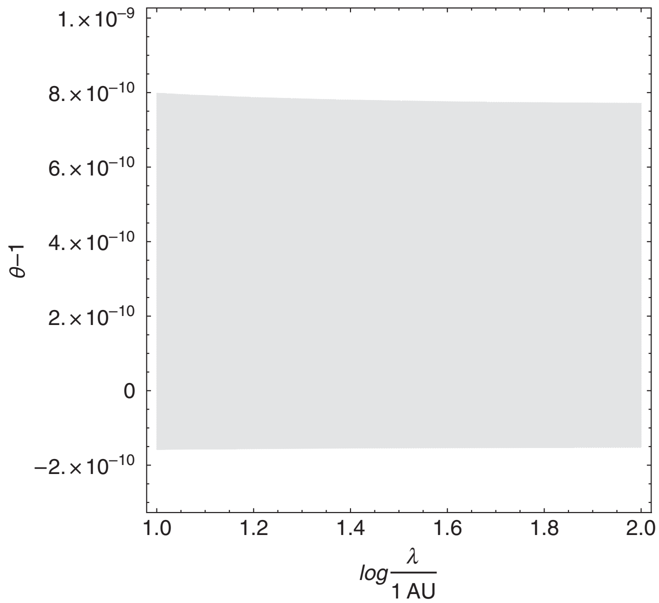

FIG. 3. Exclusion plot for the model parameters (λ, θ), for $\mu = \nu = 0$.

Eq. (97) for $\delta\phi_P$, further exclusion plots for the four independent quantities θ, μ, ν and λ are depicted on Figs. 1–5, using the previously considered experimental bounds for β and γ. The admissible region corresponds to the grey areas in the plots.

Starting in the mid 2020s, the BepiColombo mission will offer the best possibility for tightening current constraints on the PPN parameters, shown in Eq. (99): indeed,

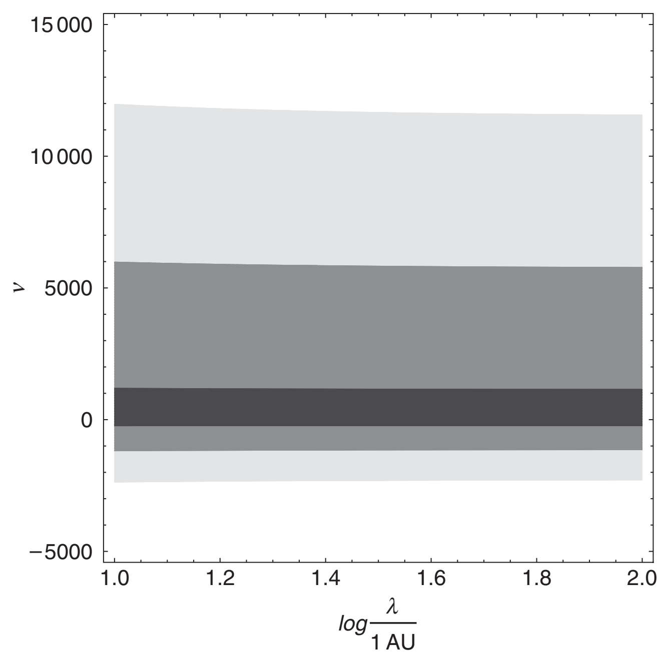

FIG. 4. Exclusion plot for the model parameters (λ, ν), for $\mu = 0$ and $\theta = \{1 + 10^{-13}, 1 + 2 \times 10^{-13}, 1 + 10^{-12}\}$ (light, medium, dark grey).

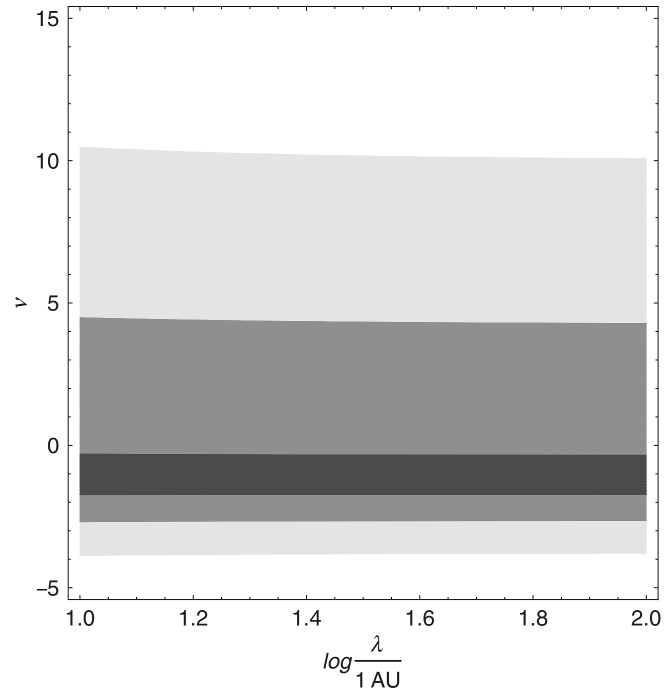

FIG. 5. Exclusion plot for the model parameters (λ, ν), for $\mu = 0$ and $\theta = \{1 + 10^{-10}, 1 + 2 \times 10^{-10}, 1 + 10^{-9}\}$ (light, medium, dark grey).

the radioscience experiment onboard the spacecraft is expected to yield an order of magnitude improvement on β [54] and γ [55],

$$|\beta - 1| \leq 7.81 \times 10^{-6},$$
$$|\gamma - 1| \leq 5.07 \times 10^{-6}. \quad (104)$$

Using these figures to derive the allowed range for the model parameters mentioned above does not change the corresponding exclusion plots qualitatively, but naturally leads to a reduction on their admissible bounds of approximately one order of magnitude.

VII. CONCLUSIONS

In this work, we have computed the metric solutions for a NMC gravity model around a Minkowski background. It is shown that, up to order $O(1/c^4)$, the corrections depend on the $f^1(R)$ and $f^2(R)$ functions and cannot be expressed in terms of powers of $1/r$: indeed, it is found that the obtained solutions must be expressed in the PPNY approximation, as first proposed in Ref. [38].

Comparison with experimental results on the precession of the perihelion of Mercury allows us to establish constraints on the model parameters, and further Solar System tests using the obtained results for the PPNY metric could yield more stringent bounds on the latter: in principle, this opens up the possibility of addressing a wider class of physical situations with great accuracy, as the results obtained in this work might be relevant for distinguishing between GR, $f(R)$ and nonminimally coupled theories from the analysis of detailed observations data in the future.

As discussed in Sec. V, the results reported here depend crucially on the assumption of a vanishing density away from the central body: following the interest around the possibility of implementing a chameleon effect in $f(R)$ theories that suppresses the dynamical impact of the latter at Solar System scales, we expect that an additional NMC should modify this effect, displaying the impact of non-minimally coupling matter with geometry.

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APPENDIX A: EQUATIONS FOR $h_{00}^{(4)}$

In order to compute $h_{00}^{(4)}$ we need the corresponding term $R^{(4)}$ in the expansion Eq. (13) of the Ricci scalar; this can be obtained by solving the trace Eq. (4) at order $O(1/c^4)$.

In the following, in order to avoid a cumbersome notation, we replace the symbol $R^{(2)}$ by R. Using the gauge conditions Eq. (9) and the trace of the field equations at order $O(1/c^4)$ leads to the following equation for $R^{(4)}$:

$$\begin{aligned}
&\nabla^2 R^{(4)} - \frac{1}{6a_2}R^{(4)} - \frac{1}{c^2}R_{,00} + \frac{3a_3}{2a_2}\nabla^2 R^2 + 2a_2 R\nabla^2 R - \frac{2}{c^2}U\nabla^2 R + \frac{64\pi G}{c^4}(a_2 - q_1)\frac{q_1}{a_2}U_{,ij}\rho_{,ij} \\
&- \frac{16\pi G}{c^2}\left[q_1(\rho\nabla^2 R + R\nabla^2\rho) + \frac{q_2}{a_2}\nabla^2(\rho R)\right] + 24a_2^2 R_{,ij}R_{,ij} - \frac{8}{c^2}(a_2 - q_1)U_{,ij}R_{,ij} \\
&- \frac{384\pi G}{c^2}a_2 q_1 \rho_{,ij}R_{,ij} - \frac{4\pi G}{c^2}\frac{q_1}{a_2}\rho R + \frac{8\pi G}{c^4}\frac{q_1}{a_2}\rho_{,00} + \frac{16\pi G}{c^4}\frac{q_1}{a_2}U\nabla^2\rho \\
&+ \frac{128\pi^2 G^2}{c^4}\frac{q_1^2}{a_2}\rho\nabla^2\rho + \frac{1536\pi^2 G^2}{c^4}q_1^2\rho_{,ij}\rho_{,ij} - \frac{8\pi G}{c^4}\frac{q_1}{a_2}\nabla^2(\rho\Pi) = -\frac{4\pi G}{3a_2 c^4}(\rho\Pi - 3p).
\end{aligned} \tag{A1}$$

In the case of pure $f(R)$ gravity ($q_1 = q_2 = 0$) this equation differs from the equation found by Clifton in Ref. [42]. More precisely, in Eq. (A1), the terms proportional to

$$R\nabla^2 R, \qquad U\nabla^2 R, \qquad R_{,ij}R_{,ij}, \qquad U_{,ij}R_{,ij} \tag{A2}$$

have opposite sign with respect to the equation given in Ref. [42]. The reason is the following. The trace of the field equations, Eq. (4), contains a term proportional to $\Box f_R^1$, which requires the computation of $\Box R^{(2)}$ at order $O(1/c^4)$. Using the gauge conditions Eq. (9), we find (setting $R = R^{(2)}$)

$$\Box R = g^{\mu\nu}R_{,\mu\nu} = -\frac{1}{c^2}R_{,00} + \nabla^2 R + h^{ij(2)}R_{,ij}. \tag{A3}$$

Now the terms given in Eq. (A2) come from $h^{ij(2)}R_{,ij}$ and their sign in our equation is determined by the property $h^{ij(2)} = -h_{ij}^{(2)}$. An analogous computation in Ref. [39] is in agreement with the sign we have found for the above terms. Such changes of sign subsequently determine differences of order of unity in the coefficients of the solution for $h_{00}^{(4)}$ with respect to the result in Ref. [42].

Next, we rewrite Eq. (A1) in the form of a Yukawa-type equation,

$$(\nabla^2 - m^2)(R^{(4)} + \cdots) = -4\pi Q, \tag{A4}$$

where we recall that $m^2 = 1/6a_2$ and we introduce the potential [42]

$$X(Q) = \int Q(t,\mathbf{y})\frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}d^3y, \tag{A5}$$

which solves the equation

$$(\nabla^2 - m^2)X(Q) = -4\pi Q. \tag{A6}$$

In order to put Eq. (A1) into the form (A4), we make use of the following identity for two arbitrary potentials $\tilde{U}$ and $\tilde{V}$:

$$\begin{aligned}
\tilde{U}_{,ij}\tilde{V}_{,ij} = \frac{1}{2}[&\nabla^2(\nabla\tilde{U}\cdot\nabla\tilde{V}) - \nabla\tilde{U}\cdot\nabla(\nabla^2\tilde{V}) \\
&- \nabla\tilde{V}\cdot\nabla(\nabla^2\tilde{U})].
\end{aligned} \tag{A7}$$

Using this identity, the trace Eq. (23), and the Poisson equation for the Newtonian potential, $\nabla^2 U = -4\pi G\rho$, we get the following relations:

$$
\begin{aligned}
U_{,ij}R_{,ij} &= \frac{1}{2}\left(\nabla^2 - \frac{1}{6a_2}\right)\nabla U\cdot\nabla R + 2\pi G\nabla\rho\cdot\nabla R + \frac{2\pi G}{3a_2c^2}\nabla U\cdot\nabla(\rho - 6q_1\nabla^2\rho),\\
R_{,ij}R_{,ij} &= \frac{1}{2}\left(\nabla^2 - \frac{1}{6a_2}\right)\left(|\nabla R|^2 - \frac{R^2}{12a_2}\right) + \frac{R^2}{144a_2^2} + \frac{\pi G}{3a_2^2c^2}\left[2q_1R\nabla^2\rho - \frac{\rho R}{3} + 4a_2\nabla R\cdot\nabla(\rho - 6q_1\nabla^2\rho)\right],\\
R_{,ij}\rho_{,ij} &= \frac{1}{2}\left(\nabla^2 - \frac{1}{6a_2}\right)\nabla\rho\cdot\nabla R - \frac{1}{2}\nabla R\cdot\nabla(\nabla^2\rho) + \frac{2\pi G}{3a_2c^2}\nabla\rho\cdot\nabla(\rho - 6q_1\nabla^2\rho),\\
U_{,ij}\rho_{,ij} &= \frac{1}{2}\left(\nabla^2 - \frac{1}{6a_2}\right)\nabla\rho\cdot\nabla U + \frac{1}{12a_2}\nabla\rho\cdot\nabla U + 2\pi G|\nabla\rho|^2 - \frac{1}{2}\nabla U\cdot\nabla(\nabla^2\rho),\\
\rho_{,ij}\rho_{,ij} &= \frac{1}{2}\left(\nabla^2 - \frac{1}{6a_2}\right)|\nabla\rho|^2 + \frac{1}{12a_2}|\nabla\rho|^2 - \nabla\rho\cdot\nabla(\nabla^2\rho).
\end{aligned}
\tag{A8}
$$

Now we introduce the potential [42]

$$\hat{\chi} = G\int \rho(t,\mathbf{y})e^{-m|\mathbf{x}-\mathbf{y}|}d^3y. \tag{A9}$$

Using the solution for the trace equation at second order, Eq. (28), one can show that the potential $\hat{\chi}$ satisfies the equation

$$\left(\nabla^2 - \frac{1}{6a_2}\right)\hat{\chi} = -c^2\sqrt{6a_2}\left(1 - \frac{q_1}{a_2}\right)^{-1}\left(R - \frac{8\pi G}{c^2}\frac{q_1}{a_2}\rho\right). \tag{A10}$$

Then, using the relations (A8), the trace equation (23), and transforming the quantities $R\nabla^2\rho$, $U\nabla^2\rho$ and $\rho\nabla^2\rho$ by means of the identity $\nabla^2(ab) = a\nabla^2 b + b\nabla^2 a + 2\nabla a\cdot\nabla b$, we put Eq. (A1) for $R^{(4)}$ in the Yukawa form Eq. (A4). The solution of the resulting equation is

$$
\begin{aligned}
R^{(4)} = &-\frac{1}{c^4\sqrt{6a_2}}\left(1 - \frac{q_1}{a_2}\right)\hat{\chi}_{,00} - \left(\frac{3a_3}{2a_2} - a_2\right)R^2 - \frac{16\pi G}{c^2}\left(q_1 - \frac{q_2}{a_2}\right)\rho R + \frac{8\pi G}{c^4}\frac{q_1}{a_2}\rho\Pi + \frac{64\pi^2G^2}{c^4}\frac{q_1^2}{a_2}\rho^2 - 12a_2^2|\nabla R|^2\\
&+ \frac{4}{c^2}(a_2 - q_1)\nabla U\cdot\nabla R + \frac{192\pi G}{c^2}a_2q_1\nabla\rho\cdot\nabla R - \frac{32\pi G}{c^4}\frac{q_1}{a_2}(a_2 - q_1)\nabla\rho\cdot\nabla U - \frac{768\pi^2G^2}{c^4}q_1^2|\nabla\rho|^2\\
&- \frac{1}{12\pi a_2c^2}X(UR) + \frac{1}{8\pi}\left(\frac{a_3}{2a_2^2} + 1\right)X(R^2) + \frac{2G}{3a_2c^4}X(\rho U) - \frac{G}{12c^2}\left(16 + 20\frac{q_1}{a_2} + 8\frac{q_2}{a_2^2}\right)X(\rho R)\\
&+ \frac{8\pi G^2}{3c^4}\frac{q_1}{a_2}\left(4 - \frac{q_1}{a_2}\right)X(\rho^2) - \frac{4G}{3c^4}\left[1 - \frac{q_1}{a_2}\left(2 - \frac{q_1}{a_2}\right)\right]X(\nabla\rho\cdot\nabla U) + \frac{4G}{c^2}(a_2 - q_1)X(\nabla\rho\cdot\nabla R)\\
&- \frac{32\pi G^2}{c^4}q_1\left(1 - \frac{q_1}{a_2}\right)X(|\nabla\rho|^2) - \frac{G}{a_2c^4}X(p) + \frac{G}{3a_2c^4}\left(1 - \frac{q_1}{a_2}\right)X(\rho\Pi).
\end{aligned}
\tag{A11}
$$

We can now write the $0-0$ component of the field Eqs. (3) at order $O(1/c^4)$. Using the expressions for R_{00} and T_{00} given by Eqs. (10) and (14), respectively, and Eq. (A1) to eliminate the term proportional to $R^{(4)}$, we find that $h_{00}^{(4)}$ obeys the following:

$$
\begin{aligned}
&-\frac{1}{2}\nabla^2 h_{00}^{(4)} - \frac{1}{c^4}\nabla^2U^2 + \left(\frac{3}{2}a_3 - 2a_2^2\right)\nabla^2R^2 - 3\frac{a_2}{c^2}\nabla^2(UR) + \frac{24\pi G}{c^4}q_1\nabla^2(\rho U) + \frac{16\pi G}{c^2}(2a_2q_1 - q_2)\nabla^2(\rho R)\\
&- \frac{128\pi^2G^2}{c^4}q_1^2\nabla^2\rho^2 - 18\frac{a_2^2}{c^2}\nabla^2R_{,00} + \frac{6}{c^4}(a_2 - q_1)\nabla^2U_{,00} + \frac{144\pi G}{c^4}a_2q_1\nabla^2\rho_{,00} + a_2\nabla^2R^{(4)} - \frac{8\pi G}{c^4}q_1\nabla^2(\rho\Pi)\\
&+ \frac{8}{c^4}(a_2 - q_1)U_{,ij}U_{,ij} - 24\frac{a_2^2}{c^2}U_{,ij}R_{,ij} + \frac{192\pi G}{c^4}a_2q_1U_{,ij}\rho_{,ij} + \frac{1}{6c^2}UR + \frac{2}{3}a_2R^2 - \frac{28\pi G}{3c^4}\rho U + \frac{20\pi G}{c^2}\left(\frac{a_2}{3} - q_1\right)\rho R\\
&- \frac{96\pi^2G^2}{c^4}q_1\rho^2 = \frac{4\pi G}{c^4}[\rho(\Pi + 2v^2) + 3p].
\end{aligned}
\tag{A12}
$$

This can be written in the form of a Poisson-type equation,

$$\nabla^2(h_{00}^{(4)} + \cdots) = -4\pi Q. \tag{A13}$$

Moreover, we denote by $\mathcal{V}$ the Poisson integral

$$\mathcal{V}(Q) = \int \frac{Q(t,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d^3y, \tag{A14}$$

so that $\nabla^2\mathcal{V}(Q) = -4\pi Q$. We proceed as in the computation of $R^{(4)}$: using the identity (A7), the trace Eq. (23), and the Poisson equation for the Newtonian potential, $\nabla^2 U = -4\pi G\rho$, we get the following relations,

$$\begin{aligned} U_{,ij}U_{,ij} &= \frac{1}{2}\nabla^2(|\nabla U|^2) + 4\pi G\nabla\rho\cdot\nabla U, \\ U_{,ij}R_{,ij} &= \frac{1}{2}\nabla^2(\nabla U\cdot\nabla R) - \frac{1}{24a_2}\nabla^2(UR) + 2\pi G\nabla\rho\cdot\nabla R + \frac{\pi G}{3c^2}\frac{q_1}{a_2^2}\nabla^2(\rho U) + \frac{2\pi G}{3a_2c^2}\left(1-\frac{q_1}{a_2}\right)\nabla\rho\cdot\nabla U \\ &\quad - \frac{4\pi G}{c^2}\frac{q_1}{a_2}\nabla U\cdot\nabla(\nabla^2\rho) - \frac{\pi G}{6a_2}\rho R + \frac{1}{144a_2^2}UR - \frac{\pi G}{18a_2^2c^2}\rho U + \frac{4\pi^2G^2}{3c^2}\frac{q_1}{a_2^2}\rho^2, \\ U_{,ij}\rho_{,ij} &= \frac{1}{2}\nabla^2(\nabla\rho\cdot\nabla U) + 2\pi G|\nabla\rho|^2 - \frac{1}{2}\nabla U\cdot\nabla(\nabla^2\rho). \end{aligned} \tag{A15}$$

Then, using relations (A15), Eq. (A12) for $h_{00}^{(4)}$ can be recast in the Poisson form Eq. (A13), with solution

$$\begin{aligned} h_{00}^{(4)} = &-\frac{2}{c^4}U^2 + (3a_3 - 4a_2^2)R^2 - 4\frac{a_2}{c^2}UR + \frac{32\pi G}{c^4}q_1\rho U + \frac{32\pi G}{c^2}(2a_2q_1 - q_2)\rho R - \frac{16\pi G}{c^4}q_1\rho\Pi - \frac{256\pi^2G^2}{c^4}q_1^2\rho^2 \\ &- 36\frac{a_2^2}{c^2}R_{,00} + \frac{12}{c^4}(a_2-q_1)U_{,00} + \frac{288\pi G}{c^4}a_2q_1\rho_{,00} + \frac{8}{c^4}(a_2-q_1)|\nabla U|^2 - 24\frac{a_2^2}{c^2}\nabla U\cdot\nabla R + \frac{192\pi G}{c^4}a_2q_1\nabla\rho\cdot\nabla U \\ &- \frac{a_2}{3\pi}\mathcal{V}(R^2) + \frac{4G}{c^4}\mathcal{V}(\rho U) - \frac{2G}{c^2}\left(\frac{8}{3}a_2 - 5q_1\right)\mathcal{V}(\rho R) + \frac{64\pi G^2}{c^4}q_1\mathcal{V}(\rho^2) - \frac{8G}{c^4}(a_2-q_1)\mathcal{V}(\nabla\rho\cdot\nabla U) \\ &+ 24\frac{G}{c^2}a_2^2\mathcal{V}(\nabla\rho\cdot\nabla R) - \frac{192\pi G^2}{c^4}a_2q_1\mathcal{V}(|\nabla\rho|^2) + \frac{2G}{c^4}\mathcal{V}(\rho\Pi) + \frac{4G}{c^4}\mathcal{V}(\rho v^2) + \frac{6G}{c^4}\mathcal{V}(p) + 2a_2R^{(4)}. \end{aligned} \tag{A16}$$

Substituting in the above expression Eq. (A11) for $R^{(4)}$, we finally obtain the solution for $h_{00}^{(4)}$ given in Eq. (47) of Sec. IV C.

APPENDIX B: TRANSFORMATIONS OF THE METRIC

Following the discussion preceding Eq. (50), for the gauge transformation, we adopt the form

$$\begin{aligned} \xi_0 &= \frac{6}{c^3}(a_2-q_1)U_{,0} - 18\frac{a_2^2}{c}R_{,0} + \frac{144\pi G}{c^3}a_2q_1\rho,0 - \frac{1}{c^3}\sqrt{\frac{a_2}{6}}\left(1-\frac{q_1}{a_2}\right)\hat{\chi}_{,0}, \\ \xi_i &= \frac{4}{c^2}(a_2-q_1)U_{,i} - 12a_2^2R_{,i} + \frac{96\pi G}{c^2}a_2q_1\rho_{,i}, \end{aligned} \tag{B1}$$

so that the metric perturbation $h_{\mu\nu}$ transforms into a diagonal expression, with no time derivatives and some of the gradient terms in $h_{00}^{(4)}$ gauged away:

$$h_{ij}^{(2)} \to h_{ij}^{(2)} - \frac{8}{c^2}(a_2 - q_1)U_{,ij} + 24a_2^2 R_{,ij} - \frac{192\pi G}{c^2} a_2 q_1 \rho_{,ij},$$
$$h_{0i}^{(3)} \to h_{0i}^{(3)} - \frac{10}{c^3}(a_2 - q_1)U_{,0i} + 30\frac{a_2^2}{c} R_{,0i} - \frac{240\pi G}{c^3} a_2 q_1 \rho_{,0i} + \frac{1}{c^3}\sqrt{\frac{a_2}{6}}\left(1 - \frac{q_1}{a_2}\right)\hat{\chi}_{,0i},$$
$$h_{00}^{(4)} \to h_{00}^{(4)} + 36\frac{a_2^2}{c^2} R_{,00} - \frac{12}{c^4}(a_2 - q_1)U_{,00} - \frac{288\pi G}{c^4} a_2 q_1 \rho_{,00} + \frac{1}{c^4}\sqrt{\frac{2}{3}a_2}\left(1 - \frac{q_1}{a_2}\right)\hat{\chi}_{,00} - \frac{8}{c^4}(a_2 - q_1)|\nabla U|^2 + 24a_2^3 |\nabla R|^2$$
$$+ \frac{1536\pi^2 G^2}{c^4} a_2 q_1^2 |\nabla\rho|^2 + 8\frac{a_2}{c^2}(2a_2 + q_1)\nabla U \cdot \nabla R - \frac{64\pi G}{c^4} q_1(2a_2 + q_1)\nabla\rho \cdot \nabla U - \frac{384\pi G}{c^2} a_2^2 q_1 \nabla\rho \cdot \nabla R. \quad \text{(B2)}$$

Using the continuity Eq. (42), the quantity $\hat{\chi}_{,0i}$ appearing in the transformation law for $h_{0i}^{(3)}$ is given by

$$\hat{\chi}_{,0i} = \frac{1}{\sqrt{6a_2}}\left(GX(\rho v_i) - Y_i - \frac{1}{\sqrt{6a_2}} Z_i\right), \quad \text{(B3)}$$

where the potentials Y_i and Z_i are defined in Eq. (57).

Collecting the results from Secs. III and IV, the form of the metric after the gauge transformations is

$$g_{00} = -1 + 2\left(\frac{U}{c^2} + a_2 R - \frac{8\pi G}{c^2} q_1 \rho\right) - \frac{2}{c^4}U^2 - 2a_2^2 R^2 - 4\frac{a_2}{c^2} UR + \frac{32\pi G}{c^2} q_1 \rho\left(\frac{U}{c^2} + a_2 R\right) - \frac{128\pi^2 G^2}{c^4} q_1^2 \rho^2 - \frac{a_2}{3\pi}\mathcal{V}(R^2)$$
$$+ \frac{4G}{c^4}\mathcal{V}(\rho U) - \frac{2G}{c^2}\left(\frac{8}{3}a_2 - 5q_1\right)\mathcal{V}(\rho R) + \frac{64\pi G^2}{c^4} q_1 \mathcal{V}(\rho^2) - \frac{8G}{c^4}(a_2 - q_1)\mathcal{V}(\nabla\rho \cdot \nabla U) + 24\frac{G}{c^2} a_2^2 \mathcal{V}(\nabla\rho \cdot \nabla R)$$
$$- \frac{192\pi G^2}{c^4} a_2 q_1 \mathcal{V}(|\nabla\rho|^2) + \frac{2G}{c^4}\mathcal{V}(\rho\Pi) + \frac{4G}{c^4}\mathcal{V}(\rho v^2) + \frac{6G}{c^4}\mathcal{V}(p) - \frac{1}{6\pi c^2} X(UR) + \frac{1}{4\pi}\left(a_2 + \frac{a_3}{2a_2}\right) X(R^2)$$
$$+ \frac{4G}{3c^4} X(\rho U) - \frac{G}{6c^2}\left(16a_2 + 20q_1 + 8\frac{q_2}{a_2}\right) X(\rho R) + \frac{16\pi G^2}{3c^4} q_1\left(4 - \frac{q_1}{a_2}\right) X(\rho^2) - \frac{8G}{3c^4}\left[a_2 - q_1\left(2 - \frac{q_1}{a_2}\right)\right] X(\nabla\rho \cdot \nabla U)$$
$$+ \frac{8G}{c^2} a_2(a_2 - q_1) X(\nabla\rho \cdot \nabla R) - \frac{64\pi G^2}{c^4} q_1(a_2 - q_1) X(|\nabla\rho|^2) - \frac{2G}{c^4} X(p) + \frac{2G}{3c^4} X(\rho\Pi),$$
$$g_{0i} = -\frac{7}{2c^3} V_i - \frac{1}{2c^3} W_i + \frac{1}{6c^3}\left(1 - \frac{q_1}{a_2}\right)\left[GX(\rho v_i) - Y_i - \frac{1}{\sqrt{6a_2}} Z_i\right], \quad \text{(B4)}$$

$$g_{ij} = \left[1 + 2\left(\frac{U}{c^2} - a_2 R + \frac{8\pi G}{c^2} q_1 \rho\right)\right]\delta_{ij}. \quad \text{(B5)}$$

The spatial part of the metric g_{ij} is now diagonal, as in the standard post-Newtonian gauge. However, although time derivatives have been eliminated from g_{00}, the latter is not yet in the usual PPN form, since it contains contributions with the potentials $\mathcal{V}$ and X depending on the gradient terms $\nabla\rho \cdot \nabla U$, $\nabla\rho \cdot \nabla R$ and $|\nabla\rho|^2$.

Once again, following Ref. [42], we transform such potentials into expressions without gradient terms: below we show, for instance, how the gradient terms can be eliminated from the contribution $X(\nabla\rho \cdot \nabla U)$. We have

$$X(\nabla\rho \cdot \nabla U) = \int \frac{\nabla\rho(t,\mathbf{y}) \cdot \nabla U(t,\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} e^{-m|\mathbf{x}-\mathbf{y}|} d^3y. \quad \text{(B6)}$$

Introducing the vector field

$$\mathbf{A}(t,\mathbf{x},\mathbf{y}) = \frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|}\nabla U(t,\mathbf{y}), \quad \text{(B7)}$$

and using the divergence theorem and the boundary conditions (18) yields

$$X(\nabla\rho \cdot \nabla U) = \int \nabla\rho(t,\mathbf{y}) \cdot \mathbf{A}(t,\mathbf{x},\mathbf{y}) d^3y$$
$$= -\int \rho(t,\mathbf{y}) \nabla_y \cdot \mathbf{A}(t,\mathbf{x},\mathbf{y}) d^3y, \quad \text{(B8)}$$

where the operator ∇_y denotes the divergence with respect to the coordinates $\mathbf{y}$. The evaluation of the divergence of the vector field $\mathbf{A}$, using the Poisson equation $\nabla_y^2 U(t,\mathbf{y}) = -4\pi G\rho(t,\mathbf{y})$, yields

$$\nabla_y \cdot \mathbf{A}(t,\mathbf{x},\mathbf{y}) = -4\pi G\rho(t,\mathbf{y})\frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} - Gm\int \frac{\rho(t,\mathbf{z})(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}|^2|\mathbf{y}-\mathbf{z}|^3}e^{-m|\mathbf{x}-\mathbf{y}|}d^3z - G\int \frac{\rho(t,\mathbf{z})(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}|^3|\mathbf{y}-\mathbf{z}|^3}e^{-m|\mathbf{x}-\mathbf{y}|}d^3z, \tag{B9}$$

from which we obtain the following expression with the gradient terms expunged:

$$X(\nabla\rho\cdot\nabla U) = 4\pi G X(\rho^2) + \frac{\psi_4}{G\sqrt{6a_2}} + \frac{\psi_5}{G}, \tag{B10}$$

where the potentials ψ_4 and ψ_5 are defined by

$$\psi_4 = G^2\int \frac{\rho(t,\mathbf{y})\rho(t,\mathbf{z})(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}|^2|\mathbf{y}-\mathbf{z}|^3}e^{-m|\mathbf{x}-\mathbf{y}|}d^3y d^3z,$$
$$\psi_5 = G^2\int \frac{\rho(t,\mathbf{y})\rho(t,\mathbf{z})(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}|^3|\mathbf{y}-\mathbf{z}|^3}e^{-m|\mathbf{x}-\mathbf{y}|}d^3y d^3z. \tag{B11}$$

Following Ref. [42] and the argument above, we transform all the potentials involving gradients into expressions without gradient terms. We find the following identities:

$$G\mathcal{V}(\nabla\rho\cdot\nabla U) = 4\pi G^2\mathcal{V}(\rho^2) + \psi_1,$$
$$G\mathcal{V}(\nabla\rho\cdot\nabla R) = \frac{8\pi G^2}{c^2}\frac{q_1}{a_2}\mathcal{V}(|\nabla\rho|^2) + \frac{4\pi G^2}{3c^2}\frac{a_2-q_1}{a_2^2}\mathcal{V}(\rho^2) - \frac{(a_2-q_1)}{3c^2a_2^2}\left(\frac{1}{6a_2}\psi_0 - \frac{1}{\sqrt{6a_2}}\psi_2 - \psi_3\right),$$
$$GX(\nabla\rho\cdot\nabla U) = 4\pi G^2 X(\rho^2) + \frac{1}{\sqrt{6a_2}}\psi_4 + \psi_5,$$
$$GX(\nabla\rho\cdot\nabla R) = \frac{8\pi G^2}{c^2}\frac{q_1}{a_2}X(|\nabla\rho|^2) + \frac{4\pi G^2}{3c^2}\frac{a_2-q_1}{a_2^2}X(\rho^2) + \frac{(a_2-q_1)}{3c^2a_2^2}\left[\psi_6 + \frac{\psi_7+\psi_8}{\sqrt{6a_2}} + \frac{\psi_9-\psi_{10}}{6a_2}\right]. \tag{B12}$$

Substituting these identities into Eq. (B4) for g_{00}, the terms proportional to $\mathcal{V}(|\nabla\rho|^2)$ and $X(|\nabla\rho|^2)$ cancel exactly. The eleven potentials $\psi_0,\ldots,\psi_{10}$ appearing in the previous identities are given by

$$\psi_i(t,\mathbf{x}) = G^2\int \frac{\rho(t,\mathbf{y})\rho(t,\mathbf{z})}{|\mathbf{x}-\mathbf{y}||\mathbf{y}-\mathbf{z}|}\Psi_i(\mathbf{x},\mathbf{y},\mathbf{z})d^3y d^3z, \tag{B13}$$

with

$$\Psi_0(\mathbf{x},\mathbf{y},\mathbf{z}) = e^{-m|\mathbf{y}-\mathbf{z}|},$$
$$\Psi_1(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}|^2|\mathbf{y}-\mathbf{z}|^2},$$
$$\Psi_2(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}|^2|\mathbf{y}-\mathbf{z}|}e^{-m|\mathbf{y}-\mathbf{z}|},$$
$$\Psi_3(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}|^2|\mathbf{y}-\mathbf{z}|^2}e^{-m|\mathbf{y}-\mathbf{z}|},$$
$$\Psi_4(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}||\mathbf{y}-\mathbf{z}|^2}e^{-m|\mathbf{x}-\mathbf{y}|},$$
$$\Psi_5(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}|^2|\mathbf{y}-\mathbf{z}|^2}e^{-m|\mathbf{x}-\mathbf{y}|},$$
$$\Psi_6(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}|^2|\mathbf{y}-\mathbf{z}|^2}e^{-m(|\mathbf{x}-\mathbf{y}|+|\mathbf{y}-\mathbf{z}|)},$$
$$\Psi_7(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}|^2|\mathbf{y}-\mathbf{z}|}e^{-m(|\mathbf{x}-\mathbf{y}|+|\mathbf{y}-\mathbf{z}|)},$$
$$\Psi_8(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}||\mathbf{y}-\mathbf{z}|^2}e^{-m(|\mathbf{x}-\mathbf{y}|+|\mathbf{y}-\mathbf{z}|)},$$
$$\Psi_9(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{y}-\mathbf{z})}{|\mathbf{x}-\mathbf{y}||\mathbf{y}-\mathbf{z}|}e^{-m(|\mathbf{x}-\mathbf{y}|+|\mathbf{y}-\mathbf{z}|)},$$
$$\Psi_{10}(\mathbf{x},\mathbf{y},\mathbf{z}) = e^{-m(|\mathbf{x}-\mathbf{y}|+|\mathbf{y}-\mathbf{z}|)}. \tag{B14}$$

The potentials $\psi_1,\ldots,\psi_9$ coincide with those found in Ref. [42].

Substituting into Eq. (B4) the expression for the curvature (53), the potentials given in Eqs. (54), (55) and (56), and the potentials ψ_i ($i = 0,\ldots,10$) listed above, we obtain the final expression for the metric tensor, as shown in Eq. (58).

APPENDIX C: EXPRESSIONS FOR THE POTENTIALS

We list the expressions of the potentials appearing in the g_{00} coefficient of the PPNY metric, evaluated for $r > R_S$ under the assumptions given in Sec. V and subsection V B.

For the Yukawa potential, we have

$$\mathcal{Y} = GM_N\frac{e^{-r/\lambda}}{r}\left[1 + \frac{1}{10}\left(\frac{R_S}{\lambda}\right)^2 + \frac{1}{280}\left(\frac{R_S}{\lambda}\right)^4\right], \tag{C1}$$

where, in order to express g_{00} by means of the effective mass M_S, using Eq. (60) we replace the Newtonian mass M_N with the following expression:

$$M_N = M_S - \frac{GM_S^2}{R_Sc^2}\sum_{n=-2}^{2}B_n\left(\frac{R_S}{\lambda}\right)^n + O\left(\frac{1}{c^4}\right). \tag{C2}$$

For the potential Σ_2, we find

$$\Sigma_2 = \frac{6}{5}\frac{(GM_S)^2}{R_S}\frac{e^{-r/\lambda}}{r}\left[1 + \frac{2}{21}\left(\frac{R_S}{\lambda}\right)^2\right]. \tag{C3}$$

In the case of uniform density ρ, we have $\Pi = 0$; hence, $\Sigma_3 = 0$. At the required order the pressure is given by Newtonian equilibrium: $p(r) = p(0)(1 - r^2/R_S^2)$ where the pressure $p(0)$ at the center of the body is $p(0) = G(\pi/6)^{1/3} M_S^{2/3} \rho^{4/3}$. That yields for the potential Σ_4

$$\Sigma_4 = \frac{e^{-r/\lambda}}{r}\frac{1}{5}\frac{(GM_S)^2}{R_S}\left[1 + \frac{1}{14}\left(\frac{R_S}{\lambda}\right)^2\right],$$
$$8\pi\theta\left(-2q_1 + \frac{a_3 q_1}{a_2^2} - \frac{4}{3}\frac{q_2}{a_2}\right)\Sigma_5 = \frac{e^{-r/\lambda}}{r}\frac{(GM_S)^2}{R_S}\theta\left[\theta(\mu - 2) - \frac{4}{3}\nu\right]\left[\left(\frac{\lambda}{R_S}\right)^2 + \frac{1}{10} + \frac{1}{280}\left(\frac{R_S}{\lambda}\right)^2\right]. \tag{C4}$$

The following potentials contain both a Yukawa term and other functions of r:

$$\frac{1}{18\pi}\frac{(1-\theta)}{c^4}\frac{1}{a_2}X(U\mathcal{Y}) = \frac{2}{3c^2}GM_S\frac{e^{-r/\lambda}}{r}(1-\theta)\frac{GM_S}{c^2R_S}\left[\left(\frac{R_S}{\lambda}\right)\mathrm{Ei}\left(-2\frac{R_S}{\lambda}\right) + \frac{34}{35}\left(\frac{R_S}{\lambda}\right)^2\right]$$
$$+ \frac{2}{3c^2}\frac{GM_S}{r}\left[e^{-r/\lambda}\ln\left(\frac{r}{R_S}\right) - e^{r/\lambda}\mathrm{Ei}\left(-2\frac{r}{\lambda}\right)\right](1-\theta)\frac{GM_S}{c^2R_S}\left(\frac{R_S}{\lambda}\right), \tag{C5}$$

$$\frac{1}{36\pi}\frac{(1-\theta)^2}{c^4}\left(1 + \frac{a_3}{2a_2^2}\right)\frac{1}{a_2}X(\mathcal{Y}^2) = \frac{2}{3c^2}GM_S\frac{e^{-r/\lambda}}{r}(1-\theta)^2\left(1+\frac{\mu}{2}\right)\frac{GM_S}{c^2R_S}$$
$$\times\left\{\frac{1}{2}\left(\frac{R_S}{\lambda}\right)\left[\mathrm{Ei}\left(-3\frac{R_S}{\lambda}\right) - \mathrm{Ei}\left(-\frac{R_S}{\lambda}\right)\right] + \frac{17}{35}\left(\frac{R_S}{\lambda}\right)^2\right\}$$
$$+ \frac{1}{3c^2}\frac{GM_S}{r}\left[e^{-r/\lambda}\mathrm{Ei}\left(-\frac{r}{\lambda}\right) - e^{r/\lambda}\mathrm{Ei}\left(-3\frac{r}{\lambda}\right)\right](1-\theta)^2\left(1+\frac{\mu}{2}\right)\frac{GM_S}{c^2R_S}\left(\frac{R_S}{\lambda}\right), \tag{C6}$$

where $\mathrm{Ei}(x)$ denotes the exponential integral function, Eq. (66).

The potential $\mathcal{V}(\mathcal{Y}^2)$ is decomposed into the sum of two potentials $\mathcal{V}_1(\mathcal{Y}^2) + \mathcal{V}_2(\mathcal{Y}^2)$ where $\mathcal{V}_1$ is proportional to $1/r$ for $r > R_S$ (it is absorbed into the effective mass term), while $\mathcal{V}_2$ contains the following functions of r:

$$\frac{1}{27\pi}\frac{(1-\theta)^2}{c^4}\frac{1}{a_2}\mathcal{V}_2(\mathcal{Y}^2)$$
$$= \frac{4}{9c^2}GM_S(1-\theta)^2\frac{GM_S}{c^2R_S}\left(\frac{R_S}{\lambda}\right)\left[\frac{e^{-2r/\lambda}}{r} + \frac{2}{\lambda}\mathrm{Ei}\left(-2\frac{r}{\lambda}\right)\right]. \tag{C7}$$

For a static, spherically symmetric mass density $\rho = \rho(r)$ we find that, for $r > R_S$,

$$\psi_1(r) = \psi_2(r) = \psi_3(r) = 0. \tag{C8}$$

Using the results in Appendix B, the linear combination of potentials ψ_4 and ψ_5 in g_{00} is proportional to a Yukawa integral of the type $X(Q)$, with Q supported inside the spherical body, so that such a linear combination is proportional to a Yukawa term:

$$-\frac{4}{3}\frac{(1-\theta)^2}{c^4}a_2\left(\sqrt{\frac{2}{3a_2}}\psi_4 + 2\psi_5\right)$$
$$= -\frac{4}{45c^2}GM_S\frac{e^{-r/\lambda}}{r}(1-\theta)^2\frac{GM_S}{c^2R_S}\left[1 + \frac{1}{14}\left(\frac{R_S}{\lambda}\right)^2\right]. \tag{C9}$$

Analogously, the linear combination of potentials $\psi_6, \ldots, \psi_9$ is also proportional to a Yukawa term:

$$\frac{4}{3}\frac{(1-\theta)^2}{c^4}a_2\left[2\psi_6 + \sqrt{\frac{2}{3a_2}}(\psi_7 + \psi_8) + \frac{1}{3a_2}\psi_9\right]$$
$$= \frac{2}{15c^2}GM_S\frac{e^{-r/\lambda}}{r}(1-\theta)^2\frac{GM_S}{c^2R_S}$$
$$\times\left[1 - \frac{5}{9}\left(\frac{R_S}{\lambda}\right) + \frac{43}{210}\left(\frac{R_S}{\lambda}\right)^2\right]. \tag{C10}$$

Eventually for the potential ψ_{10} we find:

$$\frac{2}{3a_2}\frac{(1-\theta)}{c^4}\left(-2a_2 + q_1 + \frac{a_3q_1}{a_2^2} - \frac{2}{3}\frac{q_2}{a_2}\right)\psi_{10}$$
$$= \frac{4}{5c^2}GM_S\frac{e^{-r/\lambda}}{r}(1-\theta)\frac{GM_S}{c^2R_S}\left(-2 + \theta + \theta\mu - \frac{2}{3}\nu\right)$$
$$\times\left[1 - \frac{5}{6}\left(\frac{R_S}{\lambda}\right) + \frac{11}{21}\left(\frac{R_S}{\lambda}\right)^2\right]. \tag{C11}$$

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