

Inflation and cyclotron motion

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We consider, in the context of a braneworld cosmology, the motion of the Universe coupled to a four-form gauge field, with constant field strength, defined in higher dimensions. It is found, under rather general initial conditions, that in this situation there is a period of exponential inflation combined with cyclotron motion in the inflaton field space. The main effect of the cyclotron motion is that slow roll conditions on the inflaton potential, which are typically necessary for exponential inflation, can be evaded. There are Landau levels associated with the four-form gauge field, and these correspond to quantum excitations of the inflaton field satisfying unconventional dispersion relations.

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I. INTRODUCTION

Braneworld cosmology is a concept that exists in many variations. There are versions in which the higher dimensions are compactified, as in the Arkani-Hamed, Dimopoulos, Dvali proposal [1], or large but warped, as in the Randall-Sundrum model [2] and string-motivated Dirac-Born-Infeld inflation [3,4]. There is also the intriguing Dvali-Gabadadze-Porrati (DGP) version where the extra dimension is large but nearly flat [5]. Consideration of the four-dimensional effective theory in the DGP model has led to a very general class of four-dimensional Galileon models [6] with powers of derivative terms greater than two, for which there now exists extensive literature (see, e.g., Refs. [7–10] and references therein).

In this article I would like to describe some interesting features of the following action, describing a brane with standard model particle content evolving in a flat higher-dimensional background, with a coupling of the brane to an external four-form gauge field in the bulk:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + S_{\text{SM}} - \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi^s \partial_\nu \varphi^s + V(\varphi) \right) + \frac{q_0}{4!} \int d^4x A_{abcd} [\phi(x)] \epsilon^{\alpha\beta\gamma\delta} \partial_\alpha \phi^a \partial_\beta \phi^b \partial_\gamma \phi^c \partial_\delta \phi^d, \quad (1)$$

where S_{SM} is the action of standard model (and possibly beyond-standard-model) fields, and

$$g_{\mu\nu} = \partial_\mu \phi^A \eta_{AB} \partial_\nu \phi^B, \quad A, B = 0, 1, \dots, D \quad (2)$$

is the induced metric of a three-brane in a $D + 1$ -dimensional Minkowski space. We adopt the convention that uppercase Latin indices run from 0 to D , indices r, s run from $D + 1$ to $D + N$, and all other lowercase Latin indices run from 0 to $D + N$. We also define

$$\phi^s = \frac{1}{\sigma^2} \varphi^s, \quad (3)$$

where σ is a constant with dimensions of mass. The φ^s fields are a set of N inflaton fields, with $V(\varphi)$ the inflaton potential, and A_{abcd} is a potential which is totally antisymmetric in the indices. It can be thought of as a four-form gauge field in $D + 1 + N$ dimensions. The induced metric corresponds to $D + 1$ dimensions, however.

The main novelty of this formulation is the interaction of the braneworld with an external four-form gauge field in the bulk, and it is the purpose of this article to describe some possible consequences in an inflationary scenario. Like the DGP model there are large flat extra dimensions, but unlike that model there is no Einstein-Hilbert action in the bulk. Unlike Galileon models, in general, there is no Galilean invariance, and the external four-form gauge field singles out special directions in the bulk. Inflation, in the scenario suggested below, is driven by inflaton fields with an ordinary $V(\varphi)$ potential in the inflaton action, rather than by Galileon fields.

Without the external gauge field, a model with an Einstein-Hilbert action and other fields on the brane was first considered long ago by Regge and Teitelboim [11]. The first question to ask of a model of this type is whether the equations of motion are equivalent, at the classical level, to standard general relativity at $A_{abcd} = 0$. The answer is, not quite. Denote

$$E^{\mu\nu} \equiv \frac{\delta S[A = 0]}{\delta g_{\mu\nu}} = \frac{1}{2} \sqrt{-g} \left\{ -\frac{1}{8\pi G} G^{\mu\nu} + T^{\mu\nu} \right\}, \quad (4)$$

where $T^{\mu\nu}$ is the stress-energy tensor of the standard model and inflaton fields. Then the field equations resulting from variation of the ϕ^A at $A_{abcd} = 0$ are

$$\eta_{AB}\partial_\mu(E^{\mu\nu}\partial_\nu\phi^B) = 0. \quad (5)$$

These equations are obviously satisfied by the Einstein field equations $E^{\mu\nu} = 0$. Moreover, any solution of $E^{\mu\nu} = 0$ can be embedded locally in a ten-dimensional flat Minkowski space, although globally an embedding may require still higher dimensions [12]. But of course there may also be solutions of (5) which are not solutions of the Einstein equations. A simple (and intriguing) example is pure gravity with a cosmological constant, in which case

$$E^{\mu\nu} = \frac{1}{2}\sqrt{-g}\left\{-\frac{1}{8\pi G}G^{\mu\nu} - \lambda g^{\mu\nu}\right\}. \quad (6)$$

In this case the equations of motion are certainly solved by de Sitter space, for which $E^{\mu\nu} = 0$. But flat Minkowski space is also a solution: just choose $\phi^\mu = x^\mu$, $\mu = 0-3$, and $\phi^{A>3} = \text{constant}$. Then $g_{\mu\nu} = \eta_{\mu\nu}$, $G_{\mu\nu} = 0$, and the equations of motion boil down to $\square\phi^A = 0$, which is satisfied trivially.

A criticism of Deser *et al.* [13] is that the embedding of a four-manifold is not unique. Some embeddings of a four-manifold may satisfy the equations of motion (5), and some may not. This fact does not necessarily rule out the embedding formulation of general relativity on experimental grounds; it could simply be that the $E^{\mu\nu} = 0$ alternative is selected by initial conditions on the ϕ^a .

When the four-form gauge field is included, there will, in general, be some deviation from the standard Einstein field equations. The equations of motion in this case are

$$2\eta_{AB}\partial_\mu(E^{\mu\nu}\partial_\nu\phi^B) - \frac{q_0}{4!}F_{ABCD}\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha\phi^A\partial_\beta\phi^B\partial_\gamma\phi^C\partial_\delta\phi^D = 0, \quad (7)$$

and

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi^s) - \sqrt{-g}\frac{\partial V}{\partial\phi^s} + \frac{q_0}{4!\sigma^2}F_{abcd}\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha\phi^a\partial_\beta\phi^b\partial_\gamma\phi^c\partial_\delta\phi^d = 0, \quad (8)$$

where F_{abcd} is the field strength

$$F_{abcd} = \frac{\partial A_{abcd}}{\partial\phi^f} - \frac{\partial A_{fbcd}}{\partial\phi^a} + \frac{\partial A_{facd}}{\partial\phi^b} - \frac{\partial A_{fabd}}{\partial\phi^c} + \frac{\partial A_{fabc}}{\partial\phi^d} \quad (9)$$

corresponding to the four-form gauge field. These are supplemented by the usual equations of motions of the standard model fields.

In this article I explore the cosmological consequences of these equations of motion in the simplest nontrivial case, namely, a constant field strength F_{abcd} in a homogeneous

isotropic spacetime. For this purpose it will be sufficient to work in a five-dimensional embedding space, $A = 0, \dots, 4$, with two inflaton fields $\phi^{5,6}$, and ignoring, at the classical level, all standard model fields.

II. INFLATION

It is well known that a four-dimensional manifold described by a Friedmann-Lemaître metric can be embedded in five-dimensional space, and for simplicity we adopt the version with zero spatial curvature. We take the embedding to be [14,15]

$$\begin{aligned} \phi^0 &= \frac{1}{2}\left\{a(t) + \int^t \frac{dt'}{da/dt'} + a(t)r^2\right\} \\ \phi^1 &= a(t)r\cos(\theta) \\ \phi^2 &= a(t)r\sin(\theta)\cos(\chi) \\ \phi^3 &= a(t)r\sin(\theta)\sin(\chi) \\ \phi^4 &= \frac{1}{2}\left\{a(t) - \int^t \frac{dt'}{da/dt'} - a(t)r^2\right\}, \end{aligned} \quad (10)$$

and it is not hard to see that

$$\begin{aligned} ds^2 &= \eta_{AB}d\phi^A d\phi^B, \quad A, B = 0, 1, 2, 3, 4 \\ &= -dt^2 + a^2(t)(dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\chi^2)) \end{aligned} \quad (11)$$

is the Friedmann-Lemaître metric. But let us also suppose that there is a four-form gauge field dependent on the coordinates ϕ^a , whose nonzero components are

$$\begin{aligned} A_{5123}[\phi] &= -\frac{1}{2}B\phi^6, \\ A_{6123}[\phi] &= \frac{1}{2}B\phi^5. \end{aligned} \quad (12)$$

The four-form gauge field A_{abcd} is antisymmetric under permutations of indices, but apart from (12) and components obtained from (12) by permutation, it is assumed that all other components vanish. This choice leads to a constant nonzero field strength $F_{56123} = B$, and we are interested in exploring the consequences for early-Universe dynamics in a situation of this kind. In this context we also assume the simplest possible inflaton potential

$$V[\phi] = \frac{1}{2}m^2\phi^s\phi^s. \quad (13)$$

We begin with the usual simplifying assumptions of spatial homogeneity and isotropy, taking, in particular,

$$\phi^{5,6}(x, y, z, t) = \phi^{5,6}(t), \quad (14)$$

and $\phi^a = 0$ for $a > 6$. In conjunction with (12), this has the consequence that

$$F_{Abcd}\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha\phi^a\partial_\beta\phi^b\partial_\gamma\phi^c\partial_\delta\phi^d = 0. \quad (15)$$

This is because two of the indices $abcd$ must be 5 and 6, so the expression necessarily includes at least one space derivative of φ^s , which vanishes according to (14). Then the equation of motion (7) is satisfied by $E^{\mu\nu} = 0$, which are the standard Einstein field equations. For a Friedmann-Lemaitre metric, disregarding the other standard model fields, the Einstein equations are just the conventional expressions for the $a(t)$ scale factor coupled to a pair of scalar fields:

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= \frac{8\pi G}{3} \left(\frac{1}{2} \partial_t \varphi^s \partial_t \varphi^s + \frac{1}{2} m^2 \varphi^s \varphi^s \right), \\ \frac{\ddot{a}}{a} &= \frac{8\pi G}{3} \left(-\partial_t \varphi^s \partial_t \varphi^s + \frac{1}{2} m^2 \varphi^s \varphi^s \right). \end{aligned} \quad (16)$$

The equations of motion for the φ^s , however, involve the field strength

$$\begin{aligned} \partial_t^2 \varphi^5 - qB \partial_t \varphi^6 + 3 \frac{\dot{a}}{a} \partial_t \varphi^5 + m^2 \varphi^5 &= 0, \\ \partial_t^2 \varphi^6 + qB \partial_t \varphi^5 + 3 \frac{\dot{a}}{a} \partial_t \varphi^6 + m^2 \varphi^6 &= 0, \end{aligned} \quad (17)$$

where $q = q_0/\sigma^4$. It is not hard to verify consistency of (16) and (17).

If we set $\dot{a}/a = 0$ and $m^2 = 0$ in Eq. (17), then these equations are obviously the equations of motion of a charged particle moving, in the $\varphi^5 - \varphi^6$ plane, under the influence of a magnetic field orthogonal to that plane; i.e., this is cyclotron motion. If we instead set $qB = 0$, then these are the equations used in simple models of inflation. In models of that type it is normally important to impose slow roll conditions, which imply either a large initial value for the inflaton field, or else, unlike (13), a very flat potential (see, e.g., Chapter 8 in [16]). For the simple potential (13) these slow roll conditions boil down to

$$\varphi^s \varphi^s \gg \frac{1}{6\pi G}, \quad (18)$$

i.e., a large initial field.

The model we are discussing has a fairly large space of parameters and initial conditions $\{qB, m^2, \varphi^s(0), \partial_t \varphi^s(0)\}$, but the time development is typically a spiral in the $\varphi^5 - \varphi^6$ plane. What may be of interest is the fact that for $qB \neq 0$ it is possible to have a period of approximately exponential inflation, with a large number of e -foldings, even when the slow-roll condition (18) is strongly violated.¹ A single example should suffice. Working in Planck units, we choose parameters and initial conditions

¹It should be noted, however, that there are other mechanisms for easing the slow roll conditions in the context of a braneworld cosmology, cf. [17].

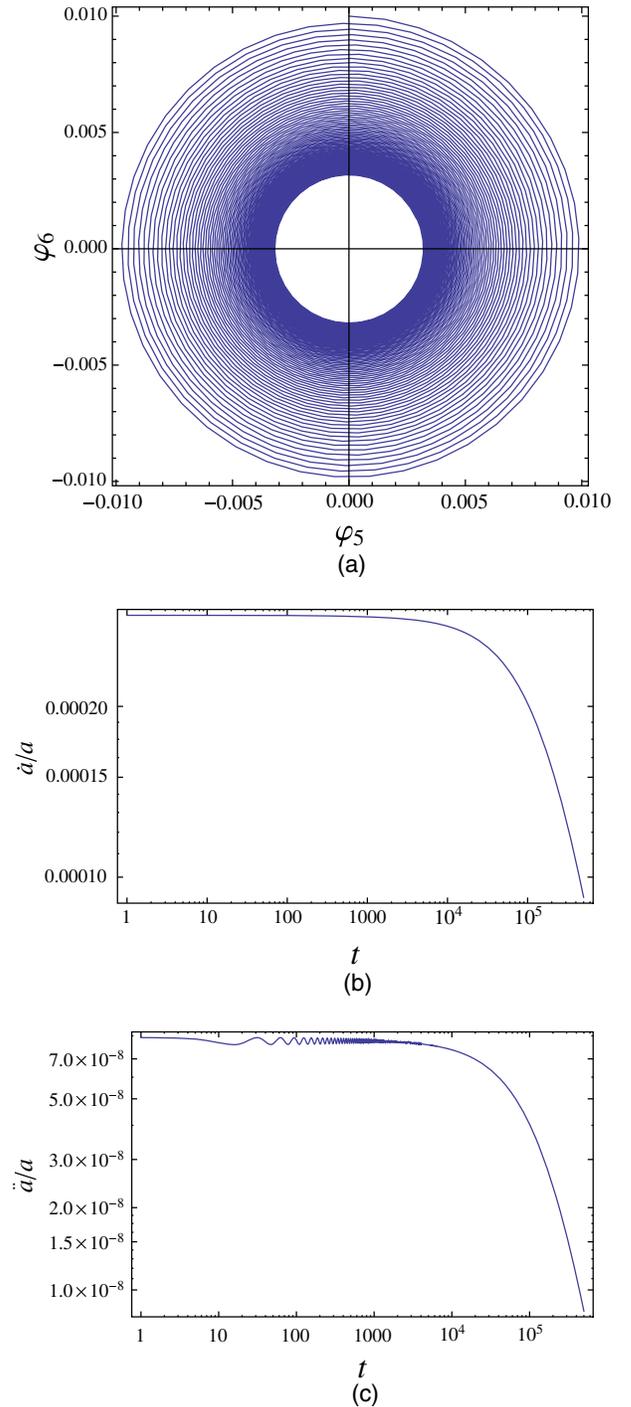


FIG. 1. Numerical solution of the evolution equations (16) and (17), with parameters and initial conditions (19). (a) Trajectory in the $\varphi^5 - \varphi^6$ plane; (b) log-log plot of \dot{a}/a vs time t ; (c) log-log plot \ddot{a}/a vs time t . Note that the log-log plots of \dot{a}/a and \ddot{a}/a vs time t are almost flat in the period $1 < t < 10^4$, indicating a period of exponential expansion, in this case with about 100 e -foldings.

$$\begin{aligned} qB &= 0.2, & m^2 &= 2 \times 10^{-4}, \\ \varphi^5(0) &= 0, & \varphi^6(0) &= 10^{-2}, \\ (\partial_t \varphi^5)_{t=0} &= 0, & (\partial_t \varphi^6)_{t=0} &= 0. \end{aligned} \quad (19)$$

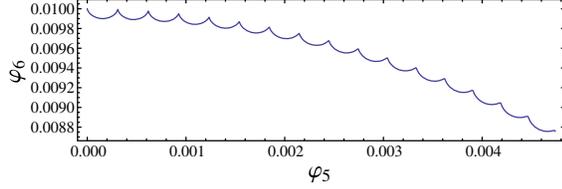


FIG. 2. The trajectory of Fig. 1(a) at the beginning of the time evolution, in the period $0 < t < 500$, showing the effect of the ‘‘Lorentz force,’’ directed away from the center of the ϕ^5 - ϕ^6 plane.

The resulting spiral evolution in the ϕ^5 - ϕ^6 plane is shown in Fig. 1(a), with \dot{a}/a and \ddot{a}/a vs cosmic time t shown in Figs. 1(b) and 1(c), respectively. The expansion is very nearly a simple exponential up to $t \approx 10^4$ in Planckian units, which is evident in the rather flat curves on the log-log plots, and the fact that

$$\frac{\ddot{a}}{a} \approx \left(\frac{\dot{a}}{a}\right)^2 \quad (20)$$

in this period. Expansion continues after this period, however, resulting in a total of about 100 e -foldings by $t = 10^6$.

The potential $V(\phi)$ is responsible for a force towards the origin of the ϕ^5 - ϕ^6 plane, while the ‘‘Lorentz force’’ due to the four-form gauge field is directed away from the origin. Eventually these forces balance to produce a circular motion, spiraling towards the center. To see this, we plot the initial stage of the evolution in Fig. 2. In the absence of the gauge field, the system simply falls to the center, oscillating around

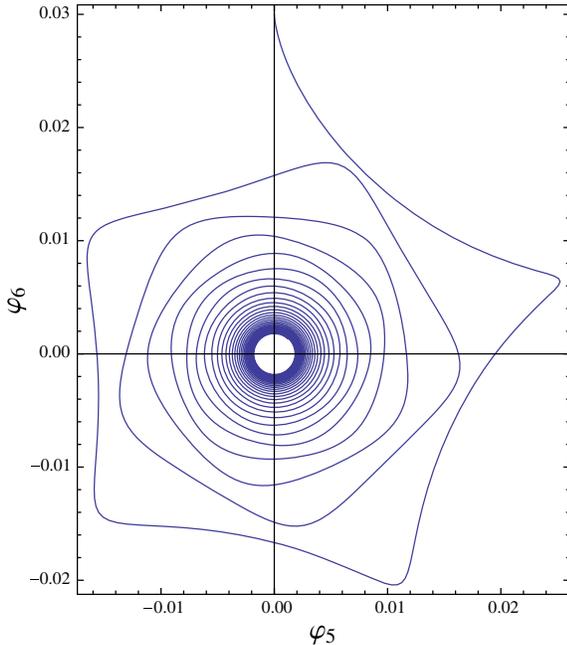


FIG. 3. Trajectory in the ϕ^5 - ϕ^6 plane for parameters $qB = -1$, $m^2 = 0.5$, $\phi^6(0) = 0.03$.

the ϕ^6 axis, and, because slow roll conditions are not satisfied, there is no inflationary period. The Lorentz force, however, deflects the initial fall to the center into an arc, and this interplay between the central potential, the Lorentz force, and gravitational friction continues until the inward and outward forces sum to a centripetal force for (roughly) circular motion, with gravitational friction causing a gradual spiral to the origin. The trajectory resulting from a quite different set of parameters is shown in Fig. 3. While this last example does not lead to many e -foldings, it does very clearly display the initial interplay of forces, leading to an eventual spiral towards the origin.

III. LANDAU LEVELS

After inflation, the constant field strength of the four-form gauge field still has an effect at the quantum level, in the form of Landau excitation levels of the quantized φ fields. We will see that these excitations satisfy a rather unusual dispersion relation.

We consider the post-inflationary period at some time t_0 where \dot{a}/a is negligible, $a(t) \approx R$. With ϕ^A given by the embedding (10), and A_{abcd} as in Eq. (12), we have

$$\begin{aligned} \frac{q_0}{4!} A_{abcd} [\phi(x)] \epsilon^{\alpha\beta\gamma\delta} \partial_\alpha \phi^a \partial_\beta \phi^b \partial_\gamma \phi^c \partial_\delta \phi^d \\ = q A_{s123} \epsilon^{0ijk} \partial_i \varphi^s \partial_j \phi^1 \partial_k \phi^2 \partial_l \phi^3 \\ = q A_s \partial_t \varphi^s (R^3 r^2 \sin \theta), \end{aligned} \quad (21)$$

where $A_s \equiv \sigma^2 A_{s123}$. The factor of R can be absorbed into a coordinate redefinition, and we then consider quantizing the action

$$\begin{aligned} S_\varphi = \int d^4x \left(\frac{1}{2} \partial_t \varphi^s \partial_t \varphi^s - \frac{1}{2} \nabla \varphi^s \cdot \nabla \varphi^s \right. \\ \left. - \frac{1}{2} m^2 \varphi^s \varphi^s + q A_s(\varphi) \partial_t \varphi^s \right), \end{aligned} \quad (22)$$

where again the index $s = 5, 6$. The corresponding Hamiltonian is

$$H = \frac{1}{2} \int d^3x \{ (p_s - q A_s)^2 + (\nabla \varphi^s)^2 + m^2 \varphi^s \varphi^s \}, \quad (23)$$

and φ^s, p_s have standard quantization conditions. Define

$$\begin{aligned} \omega_k &= \sqrt{k^2 + \frac{1}{4} q^2 B^2 + m^2} \\ \varphi^s(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} (a_s(k) e^{ik \cdot x} + a_s^\dagger(k) e^{-ik \cdot x}) \\ p_s(x) &= \int \frac{d^3k}{(2\pi)^3} \sqrt{2\omega_k} \frac{1}{2i} (a_s(k) e^{ik \cdot x} - a_s^\dagger(k) e^{-ik \cdot x}), \end{aligned} \quad (24)$$

with the usual commutation relations

$$[a_s(k_1), a_r^\dagger(k_2)] = (2\pi)^3 \delta^3(\mathbf{k}_1 - \mathbf{k}_2) \delta_{rs}. \quad (25)$$

Then

$$H = \int \frac{d^3k}{(2\pi)^3} \left\{ \omega_k (a_s^\dagger(k) a_s(k) + \delta^3(0)) + i \frac{1}{2} qB (a_5^\dagger(k) a_6(k) - a_6^\dagger(k) a_5(k)) \right\}. \quad (26)$$

Introduce

$$\begin{aligned} b_1(k) &= \frac{1}{\sqrt{2}} (a_5(k) + i a_6(k)) \\ b_2(k) &= \frac{1}{\sqrt{2}} (a_5(k) - i a_6(k)), \end{aligned} \quad (27)$$

which again have the usual commutation relations

$$[b_i(k_1), b_j^\dagger(k_2)] = (2\pi)^3 \delta^3(\mathbf{k}_1 - \mathbf{k}_2) \delta_{ij} \quad (28)$$

with indices $i, j = 1, 2$. The Hamiltonian takes the form

$$H = \int d^3k \left\{ \omega_k (b_i^\dagger(k) b_i(k) + \delta^3(0)) + \frac{1}{2} qB (b_1^\dagger(k) b_1(k) - b_2^\dagger(k) b_2(k)) \right\}, \quad (29)$$

and the corresponding spectrum is

$$E = \sum_k \left\{ \sqrt{k^2 + \frac{1}{4} q^2 B^2 + m^2} (n_1(k) + n_2(k)) + \frac{1}{2} qB (n_1(k) - n_2(k)) \right\} + E_0, \quad (30)$$

where $n_1(k), n_2(k)$ are occupation numbers, E_0 is the ground state energy, and the sum runs over momenta with nonzero occupation numbers. We also find, by standard manipulations, the conserved total momentum

$$P_i = \sum_k k_i (n_1(k) + n_2(k)). \quad (31)$$

Were it not for the term proportional to qB in Eq. (30), the spectrum would simply consist of two types of particles of mass

$$M' = \sqrt{\frac{1}{4} q^2 B^2 + m^2}. \quad (32)$$

Instead, defining $M = \frac{1}{2} qB$, it is seen that excitations which are eigenstates of both H and P_i (with momentum eigenvalues k_i) satisfy dispersion relations

$$E_1(k) = \sqrt{k^2 + M^2 + m^2} + M \quad \text{and}$$

$$E_2(k) = \sqrt{k^2 + M^2 + m^2} - M, \quad (33)$$

respectively, which is clearly at odds with the relativistic expression for a free particle. But of course these excitations are not free particles, and the Lagrangian (22) they derive from is not Lorentz invariant, or even (unlike Newtonian mechanics) boost invariant. It is the external four-form gauge field which singles out a preferred time direction (much as, e.g., an ordinary background magnetic field along the z -axis would introduce a preferred spatial direction for objects sensitive to that field), and the only remaining space-time symmetries are rotation and space-time translation invariance. Therefore the breaking of both Lorentz and boost invariance, so far as these inflaton excitations are concerned, is not a surprise. The question is how this breaking might manifest itself.

IV. PROPERTIES OF LANDAU LEVEL EXCITATIONS

A. Group velocity

To begin with, consider how a wave packet corresponding to a single ‘‘heavy’’ Landau excitation of energy $E_1(k)$, or a ‘‘light’’ Landau excitation of energy $E_2(k)$, and momentum \mathbf{k} , will propagate in time. Let $|\mathbf{k}, j\rangle$ correspond to a particle eigenstate of energy and momentum $E_j(k), \mathbf{k}$, respectively, with conventional normalization

$$\begin{aligned} |\mathbf{k}, j\rangle &= \sqrt{2\omega_k} b_j(\mathbf{k}) |0\rangle \\ |\mathbf{x}, j\rangle &= \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} |\mathbf{k}, j\rangle, \end{aligned} \quad (34)$$

and consider initial wave packets of the form

$$\begin{aligned} |\psi_j\rangle_{t=0} &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} f(k) |\mathbf{k}, j\rangle \\ \psi_j(\mathbf{x}, t=0) &= \langle \mathbf{x}, j | \psi_j \rangle_{t=0} \\ &= \int \frac{d^3k}{(2\pi)^3} f(k) e^{i\mathbf{k}\cdot\mathbf{x}}. \end{aligned} \quad (35)$$

Then at a later time

$$\begin{aligned} \psi_j(\mathbf{x}, t) &= \langle \mathbf{x}, j | e^{-iHt} | \psi_j \rangle_{t=0} \\ &= e^{-i(3-2j)Mt} \int \frac{d^3k}{(2\pi)^3} f(k) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_k t)}. \end{aligned} \quad (36)$$

From this we conclude that wave packets of both heavy and light Landau excitations (we might as well call them ‘‘landons’’) propagate with a group velocity $\nu = d\omega_p/dp$ appropriate to a particle of mass $M' \approx M$ (for $m \ll M$). On the other hand, at low momenta in the frame singled out by the external four-form gauge field,

$$\begin{aligned}
E_1(k) &\approx \frac{k^2}{2M} + 2M + \frac{m^2}{2M} \\
E_2(k) &\approx \frac{k^2}{2M} + \frac{m^2}{2M},
\end{aligned} \tag{37}$$

which means that the rest energy of the heavy landons is approximately $2M$, while that of the light landons is approximately $m^2/2M$.

B. Scattering in a gravitational field

Because of the mismatch between the inertial mass in the momentum-dependent $k^2/2M$ term and the rest energy, we may expect an apparent violation of the principle of equivalence, if it was possible to somehow observe the motion of these excitations in a gravitational field. This can be verified by calculating the differential scattering cross section of heavy and light landons in the weak gravitational field of a static massive object of mass \mathcal{M} .

Let $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with

$$\begin{aligned}
g_{00} &= -\left(1 - \frac{2G\mathcal{M}}{r}\right), & g_{ii} &= \left(1 + \frac{2G\mathcal{M}}{r}\right) \\
g_{\mu\nu} &= 0 \quad (\mu \neq \nu),
\end{aligned} \tag{38}$$

be the metric corresponding to the massive object at the origin, at distances r such that the gravitational field is weak. For our purposes it is sufficient to ignore this restriction on r , unless we are interested in large angle scattering. We first need the interaction Hamiltonian to lowest order in $G\mathcal{M}$. For this we consider the part of the total action $S' = S_\varphi + S_A$ containing φ , where

$$\begin{aligned}
S_\varphi &= -\int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi^s \partial_\nu \varphi^s + \frac{1}{2} m^2 \varphi^s \varphi^s \right) \\
S_A &= \int d^4x q A_s \epsilon^{0ijk} \partial_i \varphi^s \partial_j \phi^1 \partial_k \phi^2 \partial_l \phi^3.
\end{aligned} \tag{39}$$

Expanding S_φ to first order in $G\mathcal{M}$ we have

$$\begin{aligned}
S_\varphi &= \int d^4x \left\{ \frac{1}{2} \left(1 + \frac{4G\mathcal{M}}{r}\right) \partial_t \varphi^s \partial_t \varphi^s - \frac{1}{2} (\nabla \varphi^s) \cdot (\nabla \varphi^s) \right. \\
&\quad \left. - \frac{1}{2} \left(1 + \frac{2G\mathcal{M}}{r}\right) m^2 \varphi^s \varphi^s \right\}.
\end{aligned} \tag{40}$$

To compute S_A to leading order in $h_{\mu\nu}$ we use

$$S_A \approx S_A(h=0) + \int d^4x \frac{\delta S_A}{\delta g_{\mu\nu}} h_{\mu\nu}. \tag{41}$$

Now S_A depends on the metric through the $\partial_\mu \phi^A$. As noted already, there is no unique mapping from the metric to the three-brane coordinates, but this turns out not to be a problem. Choose any mapping $g_{\mu\nu} \rightarrow \partial_\mu \phi^A$ and observe that, acting on any functional of the metric,

$$\begin{aligned}
\frac{\delta}{\delta(\partial_\mu \phi^A)} &= \frac{\partial g_{\alpha\beta}}{\partial(\partial_\mu \phi^A)} \frac{\delta}{\delta g_{\alpha\beta}} \\
&= 2\eta_{AB} \partial_\alpha \phi^B \frac{\delta}{\delta g_{\alpha\mu}},
\end{aligned} \tag{42}$$

which can be inverted to give

$$\frac{\delta}{\delta g_{\mu\nu}} = \frac{1}{2} g^{\mu\alpha} \partial_\alpha \phi^A \frac{\delta}{\delta(\partial_\nu \phi^A)}. \tag{43}$$

Applying this operator to S_A in Eq. (39), we find

$$\begin{aligned}
\delta S_A &= \int d^4x \left(\frac{\delta S_A}{\delta g_{\mu\nu}} \right)_{g_{\alpha\beta}=\eta_{\alpha\beta}} h_{\mu\nu} \\
&= \frac{3}{2} \int d^4x \frac{2G\mathcal{M}}{r} q A_s \partial_t \varphi^s.
\end{aligned} \tag{44}$$

Altogether

$$\begin{aligned}
S' &= \int d^4x \left\{ \frac{1}{2} \left(1 + \frac{4G\mathcal{M}}{r}\right) \partial_t \varphi^s \partial_t \varphi^s - \frac{1}{2} (\nabla \varphi^s) \cdot (\nabla \varphi^s) \right. \\
&\quad \left. - \frac{1}{2} \left(1 + \frac{2G\mathcal{M}}{r}\right) m^2 \varphi^s \varphi^s \right. \\
&\quad \left. + \left(1 + \frac{3G\mathcal{M}}{r}\right) q A_s \partial_t \varphi^s \right\}.
\end{aligned} \tag{45}$$

We go to the Hamiltonian formulation, introducing canonical momenta conjugate to the φ^s ,

$$p_s = \left(1 + \frac{4G\mathcal{M}}{r}\right) \partial_t \varphi^s + \left(1 + \frac{3G\mathcal{M}}{r}\right) q A_s \partial_t \varphi^s, \tag{46}$$

leading to a Hamiltonian operator

$$\begin{aligned}
H &= \int d^3x \left\{ \frac{1}{2} \left(1 + \frac{4G\mathcal{M}}{r}\right)^{-1} \left(p_s - \left(1 + \frac{3G\mathcal{M}}{r}\right) q A_s \right) \right. \\
&\quad \times \left(p_s - \left(1 + \frac{3G\mathcal{M}}{r}\right) q A_s \right) \\
&\quad \left. + \frac{1}{2} (\nabla \varphi^s) \cdot (\nabla \varphi^s) + \frac{1}{2} \left(1 + \frac{2G\mathcal{M}}{r}\right) m^2 \varphi^s \varphi^s \right\} \\
&= H_0 + \int d^3x \left\{ -\frac{2G\mathcal{M}}{r} (p_s - g A_s) (p_s - g A_s) \right. \\
&\quad \left. + \frac{G\mathcal{M}}{r} m^2 \varphi^s \varphi^s - \frac{3G\mathcal{M}}{r} q A_s (p_s - g A_s) \right\}.
\end{aligned} \tag{47}$$

Then the Hamiltonian density in the interaction picture, to first order in $G\mathcal{M}$, is²

²Note that in the interaction picture the $G\mathcal{M} = 0$ operator identification $p_s = \partial_t \phi_s + q A_s$ must be used for the interaction Hamiltonian density.

$$\mathcal{H}_I = -\frac{2GM}{r} \left\{ \partial_t \varphi^s \partial_t \varphi^s - \frac{1}{2} m^2 \varphi^s \varphi^s + \frac{3}{2} M (\varphi^5 \partial_t \varphi^6 - \varphi^6 \partial_t \varphi^5) \right\}. \quad (48)$$

Using interaction picture operators

$$\begin{aligned} \varphi^5(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \frac{1}{\sqrt{2}} (b_1(k) e^{i(k \cdot x - E_1(k)t)} \\ &\quad + b_1^\dagger(k) e^{-i(k \cdot x - E_1(k)t)} + b_2(k) e^{i(k \cdot x - E_2(k)t)} \\ &\quad + b_2^\dagger(k) e^{-i(k \cdot x - E_2(k)t)}) \\ \varphi^6(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \frac{1}{\sqrt{2}i} (b_1(k) e^{i(k \cdot x - E_1(k)t)} \\ &\quad - b_1^\dagger(k) e^{-i(k \cdot x - E_1(k)t)} - b_2(k) e^{i(k \cdot x - E_2(k)t)} \\ &\quad + b_2^\dagger(k) e^{-i(k \cdot x - E_2(k)t)}), \end{aligned} \quad (49)$$

we can compute matrix elements

$$\langle \mathbf{p}_2, j | \int d^4x \mathcal{H}_I | \mathbf{p}_1, j \rangle, \quad (50)$$

and from there it is a standard exercise to calculate the differential cross sections for the heavy/light ($j = 1, 2$) Landau excitations in the specified gravitational field. The answer is

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{type } j}^{\text{grav}} = (GM)^2 \frac{(E_j^2(p) - \frac{1}{2}m^2 \mp \frac{3}{2}ME_j(p))^2}{p^4 \sin^4(\theta/2)}, \quad (51)$$

where the minus sign is for type 1 and the plus sign for type 2 landons. The type-changing cross sections, in which an initial type 1 landon scatters into a type 2 final state or vice versa, both vanish. We note that for normal scalar fields, i.e., $E_p = \sqrt{p^2 + m^2}$, $A_s = M = 0$, Eq. (51) agrees with the gravitational cross section previously obtained by Golowich *et al.* [18].

Now let us look at the low momentum $p^2 \ll m^2 \ll M^2$ limit. For comparison, the differential cross section for a particle of mass m in a potential

$$V(r) = -\frac{\lambda}{r}, \quad (52)$$

computed via the Born approximation in nonrelativistic quantum mechanics, is the familiar Rutherford result

$$\left(\frac{d\sigma}{d\Omega} \right)^{\text{Ruth}} = \frac{1}{4} \lambda^2 \frac{1}{m^2 v^4 \sin^4(\theta/2)}. \quad (53)$$

For normal scalar particles ($\frac{1}{2}qB = M = 0$), using (51) with the approximations (37) in the low momentum limit, the gravitational cross section can be expressed as

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{normal}}^{\text{grav}} = \frac{1}{4} (GMm)^2 \frac{1}{m^2 v^4 \sin^4(\theta/2)}, \quad (54)$$

which, comparing to the Rutherford potential, corresponds to scattering from the potential

$$V(r) = -\frac{GMm}{r}. \quad (55)$$

In other words, the gravitational mass and the inertial mass are the same. In contrast, for landons of types 1 and 2, Eq. (51), becomes in the limit $p^2 \ll m^2 \ll M^2$,

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega} \right)_{\text{type 1}}^{\text{grav}} &= \frac{1}{4} (GM2M)^2 \frac{1}{M^2 v^4 \sin^4(\theta/2)} \\ \left(\frac{d\sigma}{d\Omega} \right)_{\text{type 2}}^{\text{grav}} &= \frac{1}{4} \left(GM \frac{m^2}{2M} \right)^2 \frac{1}{M^2 v^4 \sin^4(\theta/2)}. \end{aligned} \quad (56)$$

This is a result that we might have guessed. By comparison to the Rutherford cross section, the gravitational masses of both types 1 and 2 landons are equal to their rest energies, which (for $m \ll M$) are $2M$ and $m^2/2M$, respectively, while the inertial mass, in accordance with its appearance in group velocity, is approximately M in both cases.

The principle of equivalence, of course, asserts the identity of gravitational and inertial mass, which would seem to be badly violated for both heavy and light landons. Indeed, in the present scenario, if it were possible to drop a heavy and a light landon from the top of a tall building and observe how they propagate, the heavy landon would accelerate at $2g$, while the light landon would drift downwards (assuming $m \ll M$) only very slowly, with acceleration $(m^2/2M^2)g$. These odd effects should be viewed as only an apparent violation of the equivalence principle, arising due to interaction with an external four-form gauge field that singles out a particular time direction. A rough analogy might be the retardation in the gravitational acceleration of a falling conducting ring in the presence of a constant magnetic field directed parallel to the gravitational field. If we were unaware of the external field, this might also seem like a violation of the principle of equivalence, rather than a manifestation of Lenz's law. In the present situation, the external four-form gauge field makes a contribution to the landon rest energies which cannot be absorbed into the inertial masses, resulting in both an unusual dispersion relation and a seeming violation of the equivalence principle.

C. Energy density in the early Universe

If the inflaton field couples only to gravity and the external four-form gauge field, as assumed from the beginning in Eq. (1), then observations of the sort just mentioned would be difficult to carry out, and it may be more useful to look for signatures of the unconventional

dispersion relations in the early Universe, due to an unconventional equation of state. Since it requires an energy of at least $4M$ to pair-create the heavy excitations, and assuming M is $O(1)$ in Planck units, then after inflation the number density of these objects is fixed. Assuming a dilute ideal gas, the equation of state is conventional:

$$\rho = n \left(2M + \frac{m^2}{2M} \right) + \frac{3}{2} P, \quad (57)$$

where ρ , n are the energy and number density, respectively, and P is pressure. The result follows from Boltzmann statistics, plus the fact that, in a nonrelativistic regime where (37) applies, momentum degrees of freedom enter quadratically. Hence the equipartition theorem applies, and the result is no different than that of a monatomic ideal gas, with particles of rest energy $2M + m^2/2M$. Heavy excitations would contribute to deceleration in the matter-dominated era, but their contribution cannot be easily distinguished from that of other types of matter.

The situation is more interesting with respect to light excitations. It is assumed that the rest energy m^2/M is so small that the number of these excitations is not fixed in the hot environment of the early Universe³ and the chemical potential can be taken to be zero. In that situation, as with photons, it is necessary to carry out the analysis in a grand canonical ensemble. Following the usual analysis, the logarithm of the grand canonical partition function \mathcal{Z} is

$$\begin{aligned} \log \mathcal{Z} &= -V \int \frac{d^3 k}{(2\pi)^3} \ln(1 - e^{-\beta E_2(k)}) \\ &= \beta V P, \end{aligned} \quad (58)$$

with $E_2(k)$ defined in Eq. (37). The energy density is

$$\rho = \int \frac{d^3 k}{(2\pi)^3} \frac{E_2(k)}{e^{\beta E_2(k)} - 1}. \quad (59)$$

We assume that in the early Universe $m \ll k \ll M$, and observe that

$$\begin{aligned} \frac{d}{dk} \ln(1 - e^{-\beta E_2(k)}) &= \frac{\beta}{e^{\beta E_2(k)} - 1} \frac{d}{dk} \left(\frac{k^2 + m^2}{2M} \right) \\ &= \frac{\beta k}{M} \frac{1}{e^{\beta E_2(k)} - 1}. \end{aligned} \quad (60)$$

Applying this identity we have

³At least the number is not fixed if there are any interaction terms in the inflaton potential. If this is not the case and the number is fixed, then the analysis is the same as for an ideal gas with particle rest mass $\frac{m^2}{2M}$. Taking $m^2/M \ll P$, the result is $\rho \approx \frac{3}{2}P$, which, will be seen, is the same as the grand canonical result derived below.

$$\begin{aligned} \rho &= \frac{4\pi}{(2\pi)^3} \int_0^\infty dk k^2 E_2(k) \frac{M}{\beta k} \frac{d}{dk} \ln(1 - e^{-\beta E_2(k)}) \\ &= \frac{4\pi}{(2\pi)^3} \frac{M}{\beta} k E_2(k) \ln(1 - e^{-\beta E_2(k)}) \Big|_0^\infty \\ &\quad - \frac{4\pi}{(2\pi)^3} \frac{M}{\beta} \int_0^\infty dk \left(\frac{d}{dk} k E_2(k) \right) \ln(1 - e^{-\beta E_2(k)}). \end{aligned} \quad (61)$$

The boundary terms go to zero linearly with k as $k \rightarrow 0$, and exponentially to zero like $\exp(-\beta k^2/2M)$ as $k \rightarrow \infty$. Carrying out the derivative inside the integral we have

$$\begin{aligned} \rho &= -\frac{4\pi}{(2\pi)^3} \frac{3}{2\beta} \int_0^\infty dk k^2 \ln(1 - e^{-\beta E_2(k)}) \\ &\quad - \frac{4\pi}{(2\pi)^3} \frac{1}{2\beta} \int_0^\infty dk m^2 \ln(1 - e^{-\beta E_2(k)}). \end{aligned} \quad (62)$$

The magnitude of the integrand of the second integral only exceeds the magnitude of the integrand of the first integral for $k < m/\sqrt{3}$. However, the logarithm is $O(1)$ up to $k \approx \sqrt{2M/\beta}$, after which it falls exponentially. Therefore, if $m^2/2M \ll 1/\beta$, then the interval $m/\sqrt{3} < k < \sqrt{2M/\beta}$ is far larger than the interval $0 < k < m/\sqrt{3}$. The second integral is therefore negligible compared to the first, and, comparing to (58), we have

$$\begin{aligned} \rho &= -\frac{3}{2\beta} \int \frac{d^3 k}{(2\pi)^3} \ln(1 - e^{-\beta E_2(k)}) \\ &= \frac{3}{2} P. \end{aligned} \quad (63)$$

An equation of state with $P = w\rho$ leads, in a FRW metric, to a dependence $\rho \sim a^{-3(1+w)}$. In our case, with $w = \frac{2}{3}$, that implies $\rho \sim a^{-5}$. This raises the interesting possibility, since ordinary radiation energy density goes as a^{-4} , that following inflation there might have been a ‘‘Landau level-dominated’’ era, just prior to the radiation-dominated era. Of course, to pin down the time of transition between these two eras, it would be necessary to know an additional cosmological parameter Ω_{Landau} in the Friedmann equation, and at the moment this number is unknown. It is understood that since the light landons only manifest their effects through gravitation, they could only be in thermal equilibrium with other particles when gravity is relatively strong, i.e., near the Planck time.

D. Causality

On a flat $g_{\mu\nu} = \eta_{\mu\nu}$ background, the field commutators are

$$\begin{aligned}
[\varphi^5(x), \varphi^5(y)] &= \cos(M(x_0 - y_0)) \\
&\quad \times \{D_{M'}(x - y) - D_{M'}(y - x)\} \\
[\varphi^6(x), \varphi^6(y)] &= \cos(M(x_0 - y_0)) \\
&\quad \times \{D_{M'}(x - y) - D_{M'}(y - x)\} \\
[\varphi^5(x), \varphi^6(y)] &= \sin(M(x_0 - y_0)) \\
&\quad \times \{D_{M'}(x - y) - D_{M'}(y - x)\}, \quad (64)
\end{aligned}$$

where

$$D_{M'}(x - y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{i(k \cdot (x-y) - \omega_k(x_0 - y_0))} \quad (65)$$

and ω_k, M' were defined in Eqs. (24) and (32), respectively. For spacelike separations $x - y$, the difference

$$\Delta D_{M'} = D_{M'}(x - y) - D_{M'}(y - x) \quad (66)$$

vanishes, and hence the field commutators vanish, consistent with causality. It has been assumed that the φ^s fields are only observable via their coupling to gravity, i.e., through the stress-energy tensor. The commutation relations (64) also imply that spacelike separated stress-energy operators commute.

V. CONCLUSIONS

It has been shown that, within a braneworld scenario in which the three-brane is coupled to a four-form gauge field, a cosmological version of cyclotron motion can result in a period of exponential inflation with an appropriate number of e -foldings, even in the absence of the usual slow-roll conditions on the inflaton potential. The mechanism is that the tendency of the inflaton field to fall to the minimum of the potential is countered by a Lorentz force in the inflaton field space. We also find a spectrum of quantum excitations of the inflaton fields, essentially a cosmological version of Landau levels, satisfying unusual dispersion relations. One consequence of the unconventional dispersion relations is the possible existence of a Landau level-dominated era, with energy density $\rho \sim a^{-5}$, preceding the radiation-dominated era.

So far only the simplest aspects of this scenario have been discussed. The fluctuation spectrum, production of standard model particles, and possible observational signatures in the Cosmic Microwave Background call for further investigation.

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APPENDIX: NO-BRANE VERSION

We may also consider the action (1) without the assumption of an embedding (2) and corresponding brane-world cosmology. In other words, the ϕ^a are simply taken to be ordinary scalar fields, which may have a potential of some kind, and are degrees of freedom completely distinct from the metric, which is fundamental rather than induced. While this alternative setup may not be so relevant to inflationary cosmology, the formulation may still be interesting as a generalization of the Lorentz force law to Wheeler-DeWitt superspace.

The action of a charged spinless point particle interacting with an electromagnetic field is

$$S = -m \int d\tau \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} + q \int dx^\mu A_\mu, \quad (A1)$$

leading to the equation of motion

$$\begin{aligned}
g_{\mu\nu} \frac{d^2 x^\nu}{ds^2} + \frac{1}{2} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \right) \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = \frac{q}{m} F_\mu, \\
\text{where } F_\mu = \mathcal{F}_{\mu\nu} \frac{dx^\nu}{ds}, \quad (A2)
\end{aligned}$$

and $\mathcal{F}_{\mu\nu}$ is the electromagnetic field strength tensor. This is simply the Lorentz force law in curved spacetime.

We restrict the discussion to purely bosonic fields, including gravity. To fix notation, let $\{q^A(\mathbf{x}), p_A(\mathbf{x}), A = 1, 2, \dots, n_f\}$ denote the canonical conjugate variables with the nongravitational fields scaled by an appropriate power of Newton's constant so as to be dimensionless. The index A now runs over all spatial indices and quantum numbers carried by the fields. In the absence of the four-form field A_{abcd} , the first-order Arnowitt-Deser-Misner action has the form

$$\begin{aligned}
S_{\text{ADM}} &= \int d^4x [p_A \partial_t q^A - N \mathcal{H}_x - N_i \mathcal{H}_x^i], \\
\mathcal{H}_x &= \kappa^2 G^{AB} p_A p_B + \sqrt{g} U(q), \\
\mathcal{H}_x^i &= O^{iA} [q, \partial_x] p_A, \quad (A3)
\end{aligned}$$

and the dynamics is given by Hamilton's equations plus the constraints $\mathcal{H}_x = \mathcal{H}_x^i = 0$. In the case of pure gravity, the correspondence with standard notation is

$$\begin{aligned}
\{A = 1 - 6\} &\leftrightarrow \{(i, j), i \leq j\} \\
q^A(x) &\leftrightarrow g_{ij}(x) \\
p_A(x) &\leftrightarrow \begin{cases} p^{ij}(x) & (i = j) \\ 2p^{ij}(x) & (i < j) \end{cases} \\
G_{AB}(x) &\leftrightarrow G^{ijnm}(x) \\
\sqrt{g}U &= -\frac{1}{\kappa^2} \sqrt{g}^{(3)} R \\
\mathcal{H}^i &= -2p_{;k}^{ik}, \quad (A4)
\end{aligned}$$

where \sqrt{g} is the determinant of the three-metric g_{ij} , $\kappa^2 = 16\pi G$, ${}^{(3)}R$ is the three-dimensional scalar curvature, G_{ijkl} is the DeWitt superspace metric

$$G_{ijkl} = \frac{1}{2\sqrt{g}}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}), \quad (\text{A5})$$

and of course Hamilton's equations plus constraints are equivalent to the Einstein field equations for pure gravity.

Now let $q^A(x) = \phi^A(x)$ for indices $A \in \mathcal{C}$, where \mathcal{C} is a subset of indices. We will denote indices in this subset by lowercase Latin letters, and the ϕ^a are a set of scalar fields. Adding the term

$$\frac{q_0}{4!} \int d^4x A_{abcd}[\phi(x)] e^{\alpha\beta\gamma\delta} \partial_\alpha \phi^a \partial_\beta \phi^b \partial_\gamma \phi^c \partial_\delta \phi^d \quad (\text{A6})$$

to the action, and using the Hamiltonian formulation, it is readily verified that the expressions for $\mathcal{H}_x, \mathcal{H}_x^i$ are changed by minimal substitution

$$p_a(x) \rightarrow p_a(x) - q_0 A_a(x), \quad (\text{A7})$$

where

$$A_a(x) \equiv \frac{1}{3!} A_{abcd} e^{0ijk} \partial_i \phi^b \partial_j \phi^c \partial_k \phi^d. \quad (\text{A8})$$

In order that the constraint algebra is satisfied, it is necessary that terms involving $\delta A_a / \delta \phi^f$, which now arise in the usual Poisson brackets among the $\mathcal{H}_x, \mathcal{H}_x^i$, all cancel. With a little more effort, those cancellations can also be verified.

In the case of a standard action containing only bosonic fields, i.e., including the metric tensor but not the four-form A_{abcd} field, it has been shown [19] that the geodesic equation derived from the following action,

$$\begin{aligned} S_q &= - \int d\tau \sqrt{-\mathcal{G}_{(Ax)(By)} \frac{dq^{(Ax)}}{d\tau} \frac{dq^{(By)}}{d\tau}} \\ &= -M \int ds \end{aligned} \quad (\text{A9})$$

(reminiscent in some ways of the Baierlein-Sharp-Wheeler action [20]), is equivalent to the equations of motion of the standard action in a certain gauge. In other words, bosonic field equations, in general, relativity can be expressed as the geodesic motion of a point particle in Wheeler-DeWitt superspace with a nonstandard supermetric \mathcal{G} . This is of course in close analogy to Jacobi's principle in mechanics. The notation is as follows: We define a mixed discrete/continuous index (Ax) as a "coordinate index" in superspace

$$q^{(Ax)} = \begin{cases} \mathcal{N}(x) & A = 0 \\ q^A(x) & A \neq 0 \end{cases}, \quad (\text{A10})$$

with summation convention

$$V_{..(Ax)..} W^{..(Ax)..} \equiv \sum_{A=0}^{n_f} \int d^3x V_{..(Ax)..} W^{..(Ax)..}, \quad (\text{A11})$$

and the nonstandard supermetric is taken to be

$$\mathcal{G}_{(Ax)(By)} = \left[\int d^3x' \mathcal{N} \sqrt{g} U \right] \frac{1}{4\mathcal{N}(x)\kappa^2} G_{AB}(x) \delta^3(x-y), \quad (\text{A12})$$

while $\mathcal{G}_{(Ax)(By)} = 0$ for $A = 0$ and/or $B = 0$. With these definitions, it is found [19] that the equations of motion which follow from (A9) are the same as those for the standard action in a shift gauge $N_i = 0$, with lapse function

$$N = M \frac{\mathcal{N}}{\int d^3x \mathcal{N} \sqrt{g} U(q)}, \quad (\text{A13})$$

and M is any constant with dimensions of mass. The choice of M is essentially a choice of affine parameter.

Adding (A6) to (A9), the equations of motion are

$$\begin{aligned} \mathcal{G}_{(Ax)(By)} \frac{d^2 q^{(By)}}{ds^2} + \frac{1}{2} \left(\frac{\delta \mathcal{G}_{(Ax)(By)}}{\delta q^{(Cz)}} + \frac{\delta \mathcal{G}_{(Ax)(Cz)}}{\delta q^{(By)}} - \frac{\delta \mathcal{G}_{(By)(Cz)}}{\delta q^{(Ax)}} \right) \\ \times \frac{dq^{(By)}}{ds} \frac{dq^{(Cz)}}{ds} = q_0 F_{(Ax)}, \end{aligned} \quad (\text{A14})$$

where $F_{(Ax)} = 0$ for indices $A \notin \mathcal{C}$, while for $A = f \in \mathcal{C}$

$$F_{(fx)} = \frac{1}{3!} F_{abcd}[\phi(x)] e^{ijk0} \partial_i \phi^a \partial_j \phi^b \partial_k \phi^c \frac{\partial \phi^d}{\partial s}, \quad (\text{A15})$$

and F_{abcd} is given in Eq. (9). Inserting the supermetric (A12) in (A14), one finds that these are the equations of motion that follow from the standard action (1) (excluding fermionic fields) in the shift gauge $N_i = 0$ and lapse function (A13).

Equations (A14) and (A15) are the suggested extension of the Lorentz force law to Wheeler-DeWitt superspace, reducing to the usual bosonic field equations (including gravity) for $A_{abcd} = 0$. Of course these equations of motion are no different from those obtained from the action (1), only dispensing with (2) and treating the metric components as fundamental degrees of freedom. It should be noted, however, that the ϕ^a fields in this no-brane formulation have no particular correlation with coordinates in a Friedmann-Lemaitre metric, and for this reason we do not expect the kind of cyclotron motion and inflation that is seen in the braneworld version.

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