Pseudoscalar condensation induced by chiral anomaly and vorticity for massive fermions

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We derive the pseudoscalar condensate induced by anomaly and vorticity from the Wigner function for massive fermions in homogeneous electromagnetic fields. It has an anomaly term and a force-vorticity coupling term. As a mass effect, the pseudoscalar condensate is linearly proportional to the fermion mass in small mass expansion. By a generalization to two-flavor and three-flavor cases, the neutral pion and eta meson condensates are calculated from the Wigner function and have anomaly parts as well as force-vorticity parts, in which the anomaly part of the neutral pion condensate is consistent with the previous result. We also discuss the possible observables of the condensates in heavy-ion collisions such as collective flows of neutral pions and eta mesons which may be influenced by the electromagnetic field and vorticity profiles.

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I. INTRODUCTION

The chiral or axial vector anomaly is the anomalous nonconservation of a chiral or axial vector current of fermions arising from quantum effects. The chiral anomaly is also called the Adler-Bell-Jackiw (ABJ) anomaly after the names of the three founders [1,2]. In quantum electrodynamics the anomalous nonconservation of the chiral or axial vector current can be written as

$$\partial_\mu j_5^\mu = -2mP - \frac{Q^2}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{1}$$

where m and Q are the fermion mass and charge respectively, $F_{\mu\nu}$ is the strength tensor of the electromagnetic field with $\tilde{F}^{\rho\lambda} = \frac{1}{2}\epsilon^{\rho\lambda\mu\nu}F_{\mu\nu}$ being its dual, the chiral or axial vector current is defined by $j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$, the pseudoscalar is defined by $P = -i\bar{\psi}\gamma_5\psi$, where ψ and $\bar{\psi}$ are fermionic fields, γ^μ ($\mu = 0$, 1, 2, 3) are Dirac matrices and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the chiral matrix. The most successful test of chiral anomaly is in the decay of a neutral pion into two photons, which had been a puzzle for some time in the 1960s whose solution led to the discovery of the ABJ anomaly. For neutral pions, one can define the chiral current as $j_{5,\pi}^\mu = \bar{\psi}\gamma^\mu\gamma_5(\sigma_3/2)\psi$ and pseudoscalar as $P_\pi = -i\bar{\psi}\gamma_5(\sigma_3/2)\psi$, where $\psi = (u,d)^T$ and $\bar{\psi} = (\bar{u},\bar{d})$ are quark fields of two flavors and $\sigma_3 = \mathrm{diag}(1,-1)$ is the third Pauli matrix. The anomaly equation (1) now becomes

$$\partial_\mu j_{5,\pi}^\mu = f_\pi m_\pi^2 \phi_\pi - \frac{Q_e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{2}$$

where Q_e is the absolute value of the electron's charge. Here we have used the PCAC (partially conserved axial current) hypothesis [3] to relate the pseudoscalar P_π to the neutral pion field ϕ_π, $2m_q P_\pi = -f_\pi m_\pi^2 \phi_\pi$, where f_π is the pion decay constant and m_q and m_π are the quark and pion mass respectively. In the chiral limit with zero quark mass we have $m_\pi = 0$ indicating pions as Goldstone bosons.

The chiral magnetic effect (CME) is an effect closely related to the chiral anomaly [4–7]. It is about the generation of an electric current along the magnetic field resulting from an imbalance of the population of chiral fermions. Another accompanying effect is the vortical effect in which an electric current is induced by the vorticity in a system of charged particles [8–10]. For chiral fermions it is called the chiral vortical effect (CVE) [11–13]. It has been demonstrated that the electric current from CME and CVE must coexist in order to guarantee the second law of thermodynamics in a chiral fluid [11,14,15]. The CVE can be regarded as a quantum effect in hydrodynamics related to the chiral anomaly.

The CME, CVE and other related effects such as the chiral magnetic wave [16,17] have been extensively studied in the quark-gluon plasma produced in high-energy heavy-ion collisions in which very strong magnetic fields [5,18–26] and huge global angular momenta [27–30] are produced in noncentral collisions. The charge separation effect observed in the STAR [31,32] and ALICE [33] experiments is consistent with the CME prediction. But there were debates that the charge separation might arise from other effects such as cluster particle correlations [34] or local charge conservation [35], so it is not conclusive that the charge separation effect would be the evidence of the CME. The charge asymmetry dependence of pion elliptic flow was observed in heavy-ion collisions by STAR and is considered as the possible consequences of the chiral magnetic wave [36]. The CME has recently been confirmed to exist in materials such as Dirac and Weyl semimetals [37–39]. Recently the STAR Collaboration has measured nonvanishing hyperon polarization in the beam energy scan program [40]. This is a piece of evidence for the local polarization effect from vorticity in collisions at lower energy and was first predicted in Ref. [41].

Quantum kinetic theory in terms of the Wigner function [42–45] is a useful tool to study the CME, CVE and other related effects [13,46–48]. The axial vector component of the Wigner function for massless fermions can be generalized to massive fermions and gives their phase-space density of the spin vector. The spin vector arises from nonzero fermion mass [49]. Therefore one can calculate the polarization of massive fermions from the axial vector component [48]. The polarization density is found to be proportional to the local vorticity $\boldsymbol{\omega}$ as well as the magnetic field. The polarization per particle for fermions is always smaller than that for antifermions as the result of more Pauli blocking effect for fermions than antifermions. This is consistent with STAR's preliminary result on the Λ polarization [40].

In this paper we give another important feature of massive fermions from the axial vector component of the Wigner function, namely, the thermal average of pseudoscalar quantity P in Eq. (1). In low-energy QCD, it is proportional to the pion field from the PCAC hypothesis, so we call it the pseudoscalar condensate. We will show that such a pseudoscalar condensate is induced by anomaly and force-vorticity coupling and depends on the fermion mass, fermion chemical potential and temperature. It is a mass effect in a plasma of fermions: for massless fermions the pseudoscalar is vanishing. There were many previous studies on pseudoscalar condensates and related topics in nuclear and quark matter in hot and dense environments. These include, e.g., the condensate of negatively charged pions in cold and dense nuclear matter [50,51], metastable states in the limit of a large number of colors for hot QCD in which parity is spontaneously broken in hadronic phase leading to global parity asymmetries of charged pions [52], and pion condensate in isospin asymmetric matter [53,54].

The paper is organized as follows. In Sec. II, we summarize the properties of the Wigner function for massless fermions in electromagnetic fields. In Sec. III we present the equation for the pseudoscalar and axial vector component of the Wigner function for massive fermions, the Wigner function counterpart of Eq. (1). In Sec. IV, we analyze the axial vector component at leading order from which the polarization vector can be obtained. We also derive the polarization vector of a fermion in the lab frame with its 3-momentum $\mathbf{p}$ in the fluid cell's comoving frame and the polarization vector $\mathbf{n}$ in the particle's rest frame. This is useful to connect the experimental observable to the theoretical prediction. In Sec. V we derive nonconservation of the chiral current with anomaly and fermion mass by taking space-time divergence of the chiral current derived from the axial vector component. We also calculate the condensates of neutral pions and eta mesons. Then we derive the pseudoscalar condensate induced by anomaly and vorticity for massive fermions. The summary is made in the last section.

We adopt the same sign conventions for fermion charge Q as in Refs. [13,44,46,47], and the same sign convention for the axial vector component $\mathcal{A}^\mu \sim \langle \bar{\psi}\gamma^\mu\gamma^5\psi\rangle$ as in Refs. [13,46,47] but different sign convention from Ref. [44].

II. WIGNER FUNCTION FOR MASSLESS FERMIONS IN ELECTROMAGNETIC FIELDS

The gauge invariant Wigner function is the quantum mechanical analogue of a classical phase-space distribution. In a background electromagnetic field, the Wigner function $W_{\alpha\beta}(x,p)$ is defined by

$$W_{\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \left\langle \bar{\psi}_\beta\left(x+\frac{1}{2}y\right) \times \mathrm{P}U\left(G, x+\frac{1}{2}y, x-\frac{1}{2}y\right)\psi_\alpha\left(x-\frac{1}{2}y\right)\right\rangle, \tag{3}$$

where ψ_α and $\bar{\psi}_\beta$ are fermionic quantum fields with Dirac indices α and β, $\langle\hat{O}\rangle$ denotes the grand canonical ensemble average of the normal ordered operator, $x = (x_0, \mathbf{x})$ and $p = (p_0, \mathbf{p})$ are time-space and energy-momentum 4-vectors respectively, and the gauge link $\mathrm{P}U(G, x_1, x_2)$ is to ensure the gauge invariance of the Wigner function where G^μ is the gauge potential of the classical electromagnetic field and P denotes the path-ordered product. The 4×4 matrix $W_{\alpha\beta}(x,p)$ can be decomposed by 16 independent generators of Clifford algebra, namely, 1, γ^5, γ^μ, $\gamma^5\gamma^\mu$, $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, into the scalar, pseudoscalar, vector, axial vector and tensor components, respectively. From the Dirac equation for the fermionic field, one can derive the equation for $W_{\alpha\beta}(x,p)$ which leads to a set of coupled equations for all components. Finding a general solution to the Wigner function is very difficult. However it is much simplified for massless fermions for which the set of equations for the vector and axial vector components are decoupled from the rest of the components. Assuming that the electromagnetic field is homogeneous and weak and is in the same order as the space-time derivative, one can solve the vector and axial vector components perturbatively. To the linear order in the field strength and vorticity, one can obtain the vector and axial vector components [13], which give the charge current j^μ and chiral charge current j_5^μ by integration over 4-momenta: $j^\mu = nu^\mu + \xi\omega^\mu + \xi_B B^\mu$ and $j_5^\mu = n_5 u^\mu + \xi_5\omega^\mu + \xi_{5B}B^\mu$. Here $\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho u_\sigma$ is the vorticity 4-vector, $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma}$ is the magnetic field 4-vector with the fluid 4-velocity u_ν, and n and n_5 are charge and chiral charge density respectively. In the charge current j^μ one obtains the CME and CVE coefficients $\xi = \mu\mu_5/\pi^2$ and $\xi_B = Q\mu_5/(2\pi^2)$ respectively, where μ and μ_5 are chemical potentials for the charge and chiral charge respectively. One can also derive the coefficients of vorticty and magnetic field in j_5^μ: $\xi_5 = T^2/6 + (\mu^2+\mu_5^2)/(2\pi^2)$ and $\xi_{5B} = Q\mu/(2\pi^2)$. The conservation and anomalous nonconservation laws for the charge and chiral charge current respectively can be verified, $\partial_\mu j^\mu = 0$ and $\partial_\mu j_5^\mu = -[Q^2/(8\pi^2)]F_{\mu\nu}\tilde{F}^{\mu\nu}$. The covariant chiral kinetic equation which is related to the Berry phase in four dimensions can also be derived from the first order solution to the vector and axial vector components of the Wigner function [46].

III. EQUATION FOR THE PSEUDOSCALAR AND AXIAL VECTOR COMPONENT OF THE WIGNER FUNCTION

In this section we look at the equation for the pseudoscalar and axial vector component of the Wigner function. From the Dirac equation for the fermionic field, one can derive the equation for the Wigner function in (3) in a constant electromagnetic field,

$$\left[\gamma_\mu\left(p^\mu + i\hbar\frac{1}{2}\nabla^\mu\right) - m\right]W(x,p) = 0, \tag{4}$$

where the phase-space derivative is defined by $\nabla^\mu = \partial_x^\mu - QF^{\mu\nu}\partial_{p,\nu}$ and we suppressed Dirac indices of the Wigner function. The Wigner function as a 4×4 matrix in Dirac space can be decomposed into the scalar, pseudoscalar, vector, axial vector and tensor components as

$$W = \frac{1}{4}\left[\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right]. \tag{5}$$

The components in the decomposition (5) can be obtained by the projection of corresponding Dirac matrices on the Wigner function and taking traces. Equation (4) for the Wigner function can be converted to a set of coupled equations for all components. There is an equation that relates the pseudoscalar to the axial vector component which is of special interest,

$$\hbar\nabla^\mu\mathcal{A}_\mu = -2m\mathcal{P}. \tag{6}$$

An interesting observation of the above equation is that the pseudoscalar component $\mathcal{P}$ is of quantum origin since it is proportional to the Planck constant $\hbar$. We note that Eq. (6) is nothing but the Wigner function counterpart of Eq. (1). We will show that the integration of Eq. (6) over the 4-momentum gives Eq. (1).

IV. AXIAL VECTOR COMPONENT AT LEADING ORDER AND POLARIZATION VECTOR

At leading (zeroth) order of electromagnetic interaction, the gauge link in the Wigner function in Eq. (3) can be set to 1, and we denote the Wigner function at this order as $W_{(0)}$. We can expand the fermionic fields in momentum space with creation and destruction operators, which we insert into Eq. (3). After taking the ensemble average of normal ordered operators, we obtain

$$\begin{aligned} W_{(0)}(x,p) = {} & \frac{1}{(2\pi)^3}\delta(p^2-m^2) \\ & \times\Big\{\theta(p^0)\sum_s f_{\rm FD}(E_p-\mu_s)u(\mathbf{p},s)\bar{u}(\mathbf{p},s) \\ & - \theta(-p^0)\sum_s f_{\rm FD}(E_p+\mu_s)v(-\mathbf{p},s)\bar{v}(-\mathbf{p},s)\Big\}, \end{aligned} \tag{7}$$

where $u(\mathbf{p},s)$ and $v(-\mathbf{p},s)$ are Dirac spinors of positive and negative energy respectively, $s=\pm$ denotes the spin state parallel or antiparallel to the spin quantization direction $\mathbf{n}$ in the rest frame of the particle. We have also used $\langle a^\dagger(\mathbf{p},s)a(\mathbf{p},s)\rangle = f_{\rm FD}(E_p-\mu_s)$ and $\langle b^\dagger(-\mathbf{p},s)b(-\mathbf{p},s)\rangle = f_{\rm FD}(E_p+\mu_s)$ with the Fermi-Dirac distribution defined by $f_{\rm FD}(x) = 1/(e^{\beta x}+1)$ ($\beta \equiv 1/T$, T is the temperature) and μ_s is the chemical potential for fermions in the spin state s.

The axial vector component at the leading order is given by

$$\begin{aligned} \mathcal{A}^\mu_{(0)} & = {\rm Tr}[\gamma^\mu\gamma^5 W_{(0)}] \\ & = m[\theta(p_0)n^\mu(\mathbf{p},\mathbf{n}) - \theta(-p_0)n^\mu(-\mathbf{p},-\mathbf{n})]\delta(p^2-m^2)A, \end{aligned} \tag{8}$$

where A is defined by

$$\begin{aligned} A \equiv \frac{2}{(2\pi)^3}\sum_s s[&\theta(p^0)f_{\rm FD}(p_0-\mu_s) \\ & + \theta(-p^0)f_{\rm FD}(-p_0+\mu_s)], \end{aligned} \tag{9}$$

and we have used $\bar{u}(\mathbf{p},s)\gamma^\mu\gamma^5 u(\mathbf{p},s) = 2msn^\mu(\mathbf{p},\mathbf{n})$ and $\bar{v}(-\mathbf{p},s)\gamma^\mu\gamma^5 v(-\mathbf{p},s) = 2msn^\mu(-\mathbf{p},-\mathbf{n})$ with $n^\mu(\mathbf{p},\mathbf{n})$ given by

$$n^\mu(\mathbf{p},\mathbf{n}) = \Lambda^\mu_\nu(-\mathbf{v}_p)n^\nu(\mathbf{0},\mathbf{n}) = \left(\frac{\mathbf{n}\cdot\mathbf{p}}{m}, \mathbf{n} + \frac{(\mathbf{n}\cdot\mathbf{p})\mathbf{p}}{m(m+E_p)}\right). \tag{10}$$

Here $\Lambda^\mu_\nu(-\mathbf{v}_p)$ is the Lorentz transformation for $\mathbf{v}_p = \mathbf{p}/E_p$ and $n^\nu(\mathbf{0},\mathbf{n}) = (0,\mathbf{n})$ is the 4-vector of the spin quantization direction in the rest frame of the fermion. One can check that $n^\mu(\mathbf{p},\mathbf{n})$ satisfies $n^2=-1$ and $n\cdot p=0$, so it behaves like a spin 4-vector up to a factor of $1/2$. For Pauli spinors χ_s and $\chi_{s'}$ in $u(\mathbf{p},s)$ and $v(-\mathbf{p},s')$ respectively, we have $\chi_s^\dagger\boldsymbol{\sigma}\chi_s=s\mathbf{n}$ and $\chi_{s'}^\dagger\boldsymbol{\sigma}\chi_{s'}=-s'\mathbf{n}$. We can take the massless limit by setting $\mathbf{n}=\hat{\mathbf{p}}$, then we have $mn^\mu(\mathbf{p},\mathbf{n})\to(|\mathbf{p}|,\mathbf{p})$ and $mn^\mu(-\mathbf{p},-\mathbf{n})\to(|\mathbf{p}|,-\mathbf{p})$. This way we can recover the previous result of the axial vector component for massless fermions [13,46] where $s=\pm$ denotes the right-handed and left-handed fermions.

We implied that $n^\mu(\mathbf{p},\mathbf{n})$ in (10) is the form in the comoving or local rest frame of a fluid cell. We can boost all quantities in Eq. (8) to the lab frame in which the fluid cell is moving with a 4-velocity $u^\alpha=\gamma(1,\mathbf{v})$ where $\gamma=1/\sqrt{1-v^2}$ is the Lorentz factor. We boost $n^\mu(\mathbf{p},\mathbf{n})$ and $n^\mu(-\mathbf{p},-\mathbf{n})$ to the lab frame as

$$
\begin{aligned}
n'^\alpha_{\rm lab}(\mathbf{p},\mathbf{n})&=\Lambda^\alpha_\beta(-\mathbf{v})n^\beta(\mathbf{p},\mathbf{n}),\\
n'^0_{\rm lab}(\mathbf{p},\mathbf{n})&=\gamma\left[\frac{\mathbf{n}\cdot\mathbf{p}}{m}+\mathbf{v}\cdot\mathbf{n}+\frac{(\mathbf{n}\cdot\mathbf{p})(\mathbf{v}\cdot\mathbf{p})}{m(m+E_p)}\right],\\
\mathbf{n}'_{\rm lab}(\mathbf{p},\mathbf{n})&=\mathbf{n}+\frac{(\mathbf{n}\cdot\mathbf{p})\mathbf{p}}{m(m+E_p)}\\
&\quad+\frac{\gamma-1}{v^2}\mathbf{v}\left[\mathbf{n}\cdot\mathbf{v}+\frac{(\mathbf{n}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{v})}{m(m+E_p)}\right]+\frac{\gamma}{m}\mathbf{v}(\mathbf{n}\cdot\mathbf{p}).
\end{aligned}
\tag{11}
$$

Note that $\mathbf{n}$ is the spin quantization direction in the fermion's rest frame and $\mathbf{p}$ is the fermion's momentum in the local rest frame of the fluid cell. The fermion momentum in the lab frame is $p'^\alpha=\Lambda^\alpha_\beta(-\mathbf{v})p^\beta$ with $p^\beta=(E_p,\mathbf{p})$ or explicitly

$$
\begin{aligned}
p'^0&=\gamma[E_p+(\mathbf{p}\cdot\mathbf{v})],\\
\mathbf{p}'&=\mathbf{p}+\frac{\gamma-1}{v^2}\mathbf{v}(\mathbf{p}\cdot\mathbf{v})+\gamma\mathbf{v}E_p.
\end{aligned}
\tag{12}
$$

One can check $p'\cdot u=p^0=E_p$ and $n'\cdot p'=0$ in the lab frame. So Eqs. (8) and (9) can be written in the lab frame by making the replacement $n^\alpha(\mathbf{p},\mathbf{n})\to n'^\alpha_{\rm lab}(\mathbf{p},\mathbf{n})$; also we have $p_0=p'\cdot u$ and $p'^2=p^2$ in the two formulas.

We note that Eq. (11) is the polarization vector in the lab frame (the fluid cell moves with a velocity $\mathbf{v}$) of a fermion with the 3-momentum $\mathbf{p}$ in the fluid cell's comoving frame and the polarization vector $\mathbf{n}$ in the particle's rest frame. After taking integration of $n'^\alpha_{\rm lab}(\mathbf{p},\mathbf{n})$ over $\mathbf{p}$ which follows the Fermi-Dirac distribution, one can obtain the thermal average of the polarization vector in the lab frame. The Λ polarization can be measured in experiments by its decay to a proton and pion in its rest frame. It is observed that Λ is polarized along the global angular momentum in the beam energy scan program at the RHIC [40]. From Eq. (11) one can calculate the Λ polarization along a fixed direction in the lab frame, e.g. the direction of the global angular momentum, and compare with data.

In this paper, we assume that the chemical potential does not depend on the spin state $\mu_s=\mu$ for $s=\pm$, so $A=0$ and then $\mathcal{A}^\mu_{(0)}$ is vanishing. The nonvanishing contribution comes from the axial vector component at the next-to-leading or first order even with spin-independent chemical potentials.

V. CHIRAL CURRENT NONCONSERVATION LAW AND PSEUDOSCALAR CONDENSATION

In this section we will derive the nonconservation law of the chiral current of massive fermions by taking space-time divergence of the chiral current derived from the axial vector component. Then we can derive the pseudoscalar condensate induced by anomaly and vorticity.

As we have discussed in the last section that the contribution should come from the axial vector component at the next-to-leading or first order. We use the following form for the axial vector component for massive fermions by generalizing the solution for massless fermions [13,46–48],

$$
\begin{aligned}
\mathcal{A}^\alpha_{(1)}(x,p)&=-\frac{1}{2}\hbar\beta\tilde{\Omega}^{\alpha\sigma}p_\sigma\frac{dV}{d(\beta p_0)}\delta(p^2-m^2)\\
&\quad-Q\hbar\tilde{F}^{\alpha\lambda}p_\lambda V\frac{\delta(p^2-m^2)}{p^2-m^2},
\end{aligned}
\tag{13}
$$

where $p_0\equiv u\cdot p$, $\tilde{\Omega}^{\rho\sigma}=\frac{1}{2}\epsilon^{\rho\sigma\mu\nu}\partial_\mu u_\nu$, and V is associated with the vector component and given by

$$
V\equiv\frac{4}{(2\pi)^3}[\theta(p^0)f_{\rm FD}(p_0-\mu)+\theta(-p^0)f_{\rm FD}(-p_0+\mu)],
\tag{14}
$$

where we have taken $\mu_s=\mu$ for $s=\pm$. Also we have assumed in Eq. (13) that $\beta=1/T$ is a constant. The chiral current can be obtained by integrating over the 4-momentum of the axial vector component, $j_5^\mu=\int d^4p\mathcal{A}^\mu_{(1)}(x,p)$, whose space-time divergence is given by

$$
\begin{aligned}
\partial_\mu j_5^\mu&=\int d^4p\partial_\mu\mathcal{A}^\mu_{(1)}\\
&=-\frac{1}{2}\hbar\beta\tilde{\Omega}^{\mu\sigma}\int d^4p\,p_\sigma\partial_\mu\left[\frac{dV}{d(\beta p_0)}\right]\delta(p^2-m^2)\\
&\quad-\hbar Q\tilde{F}^{\mu\lambda}\int d^4p\,p_\lambda(\partial_\mu V)\frac{\delta(p^2-m^2)}{p^2-m^2},
\end{aligned}
\tag{15}
$$

where we have assumed that $\tilde{F}^{\mu\lambda}$ does not depend on space-time, and used $u_\sigma\partial_\mu\tilde{\Omega}^{\mu\sigma}=0$ due to static equilibrium conditions [13]. When taking space-time divergence, we neglect the space-time derivative of $\theta(p_0)$ and $\theta(-p_0)$ in V which would be vanishing when contacting the mass-shell condition. The two terms in the last equality of Eq. (15) are evaluated in Appendix A and can be grouped into an $E\cdot B$ and $E\cdot\omega$ term,

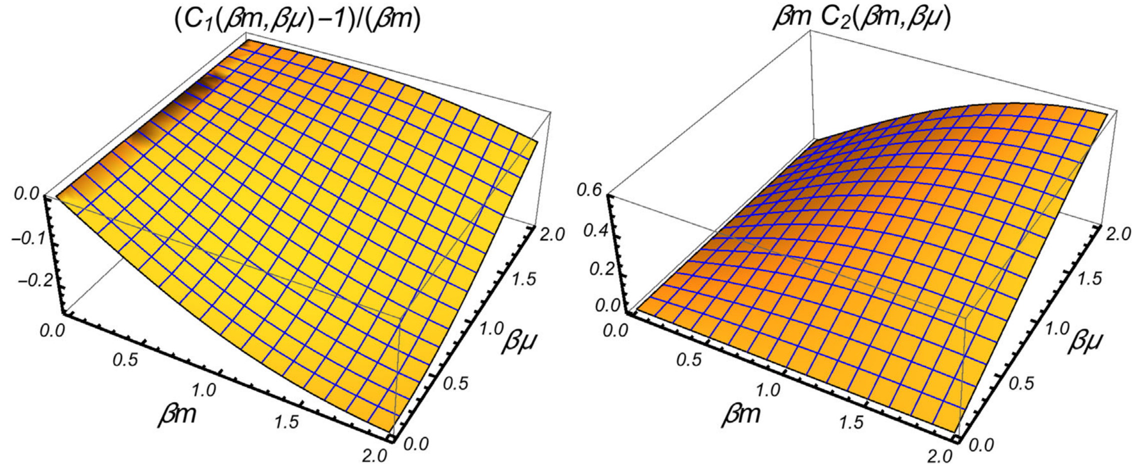

FIG. 1. $[C_1(\beta m,\beta\mu)-1]/(\beta m)$ and $(\beta m)C_2(\beta m,\beta\mu)$ as functions of βm and $\beta\mu$. They are all proportional to m in the chiral limit.

$$\partial_\mu j_5^\mu = \frac{1}{2}\hbar Q^2\beta^2(E\cdot B)\int d^4p V''_{\beta\mu,\beta p_0}\delta(p^2-m^2)+\frac{1}{2}\hbar\beta Q(E\cdot\omega)\int d^4p\left[\beta p_0 V''_{\beta p_0,\beta\mu}-\beta\frac{2\bar{p}^2}{3p_0}V''_{\beta p_0,\beta p_0}+\frac{2\bar{p}^2}{3p_0^2}V'_{\beta p_0}\right]\delta(p^2-m^2), \tag{16}$$

where $\bar{p}^\alpha\equiv p^\alpha-(p\cdot u)u^\alpha$. The integrals in Eq. (16) can be finally simplified into the following forms

$$\partial_\mu j_5^\mu = -\frac{1}{2\pi^2}\hbar Q^2(E\cdot B)C_1(\beta m,\beta\mu)-\frac{m^2}{2\pi^2}\hbar Q\beta(E\cdot\omega)C_2(\beta m,\beta\mu), \tag{17}$$

where the dimensionless functions $C_1(\beta m,\beta\mu)$ and $C_2(\beta m,\beta\mu)$ are defined by

$$C_1(\beta m,\beta\mu)=\int_0^\infty dx\left[\frac{e^{\sqrt{x^2+(\beta m)^2}-\beta\mu}}{(e^{\sqrt{x^2+(\beta m)^2}-\beta\mu}+1)^2}+\frac{e^{\sqrt{x^2+(\beta m)^2}+\beta\mu}}{(e^{\sqrt{x^2+(\beta m)^2}+\beta\mu}+1)^2}\right],$$
$$C_2(\beta m,\beta\mu)=\int_0^\infty dx\frac{1}{\sqrt{x^2+(\beta m)^2}}\left[\frac{e^{\sqrt{x^2+(\beta m)^2}-\beta\mu}}{(e^{\sqrt{x^2+(\beta m)^2}-\beta\mu}+1)^2}-\frac{e^{\sqrt{x^2+(\beta m)^2}+\beta\mu}}{(e^{\sqrt{x^2+(\beta m)^2}+\beta\mu}+1)^2}\right]. \tag{18}$$

We plot the functions $[C_1(\beta m,\beta\mu)-1]/(\beta m)$ and $(\beta m)C_2(\beta m,\beta\mu)$ in Fig. 1. One can verify asymptotic values $C_1(\beta m,\beta\mu)-1\sim O[(\beta m)^2]$ and $C_2(\beta m,\beta\mu)\sim O(1)$ for $\beta m\to 0$; see Eq. (B19) in Appendix B.

On the other hand, taking an integration over the 4-momentum of Eq. (6) leads to

$$-2mP=\hbar\partial_\mu j_5^\mu+\frac{1}{2\pi^2}\hbar^2Q^2(E\cdot B), \tag{19}$$

where we have used $P(x)=\int d^4p\mathcal{P}(x,p)$ and the integral

$$QF_{\mu\nu}\int d^4p\partial_p^\nu\mathcal{A}^\mu(x,p)=-\frac{1}{2\pi^2}\hbar Q^2(E\cdot B), \tag{20}$$

whose derivation is given in Appendix C. Inserting Eq. (17) into Eq. (19) we obtain the pseudoscalar condensate

$$P=\frac{1}{4\pi^2}\hbar^2Q^2(E\cdot B)\frac{1}{m}[C_1(\beta m,\beta\mu)-1]$$
$$+\frac{1}{4\pi^2}\hbar^2Q(E\cdot\omega)\beta mC_2(\beta m,\beta\mu). \tag{21}$$

From the small mass behavior $C_1(\beta m,\beta\mu)-1\sim(\beta m)^2$ and $C_2(\beta m,\beta\mu)\sim O(1)$, the pseudoscalar is proportional to the fermion mass. This feature is similar to the PCAC hypothesis where the pseudoscalar is proportional to m_π^2/m_q. We see in $C_2(\beta m,\beta\mu)$ that there is a sign difference between the fermion and antifermion terms, so the $QE\cdot\omega$ term can also be regarded as the *force-vorticity* coupling term. Furthermore the $E\cdot\omega$ term is vanishing for $\beta\mu=0$.

The above discussions are about single fermion species. Let us consider a quark plasma with two quark flavors u and d. For u or d quarks, Eqs. (17) and (19) become

$$\partial_\mu j^\mu_{5,a} = -\frac{N_C}{2\pi^2}\hbar Q_a^2 (E\cdot B) C_1(\beta m_a, \beta\mu_a) - \frac{N_C m_a^2}{2\pi^2}\hbar Q_a \beta (E\cdot\omega) C_2(\beta m_a, \beta\mu_a),$$
$$-2m_a P_a = \hbar \partial_\mu j^\mu_{5,a} + \frac{N_C}{2\pi^2}\hbar^2 Q_a^2 (E\cdot B), \tag{22}$$

where $N_C=3$ is the number of colors, $a=u$, d, and $P_a = -i\bar{a}\gamma_5 a$, $j^\mu_{5,a} = \bar{a}\gamma^\mu\gamma_5 a$. We know from the Introduction that the pseudoscalar and chiral current for neutral pions are given by

$$P_\pi = -i\bar\psi\gamma_5(\sigma_3/2)\psi = \frac{1}{2}(P_u - P_d),$$
$$j^\mu_{5,\pi} = \bar\psi\gamma^\mu\gamma_5(\sigma_3/2)\psi = \frac{1}{2}(j^\mu_{5,u} - j^\mu_{5,d}). \tag{23}$$

In the first case we assume that the masses and chemical potentials of the u and d quarks are the same, $m_u = m_d = m_q$ and $\mu_u = \mu_d = \mu_q$. Then the pion condensate is

$$\begin{aligned} P_\pi &= \frac{N_C}{8\pi^2}\hbar^2 (Q_u^2 - Q_d^2)(E\cdot B)\frac{1}{m_q}[C_1(\beta m_q, \beta\mu_q) - 1] \\ &\quad + \frac{N_C}{8\pi^2}\hbar^2 (Q_u - Q_d)(E\cdot\omega)\beta m_q C_2(\beta m_q, \beta\mu_q) \\ &= \frac{N_C}{24\pi^2}\hbar^2 Q_e^2 (E\cdot B)\frac{1}{m_q}[C_1(\beta m_q, \beta\mu_q) - 1] \\ &\quad + \frac{N_C}{8\pi^2}\hbar^2 Q_e (E\cdot\omega)\beta m_q C_2(\beta m_q, \beta\mu_q). \end{aligned} \tag{24}$$

In the second case we assume that the masses of the u and d quarks are the same, $m_u = m_d = m_q$, but the chemical potentials are different; the pion condensate is

$$\begin{aligned} P_\pi &= \frac{N_C}{8\pi^2}\hbar^2 (E\cdot B)\frac{1}{m_q}[Q_u^2 C_1(\beta m_q, \beta\mu_u) \\ &\quad - Q_d^2 C_1(\beta m_q, \beta\mu_d) - (Q_u^2 - Q_d^2)] \\ &\quad + \frac{N_C}{8\pi^2}\hbar^2 (E\cdot\omega)\beta m_q [Q_u C_2(\beta m_q, \beta\mu_u) \\ &\quad - Q_d C_2(\beta m_q, \beta\mu_d)]. \end{aligned} \tag{25}$$

In vacuum at zero temperature and chemical potentials, both functions C_1 and C_2 are vanishing and we obtain from Eqs. (24) and (25)

$$P_\pi^{\mathrm{vac}} = -\frac{\hbar^2}{8\pi^2 m_q} Q_e^2 (E\cdot B), \tag{26}$$

which is consistent with the result derived in the Nambu-Jona-Lasinio model and chiral perturbation theory [55]. We see that the pion condensate in vacuum comes out quite natural as a mass effect of charged fermions and is not subject to any additional constraint such as a critical value for $|\mathbf{E}\cdot\mathbf{B}|$ in [55]. Recently the chiral current and pseudoscalar condensate for massive fermions have also been calculated in a holographic model in finite density and magnetic field [56].

We can also generalize to quark matter with three quark flavors u, d and s. So Eq. (22) applies to $a = u, d, s$. Now we look at the pseudoscalar and chiral current for the flavor octet η_8 and singlet η_1 which can be defined by

$$\begin{aligned} P_{\eta 8} &= -i\bar\psi\gamma_5\frac{\lambda_8}{2}\psi = \frac{1}{2\sqrt{3}}(P_u + P_d - 2P_s), \\ P_{\eta 1} &= -i\bar\psi\gamma_5\frac{1}{\sqrt{3}}\mathbf{1}\psi = \frac{1}{\sqrt{3}}(P_u + P_d + P_s), \\ j^\mu_{5,\eta 8} &= \bar\psi\gamma^\mu\gamma_5\frac{\lambda_8}{2}\psi = \frac{1}{2\sqrt{3}}(j^\mu_{5,u} + j^\mu_{5,d} - 2j^\mu_{5,s}), \\ j^\mu_{5,\eta 1} &= \bar\psi\gamma^\mu\gamma_5\frac{1}{\sqrt{3}}\mathbf{1}\psi = \frac{1}{\sqrt{3}}(j^\mu_{5,u} + j^\mu_{5,d} + j^\mu_{5,s}). \end{aligned} \tag{27}$$

As an example for the η meson condensate, we consider the simplest case for quark masses and chemical potentials, $m_u = m_d = m_s = m_q$ and $\mu_u = \mu_d = \mu_s = \mu_q$, which can be considered as the case approaching the chiral limit. In this case, we have

$$\begin{aligned} P_{\eta 8} &= \frac{N_C}{8\sqrt{3}\pi^2}\hbar^2 (Q_u^2 + Q_d^2 - 2Q_s^2)(E\cdot B)\frac{1}{m_q}[C_1(\beta m_q, \beta\mu_q) - 1] + \frac{N_C}{8\sqrt{3}\pi^2}\hbar^2 (Q_u + Q_d - 2Q_s)(E\cdot\omega)\beta m_q C_2(\beta m_q, \beta\mu_q) \\ &= \frac{N_C}{24\sqrt{3}\pi^2}\hbar^2 Q_e^2 (E\cdot B)\frac{1}{m_q}[C_1(\beta m_q, \beta\mu_q) - 1] + \frac{N_C}{8\sqrt{3}\pi^2}\hbar^2 Q_e (E\cdot\omega)\beta m_q C_2(\beta m_q, \beta\mu_q), \\ P_{\eta 1} &= \frac{N_C}{4\sqrt{3}\pi^2}\hbar^2 (Q_u^2 + Q_d^2 + Q_s^2)(E\cdot B)\frac{1}{m_q}[C_1(\beta m_q, \beta\mu_q) - 1] + \frac{N_C}{4\sqrt{3}\pi^2}\hbar^2 (Q_u + Q_d + Q_s)(E\cdot\omega)\beta m_q C_2(\beta m_q, \beta\mu_q) \\ &= \frac{N_C}{6\sqrt{3}\pi^2}\hbar^2 Q_e^2 (E\cdot B)\frac{1}{m_q}[C_1(\beta m_q, \beta\mu_q) - 1]. \end{aligned} \tag{28}$$

The main difference between $P_{\eta 8}$ and $P_{\eta 1}$ in the above is that $P_{\eta 1}$ does not have the $E\cdot\omega$ term due to the cancellation of electric charges of quarks with different flavors. Of course more realistic cases in heavy-ion collisions should be like $m_u = m_d = m_q \neq m_s$ and $\mu_u = \mu_d = \mu_q \neq \mu_s$, for which the calculation is straightforward. There are possible observables of η meson condensates in heavy-ion collisions [52], but it is beyond the scope of the current paper and will be addressed in a future study.

The pseudoscalar condensate for charged fermions in Eq. (21) is our main result. It is a quite general formula. Such a pseudoscalar condensate is charge neutral and induced by anomaly and vorticity in a thermal and dense environment, which is a natural consequence of the nonconservation law of the chiral current in electromagnetic fields and is not subject to any additional constraints. For example, the neutral pion condensate is always in the form of Eqs. (24) and (25) without further condition about the value of $|\mathbf{E}\cdot\mathbf{B}|$ if anomaly and vorticity are there. It is also worth mentioning a new electric-field-vorticity coupling term in the condensate which has not been derived in the literature to our knowledge. Such pion and eta meson condensates may have observables related to the electromagnetic field and vorticity in heavy-ion collisions. For example they may have effects on the collective flows of neutral pions and eta mesons. In this sense there may be some connection of this work to the disoriented chiral condensates [57–61]. This is a topic that we plan to address in the future.

VI. SUMMARY

We derive the pseudoscalar condensate induced by anomaly and vorticity from the Wigner function for massive fermions in homogeneous electromagnetic fields. The pseudoscalar component of the Wigner function is determined from the axial vector component by Eq. (6). Taking an integration over the 4-momentum for Eq. (6) we obtain the anomalous nonconservation of the chiral current by the anomalous term and a product of mass and pseudoscalar. By directly calculating the space-time divergence of the chiral current, we can determine the pseudoscalar condensate which has an anomalous $E\cdot B$ term and an $E\cdot\omega$ term. The $E\cdot\omega$ term can also be regarded as a force-vorticity coupling since there is a sign difference in its prefactor between the fermion and antifermion sector. The force-vorticity part of the pseudoscalar condensate is the new term. As a mass effect, the pseudoscalar condensate is linearly proportional to the fermion mass when the mass is small. Such a pseudoscalar condensate is a general feature for a fluid of massive and charged fermions in a thermal and dense plasma with anomaly and vorticity. The neutral pion and eta meson condensates can also be derived from generalization of the single flavor to multiflavor case, which depend on quark masses, quark chemical potentials and temperature. We can reproduce the previous result of the neutral pion condensate in vacuum induced by the anomaly, but our result also has a force-vorticity part which has not been derived in previous literature to our knowledge. There are possible observables of pseudoscalar condensates related to the electromagnetic field and vorticity in heavy-ion collisions such as collective flows of neutral pions and eta mesons.

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APPENDIX A: DERIVATION OF EQ. (15)

Let us treat the vorticity term in Eq. (15) as

$$\begin{aligned} I_\omega &= -\frac{1}{2}\hbar\beta\tilde{\Omega}^{\mu\sigma}\partial_\mu(\beta\mu)\int d^4p\,p_\sigma V''_{\beta p_0,\beta\mu}\delta(p^2-m^2) \\ &\quad -\frac{1}{2}\hbar\beta\tilde{\Omega}^{\mu\sigma}\partial_\mu(\beta u_\rho)\int d^4p\,p_\sigma p^\rho V''_{\beta p_0,\beta p_0}\delta(p^2-m^2) \\ &= \frac{1}{2}\hbar\beta^2 Q(E\cdot\omega)\int d^4p\,p_0 V''_{\beta p_0,\beta\mu}\delta(p^2-m^2), \end{aligned} \tag{A1}$$

where the second term in the first equality is vanishing. This can be seen by

$$\begin{aligned} &\tilde{\Omega}^{\mu\sigma}\partial_\mu(\beta u_\rho)\int d^4p\,p_\sigma p^\rho V''_{\beta p_0,\beta p_0}\delta(p^2-m^2) \\ &\quad = \tilde{\Omega}^{\mu\sigma}\partial_\mu(\beta u_\rho)u^\rho u_\sigma\int d^4p\,p_0^2 V''_{\beta p_0,\beta p_0}\delta(p^2-m^2) \\ &\qquad +\frac{1}{3}\tilde{\Omega}^{\mu\sigma}\partial_\mu(\beta u_\rho)\Delta^\rho_\sigma\int d^4p\,\bar{p}^2 V''_{\beta p_0,\beta p_0}\delta(p^2-m^2) \\ &\quad = 0, \end{aligned} \tag{A2}$$

where we have assumed that β is constant and used $\Delta^\rho_\sigma = g^\rho_\sigma - u_\sigma u^\rho$, $u^\rho\partial_\mu u_\rho = 0$ and

$$\tilde{\Omega}^{\mu\sigma}\partial_\mu(\beta u_\rho)\Delta^\rho_\sigma = \beta\tilde{\Omega}^{\mu\sigma}\partial_\mu u_\sigma = 2\beta\partial_\alpha\omega^\alpha = 0. \tag{A3}$$

Then we look at the second term in Eq. (15) which is related to the electromagnetic field,

$$
\begin{aligned}
I_F &= \hbar\beta Q\tilde{F}^{\mu\lambda}\int d^4p[-p_\lambda(\partial_\mu\mu)+(\partial_\mu u_\rho)p_\lambda p^\rho]V'_{\beta p_0}\delta'(p^2-m^2)\\
&= \hbar\beta Q^2(E\cdot B)\int d^4p\,p_0V'_{\beta p_0}\delta'(p^2-m^2)+\frac{1}{3}\hbar\beta Q\tilde{F}^{\mu\lambda}(\partial_\mu u_\lambda)\int d^4p\,\bar{p}^2V'_{\beta p_0}\delta'(p^2-m^2)\\
&= \hbar\beta Q^2(E\cdot B)\int d^4p\,p_0V'_{\beta p_0}\delta'(p^2-m^2)+\frac{2}{3}\hbar\beta Q(E\cdot\omega)\int d^4p\,\bar{p}^2V'_{\beta p_0}\delta'(p^2-m^2)\\
&= \frac{1}{2}\hbar Q^2\beta^2(E\cdot B)\int d^4p\,V''_{\beta\mu,\beta p_0}\delta(p^2-m^2)+\hbar Q\beta(\omega\cdot E)\int d^4p\frac{1}{3p_0^2}\bar{p}^2V'_{\beta p_0}\delta(p^2-m^2)\\
&\quad-\hbar Q\beta^2(\omega\cdot E)\int d^4p\frac{1}{3p_0}\bar{p}^2V''_{\beta p_0,\beta p_0}\delta(p^2-m^2),
\end{aligned}
\tag{A4}
$$

where we have used $\delta'(x)=-\delta(x)/x$, $\delta'(p^2-m^2)\equiv d\delta(p^2-m^2)/dp_0^2$, $\partial_\mu\mu=-QE_\mu$, and

$$
\begin{aligned}
\partial_\mu V &= \beta[(\partial_\mu\mu)V'_{\beta\mu}+(\partial_\mu u_\rho)p^\rho V'_{\beta p_0}],\\
\frac{d}{dp_0}(\partial_\mu V) &= \frac{d}{dp_0}[\partial_\mu(\beta\mu)V'_{\beta\mu}+\partial_\mu(\beta u_\rho)p^\rho V'_{\beta p_0}]\\
&= \beta\partial_\mu(\beta\mu)V''_{\beta\mu,\beta p_0}+\partial_\mu(\beta u_\rho)u^\rho V'_{\beta p_0}+\beta\partial_\mu(\beta u_\rho)p^\rho V''_{\beta p_0,\beta p_0}\\
&= \beta^2[(\partial_\mu\mu)V''_{\beta\mu,\beta p_0}+(\partial_\mu u_\rho)p^\rho V''_{\beta p_0,\beta p_0}],\\
(\partial_\mu u_\lambda)\tilde{F}^{\mu\lambda} &= \frac{1}{2}\epsilon^{\mu\lambda\alpha\beta}(\partial_\mu u_\lambda)F_{\alpha\beta}\\
&= \frac{1}{2}\epsilon^{\mu\lambda\alpha\beta}(\partial_\mu u_\lambda)(E_\alpha u_\beta-E_\beta u_\alpha+\epsilon_{\alpha\beta\rho\sigma}u^\rho B^\sigma)\\
&= 2\omega\cdot E+\frac{1}{2}\epsilon^{\mu\lambda\alpha\beta}\epsilon_{\alpha\beta\rho\sigma}(\partial_\mu u_\lambda)u^\rho B^\sigma\\
&= 2\omega\cdot E-(\partial_\rho u_\sigma-\partial_\sigma u_\rho)u^\rho B^\sigma\\
&= 2\omega\cdot E.
\end{aligned}
\tag{A5}
$$

We can add I_ω and I_F from Eqs. (A1) and (A4) to obtain the right-hand side of Eq. (16), which we denote as $I=I_\omega+I_F=I_{E\cdot B}+I_{E\cdot\omega}$ where $I_{E\cdot B}$ and $I_{E\cdot\omega}$ denote the $E\cdot B$ and $E\cdot\omega$ terms respectively. We now work on $I_{E\cdot B}$ and $I_{E\cdot\omega}$. We now evaluate $I_{E\cdot B}$ as

$$
\begin{aligned}
I_{E\cdot B} &= -\frac{1}{2}\hbar Q^2\beta^2(E\cdot B)\int d^4p\,V''_{\beta p_0,\beta p_0}\delta(p^2-m^2)\\
&= \hbar Q^2\beta^2(E\cdot B)\int\frac{d^3p}{(2\pi)^3}\frac{1}{E_p}\frac{d}{d(\beta E_p)}\{f_{\mathrm{FD}}(E_p-\mu)[1-f_{\mathrm{FD}}(E_p-\mu)]+f_{\mathrm{FD}}(E_p+\mu)[1-f_{\mathrm{FD}}(E_p+\mu)]\}\\
&= -\hbar Q^2\beta(E\cdot B)\int\frac{d^3p}{(2\pi)^3}\frac{1}{E_p^2-m^2}\{f_{\mathrm{FD}}(E_p-\mu)[1-f_{\mathrm{FD}}(E_p-\mu)]+f_{\mathrm{FD}}(E_p+\mu)[1-f_{\mathrm{FD}}(E_p+\mu)]\}\\
&= -\frac{1}{2\pi^2}\hbar Q^2(E\cdot B)C_1(\beta m,\beta\mu),
\end{aligned}
\tag{A6}
$$

where we have used $d^3p=dE_pE_p\sqrt{E_p^2-m^2}$ and $C_1(\beta m,\beta\mu)$ is given in Eq. (18). Then the result of $I_{E\cdot\omega}$ is

$$
\begin{aligned}
I_{E\cdot\omega} &= \frac{1}{2}\hbar\beta Q(E\cdot\omega)\int d^4p\left[\frac{2\bar{p}^2}{3p_0^2}V'_{\beta p_0}-\beta\left(p_0+\frac{2\bar{p}^2}{3p_0}\right)V''_{\beta p_0,\beta p_0}\right]\delta(p^2-m^2)\\
&= \frac{2}{3}\hbar\beta Q(E\cdot\omega)\int\frac{d^3p}{(2\pi)^3E_p}\left(1-\frac{m^2}{E_p^2}\right)\{f_{\mathrm{FD}}(E_p-\mu)[1-f_{\mathrm{FD}}(E_p-\mu)]-f_{\mathrm{FD}}(E_p+\mu)[1-f_{\mathrm{FD}}(E_p+\mu)]\}\\
&\quad+\frac{1}{3}\hbar\beta^2Q(E\cdot\omega)\int\frac{d^3p}{(2\pi)^3}\left(1+\frac{2m^2}{E_p^2}\right)\{[f_{\mathrm{FD}}(E_p-\mu)][1-f_{\mathrm{FD}}(E_p-\mu)][2f_{\mathrm{FD}}(E_p-\mu)-1]\\
&\quad-[f_{\mathrm{FD}}(E_p+\mu)][1-f_{\mathrm{FD}}(E_p+\mu)][2f_{\mathrm{FD}}(E_p+\mu)-1]\}.
\end{aligned}
\tag{A7}
$$

The last term in Eq. (A7) can be further simplified by using the formula and the integral by part for E_p,

$$\frac{d}{d(\beta x)}\{f_{\mathrm{FD}}(x)[1-f_{\mathrm{FD}}(x)]\} = f_{\mathrm{FD}}(x)[1-f_{\mathrm{FD}}(x)][2f_{\mathrm{FD}}(x)-1],$$
$$\frac{d}{dE_p}\left[E_p\sqrt{E_p^2-m^2}\left(1+\frac{2m^2}{E_p^2}\right)\right] = E_p\sqrt{E_p^2-m^2}\left[\left(\frac{1}{E_p}+\frac{E_p}{E_p^2-m^2}\right)\left(1+\frac{2m^2}{E_p^2}\right)-4\frac{m^2}{E_p^3}\right]. \tag{A8}$$

Then Eq. (A7) can be further simplified as

$$I_{E\cdot\omega} = -\hbar\beta Q(E\cdot\omega)\int\frac{d^3p}{(2\pi)^3E_p}\cdot\frac{m^2}{E_p^2-m^2}\{f_{\mathrm{FD}}(E_p-\mu)[1-f_{\mathrm{FD}}(E_p-\mu)]-f_{\mathrm{FD}}(E_p+\mu)[1-f_{\mathrm{FD}}(E_p+\mu)]\}$$
$$= -\frac{m^2}{2\pi^2}\hbar Q\beta(E\cdot\omega)C_2(\beta m,\beta\mu), \tag{A9}$$

where $C_2(\beta m,\beta\mu)$ is given by Eq. (18). The right-hand side of Eq. (15) is just $I_{E\cdot B}+I_{E\cdot\omega}$, where $I_{E\cdot B}$ and $I_{E\cdot\omega}$ are in Eqs. (A6) and (A9) respectively, which finally gives Eq. (17).

APPENDIX B: SMALL MASS EXPANSION OF $C_1(\beta m,\beta\mu)$ AND $C_2(\beta m,\beta\mu)$

In this appendix, we expand $C_1(\beta m,\beta\mu)$ and $C_2(\beta m,\beta\mu)$ in small βm. For simplicity of notations, we use new variables $\bar\mu\equiv\beta\mu$, $\bar m\equiv\beta m$ and define two dimensionless functions

$$h(\bar m,\bar\mu) = \int_0^\infty dx\frac{1}{e^{\sqrt{x^2+\bar m^2}+\bar\mu}+1},$$
$$g(\bar m,\bar\mu) = \int_0^\infty dx\frac{1}{\sqrt{x^2+\bar m^2}}\frac{1}{e^{\sqrt{x^2+\bar m^2}+\bar\mu}+1}. \tag{B1}$$

We can express $C_1(\bar m,\bar\mu)$ and $C_2(\bar m,\bar\mu)$ in terms of $h(\bar m,\bar\mu)$ and $g(\bar m,\bar\mu)$,

$$C_1(\bar m,\bar\mu) = \frac{d}{d\bar\mu}[h(\bar m,-\bar\mu)-h(\bar m,\bar\mu)],$$
$$C_2(\bar m,\bar\mu) = \frac{d}{d\bar\mu}[g(\bar m,-\bar\mu)+g(\bar m,\bar\mu)]. \tag{B2}$$

The Fermi-Dirac distribution can be expanded as

$$\frac{1}{e^{\sqrt{x^2+\bar m^2}+\bar\mu}+1} = \sum_{n=1}^\infty(-1)^{n+1}e^{-n(\sqrt{x^2+\bar m^2}+\bar\mu)}, \tag{B3}$$

and then we have

$$h(\bar m,\bar\mu) = \sum_{n=1}^\infty(-1)^{n+1}e^{-n\bar\mu}\int_0^\infty dx e^{-n\sqrt{x^2+\bar m^2}}$$
$$= \bar m\sum_{n=1}^\infty(-1)^{n+1}e^{-n\bar\mu}K_1(n\bar m),$$
$$g(\bar m,\bar\mu) = \sum_{n=1}^\infty(-1)^{n+1}e^{-n\bar\mu}\int_0^\infty dx\frac{1}{\sqrt{x^2+\bar m^2}}e^{-n\sqrt{x^2+\bar m^2}}$$
$$= \sum_{n=1}^\infty(-1)^{n+1}e^{-n\bar\mu}K_0(n\bar m), \tag{B4}$$

where $K_0(x)$ and $K_1(x)$ are modified Bessel functions of the second kind whose expansion forms are

$$K_0(x) = -\left(\ln\frac{x}{2}+\gamma\right)\sum_{k=0}^\infty\frac{1}{(k!)^2}\left(\frac{x}{2}\right)^{2k}+\sum_{k=1}^\infty\frac{1}{(k!)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)\left(\frac{x}{2}\right)^{2k},$$
$$K_1(x) = \frac{1}{x}+\left(\ln\frac{x}{2}+\gamma\right)\sum_{k=0}^\infty\frac{1}{k!(k+1)!}\left(\frac{x}{2}\right)^{2k+1}-\frac{1}{2}\sum_{k=0}^\infty\frac{1}{k!(k+1)!}\left(1+\frac{1}{2}+\cdots+\frac{1}{k+1}\right)\left(\frac{x}{2}\right)^{2k+1}$$
$$-\frac{1}{2}\sum_{k=0}^\infty\frac{1}{(k+1)!(k+2)!}\left(1+\frac{1}{2}+\cdots+\frac{1}{k+1}\right)\left(\frac{x}{2}\right)^{2k+3}, \tag{B5}$$

with the Euler constant $\gamma = 0.577\ldots$.

Then one can verify that $h(\bar m,\bar\mu)$ can be cast into the following form

$$h(\bar{m},\bar{\mu}) = \phi(\bar{\mu}) - \bar{m}\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}\left(\frac{\bar{m}}{2}\frac{d}{d\bar{\mu}}\right)^{2k+1}\left[\chi(\bar{\mu}) + \frac{1}{e^{\bar{\mu}}+1}\ln\frac{\bar{m}e^{\gamma}}{2}\right] + \frac{\bar{m}}{2}\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}\left(1+\frac{1}{2}+\cdots+\frac{1}{k+1}\right)\left(\frac{\bar{m}}{2}\frac{d}{d\bar{\mu}}\right)^{2k+1}\frac{1}{e^{\bar{\mu}}+1} + \frac{\bar{m}}{2}\sum_{k=0}^{\infty}\frac{1}{(k+1)!(k+2)!}\left(1+\frac{1}{2}+\cdots+\frac{1}{k+1}\right)\left(\frac{\bar{m}}{2}\frac{d}{d\bar{\mu}}\right)^{2k+3}\frac{1}{e^{\bar{\mu}}+1}, \tag{B6}$$

where we have defined two series

$$\chi(\bar{\mu}) = \sum_{n=1}^{\infty}(-1)^{n+1}e^{-n\bar{\mu}}\ln n, \qquad \phi(\bar{\mu}) = \sum_{n=1}^{\infty}(-1)^{n+1}e^{-n\bar{\mu}}\frac{1}{n}. \tag{B7}$$

Obviously we have following relation for $\phi(\bar{\mu})$,

$$\frac{d}{d\bar{\mu}}\phi(\bar{\mu}) = -\sum_{n=1}^{\infty}(-1)^{n+1}e^{-n\bar{\mu}} = -\frac{1}{e^{\bar{\mu}}+1}, \qquad \frac{d}{d\bar{\mu}}\phi(-\bar{\mu}) = -\frac{d}{d(-\bar{\mu})}\phi(-\bar{\mu}) = \frac{e^{\bar{\mu}}}{e^{\bar{\mu}}+1}. \tag{B8}$$

Then we obtain

$$h(\bar{m},-\bar{\mu}) - h(\bar{m},\bar{\mu}) = \phi(-\bar{\mu}) - \phi(\bar{\mu}) + \bar{m}I_1\left(\bar{m}\frac{d}{d\bar{\mu}}\right)[\chi(\bar{\mu}) + \chi(-\bar{\mu})],$$
$$\frac{d}{d\bar{\mu}}[h(\bar{m},-\bar{\mu}) - h(\bar{m},\bar{\mu})] = 1 + \bar{m}I_1\left(\bar{m}\frac{d}{d\bar{\mu}}\right)\frac{d}{d\bar{\mu}}[\chi(\bar{\mu}) + \chi(-\bar{\mu})], \tag{B9}$$

with the modified Bessel function $I_1(x)$ of the first kind given by

$$I_1(x) = \sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}\left(\frac{x}{2}\right)^{2k+1}. \tag{B10}$$

We can also treat $g(\bar{m},\bar{\mu})$ in the same way

$$g(\bar{m},\bar{\mu}) = -\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{\bar{m}}{2}\frac{d}{d\bar{\mu}}\right)^{2k}\left[\chi(\bar{\mu}) + \frac{1}{e^{\bar{\mu}}+1}\ln\frac{\bar{m}e^{\gamma}}{2}\right] + \sum_{k=1}^{\infty}\frac{1}{(k!)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)\left(\frac{\bar{m}}{2}\frac{d}{d\bar{\mu}}\right)^{2k}\frac{1}{e^{\bar{\mu}}+1}, \tag{B11}$$

and we obtain

$$g(\bar{m},-\bar{\mu}) + g(\bar{m},\bar{\mu}) = -\ln\frac{\bar{m}e^{\gamma}}{2} - I_0\left(\bar{m}\frac{d}{d\bar{\mu}}\right)[\chi(\bar{\mu}) + \chi(-\bar{\mu})],$$
$$\frac{d}{d\bar{\mu}}[g(\bar{m},-\bar{\mu}) + g(\bar{m},\bar{\mu})] = -I_0\left(\bar{m}\frac{d}{d\bar{\mu}}\right)\frac{d}{d\bar{\mu}}[\chi(\bar{\mu}) + \chi(-\bar{\mu})], \tag{B12}$$

with the modified Bessel function $I_0(x)$ of the first kind given by

$$I_0(x) = \sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{x}{2}\right)^{2k}. \tag{B13}$$

Finally we obtain the expressions for $C_1(\bar{m},\bar{\mu})$ and $C_2(\bar{m},\bar{\mu})$,

$$C_1(\bar{m},\bar{\mu}) = 1 + \bar{m}I_1\left(\bar{m}\frac{d}{d\bar{\mu}}\right)\frac{d}{d\bar{\mu}}[\chi(\bar{\mu}) + \chi(-\bar{\mu})],$$
$$C_2(\bar{m},\bar{\mu}) = -I_0\left(\bar{m}\frac{d}{d\bar{\mu}}\right)\frac{d}{d\bar{\mu}}[\chi(\bar{\mu}) + \chi(-\bar{\mu})]. \tag{B14}$$

The series $\chi(\bar{\mu})$ can also be written in the form,

$$\chi(\bar{\mu}) = \sum_{n=1}^{\infty}(-1)^{n+1}e^{-n\bar{\mu}}\ln n = \left[\frac{\partial}{\partial y}\sum_{n=1}^{\infty}\frac{(-e^{-\bar{\mu}})^n}{n^y}\right]_{y=0} = \left[\frac{\partial}{\partial y}\mathrm{Li}(y,-e^{-\bar{\mu}})\right]_{y=0}, \tag{B15}$$

where $\mathrm{Li}(y,z)$ is the polylogarithm function,

$$\mathrm{Li}(y,z) = \sum_{n=1}^{\infty}\frac{z^n}{n^y}, \qquad |z| < 1. \tag{B16}$$

The analytic continuation of the polylogarithm beyond the circle of convergence $|z| < 1$ can be furnished by the following integral representation,

$$\mathrm{Li}(y,z) = \frac{1}{\Gamma(y)}\int_0^{\infty}\frac{t^{y-1}}{z^{-1}e^t - 1}dt, \qquad \mathrm{Re}(y) > 0, \qquad z \in \mathbb{C}/[1,\infty), \tag{B17}$$

with the Gamma function $\Gamma(y)$ is defined by

$$\Gamma(y) = \int_0^\infty e^{-t} t^{y-1} dt, \qquad \mathrm{Re}(y) > 0. \tag{B18}$$

The asymptotic behaviors of $I_0(x)$ and $I_1(x)$ at $x \to 0$ are $I_0(x) \approx 1$ and $I_1(x) \approx x/2$. So at $\bar{m} \to 0$, we obtain the asymptotic values of $C_1(\bar{m}, \bar{\mu})$ and $C_2(\bar{m}, \bar{\mu})$,

$$C_1(\bar{m}, \bar{\mu}) \approx 1 + \frac{\bar{m}^2}{2} \frac{d^2}{d\bar{\mu}^2} \frac{d}{dy} [\mathrm{Li}(y, -e^{-\bar{\mu}}) + \mathrm{Li}(y, -e^{\bar{\mu}})]_{y=0},$$
$$C_2(\bar{m}, \bar{\mu}) \approx -\frac{d}{d\bar{\mu}} \frac{d}{dy} [\mathrm{Li}(y, -e^{-\bar{\mu}}) + \mathrm{Li}(y, -e^{\bar{\mu}})]_{y=0}. \tag{B19}$$

APPENDIX C: DERIVATION OF EQ. (20)

In this appendix, we give a detailed derivation of Eq. (20). From the definition of the Wigner function (3) and that of the axial vector component, we obtain

$$\begin{aligned} QF_{\mu\nu} \int d^4 p \partial_p^\nu \mathcal{A}^\mu &= QF_{\mu\nu} \int d^4 p \partial_p^\nu \int \frac{d^4 y}{(2\pi)^4} e^{-ip\cdot y} \left\langle \bar{\psi}\left(x + \frac{1}{2}y\right) \gamma^\mu \gamma^5 \mathrm{P}U\left(G, x + \frac{1}{2}y, x - \frac{1}{2}y\right) \psi\left(x - \frac{1}{2}y\right) \right\rangle \\ &= QF_{\mu\nu} \int d^4 y (-iy^\nu) \delta^{(4)}(y) \left\langle \bar{\psi}\left(x + \frac{1}{2}y\right) \gamma^\mu \gamma^5 \mathrm{P}U\left(G, x + \frac{1}{2}y, x - \frac{1}{2}y\right) \psi\left(x - \frac{1}{2}y\right) \right\rangle \\ &= -iQF_{\mu\nu} \left(\lim_{y\to 0} y^\nu \left\langle \bar{\psi}\left(x + \frac{y}{2}\right) \gamma^\mu \gamma^5 \psi\left(x - \frac{y}{2}\right) \right\rangle \right) \\ &= -\frac{Q^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \\ &= -\frac{Q^2}{2\pi^2} B \cdot E \end{aligned} \tag{C1}$$

where we have used $4B \cdot E = F_{\mu\nu}\tilde{F}^{\mu\nu}$, $\lim_{y\to 0} \mathrm{P}U(G, x + \frac{1}{2}y, x - \frac{1}{2}y) = 1$ and [62]

$$\begin{aligned} \lim_{y\to 0} \bar{\psi}\left(x + \frac{y}{2}\right) \gamma^\mu \gamma^5 \psi\left(x - \frac{y}{2}\right) &= -\frac{i}{4\pi^2} Q \epsilon^{\alpha\beta\mu\rho} F_{\alpha\beta} \lim_{y\to 0} \frac{y_\rho}{y^2}, \\ \lim_{y\to 0} \frac{y^\nu y^\rho}{y^2} &= \frac{1}{4} g^{\nu\rho}. \end{aligned} \tag{C2}$$

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