PHYSICAL REVIEW D **95,** 014002 (2017)

Comparative analysis of analytical solutions for $F_2^P(x,t)$ in the DGLAP approach

D. K. Choudhury[1,2] and Neelakshi N. K. Borah[1,*]

[1]*Department of Physics, Gauhati University, Guwahati 781014, India*
[2]*Centre for Theoretical Physics, Pandu College, Guwahati 781012, India*

(Received 3 August 2016; published 4 January 2017)

Coupled Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equations involving singlet quark and gluon distributions are explored by a Taylor expansion at small x as two first-order partial differential equations in two variables: Bjorken x and t ($t = ln\frac{Q^2}{\Lambda^2}$). The system of equations are then solved by Lagrange's method and the method of characteristics. We obtain the proton structure function $F_2^P(x,t)$ by combining the corresponding nonsinglet and singlet structure functions with both methods. Analytical solutions for $F_2^P(x,t)$ thus obtained are compared with the recent data published by the H1 and ZEUS Collaborations as well as with NNPDF3.0 parametrization, and their compatibility is checked. Comparative analysis favors the analytical solution by Lagrange's method; the plausible reasons behind that are also discussed.

DOI: 10.1103/PhysRevD.95.014002

I. INTRODUCTION

In QCD, structure functions are defined as convolution of the universal parton momentum distributions inside the proton and coefficient functions, which contain information about the boson-parton interaction. At large momentum transfers Q^2 and not too small x, where x is the fraction of proton momentum carried by the parton, QCD allows the perturbative calculation of the coefficient functions and predicts a logarithmic dependence (evolution) of the proton structure functions with Q^2 to higher orders in α_S. Thus, measurements of structure functions allow perturbative QCD to be precisely tested. Traditionally the standard and basic tools for the theoretical investigation of Deep Inelastic Scattering structure functions are the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) [1–4] evolution equations.

The solutions of the DGLAP equation for the QCD evolution of parton distribution functions have been discussed considerably over the past years. The standard program to study the Q^2 evolution of these quark and gluon densities consists of the numerical solutions of the DGLAP equations. Methods used to solve DGLAP evolution equation include Mellin moment space [5,6] with subsequent inversion, the brute force method [7,8], the Laguerre method [9], the matrix method [10] etc. However, a few shortcomings were determined to be common to such approaches, e.g., the computing time required and decreasing accuracy for $x \to 0$.

Understanding the behavior of quarks and gluons carrying a very small fraction of the proton's momentum, i.e., the so-called small-x kinematic region, is also interesting both theoretically and phenomenologically. As an alternative to the numerical solution, there exists an alternative simpler analysis exclusively for the small-x region, yielding analytical solutions of the DGLAP equations. Some approximated analytical solutions of DGLAP evolution equations suitable at small x have been well discussed in the recent past with considerable phenomenological success [11–24]. The application of a Taylor approximation valid at small x allows one to convert the integro-differential DGLAP evolution equation into a partial differential equation (PDE). Although the conventional knowledge on DGLAP evolution does not favor such an approach, such a possibility was reported sometime previously, in Ref. [15]. An approximate solution of the DGLAP equation by using a Taylor expansion is presented and the x distribution of the deuteron structure function is calculated at small x in [15]. The main reason for using this approach [16–20,22,25] is that, the DGLAP equation is converted into a first-order differential equation in two variables, and later can be solved by adopting either Lagrange's method [26] or the method of characteristics [27,28]. The DGLAP predictions obtained by both these methods yield outcomes for both t and x evolution. Although these methods are theoretically sound, a relative study of them by comparing with the exact solution of the evolution equation and with experimental data is of significant importance, and we can understand their exact region of compatibility in x and Q^2.

In this paper, we present a comparative study of the above-mentioned analytical methods in the context of the proton structure function $F_2^P(x,t)$. We solve the integro-differential equation for the quark and gluon distribution functions in leading order (LO) and derive the analytical solutions for proton structure function $F_2^P(x,t)$ as the sum of a flavor nonsinglet F_2^{NS} and a flavor singlet F_2^S distribution. The analytical predictions for $F_2^P(x,t)$ are compared with experimental data [29,30] and recent

*nishi_indr@yahoo.co.in

NNPDF3.0 parametrization [31] to find out their relative merits as well as the validity in particular (x, Q^2) regions.

This paper is organized as follows. In Sec. II we describe the application and the main features of the analytical methods and thus obtain the results for $F_2^S(x,t)$. In Sec. III we discuss our notation relating the gluon and singlet structure function. In Sec. IV we give a detailed comparison of our results for $F_2^P(x,t)$ with data as well as with the numerical parametrization. Finally, Sec. V contains our conclusions.

II. FORMALISM

A. Singlet coupled DGLAP equations in Taylor-approximated form

The coupled DGLAP equations for quark singlet $[\Sigma(x,Q^2)]$ and gluon $[G(x,Q^2)]$ densities in the standard form are [1–4]

$$\frac{\partial}{\partial \ln Q^2}\begin{pmatrix}\Sigma(x,Q^2)\\ G(x,Q^2)\end{pmatrix} = \frac{\alpha_s(Q^2)}{2\pi}\begin{pmatrix}P_{qq} & P_{qg}\\ P_{gq} & P_{gg}\end{pmatrix} \otimes \begin{pmatrix}\Sigma(x,Q^2)\\ G(x,Q^2)\end{pmatrix}, \tag{1}$$

where $\alpha_s(Q^2)$ is the strong coupling constant, P_{qq}, P_{qg}, P_{gq}, and P_{gg} are the splitting functions, the symbol $\otimes$ stands for the usual Mellin convolution, and the notation is defined as

$$a(x)\otimes f(x) = \int_x^1 \frac{dy}{y} a(y) f\left(\frac{x}{y}\right). \tag{2}$$

Introducing the variable $t = \ln\frac{Q^2}{\Lambda^2}$ and using the explicit forms of the splitting functions in LO [32], the evolution equation for singlet distribution can be written as

$$\frac{\partial F_2^S(x,t)}{\partial t} - \frac{A_f}{t}[\{3+4\ln(1-x)\}F_2^S(x,t) + I_1^S(x,t) + I_1^G(x,t)] = 0, \tag{3}$$

where

$$I_1^S(x,t) = 2\int_x^1 \frac{dz}{1-z}\left[(1+z^2)F_2^S\left(\frac{x}{z},t\right) - 2F_2^S(x,t)\right] \tag{4}$$

$$I_1^G(x,t) = \frac{3}{2}n_f\int_x^1 dz[z^2+(1-z)^2]G\left(\frac{x}{z},t\right). \tag{5}$$

Here $A_f = \frac{4}{3\beta_0}$, $\beta_0 = 11 - \frac{2}{3}n_f$, n_f being the number of flavors considered, and $\alpha_s(t) = \frac{4\pi}{\beta_0 t}$. $F_2^S(x,t)$ is the singlet structure function of the proton. To carry out the integrations in Eqs. (4) and (5), we introduce the variable u defined as $u = 1-z$ and expand the argument $\frac{x}{z}$ as a series,

$$\frac{x}{z} = \frac{x}{1-u} = x\sum_{k=0}^{\infty}u^k = x + x\sum_{k=1}^{\infty}u^k. \tag{6}$$

Since $x < z < 1$, so $0 < u < 1-x$; hence, the series is convergent for $|u| < 1$ and we can use a Taylor expansion of $F_2^S(\frac{x}{z},t)$ and $G(\frac{x}{z},t)$ in an approximated form. As x is small in our region of discussion, the terms containing x^2 and higher powers of x can be neglected as those terms are still smaller. Thus, we can rewrite,

$$F_2^S\left(\frac{x}{z},t\right) \approx F_2^S(x,t) + x\sum_{k=1}^{\infty}u^k\frac{\partial F_2^S(x,t)}{\partial x} \tag{7}$$

$$G\left(\frac{x}{z},t\right) \approx G(x,t) + x\sum_{k=1}^{\infty}u^k\frac{\partial G(x,t)}{\partial x}. \tag{8}$$

Using the above Eqs. (7) and (8), we carry out the integrations in z in Eqs. (4) and (5). Neglecting terms $\mathcal{O}(x^2)$, which is justified at small x, we get

$$I_1^S(x,t) \approx (2x-3)F_2^S(x,t) + \left(x + 2x\ln\frac{1}{x}\right)\frac{\partial F_2^S(x,t)}{\partial x}, \tag{9}$$

$$I_1^G(x,t) \approx n_f\left(1-\frac{3}{2}x\right)G(x,t) - \frac{n_f}{2}\left(5x - 3x\ln\frac{1}{x}\right)\frac{\partial G(x,t)}{\partial x}. \tag{10}$$

The exact relation between the gluon distribution $G(x,t) = xg(x,t)$ and singlet quark distribution $F_2^S(x,t) = x\sum_i e_i^2\{q_i(x,t)+\overline{q_i}(x,t)\}$ is not derivable in QCD even in LO. However, simple forms of such a relation are available in the literature to facilitate the analytical solution of coupled DGLAP equations. In Ref. [33], it was assumed that Q^2 dependence of both the distributions are identical. In Ref. [15], on the other hand, the following simple relation was assumed:

$$G(x,Q^2) = k.F_2^S(x,Q^2), \tag{11}$$

where parameter k has to be determined from experiments.

Again, because the input singlet and gluon parametrization taken from global analysis of parton distribution functions, which incorporate different high-precision data, are also functions of x at fixed Q^2, the relation between singlet structure function and gluon parton densities may, hence, be expressed as a function of x [22].

However, a more rigorous analysis by Lopez and Yndurain [34] investigated the behavior of the singlet $F_2^S(x,Q^2)$ and gluon $G(x,Q^2)$ as $x\to 0$. They observed that

$$F_2^S(x,Q^2)_{x\to 0} = B_S(Q^2)x^{-\lambda_S} \tag{12}$$

$$G(x,Q^2)_{x\to 0} = B_G(Q^2)x^{-\lambda_G}, \tag{13}$$

where B_S and B_G are Q^2 dependent as $\lambda_G = \lambda_S$ and λ_S is strictly positive. Thus,

$$\frac{G(x,Q^2)}{F(x,Q^2)_{x\to 0}} \simeq f(Q^2). \tag{14}$$

This suggests a more general form [35] following

$$G(x,Q^2) = K(Q^2)F_2^S(x,Q^2) \tag{15}$$

rather than Eq. (11).

Now, using Eqs. (9) and (10) and the above relation given by Eq. (15), we can express Eq. (3) in a more precise form as

$$\begin{aligned}&\frac{\partial F_2^S(x,t)}{\partial t} - \frac{A_f}{t}\left[\{3+4\ln(1-x)+(2x-3)\}F_2^S(x,t)\right.\\&\left.+ n_f\left(1-\frac{3}{2}x\right)K(Q^2)F_2^S(x,t)\right]\\&-\frac{A_f}{t}\left[x+2x\ln\frac{1}{x}-\frac{n_f}{2}\left(5x-3x\ln\frac{1}{x}\right)K(Q^2)\right]\\&\times\frac{\partial F_2^S(x,t)}{\partial x} = 0,\end{aligned} \tag{16}$$

which is a partial differential equation for the singlet structure function $F_2^S(x,t)$ with respect to the variables x and t. We solve this PDE [Eq. (16)] with the two formalisms described here, Lagrange's method and the method of characteristics.

B. Solution by Lagrange's auxiliary method

To solve Eq. (16) by Lagrange's auxiliary method [26], we write the equation in the form

$$Q(x,t)\frac{\partial F_2^S(x,t)}{\partial t} + P(x,t)\frac{\partial F_2^S(x,t)}{\partial x} = R(x,t)F_2^S(x,t), \tag{17}$$

where

$$Q(x,t) = t \tag{18}$$

$$P(x,t) = -\frac{4}{3\beta_0}\left\{x+2x\ln\frac{1}{x}-\frac{n_f}{2}\left(5x-3x\ln\frac{1}{x}\right)K(Q^2)\right\} \tag{19}$$

$$\begin{aligned}R(x,t) = \frac{4}{3\beta_0}&\left\{3+4\ln(1-x)+(2x-3)\right.\\&\left.+n_f\left(1-\frac{3}{2}x\right)K(Q^2)\right\}.\end{aligned} \tag{20}$$

The general solution of Eq. (17) is obtained by solving the following auxiliary system of ordinary differential equations:

$$\frac{dx}{P(x,t)} = \frac{dt}{Q(x,t)} = \frac{dF_2^S(x,t)}{R(x,t)F_2^S(x,t)}. \tag{21}$$

If $u(x,t,F_2^S) = C_1$ and $v(x,t,F_2^S) = C_2$ are the two independent solutions of Eq. (17), then, in general, the solution of Eq. (17) is

$$F(u,v) = 0, \tag{22}$$

where F is an arbitrary function of u and v.

In this approach we try to find a specific solution that satisfies some physical conditions on the structure function. Such a solution can be extracted from the combination of u and v linear in F_2^S, the simplest possibility being

$$u + \alpha v = \beta, \tag{23}$$

where α and β are two quantities to be determined from the boundary conditions on F_2^S. Solving Eq. (21), we obtain

$$u(x,t,F_2^S) = tX^S(x) \tag{24}$$

$$v(x,t,F_2^S) = F_2^S(x,t)Y^S(x). \tag{25}$$

The functions $X^S(x)$ and $Y^S(x)$ are defined as

$$X^S(x) = \exp\left[-\int\frac{dx}{P(x,t)}\right] \tag{26}$$

$$Y^S(x) = \exp\left[-\int\frac{R(x,t)}{P(x,t)}dx\right]. \tag{27}$$

The explicit analytical form of $X^S(x)$ under the approximation [at very-small-x region $\log(\frac{1}{x}) \gg x\log(\frac{1}{x}) \gg x$] comes out to be

$$X^S(x) = \exp\left[\frac{6\beta_0}{4(4+3n_fK(Q^2)}\log[\log x]\right]. \tag{28}$$

Using the physically plausible boundary conditions for structure functions, i.e.,

$$F_2^S(x,t) = F_2^S(x,t_0), \quad \text{for } t = t_0 \tag{29}$$

$$F_2^S(1,t) = 0 \quad \text{for any } t, \tag{30}$$

and putting the values of u and v in Eq. (23), we obtain the solution for Eq. (17) as

$$F_2^S(x,t) = F_2^S(x,t_0)\left(\frac{t}{t_0}\right)\frac{[X^S(x) - X^S(1)]}{[X^S(x) - (\frac{t}{t_0})X^S(1)]}. \quad (31)$$

Equation (28) gives us

$$X^S(1) = 0 \quad (32)$$

which yields

$$F_2^S(x,t) = F_2^S(x,t_0)\left(\frac{t}{t_0}\right). \quad (33)$$

Equation (33) gives the Q^2 evolutions of singlet structure function at LO and is the solution for F_2^S by Lagrange's method.

C. Solution by the method of characteristics

The method of characteristics is an alternative technique for solving initial value problems of first-order PDEs. To solve the PDE Eq. (16) by the method of characteristics [27,28], we express it in terms of a new set of coordinates (s,τ), such that Eq. (16) becomes an ordinary differential equation with respect to one of the new variables. We know that most of the important properties of the solution of Eq. (16) depends on the principal part of the equation, i.e., the left-hand side in Eq. (17). This part is actually a total derivative along the solution of the characteristic equation,

$$\frac{dx}{dt} = \frac{P(x,t)}{t}, \quad (34)$$

which gives the characteristic curves of Eq. (16). That is, along the characteristic curve, the partial differential equation becomes an ordinary differential equation.

The characteristic equation, Eq. (34), can be written as

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}, \quad (35)$$

with

$$\frac{dt}{ds} = t \quad (36)$$

$$\frac{dx}{ds} = P(x,t). \quad (37)$$

Using Eq. (34) in Eq. (16), the left-hand side becomes an ordinary derivative with respect to s and the equation becomes an ordinary differential equation,

$$\frac{dF_2^S(s,\tau)}{ds} + c^S(s,\tau)F_2^S(s,\tau) = 0, \quad (38)$$

where

$$c^S(s,\tau) = -R(x,t). \quad (39)$$

Integrating Eq. (38) along the characteristic curve, we obtain the solution for $F_2^S(x,t)$ in (s,τ) space as

$$F_2^S(s,\tau) = F_2^S(\tau)(\frac{t}{t_0})^{n(x,t)}, \quad (40)$$

where

$$n(x,t) = -\frac{4}{3\beta_0}(\xi_1) \quad (41)$$

with

$$\begin{aligned}\xi_1 = {} & 4\log\left(1 - \tau\exp\left[-\left(\frac{t}{t_0}\right)^{\frac{1}{\alpha_1}}\right]\right) \\ & + 2\tau\exp\left[-\left(\frac{t}{t_0}\right)^{\frac{1}{\alpha_1}}\right] \\ & + n_f\left(1 - \frac{3}{2}\tau\exp\left[-\left(\frac{t}{t_0}\right)^{\frac{1}{\alpha_1}}\right]\right)K(Q^2) \end{aligned} \quad (42)$$

and

$$\alpha_1 = \frac{3\beta_0}{4\{2 + K(Q^2)\frac{9}{2}\}}. \quad (43)$$

As per the initial conditions, at $x(s=0) = \tau$ and $t(s=0) = t_0$, we get the input function $F_2^S(\tau) = F_2^S(x,t_0)$. This leads us back to the (x,t) space from the (s,τ) space, and we can express Eq. (40) in a more precise form as

$$F_2^S(x,t) = F_2^S(x,t_0)\left(\frac{t}{t_0}\right)^{n(x,t)}. \quad (44)$$

Equation (44) is the analytical solution for the singlet structure function within the present formalism and it gives the Q^2 evolution.

We observe that Eq. (44) is sensitive to gluon distribution as well as $K(Q^2)$, which is absent in the case of Eq. (33). In case of Lagrange's method, the nonsinglet structure function F_2^{NS} [25] and the singlet structure function F_2^S have identical evolution for the least approximated level. This is possible only when the gluon effect is negligible. This feature has already been well observed [15,21]; the only new observation is that it is true even when $k = K(Q^2)$, i.e., Q^2 dependent in Eq. (11).

Using our results derived in this section, we will calculate the proton structure function $F_2^P(x,t)$ using the relation

$$F_2^P = \frac{3}{18}F_2^{NS} + \frac{5}{18}F_2^S, \quad (45)$$

considering the corresponding analytical predictions for t evolution of the nonsinglet structure function $F_2^{NS}(x,t)$ by

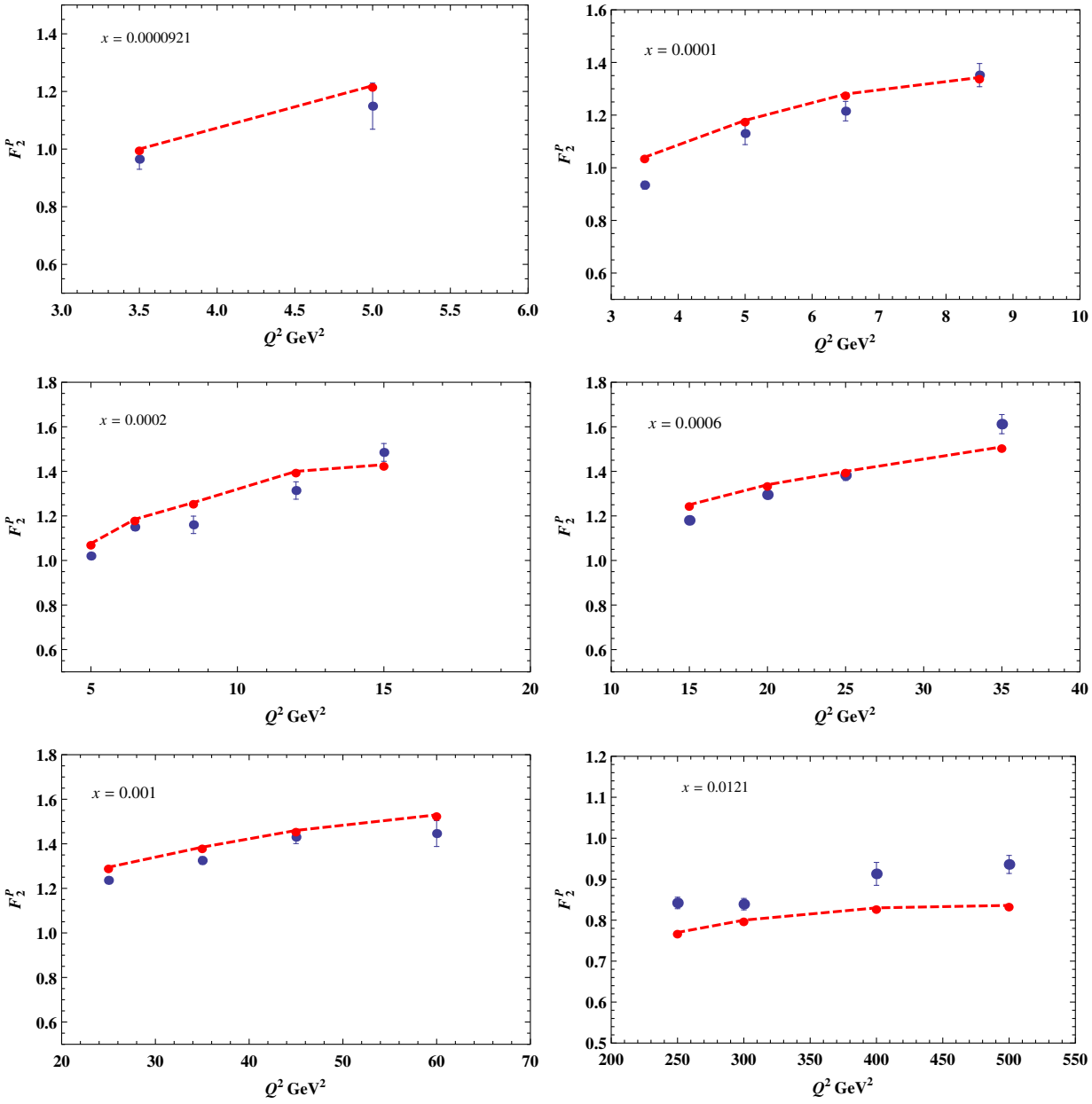

FIG. 1. Proton structure function $F_2^P(x,t)$ as a function of Q^2 for different fixed x values by Lagrange's method with H1 data. Here the dashed line represents our analytical model.

the respective analytical methods at small x from Ref. [25]. We discuss in the next section the phenomenological consequences of our results derived in this section.

III. FORM OF $K(Q^2)$

Let us now discuss the plausible forms of $K(Q^2)$ as defined in Eq. (15) above and discuss the related constraints on the parameters k and σ. The Q^2 dependence of $K(Q^2)$ is not given by the current methods under study but is based on physically plausible reasons.

A. Choice of the form of $K(Q^2)$

The important characteristics of perturbative Quantum Chromo-Dynamics (pQCD) is the $\log Q^2$ dependence, as can be seen from the definition of the running coupling constant, as well as any Q^2 evolution of structure function. A plausible theoretical form for $K(Q^2)$ compatible with perturbative QCD expectation is

$$K(Q^2) = k\left(\log\frac{Q^2}{\Lambda^2}\right)^{\sigma} = kt^{\sigma}, \tag{46}$$

where k and σ are two parameters to be fixed. In the specific case of $\sigma \to 0$, one can always recover the earlier work [33].

B. Reality constraint on the parameters k and σ

The essential condition for our analytical solution for F_2^S, Eq. (44)—obtained by the method of characteristics—to be real is that the exponent $n(x,t)$ has to be real. This imposes a reality condition on ξ_1 which qualitatively leads us to the condition that

$$0 < \tau \exp\left[-\left(\frac{t}{t_0}\right)^{\frac{1}{\alpha_1}}\right] < 1. \tag{47}$$

The new variable τ is dependent on both k and σ, as defined in α_1. The inversely proportionate feature of α_1 on both k and σ gives the realization that k and σ cannot be too large. So, for any value of x and Q^2, the choice of our parameters k and σ is bounded by the above reality condition, and they cannot be treated as free parameters.

IV. RESULTS AND DISCUSSION

The recent HERA Collaboration [29,30] data allows us to explore a wide range of Q^2 evolution, i.e. $1.5 \leq Q^2 \leq 800$ GeV^2 for $F_2^P(x,t)$. Hence, given the two parameters k and σ, we fix them for this entire region of Q^2. The reality condition allows an effective range of values for both k and σ for the considered Q^2 region, the best fitted range for k and σ being $0.001 < k < 1.45$ and $0.001 < \sigma < 0.055$, respectively. It suggests that k is not far away from unity and σ is not far away from zero.

In this work we have calculated the Q^2 evolution for singlet structure functions using two analytical approaches, Lagrange's method and method of characteristics. We have used the relation defined by Eq. (45) to derive the proton structure function $F_2^P(x,t)$. In order to perform a qualitative analysis of small-x behavior of these analytical predictions in context of the proton structure function $F_2^P(x,t)$, we have considered very recent experimental data published by H1 [29] and ZEUS [30] Collaborations and recent NNPDF3.0 [31] parametrization. Remarkable progress in data precision has made it possible to carry out the study in small-enough x, where our methods are approximated to be valid. We have used the LO Martin-Stirling-Thorne-Watt (MSTW) 2008 parametrization [36] at $Q_0^2 = 1$ GeV^2 to evolve our solutions.

In Fig. 1 we display the Q^2 evolution of our analytical solution by Lagrange's method at certain fixed x values with the small-x H1 [29] data within the range $10^{-5} \leq x \leq 10^{-2}$ and $3.5 \leq Q^2 \leq 500$ GeV^2. We have observed very good agreement between the experimental data and our analytical approach for a broad range of small-x values ($10^{-5} \leq x \leq 10^{-3}$) and $Q^2 \leq 60$ GeV^2.

ZEUS [30] has published F_2^P data for very few different kinematic bins of (x, Q^2), within the range $0.00025 \leq x \leq 0.00493$ and $9 \leq Q^2 \leq 110$ GeV^2. Figure 2 demonstrates the comparison of our analytical solution by Lagrange's method with the ZEUS data. However, because of very few data points, it is not possible to analyze the Q^2 evolution at some fixed x value. To that end, we evolved our analytical solution at two extreme x values of the experimental data (uppermost $x = 0.00025$ and lowermost $x = 0.00493$) for the entire Q^2 range and checked if the experimental data for different bins are well within theoretical bound. Here the shaded area describes the evolution of our theoretical solution for each x bin for the entire Q^2 range of $9 \leq Q^2 \leq 110$ GeV^2. It can be seen that our analytical solution for F_2^P well describes the data up to $Q^2 = 80$ GeV^2.

In Fig. 3 we show the Q^2 evolution of our analytical solution for $F_2^P(x,t)$ by the method of characteristics, along with the small-x H1 experimental data within the range $10^{-4} \leq x \leq 10^{-3}$ and $3.5 \leq Q^2 \leq 60$ GeV^2. As can be seen, the evolution of the analytical solution by the method of characteristics for $F_2^P(x,t)$ is not compatible with data; further, we observe that structure function is decreasing with increasing Q^2. This behavior has been observed to be true for other regions of small-x too, as well as for ZEUS data; hence, it is not included in the text here.

Figure 4 shows the Q^2 evolution of the analytical models with the very recent NNPDF3.0 [31] parametrization based on the HERA-II data, for different fixed values of x. We have confined our comparative study within the region $0.00025 \leq x \leq 0.0039$ and $3 \leq Q^2 \leq 85$ GeV^2. Here the vertical error bars represent uncertainties given by the standard deviations and are computed by the added-in quadrature method in our work. We note that our analytical solution for $F_2^P(x,t)$ by Lagrange's method is close to the NNPDF3.0 parametrization for the considered small-x range up to $Q^2 \leq 20$ GeV^2. However, the respective

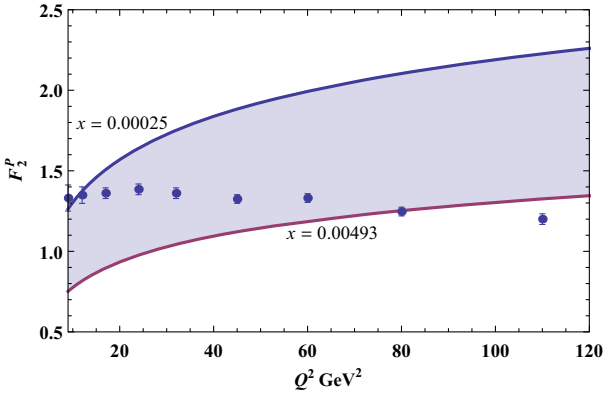

FIG. 2. Proton structure function $F_2^P(x,t)$ as function of Q^2 by Lagrange's method with ZEUS data. Each experimental data point is considered for different bins of x and Q^2. Here the lines represents our analytical model at $x = 0.00025$ and $x = 0.00493$.

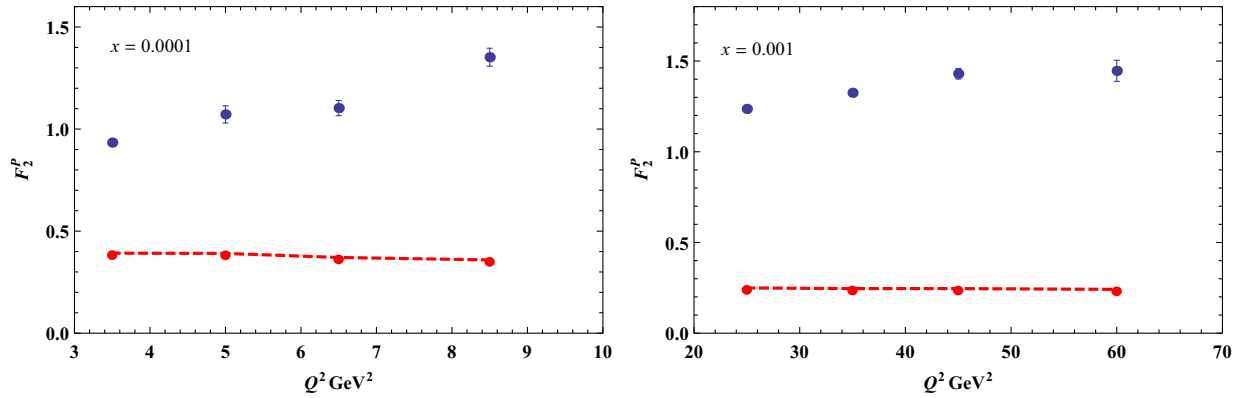

FIG. 3. Proton structure function $F_2^P(x,t)$]Proton structure function $F_2^P(x,t)$ as a function of Q^2 for different fixed x values by the method of characteristics with H1 data. Here the dashed line represents our analytical model.

analytical solution by the method of characteristics for $F_2^P(x,t)$ has not followed the general growth of evolution.

Let us discuss the possible reasons behind the failure of the method of characteristics over Lagrange's method, i.e., whether it is the method of characteristics itself, the choice of the function $K(Q^2)$, or anything else. The difference between Lagrange's method and the method of characteristics is that while Lagrange's method has undetermined parameters α and β, as defined in Eq. (23), the method of characteristics has none. Hence Lagrange's method is more flexible to accommodate data.

From the algebraic structure of the solution of the method of characteristics, given by Eq. (44), the exponent is invariably negative; this is true regardless of the numerical value of k and σ, defined in Eq. (46), so long as $K(Q^2)$ is positive. It itself does not qualitatively conform

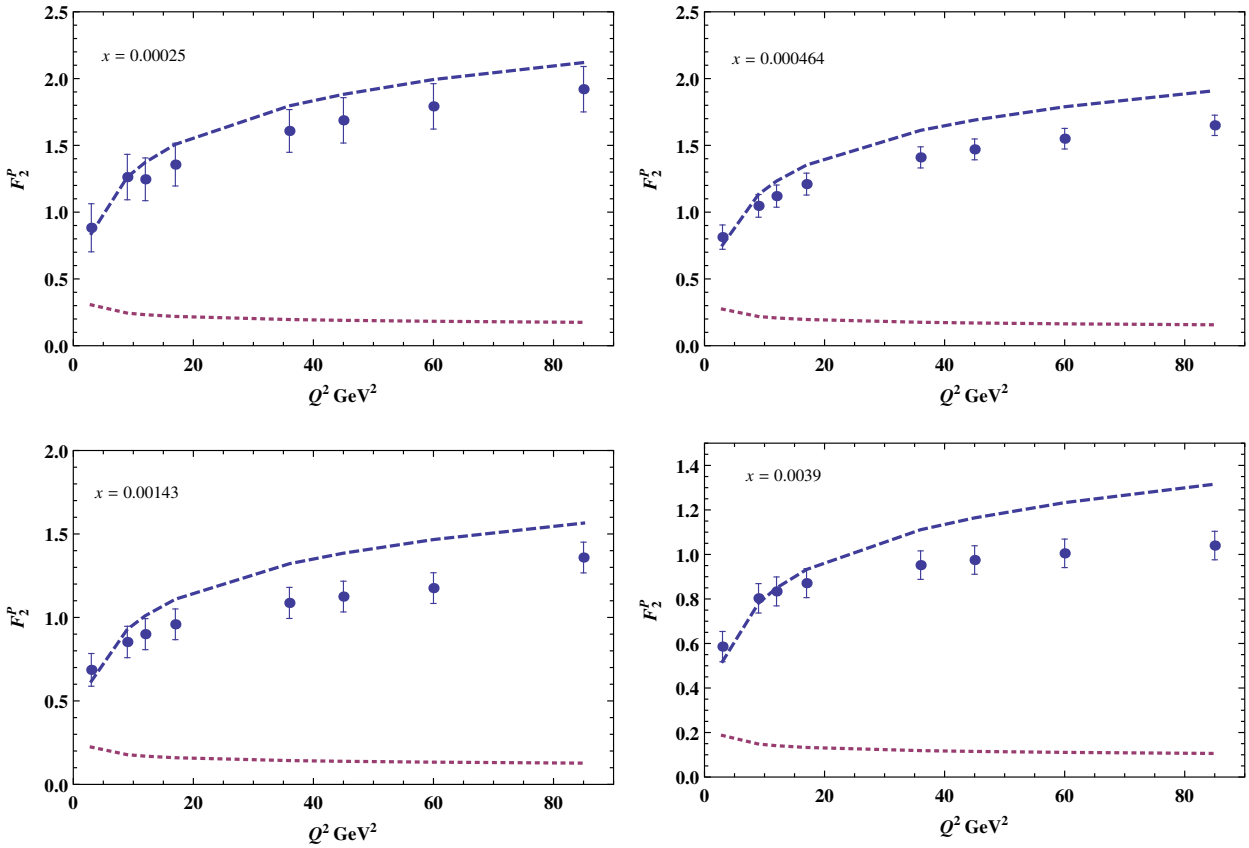

FIG. 4. Proton structure function $F_2^P(x,t)$ as a function of Q^2 for different fixed x values by Lagrange's method and the method of characteristics with NNPDF3.0 parametrization with standard deviation. Here the dashed and dotted lines represent our analytical model by Lagrange's method and the method of characteristics, respectively.

to theoretical expectation as well as data and parametrization. As per pQCD, the prediction structure function should rise with Q^2 in small x [11,37,38] and should fall at high x, a feature absent in the solution by the method of characteristics. In such a situation, the absence of any adjustable parameter from data in the method can itself be considered as theoretical limitation.

One important difference between Lagrange's method and the method of characteristics is that while in the former both singlet F_2^S and nonsinglet F_2^{NS} structure function evolve identically, in the method of characteristics it is not so. Because the effects of k and σ do not appear in the solution by Lagrange's method, it implies that quarks and antiquarks have identical Q^2 evolution, at least in the experimentally accessible region of (x, Q^2) under study. For not-so-large Q^2, it appears to be a reasonable feature of QCD, as the success of Lagrange's method shows. This particular aspect is absent in the method of characteristics.

V. CONCLUSION

In part of the relevant parameter space, the solution to the differential equation governing the proton structure function may be found by analytical means. The corresponding parameter space of the structure function is the x vs Q^2 parameter space. The present paper aims to compare the merits of two such methods: Lagrange's method and the method of characteristics. From our analysis we have observed that method of characteristics does not represent the measured features of proton structure function at small x, for which the DGLAP equations were framed. It is therefore important to find which aspect of the computation is to be blamed for the failure of method of characteristics: is it the method itself or the choice of the additional assumptions assumed during the process of derivation, like a choice of the function $K(Q^2)$ [Eq. (15)]? From our phenomenological analysis we found that the form of Q^2-dependent function $K(Q^2)$ does not seem to play crucial role in the overall qualitative behavior to change the undesirable features of the solution by the method of characteristics, as long as $K(Q^2)$ is positive. From our phenomenological study we also observe that the numerical value of σ, the exponent of $(\frac{t}{t_0})|_{t_0=1\ \mathrm{GeV}^2}$, is not very far away from zero, suggesting that the earlier assumption—i.e., $K(Q^2)$ is a constant, as used in several works [15]—is a reasonable assumption for the Q^2 range under study.

A comparative study of the two methods indicates that Lagrange's method is more flexible, having at least two quantities α and β to be determined by the boundary conditions. Such flexibility does not exist in the case of the method of characteristics. Hence, though the method of characteristics looks theoretically appealing, being parameter free, it is too rigid to accommodate experimentally known features of the structure functions.

As mentioned above, to derive the analytical solutions by Lagrange's method we had to use further assumptions like $[\log(\frac{1}{x}) \gg 1], [\log(\frac{1}{x}) \gg x \log(\frac{1}{x}) \gg x]$, in addition to special conditions like Eq. (28) curtailing the allowed parameter space. Despite all such additional assumptions, we have observed a simple physical picture that has emerged in the solution of unpolarized singlet and nonsinglet structure functions; i.e., both evolve with Q^2 identically. This can be physically realized if quark and antiquark of any flavor evolve identically. The agreement with data suggests that it perhaps approximately represents the behavior of structure functions in the low-x range under study. From the point of view of DGLAP equations, it implies that the effect of the splitting function P_{qg} is negligible compared to P_{qq}. This makes the formalism effectively much simpler. It will be interesting to see whether such effectively simpler formalism emerges at the next-to-LO and next-to-next-to-LO levels as well. Such study is under progress.

[1] Y. L. Dokshitzer, Calculation of the structure functions for deep inelastic scattering and $e+e-$ annihilation by perturbation theory in quantum chromodynamics, Zh. Eksp. Teor. Fiz. **73**, 1216 (1977) [Sov. Phys. JETP **46**, 641 (1977)].

[2] V. N. Gribov and L. N. Lipatov, Deep inelastic electron scattering in perturbation theory, Yad. Fiz. **15**, 781 (1972) [Sov. J. Nucl. Phys. **15**, 438 (1972)].

[3] L. N. Lipatov, The parton model and perturbation theory, Yad. Fiz. **20**, 181 (1974) [Sov. J. Nucl. Phys. **20**, 94 (1975)].

[4] G. Altarelli and G. Parisi, Asymptotic freedom in parton language, Nucl. Phys. **B126**, 298 (1977).

[5] C. Pascaud and F. Zomer, A fast and precise method to solve the Altarelli-Parisi equations in x space, arXiv:hep-ph/0104013.

[6] R. Toldra, A c++ code to solve the DGLAP equations applied to ultrahigh-energy cosmic rays, Comput. Phys. Commun. **143**, 287 (2002).

[7] N. Cabibbo and R. Petronzio, Two stage model of hadron structure: Parton distributions and their Q^2 dependence, Nucl. Phys. **B137**, 395 (1978).

[8] M. Stratmann and W. Vogelsang, Towards a global analysis of polarized parton distributions, Phys. Rev. D **64**, 114007 (2001).

[9] M. Gluck, E. Reya, and A. Vogt, Radiatively generated parton distributions for high-energy collisions, Z. Phys. C **48**, 471 (1990).
[10] P. G. Ratcliffe, A matrix approach to numerical solution of the DGLAP evolution equations, Phys. Rev. D **63**, 116004 (2001).
[11] R. D. Ball and S. Forte, A direct test of perturbative QCD at small x, Phys. Lett. B **336**, 77 (1994).
[12] A. V. Kotikov and G. Parente, Small x behavior of parton distributions with soft initial conditions, Nucl. Phys. **B549**, 242 (1999).
[13] L. Mankiewicz, A. Saalfeld, and T. Weigl, On the analytical approximation to the GLAP evolution at small x and moderate Q^2, Phys. Lett. B **393**, 175 (1997).
[14] A. Yu. Illarionov, A. V. Kotikov, and G. Parente Bermudez, Small x behavior of parton distributions: A study of higher twist effects, Phys. Part. Nucl. **39**, 307 (2008).
[15] J. K. Sarma, D. K. Choudhury, and G. K. Medhi, x distribution of deuteron structure function at low x, Phys. Lett. B **403**, 139 (1997).
[16] R. Baishya and J. K. Sarma, Method of characteristics and solution of DGLAP evolution equation in leading and next to leading order at small x, Phys. Rev. D **74**, 107702 (2006).
[17] R. Baishya, U. Jamil, and J. K. Sarma, Evolution of spin-dependent structure functions from DGLAP equations in leading order and next to leading order, Phys. Rev. D **79**, 034030 (2009).
[18] R. Baishya and J. K. Sarma, Semi numerical solution of non-singlet Dokshitzer-Gribov-Lipatov-Altarelli-Parisi evolution equation up to next-to-next-to-leading order at small x, Eur. Phys. J. C **60**, 585 (2009).
[19] D. K. Choudhury and P. K. Sahariah, A solution of the DGLAP equation for gluon at low x, Pramana J. Phys. **58**, 599 (2002).
[20] D. K. Choudhury and P. K. Dhar, x distribution of non-singlet structure function at low and high x, Indian Journal of chemistry Section A, Inorganic, bio-inorganic, physical, theoretical and analytical chemistry **83**, 1699 (2009).
[21] D. K. Choudhury and J. K. Sarma, Comments on structure functions at low x, Pramana J. Phys. **39**, 273 (1992).
[22] M. Devee, R. Baishya, and J. K. Sarma, Evolution of singlet structure functions from DGLAP equation at next-to-next-to-leading order at small x, Eur. Phys. J. C **72**, 2036 (2012).
[23] G. R. Boroun, Extraction of structure function and gluon distribution function at low x from cross section derivative by Regge behavior, Commun. Theor. Phys. **43**, 283 (2005).
[24] N. Ghahramany and G. R. Boroun, Extraction of the structure function $F2(x, Q^2)$ at low x from the cross-section derivative, Phys. Lett. B **528**, 239 (2002).
[25] N. N. K. Borah, D. K. Choudhury, and P. K. Sahariah, Comparison of analytical solution of DGLAP equations for $F_2^{NS}(x, t)$ at small x by two methods, Adv. High Energy Phys. **2013**, 829803 (2013).
[26] I. N. Sneddon, *Elements of Partial Differential Equations*, Dover Books on Mathematics (Dover Publications, New York, 2006).
[27] S. J. Farlow, *Partial Differential Equations for Scientists and Engineers*, Dover Books on Mathematics (Dover Publications, New York, 2012).
[28] W. E. Williams, *Partial Differential Equations*, Oxford Applied Mathematics and Computing Science Series (Clarendon Press, Oxford, 1980).
[29] V. Andreev *et al.*, Measurement of inclusive ep cross sections at high Q^2 at $\sqrt{s} = 225$ and 252 GeV and of the longitudinal proton structure function F_L at HERA, Eur. Phys. J. C **74**, 2814 (2014).
[30] H. Abramowicz *et al.*, Deep inelastic cross-section measurements at large y with the ZEUS detector at HERA, Phys. Rev. D **90**, 072002 (2014).
[31] R. D. Ball *et al.*, Parton distributions for the LHC Run II, J. High Energy Phys. 04 (2015) 040.
[32] L. F. Abbott, W. B. Atwood, and R. Michael Barnett, A QCD analysis of eN deep inelastic scattering data, Phys. Rev. D **22**, 582 (1980).
[33] D. K. Choudhury and A. Saikia, An improved approximate solution of Altarelli-Parisi equations, Pramana J. Phys. **33**, 359 (1989).
[34] C. Lopez and F. J. Yndurain, Behavior of deep inelastic structure functions near physical region endpoints from QCD, Nucl. Phys. **B171**, 231 (1980).
[35] A. Jahan and D. K. Choudhury, A lower bound on longitudinal structure function at small x from a self-similarity based model of proton, Commun. Theor. Phys. **61**, 654 (2014).
[36] A. D. Martin, W. J. Stirling, R. S. Thorne, and G. Watt, Parton distributions for the LHC, Eur. Phys. J. C **63**, 189 (2009).
[37] F. J. Yndurain, *The Theory of Quark and Gluon Interactions* (Springer, Berlin Heidelberg, 1999).
[38] R. D. Ball and S. Forte, Double asymptotic scaling at HERA, Phys. Lett. B **335**, 77 (1994).