PHYSICAL REVIEW D **94,** 126011 (2016)

Dynamical boundary for anti–de Sitter space

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(Received 22 November 2016; published 20 December 2016)

We argue that a natural boundary condition for gravity in asymptotically anti–de Sitter (AdS) spaces is to hold the *renormalized* boundary stress tensor density fixed, instead of the boundary metric. This leads to a well-defined variational problem, as well as new counterterms and a finite on-shell action. We elaborate this in various (even and odd) dimensions in the language of holographic renormalization. Even though the *form* of the new renormalized action is distinct from the standard one, once the cutoff is taken to infinity, their *values* on classical solutions coincide when the trace anomaly vanishes. For AdS_4, we compute the Arnowitt-Deser-Misner form of this renormalized action and show in detail how the correct thermodynamics of Kerr-AdS black holes emerge. We comment on the possibility of a consistent quantization with our boundary conditions when the boundary is dynamical, and make a connection to the results of Compere and Marolf. The difference between our approach and microcanonical-like ensembles in standard AdS/CFT is emphasized.

DOI: 10.1103/PhysRevD.94.126011

I. INTRODUCTION

Historically, most of the work on boundary conditions in gravity has been in the context of Dirichlet boundary conditions: the Gibbons-Hawking-York (GHY) boundary term [1] was the first boundary term to be identified that made the variational problem for gravity well defined. It also gave a formal yet compelling basis for horizon thermodynamics [2,3]. In the usual AdS/CFT correspondence [4–6], the boundary values of fields on the gravity side are identified as the sources of the fields in the field theory. Thus AdS/CFT correspondence is formulated as a Dirichlet problem as well (on the gravity side).

Recently however a boundary term for gravity [7] has been introduced (see also various related works [8–19]) which is a natural candidate for a Neumann formulation of gravity. Furthermore it was shown [20] that various thermodynamical aspects of gravity can in fact be reproduced using this Neumann boundary term as well. In light of this, in this paper, we will explore gravity in asymptotically anti–de Sitter (AdS) spacetimes with Neumann boundary conditions.

The proposal of Ref. [7] was to treat the Neumann boundary condition as holding the canonical conjugate of the boundary metric[1] fixed. In particle mechanics and field theory, holding the canonical conjugate of the boundary value of the field fixed is identical to the usual Neumann boundary conditions, but in gravity this leads to an alternative to holding the normal derivative[2] of the boundary metric fixed, and leads to a well-defined new boundary term [7]. The translation from Dirichlet to Neumann can be understood as a Legendre transform [20,21].

Typically, to get a finite action on solutions, one has to take care of infrared divergences of the Einstein-Hilbert action in both flat space and in AdS. This is true even with the addition of boundary terms that make the variational problem well defined. In flat space, this was done for the GHY boundary term in Ref. [2] and for the Neumann term in Ref. [20] via appropriate background subtraction procedures. In AdS however, for the Dirichlet problem, there exists a well-defined and quite natural way to get finite actions by the addition of counterterms [22,23], which have a very natural interpretation in the dual field theory as canceling UV divergences. Such counterterms lead to a finite action and a finite (renormalized) stress tensor.

The existence of this finite stress tensor suggests that in AdS, one can define the Neumann variational problem to be one where we hold the renormalized stress tensor density fixed, and one should get a well-defined variational principle and finite Neumann action. We can do this in two ways: we can do this via starting from the renormalized Dirichlet action in AdS (which is well known from, say, Ref. [23]) and do a Legendre transform on the boundary metric, or we can start from a Fefferman-Graham expansion as the definition of asymptotically AdS space, and systematically construct counterterms for the unrenormalized Neumann action by demanding vanishing of divergences. In the next section, we will adopt the latter strategy and write down explicit renormalized Neumann actions in AdS_{d+1} with $d = 2$, 3, 4. Remarkably, we will find in Appendix B that both these approaches yield the same results.

In a later section we will evaluate the finite actions that this leads to on classical (black hole) solutions. We will also

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[1] This turns out to be the boundary stress tensor density.
[2] To the best of our knowledge, a boundary term with the normal derivative at the boundary fixed, is not known for gravity.

find that these actions have the same numerical values as the corresponding Dirichlet actions on these solutions, up to subleading terms that vanish when the radial cutoff is taken to infinity. This is not surprising, because the Legendre transform relating the two actions is of the form

$$S_N^{\text{ren}} = S_D^{\text{ren}} - \int_{\partial\mathcal{M}} \pi^{ij}\gamma_{ij} \tag{1.1}$$

where γ_{ij} is the boundary metric and π^{ij} is its canonical conjugate, and is equal to the renormalized energy-momentum tensor density. This means that $\pi^{ij}\gamma_{ij}$ is proportional to the trace of the boundary stress tensor and so when the conformal anomaly of the boundary theory vanishes, this object is zero on classical solutions.[3]

We will see however that even though the values can be the same (as the cutoff is taken to infinity), the forms of the renormalized Dirichlet and Neumann actions can be quite different. To illustrate this in some detail, we will compute the Arnowitt-Deser-Misner (ADM) version of both these actions. We will also find that comparing the covariant and Hamiltonian ways of evaluating these actions yields the generalized Smarr formula, but in different ways. The covariant-canonical relations and the Smarr formula automatically imply the first law as well [24].

Our results are conservatively thought of merely as a new boundary condition for classical AdS gravity with suitable boundary terms, but we find it plausible that our results go beyond classical. We believe they are indicative of a possibly interesting boundary condition for quantum gravity in AdS. This might seem *a priori* impossible because consistent quantization of fields in AdS requires that they be normalizable (or have finite energy), and except in some windows of masses [21] for scalars (say), it is known that only one boundary condition leads to consistent quantization for fields in a fixed AdS background. We will argue however that this is not quite true: the reason is that the notion of energy in a fixed AdS background is different from that in an AdS background where the metric is dynamical [25]. In particular we will speculate (partly inspired by a result of Compere and Marolf) that it may be possible that our boundary conditions are consistent at the quantum level when the boundary metric in AdS is dynamical: that is, the boundary theory contains dynamical gravity. We leave a conclusive take on this problem for later work.

Holding the boundary metric fixed is the standard way of thinking about AdS familiar from AdS/CFT. To clarify some points which might cause confusion, we conclude by elaborating a little on the choice of ensembles in AdS/CFT. We relegate some of the relevant facts we need (among other things) to the Appendices.

[3]As it happens, since the conformal anomaly is related to the curvatures of the boundary surface, when these curvatures are vanishing, we will see a match for standard black hole solutions between Dirichlet and Neumann also in AdS_{d+1} with even d.

II. HOLOGRAPHIC RENORMALIZATION OF NEUMANN GRAVITY

In this section we will derive the renormalized Neumann action by directly dealing with the Fefferman-Graham (FG) expansion (2.2) and demanding that the action be finite. Typically in Dirichlet theory one imagines that the boundary conditions are set by the leading part of the FG expansion; in our case it is a combination of the g_i's [see Eq. (2.2)] that is getting fixed via the renormalized boundary stress tensor. A standard review is Ref. [26].

A. Regularized action in Fefferman-Graham coordinates

By asymptotically AdS_{d+1} space, in this paper we will mean a metric that solves the Einstein equation with a negative cosmological constant, that can be expressed asymptotically (i.e., as $z \to 0$) by a general Fefferman-Graham expansion given by

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = \frac{l^2}{z^2}(dz^2 + g_{ij}(x,z)dx^i dx^j) \tag{2.1}$$

where

$$g(x,z) = g_0 + z^2 g_2 + \cdots + z^d g_d + z^d \log z^2 h_d + O(z^{d+1}). \tag{2.2}$$

Only even powers of z appear up to $O(z^{[d-1]})$. The log term appears only for even d. In all the discussions that follow, we set $l = 1$. The cosmological constant is related to the AdS radius through the relation $\Lambda = -\frac{d(d-1)}{2l^2}$. Since only even powers appear in the expansion, we introduce a new coordinate $\rho = z^2$ in which the metric takes the form

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho)dx^i dx^j,$$
$$g(x,\rho) = g_0 + \rho g_2 + \cdots + \rho^{d/2} g_d + \rho^{d/2}\log\rho h_d. \tag{2.3}$$

Note that the condition that this metric solves the Einstein equation means that the higher-order $g_{(m)ij}$ can be determined in terms of the lower-order ones, and explicit formulas can be written down for them. We present explicit expressions in Appendix B.

We can compute[4] the Neumann action [7,20] (note that Ref. [7] worked with the bulk dimension, so our $d = D - 1$ in the notation there),

[4]In what follows, γ_{ij} is the induced metric on $\partial\mathcal{M}$ and ε takes values ± 1 depending on whether $\partial\mathcal{M}$ is time-like or space-like respectively. Θ is the trace of the extrinsic curvature of $\partial\mathcal{M}$ which is defined to be $\Theta_{ij} = \nabla_{(a}n_{b)}e^a_i e^b_j$, where n_a is the outward-pointing normal vector and $e^a_i = \frac{\partial x^a}{\partial y^i}$ is the projector arising from the bulk coordinates x^a and the boundary coordinates y^i.

$$S_N = \frac{1}{2\kappa}\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}(R-2\Lambda) - \frac{(d-3)}{2\kappa}\int_{\partial\mathcal{M}} d^d y\sqrt{|\gamma|}\epsilon\Theta \tag{2.4}$$

for Eq. (2.3) and we immediately sees that it diverges. This is not a surprise: the same thing happens for the Dirichlet action as well, and the process of adding counterterms to the Dirichlet action to make it finite is known as holographic renormalization [23]. We can adopt a similar approach here. The first step is to cut off the radial integration at a finite $\rho = \epsilon$, to regulate the action. After this regularization, the Neumann action (2.4) is given by

$$S_N^{\mathrm{reg}} = -\frac{d}{2\kappa}\int d^d x\int_\epsilon d\rho \frac{1}{\rho^{d/2+1}}\sqrt{-g} - \frac{(d-3)}{2\kappa} \times \int d^d x \frac{1}{\rho^{d/2}}(d\sqrt{-g} - 2\rho\partial_\rho\sqrt{-g})\Big|_{\rho=\epsilon}. \tag{2.5}$$

Our goal is to add counterterms so that the Neumann action becomes finite. We will find that this is indeed a natural construction and for standard black hole solutions it leads to the same on-shell action as the Dirichlet theory.

B. AdS$_3$ ($d=2$)

In $d = 2$ the regularized Neumann action takes the form,

$$S_N^{\mathrm{reg}} = -\frac{1}{\kappa}\int d^2x\left[\int_\epsilon d\rho\frac{\sqrt{-g}}{\rho^2} + \left(-\frac{\sqrt{-g}}{\epsilon} + \partial_\rho\sqrt{-g}\right)\Big|_{\rho=\epsilon}\right]. \tag{2.6}$$

Using the expansion for the determinant [Eq. (A1)] and doing the ρ integral, we arrive at the following final form for the regulated action:

$$S_N^{\mathrm{reg}} = \frac{1}{2\kappa}\int d^2x\sqrt{-g_0}\log\epsilon \mathrm{Tr} g_2. \tag{2.7}$$

In this paper, we will ignore this logarithmic divergence, because it will not be relevant for the situations we consider, like black holes. This is similar to the approach of Ref. [22] and we would like to write down counterterms parallel to theirs in terms of the induced metric. The logarithmic divergence in the Dirichlet case was presented later in Ref. [23]. We emphasize however that even though we do not use them, our presentation of logarithmic divergences is complete: the expressions for the quantities involving g_2 in Eqs. (2.7) and (2.20) in terms of curvatures of the boundary metric g_0 are presented in Appendix B. Note however that unlike the other counterterms, we cannot absorb the cutoff dependence of the logarithmic divergence entirely into expressions involving the induced metric; a logarithmic cutoff dependence will remain. This is unavoidable, and this is the form in which Ref. [23] also left their results; see the last term of their Eq. (B.4). The renormalized quantities are of course cutoff independent by construction.

Once we ignore the logarithmic term, the renormalized Neumann action is therefore identical to the original Neumann action S_N in three dimensions: no counterterms are required to render the action finite,

$$S_N^{\mathrm{ren}} = S_N. \tag{2.8}$$

This was an observation that was already made in a slightly different language in Refs. [8,27], as a special observation about three dimensions. From our perspective, the fact that the bare action is already finite in $2+1$ dimensions is the crucial reason why their construction works.

Now we come to one crucial observation. The renormalized stress tensor in $2+1$ dimensions is given by [22]

$$T_{ab}^{\mathrm{ren}} = \frac{1}{\kappa}[\Theta_{ab} - \Theta\gamma_{ab} + \gamma_{ab}]. \tag{2.9}$$

We will now show that the renormalized Neumann action (which coincidentally happens to be the same as the bare Neumann action in $2+1$ dimensions[5]) gives rise to a well-defined variational principle when we demand that the renormalized boundary stress tensor density is held fixed. This means that, given the renormalized stress tensor as our boundary data, we have a well-defined variational principle.

To show this, first note that in three dimensions,

$$\begin{aligned}\delta S_N^{\mathrm{ren}} = \delta S_N &= \delta\left[\frac{1}{2\kappa}\int_{\mathcal{M}} d^3x\sqrt{-g}(R-2\Lambda) + \frac{1}{2\kappa}\int_{\partial\mathcal{M}} d^2x\sqrt{-\gamma}\Theta\right] \\ &= \frac{1}{2\kappa}\int_{\mathcal{M}} d^3x\sqrt{-g}(G_{ab}+\Lambda g_{ab})\delta g^{ab} \\ &\quad - \int d^2x\left[\delta\left(-\frac{\sqrt{-\gamma}}{2\kappa}(\Theta^{ab} - \Theta\gamma^{ab})\right)\gamma_{ab}\right].\end{aligned} \tag{2.10}$$

The bare stress tensor is defined as

$$T_{ab}^{\mathrm{bare}} = \frac{1}{\kappa}[\Theta_{ab} - \Theta\gamma_{ab}]. \tag{2.11}$$

The surface term in Eq. (2.10) can be thus expressed as

$$\delta\left(-\frac{\sqrt{\gamma}}{2\kappa}(\Theta^{ab} - \Theta\gamma^{ab})\right)\gamma_{ab} = \delta\left(\frac{\sqrt{-\gamma}}{2}T^{\mathrm{bare}ab}\right)\gamma_{ab}. \tag{2.12}$$

[5]This coincidence of the renormalized and the bare Neumann actions is a feature of $2+1$ dimensions and does not hold in higher dimensions, but the statements we make about the renormalized action apply in higher dimensions as well.

Now by an explicit calculation, we can see that

$$\delta\left(\frac{\sqrt{-\gamma}}{2}T^{\rm bare}_{ab}\right)\gamma^{ab} = \delta\left(\frac{\sqrt{-\gamma}}{2}T^{\rm ren}_{ab}\right)\gamma^{ab}. \tag{2.13}$$

This shows that the Neumann variational problem of the renormalized action might as well be formulated by holding the renormalized boundary stress tensor density fixed. This arises because in formulating the variational problem one has the freedom to add a χ_{ab} to the stress tensor that one is holding fixed at the boundary as long as it satisfies

$$\delta(\sqrt{-\gamma}\chi_{ab})\gamma^{ab} = 0. \tag{2.14}$$

We will see that in odd d dimensions, this ambiguity[6] in practice does not arise because the variational problem of the Neumann type for the renormalized action essentially automatically leads to the renormalized stress tensor. We turn now to demonstrate this in four dimensions.

C. AdS$_4$ ($d = 3$)

In $d = 3$, the singular part of the regularized action evaluates to

$$S^{\rm reg}_N = -\frac{3}{2\kappa}\int d^3x \int_\epsilon d\rho \frac{\sqrt{-g}}{\rho^{5/2}} = -\frac{1}{\kappa}\int d^3x\sqrt{-g_0}\left(\frac{1}{\epsilon^{3/2}} + \frac{3}{2\epsilon^{1/2}}{\rm Tr}g_2\right) \tag{2.15}$$

where we have once again used the determinant expansion (A1). The determinant of the induced metric γ_{ab} can be expressed as

$$\sqrt{-\gamma} = \frac{\sqrt{-g}}{\epsilon^{d/2}}. \tag{2.16}$$

This, together with Eq. (A4) allows us to write the counterterm action

$$S^{ct} = \frac{1}{\kappa}\int d^3x\sqrt{-\gamma}\left(1 - \frac{1}{4}R[\gamma]\right). \tag{2.17}$$

The fact that this is the correct counterterm can be checked by expanding Eq. (2.17) in the Fefferman-Graham expansion order by order and using Eqs. (A1) and (A4). The renormalized Neumann action, in a notation analogous to that in Ref. [22], is thus given by

$$S^{\rm ren}_N = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R - 2\Lambda) + \frac{1}{\kappa}\int d^3x\sqrt{-\gamma}\left(1 - \frac{1}{4}R[\gamma]\right). \tag{2.18}$$

[6]We will discuss this ambiguity, together with the logarithmic divergence, elsewhere.

Including this counterterm and doing variations, we also reproduce the stress tensor of Refs. [22,23]

$$T^{\rm ren}_{ab} = \frac{1}{\kappa}[\Theta_{ab} - \Theta\gamma_{ab} + 2\gamma_{ab} - G_{ab}] \tag{2.19}$$

where $G_{ab} = R_{ab}[\gamma] - \frac{1}{2}R[\gamma]\gamma_{ab}$ is the Einstein tensor of the induced metric.[7] This stress tensor is known for empty AdS and an AdS black hole to be finite and also has the right leading fall-offs to reproduce the correct finite charges for the AdS black hole.

This shows again that the renormalized Neumann action leads to a well-defined variational problem when holding the renormalized boundary stress tensor fixed.

D. AdS$_5$ ($d = 4$)

For the case of $d = 4$, the divergent part of the action evaluates to

$$S^{\rm reg}_N = -\frac{2}{\kappa}\int d^4x\sqrt{-g_0}\left(\frac{3}{2\epsilon^2} + \frac{3}{4\epsilon}{\rm Tr}g_2 - \log\epsilon\frac{1}{8}(({\rm Tr}(g_2))^2 - {\rm Tr}(g_2)^2)\right). \tag{2.20}$$

Barring the log term, all other divergences in Eq. (2.20) can be canceled by adding a counterterm given by

$$S^{ct}_N = \frac{3}{\kappa}\int d^4x\sqrt{-\gamma}. \tag{2.21}$$

Once again, this can be explicitly checked by expanding Eq. (2.21) in Fefferman-Graham expansion and using the relations (A1) and (A4). The renormalized Neumann action is given by

$$S^{\rm ren}_N = \frac{1}{2\kappa}\int_{\mathcal{M}} d^5x\sqrt{-g}(R - 2\Lambda) - \frac{1}{2\kappa}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}\Theta + \frac{3}{\kappa}\int d^4x\sqrt{-\gamma}. \tag{2.22}$$

As in the case of $d = 2$, there is an ambiguity in the stress tensor. The renormalized stress tensor we hold fixed for the variational principle is given by

$$T^{\rm ren}_{ab} = \frac{1}{\kappa}\left[\Theta_{ab} - \Theta\gamma_{ab} + 3\gamma_{ab} - \frac{1}{2}G_{ab}\right]. \tag{2.23}$$

Once again this shows that the renormalized Neumann action (2.22) gives a well-defined variational principle with the renormalized stress tensor. We also note that Eq. (2.22), being an even-d case has an ambiguity similar to the $d = 2$ case, and we have used the fact that

[7]More precisely, what we reproduce is $\delta T^{\rm ren}_{ab}$ from the variational problem for the renormalized Neumann action. But unlike in odd d, this leads directly to Eq. (2.19) and we do not need to use the ambiguity of the type (2.14).

$$\delta(\sqrt{-\gamma}G_{ab})\gamma^{ab} = 0. \tag{2.24}$$

In what follows, we will often suppress the superscript *ren* when there is no source of ambiguity that we are indeed working with the renormalized action.

E. Comparison with standard holographic renormalization

How does all this compare with the standard discussion of holographic renormalization in the Dirichlet case?

One difference is that the counterterms that are added in the Dirichlet case do not change the variational problem: before and after their addition, the boundary metric that is held fixed is identical. This is not true in our case. Before renormalization, the quantity that is held fixed is the *unrenormalized* stress tensor density, but at the end it is the *renormalized* stress tensor density. It is of course not surprising that added terms can change the variational problem, but what is worthy of remark here is the philosophy behind it: we demanded the finiteness of the Neumann action, and that leads to a well-defined variational problem with the *renormalized* quantity held fixed. Satisfyingly, this same object can also be obtained as the Legendre transform of the *renormalized* Dirichlet action; see Appendix B. Note that the unrenormalized actions are merely a crutch and the renormalized actions are the physically relevant objects.

Let us also note that the *total* action/partition function (including counterterms *and* everything else) can only be a functional of the quantity fixed at the boundary. This is guaranteed at the level of the action because again, the Neumann action is a Legendre transform of Dirichlet and therefore (by construction) depends only on the conjugate variable. In equations, as we discuss in Appendix B, we can view our action as

$$S_N^{\rm ren}[\pi_{ab}^{\rm ren}] = S_D^{\rm ren}[\gamma^{ab}] - \int_{\partial\mathcal{M}} d^{D-1}x\pi_{ab}^{\rm ren}\gamma^{ab} \tag{2.25}$$

where

$$\pi_{ab}^{\rm ren} = \frac{\delta S_D^{\rm ren}}{\delta\gamma^{ab}}. \tag{2.26}$$

This can be viewed as the semiclassical version[8] of a Legendre transform at the level of partition functions:

$$\Gamma[\delta W/\delta\gamma^{ab}] = W[\gamma^{ab}] - \int_{\partial\mathcal{M}} d^{D-1}x\frac{\delta W}{\delta\gamma^{ab}}\gamma^{ab}. \tag{2.27}$$

At the level of the semiclassical saddle, this translates to the statement that the variational principle (while holding the conjugate quantity fixed at boundary) is well defined, which we checked explicitly earlier in this section. The separate terms (including counterterms) in the action which are integrated over can have complicated dependences, but they conspire to satisfy the above demands.

As an aside, we also note some papers in the literature which dealt with related setups. In particular, in Ref. [14] the boundary metric fluctuated but they arranged it so that the variational principle with the Dirichlet action works, by setting $T^{ij} = 0$. There are other papers, especially in three dimensions, which deal with similar setups [15,19,29]. In fact, our approach can be thought of in many ways as a general framework for dealing with some of these situations. The authors of Ref. [14] treated the boundary stress tensor as a fixed given *value* (namely, zero), so their partition function was a *number*, so they did not discuss the points we emphasized in the previous paragraph. Our work can be thought of as a generalization of theirs and our partition function is a proper functional, where instead of setting the stress tensor (density) to be zero, we treat it as arbitrary but fixed.[9]

III. FINITE ON-SHELL ACTION

In this section we present the results of on-shell action and stress energy tensor for the Neumann action in various dimensions. We also draw a comparison between our on-shell action and the on-shell Dirichlet action. Note that the precise value of the action is sensitive to the infrared cutoff of the action integral. So one cannot work abstractly at the level of the Fefferman-Graham expansion like we did so far, because we need to know the metric finitely deep into the geometry and not merely as an expansion at the boundary. So we will consider explicit solutions like black holes.

A. AdS$_3$

The Dirichlet action for gravity in AdS$_3$ is given by [22]

$$S_D = \frac{1}{2\kappa}\int_{\mathcal{M}} d^3x\sqrt{-g}(R-2\Lambda) + \frac{1}{\kappa}\int_{\partial\mathcal{M}}\sqrt{-\gamma}\Theta - \frac{1}{\kappa}\int_{\partial\mathcal{M}}\sqrt{-\gamma}. \tag{3.1}$$

We evaluate the above action on the Bañados-Teitelboim-Zanelli (BTZ) metric

$$ds^2 = -\frac{(r^2-r_+^2)(r^2-r_-^2)}{r^2}dt^2 + \frac{r^2dr^2}{(r^2-r_+^2)(r^2-r_-^2)} + r^2\left(d\phi - \frac{r_+r_-}{r^2}dt\right)^2 \tag{3.2}$$

[8]We will briefly discuss the existence of a full quantum theory further in Secs. V and VI, as well as in more detail in Ref. [28].

[9]The "arbitrariness" of the boundary stress tensor should of course still satisfy the requirement that the Fefferman-Graham expansion should satisfy the bulk equations of motion; see the discussion in Ref. [23] for details.

where r_+ and r_- are the outer and inner horizons respectively and are related to the charges through the relation $M = r_+^2 + r_-^2$ and $J = 2r_+r_-$. In the above metric we have set $l = 1$. Evaluating the action between time $-T$ to T and $r_+ < r < R$ on this solution yields

$$S_D^{\rm BTZ} = \frac{2\pi(r_+^2 + r_-^2)T}{\kappa} + O\left(\frac{1}{R^2}\right). \tag{3.3}$$

The on-shell Neumann action for the BTZ solution yields

$$S_N^{\rm BTZ} = \frac{2\pi(r_+^2 + r_-^2)T}{\kappa} \tag{3.4}$$

which matches with the Dirichlet action in the limit $R \to \infty$. The stress-energy tensor similarly takes the form

$$T_{ab} = \begin{pmatrix} -\frac{r_+^2+r_-^2}{2\kappa} & \frac{r_+r_-}{\kappa} \\ \frac{r_+r_-}{\kappa} & -\frac{r_+^2+r_-^2}{2\kappa} \end{pmatrix} + O\left(\frac{1}{R^2}\right). \tag{3.5}$$

This stress tensor has the right falloffs to reproduce finite charges M and J through the relation [30,31]

$$Q_\xi = -\int_\Sigma d^{D-1}x\sqrt{\sigma}(u^a T_{ab}\xi^b) \tag{3.6}$$

where ξ^a is the Killing vector generating the isometry of the boundary metric and u^a is the unit time-like vector. We see that the counterterm action that was chosen to make the on-shell Neumann action finite also produces a finite stress tensor. This was shown for the Dirichlet case in Ref. [22].

B. AdS$_4$

The (renormalized) Dirichlet action in $D = 4$ takes the form

$$S_D = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R - 2\Lambda) + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-\gamma}\Theta - \frac{2}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-\gamma}\left(1 + \frac{{}^{(3)}R}{4}\right). \tag{3.7}$$

The AdS-Schwarzschild black hole metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + r^2\right)dt^2 + \frac{dr^2}{(1 - \frac{2M}{r} + r^2)} + r^2 d\Omega^2. \tag{3.8}$$

The horizon is obtained by the real root of

$$1 - \frac{2M}{r_H} + r_H^2 = 0. \tag{3.9}$$

Evaluating the action for this metric yields (integrated in the region $-T < t < T$ and $r_H < r < R$)

$$S_D^{\rm AdS-BH} = -\frac{8\pi(M - r_H^3)T}{\kappa} + O\left(\frac{1}{R}\right). \tag{3.10}$$

The stress tensor computed for this metric is given by

$$T_{ab} = \begin{pmatrix} -\frac{2M}{\kappa R} & 0 & 0 \\ 0 & -\frac{M}{\kappa R} & 0 \\ 0 & 0 & -\frac{M\sin^2(\theta)}{\kappa R} \end{pmatrix} + O(1/R^2) \tag{3.11}$$

which once again has the right falloffs to obtain finite charges as described in the previous section. The Neumann action in $D = 4$ takes the form

$$S_N = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R - 2\Lambda) + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-\gamma}\left(1 - \frac{{}^{(3)}R}{4}\right) \tag{3.12}$$

which evaluates to

$$S_N^{\rm AdS-BH} = -\frac{8\pi(M - r_H^3)T}{\kappa} + O\left(\frac{1}{R}\right). \tag{3.13}$$

The subleading term here differs from the subleading term in the Dirichlet action and the two actions are same only in the $R \to \infty$ limit.

C. AdS$_5$

In $D = 5$ the Dirichlet action takes the form

$$S_D = \frac{1}{2\kappa}\int_{\mathcal{M}} d^5x\sqrt{-g}(R - 2\Lambda) + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}\Theta - \frac{3}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-\gamma}\left(1 + \frac{{}^{(4)}R}{12}\right). \tag{3.14}$$

We evaluate this action for the black hole metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2 \tag{3.15}$$

where

$$f(r) = r^2 + 1 - \frac{2M}{r^2}. \tag{3.16}$$

The horizon is once again determined by the largest positive root of

$$r_H^2 + 1 - \frac{2M}{r_H^2} = 0. \tag{3.17}$$

The action evaluates to

$$S_D^{\rm BH} = -\frac{2\pi^2 T}{\kappa}\left(2M + \frac{3}{4} - 2r_H^4\right) + O(1/R^4). \tag{3.18}$$

The stress tensor takes the form

$$T_{ab} = \begin{pmatrix} -\frac{3(8M+1)}{8R^2\kappa} & 0 & 0 & 0 \\ 0 & -\frac{(8M+1)}{8R^2\kappa} & 0 & 0 \\ 0 & 0 & -\frac{((8M+1)\sin^2(\psi))}{8R^2\kappa} & 0 \\ 0 & 0 & 0 & -\frac{((8M+1)\sin^2(\theta)\sin^2(\psi))}{8R^2\kappa} \end{pmatrix} + O(1/R^4). \quad (3.19)$$

The Neumann action in this case can be written as

$$S_N = \frac{1}{2\kappa}\int_{\mathcal{M}} d^5x\sqrt{-g}(R-2\Lambda) - \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}\Theta + \frac{3}{\kappa}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma} \quad (3.20)$$

which evaluates to

$$S_N^{\rm BH} = -\frac{2\pi^2 T}{\kappa}\left(2M + \frac{3}{4} - 2r_H^4\right) + O(1/R^2). \quad (3.21)$$

Again, we find agreement when the radial cutoff is taken to infinity.

IV. ADM FORMULATION OF RENORMALIZED AdS_4 ACTION

The ADM formulation of general relativity (GR) works by singling out the time direction from the spatial direction and reexpressing the content of GR in terms of ADM variables. Thus the spacetime is thought of as foliated by spatial slices Σ_t which are the hypersurfaces of constant t. The spacetime metric can be expressed as

$$ds^2 \equiv g_{\alpha\beta}dx^\alpha dx^\beta = -N^2dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt) \quad (4.1)$$

where N is the lapse function, N^a is the shift vector and h_{ab} is the induced metric on the hypersurface Σ_t. In what follows, we assume that the manifold is a box with finite spatial extent such that the boundary is time-like, denoted $\mathcal{B}$. The spatial section of $\mathcal{B}$ is denoted B. We will also ignore the space-like boundaries at initial and final times and work with coordinates such that the time-like boundary is orthogonal to the spatial hypersurfaces, Σ_t. Under the ADM split of the bulk metric, Eq. (4.1), the induced metric on the boundary $\mathcal{B}$, also undergoes a decomposition

$$ds^2 \equiv \gamma_{ij}dx^i dx^j = -N^2dt^2 + \sigma_{AB}(d\theta^A + N^A dt)(d\theta^B + N^B dt) \quad (4.2)$$

where σ_{AB} is the induced metric on B. We will also need the expression for the decomposition of the Ricci scalar

$${}^{(D)}R = {}^{(D-1)}R + K^{ab}K_{ab} - K^2 - 2\nabla_\alpha(u^\beta\nabla_\beta u^\alpha - u^\alpha\nabla_\beta u^\beta) \quad (4.3)$$

where K_{ab} is the extrinsic curvature of the spatial hypersurface Σ_t (not to be confused with the boundary). The point about ADM split is that N and N^a are not dynamical fields and therefore their conjugates are constraint relations. The dynamical field is the spatial metric h_{ab} and the canonical conjugate momentum is given by

$$p^{ab} \equiv \frac{\partial}{\partial\dot{h}_{ab}}(\sqrt{-g}\mathcal{L}_G) = \frac{\sqrt{h}}{2\kappa}(K^{ab} - Kh^{ab}) \quad (4.4)$$

where K_{ab} is the extrinsic curvature of Σ_t. The details of the ADM decomposition of the gravitational action can be found in Refs. [20,32]. We will work with AdS_4 in what follows, for convenience.

A. Dirichlet action

In this section, we return to the case of ADM decomposition, for the renormalized Dirichlet action in AdS_4. The renormalized action in the covariant form is given by [22,23]

$$S_D = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R-2\Lambda) + \frac{1}{\kappa}\int_{\mathcal{B}} d^3x\sqrt{-\gamma}\Theta + \frac{1}{\kappa}\int_{\mathcal{B}} d^3x\sqrt{-\gamma}\left(-\frac{2}{l}\right)\left[1 + \frac{{}^{(3)}R}{4}\right]. \quad (4.5)$$

The first two terms are the Einstein-Hilbert and the GHY pieces, and can be written in terms of the ADM variables following the steps of Ref. [20]. This gives us the following form for the action [32]:

$$S_D = S_{\rm EH} + S_{\rm GHY} + S_{ct} = \int_{\mathcal{M}} d^Dx(p^{ab}\dot{h}_{ab} - NH - N_aH^a) + \int_{\mathcal{B}} d^{D-1}y\sqrt{\sigma}(N\varepsilon - N^a j_a) + S_{ct} \quad (4.6)$$

where H and H^a are the Hamiltonian and momentum constraints,

$$H = \frac{\sqrt{h}}{2\kappa}(K^{ab}K_{ab} - K^2 - {}^{(3)}R + 2\Lambda),$$
$$H^a = -\frac{\sqrt{h}}{\kappa}D_b(K^{ab} - Kh^{ab}). \quad (4.7)$$

$\sqrt{\sigma}\varepsilon$, $\sqrt{\sigma}j_a$ and $N\sqrt{\sigma}s^{ab}/2$ are the momenta conjugate to N, N^a and σ_{ab}.. and are given by

$$\varepsilon = \frac{k}{\kappa}, \qquad j_a = \frac{2}{\sqrt{h}}r_b p_a^b,$$
$$s^{ab} = \frac{1}{\kappa}\left[k^{ab} - \left(\frac{r^a\partial_a N}{N} + k\right)\sigma^{ab}\right] \quad (4.8)$$

where k^{ab} is the extrinsic curvature of B embedded in Σ_t and $k = k^{ab}\sigma_{ab}$. The counterterm action is given in the covariant form by

$$S_{ct} = \frac{1}{\kappa}\int_{\mathcal{B}} d^3x\sqrt{-\gamma}\left(-\frac{2}{l}\right)\left[1 + \frac{{}^{(3)}R}{4}\right]. \quad (4.9)$$

Using Eq. (4.3) and the expression for the determinant $\sqrt{-\gamma} = N\sqrt{\sigma}$, we obtain the counterterm action as

$$S_{ct} = \frac{1}{\kappa}\int_{\mathcal{B}} d^3x\left(-\frac{2}{l}\right)\left[1 + \frac{l^2}{4}({}^{(2)}R + \hat{k}_{ab}\hat{k}^{ab} - \hat{k}^2)\right] \quad (4.10)$$

where $\hat{k}_{ab}$ is the extrinsic curvature of B as a hypersurface embedded in $\mathcal{B}$. For black hole geometries, we also get a contribution from the horizon which is given by decomposing the covariant Neumann action with a boundary at the horizon where no data is specified [20,30,33]. The action then takes the form

$$\begin{aligned} S_D = &\int_{\mathcal{M}} d^Dx(p^{ab}\dot{h}_{ab} - NH - N_aH^a) \\ &+ \int_{\mathcal{H}} d^{D-1}y\sqrt{\sigma}\left(\frac{r^a\partial_a N}{\kappa} + \frac{2r_aN_bp^{ab}}{\sqrt{h}}\right) \\ &+ \int_{\mathcal{B}} d^{D-1}y\sqrt{\sigma}(N\varepsilon - N^aj_a) \\ &+ \frac{1}{\kappa}\int_{\mathcal{B}} d^3x\left(-\frac{2}{l}\right)\left[1 + \frac{l^2}{4}({}^{(2)}R + \hat{k}_{ab}\hat{k}^{ab} - \hat{k}^2)\right]. \end{aligned} \quad (4.11)$$

We can further express the above action in terms of the renormalized parameters thereby absorbing the counterterm into the renormalized quantities $\varepsilon^{\mathrm{ren}} = \varepsilon + \varepsilon^{ct}$, $j_a^{\mathrm{ren}} = j_a + j_a^{ct}$ and $s_{ab}^{\mathrm{ren}} = s_{ab} + s_{ab}^{ct}$. To do so, we do a canonical decomposition of the tensor using normal and tangential projections [31]. The expressions for renormalized quantities are given by

$$\varepsilon^{\mathrm{ren}} = u_a u_b T^{ab},$$
$$j_a^{\mathrm{ren}} = -\sigma_{ab}T^{bc}u_c,$$
$$s_{ab}^{\mathrm{ren}} = \sigma_{ac}\sigma_{bd}T^{cd} \quad (4.12)$$

where T^{ab} is the renormalized stress tensor given by

$$T^{ab} = \frac{1}{\kappa}\left(\Theta^{ab} - \Theta\gamma^{ab} + \frac{2}{l}\gamma^{ab} - lG^{ab}\right). \quad (4.13)$$

Using the above expressions, we get

$$\begin{aligned} \varepsilon^{\mathrm{ren}} &= \varepsilon - \frac{1}{\kappa}\left[\frac{2}{l} + \frac{l}{2}({}^{(2)}R - \hat{k}_{ab}\hat{k}^{ab} + \hat{k}^2)\right], \\ j_a^{\mathrm{ren}} &= j_a + \frac{l}{\kappa}(d_a\hat{k} - d_b\hat{k}_a^b), \\ s_{ab}^{\mathrm{ren}} &= s_{ab} + \frac{1}{\kappa}\left[\frac{2}{l}\sigma_{ab} + \frac{l}{2}({}^{(2)}R + \hat{k}^{ab}\hat{k}_{ab} - \hat{k}^2) \right. \\ &\quad - l\left(-\frac{1}{N}\mathcal{L}_m\hat{k}_{ab} - \frac{1}{N}d_ad_bN + {}^{(2)}R_{ab} \right. \\ &\quad \left.\left. + \hat{k}\hat{k}_{ab} - 2\hat{k}_{ac}\hat{k}_b^c\right)\right]. \end{aligned} \quad (4.14)$$

In writing the above expressions, we have made use of Gauss-Codazzi relations whose exact expressions are given in Appendix B. Thus, the renormalized action can be expressed as

$$\begin{aligned} S_D = &\int_{\mathcal{M}} d^Dx(p^{ab}\dot{h}_{ab} - NH - N_aH^a) \\ &+ \int_{\mathcal{H}} d^{D-1}y\sqrt{\sigma}\left(\frac{r^a\partial_a N}{\kappa} + \frac{2r_aN_bp^{ab}}{\sqrt{h}}\right) \\ &+ \int_{\mathcal{B}} d^{D-1}y\sqrt{\sigma}(N\varepsilon^{\mathrm{ren}} - N^aj_a^{\mathrm{ren}}). \end{aligned} \quad (4.15)$$

1. Kerr-AdS: Covariant

As an illustration of our construction, we can evaluate the action on the Kerr-AdS metric in $D = 4$. Rotating black holes are better defined in AdS, than in flat space (see e.g., Refs. [34,35]). The metric in Boyer-Lindquist type coordinates is given by

$$\begin{aligned} ds^2 = &\rho^2\left(\frac{dr^2}{\Delta} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(adt - \frac{r^2 + a^2}{\Sigma}d\phi\right)^2 \\ &- \frac{\Delta}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Sigma}d\phi\right)^2 \end{aligned} \quad (4.16)$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = (r^2+a^2)\left(1+\frac{r^2}{l^2}\right) - 2Mr,$$

$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \qquad \Sigma = 1 - \frac{a^2}{l^2}. \tag{4.17}$$

The horizon is at the largest positive root of $\Delta(r_H) = 0$. The angular velocity of the black hole (for $r \geq r_H$) is given by

$$\omega = a\Sigma\left(\frac{\Delta_\theta(r^2+a^2) - \Delta}{(r^2+a^2)^2\Delta_\theta - a^2\Delta\sin^2\theta}\right). \tag{4.18}$$

The angular velocity at the horizon is given by

$$\Omega_H = \frac{a\Sigma}{r_H^2 + a^2} \tag{4.19}$$

while the angular velocity at the boundary ($r \to \infty$), is given by $\Omega_\infty = -a/l^2$. The angular velocity relevant for the thermodynamics is given by $\Omega = \Omega_H - \Omega_\infty$ [24,36]. Given the metric, the ADM variables can be read off by comparing Eq. (4.16) with the ADM form of the metric. The lapse, shift and spatial metric are given by

$$N = \sqrt{\frac{\rho^2\Delta\Delta_\theta}{(r^2+a^2)^2\Delta_\theta - a^2\Delta\sin^2\theta}},$$

$$N^\phi = a\Sigma\frac{(\Delta - \Delta_\theta(r^2+a^2))}{(r^2+a^2)^2\Delta_\theta - a^2\Delta\sin^2\theta},$$

$$h_{ab} = \begin{pmatrix} \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & \frac{\rho^2}{\Delta_\theta} & 0 \\ 0 & 0 & \frac{((r^2+a^2)^2\Delta_\theta - a^2\Delta\sin^2\theta)}{\rho^2\Sigma^2} \end{pmatrix}. \tag{4.20}$$

For the thermodynamic interpretation we must work with the complex metric associated with the black hole, which is given by the identification $N \to -i\tilde{N}$, $N^\phi \to -i\tilde{N}^\phi$ [20,30]. The periodicity of the time circle can be estimated by evaluating the $r^a\partial_a\tilde{N} \equiv 2\pi/\beta$ term on the horizon. This gives the time periodicity, β, to be

$$\beta = \frac{4\pi(r_H^2 + a^2)}{r_H(1 + \frac{a^2}{l^2} + \frac{3r_H^2}{l^2} - \frac{a^2}{r_H^2})}. \tag{4.21}$$

The expressions for various terms in the covariant action are

$$R = -\frac{12}{l^2},$$

$$\Theta = \frac{3}{l} + \frac{(-3a^2 + 2l^2 - 5a^2\cos 2\theta)}{4lR_c^2}, + O(1/R_c^4)$$

$${}^{(3)}R = \frac{2l^2 - 3a^2 - 5a^2\cos 2\theta}{l^2R_c^2} + O(1/R_c^4). \tag{4.22}$$

Evaluating the complex metric on the covariant action (4.5), and using the expression (4.21) for the periodicity, we get

$$S_D = -i\frac{\pi l^2(r_H^2 + a^2)^2(l^2 - r_H^2)}{(l^2 - a^2)(a^2l^2 - (a^2 + l^2)r_H^2 - 3r_H^4)}. \tag{4.23}$$

This is related to the free energy through the relation

$$-\beta F_D \equiv \log Z_D \approx iS_D \tag{4.24}$$

where β is the inverse temperature which can be identified with the periodicity of the time circle. This gives the free energy of the black hole to be

$$F_D = \frac{(r_H^2 + a^2)(l^2 - r_H^2)}{4(l^2 - a^2)r_H}. \tag{4.25}$$

2. Kerr-AdS: ADM

Evaluating the complex metric on the ADM decomposed action, the bulk term vanishes because the metric is stationary and satisfies Einstein's equation. The horizon term gives a contribution of

$$S_{\mathcal{H}} = -i\frac{A}{4} - i\Omega_H PJ. \tag{4.26}$$

On the boundary we can see that the renormalized ε, j^ϕ and s^{AB} have correct falloffs so as to give finite results for the integral,

$$\varepsilon_{\text{ren}} = \left(\frac{M(a^2 - 4l^2 + 3a^2\cos 2\theta)}{l\Sigma\kappa}\right)\frac{1}{R_c^3} + O\left(\frac{1}{R_c^4}\right),$$

$$j^\phi_{\text{ren}} = \frac{3aM}{\kappa}\sqrt{\frac{\Delta_\theta}{\Sigma}}\frac{1}{R_c^4} + O\left(\frac{1}{R_c^5}\right). \tag{4.27}$$

Evaluating the boundary integrals, we have

$$S_{\mathcal{B}} = iEP + i\Omega_\infty PJ \tag{4.28}$$

where E and J are calculated as

$$E = \frac{M}{\Sigma^2}, \qquad J = \frac{Ma}{\Sigma^2} \tag{4.29}$$

which are the ADM charges of the Kerr black hole. Using Eq. (4.24), we have

$$F_D = E - TS - \Omega J. \tag{4.30}$$

Now, by an explicit computation, we can verify that the free energy, F_D, in Eq. (4.25) can be expressed as

$$F_D = -T\frac{A}{4} - \Omega J + g(A, J) \tag{4.31}$$

where

$$g(A,J)=\sqrt{\frac{A}{16\pi}+\frac{4\pi}{A}J^2+\frac{J^2}{l^2}+\frac{A}{8\pi l^2}\left(\frac{A}{4\pi}+\frac{A^2}{32\pi^2 l^2}\right)}. \tag{4.32}$$

Equating Eq. (4.31) to the free energy computed using the ADM approach, Eq. (4.30), we get the generalized Smarr formula [see Eq. (41) of Ref. [24]]:

$$E^2=\frac{A}{16\pi}+\frac{4\pi}{A}J^2+\frac{J^2}{l^2}+\frac{A}{8\pi l^2}\left(\frac{A}{4\pi}+\frac{A^2}{32\pi^2 l^2}\right). \tag{4.33}$$

Following Ref. [24] we can also relate these calculations to the first law, which we will not repeat.

B. Neumann action

The renormalized Neumann action in AdS_4 is given by

$$S_N=\frac{1}{2\kappa}\int_{\mathcal{M}}d^4x\sqrt{-g}(R-2\Lambda)+\frac{1}{\kappa}\int_{\mathcal{B}}d^3x\sqrt{-\gamma}\left(\frac{1}{l}\right)\left[1-\frac{l^2}{4}{}^{(3)}R\right]. \tag{4.34}$$

The bare part of the Neumann action in ADM was derived in Ref. [20]. In $D=4$ it can be used to write

$$S_N=S_{\rm EH}+S_{ct}=\int_{\mathcal{M}}d^4x(p^{ab}\dot{h}_{ab}-NH-N_aH^a)+\int_{\mathcal{B}}d^3x\sqrt{\sigma}\left(\frac{N\varepsilon}{2}-N^aj_a+\frac{N}{2}s^{ab}\sigma_{ab}\right)+S_{ct}. \tag{4.35}$$

The counterterm action can be decomposed similarly to the Dirichlet case and we get

$$S_{ct}=\frac{1}{\kappa}\int_{\mathcal{B}}d^3x\left(\frac{1}{l}\right)\left[1-\frac{l^2}{4}({}^{(2)}R+\hat{k}_{ab}\hat{k}^{ab}-\hat{k}^2)\right]. \tag{4.36}$$

For the black hole geometries, one again has a contribution from the horizon and the action takes the form

$$S_N=\int_{\mathcal{M}}d^4x(p^{ab}\dot{h}_{ab}-NH-N_aH^a)+\int_{\mathcal{H}}d^3y\sqrt{\sigma}\left(\frac{r^a\partial_aN}{\kappa}+\frac{2r_aN_bp^{ab}}{\sqrt{h}}\right)+\int_{\mathcal{B}}d^3x\sqrt{\sigma}\left(\frac{N\varepsilon}{2}-N^aj_a+\frac{N}{2}s^{ab}\sigma_{ab}\right)+\frac{1}{\kappa}\int_{\mathcal{B}}d^3x\left(\frac{1}{l}\right)\left[1-\frac{l^2}{4}({}^{(2)}R+\hat{k}_{ab}\hat{k}^{ab}-\hat{k}^2)\right]. \tag{4.37}$$

Using the expressions for renormalized parameters, the Neumann action can be expressed as

$$S_N=\int_{\mathcal{M}}d^4x(p^{ab}\dot{h}_{ab}-NH-N_aH^a)+\int_{\mathcal{H}}d^3y\sqrt{\sigma}\left(\frac{r^a\partial_aN}{\kappa}+\frac{2r_aN_bp^{ab}}{\sqrt{h}}\right)+\int_{\mathcal{B}}d^3x\sqrt{\sigma}\left(\frac{N\varepsilon^{\rm ren}}{2}-N^aj_a^{\rm ren}+\frac{N}{2}s^{\rm renab}\sigma_{ab}\right). \tag{4.38}$$

1. Kerr-AdS: Covariant

We can evaluate the covariant Neumann action on the Kerr-AdS complex metric; we obtain

$$S_N=-i\frac{\pi l^2(r_H^2+a^2)^2(l^2-r_H^2)}{(l^2-a^2)(a^2l^2-(a^2+l^2)r_H^2-3r_H^4)}. \tag{4.39}$$

Notice that unlike the asymptotically flat case, the on-shell values of the Dirichlet and Neumann actions are equal. The on-shell action is related to the Neumann free energy through the relation

$$-\beta F_N\equiv\log Z_N\approx iS_N \tag{4.40}$$

which gives the free energy of the black hole to be

$$F_N=\frac{(r_H^2+a^2)(l^2-r_H^2)}{4(l^2-a^2)r_H}. \tag{4.41}$$

2. Kerr-AdS: ADM

Evaluating the complex metric on the ADM decomposed action, the horizon term gives a contribution of

$$S_{\mathcal{H}}=-i\frac{A}{4}-i\Omega_HPJ. \tag{4.42}$$

On the boundary we have,

$$s^{ab}_{\rm ren}=\begin{pmatrix}-\frac{lM\Delta_\theta}{\kappa} & 0\\ 0 & -\frac{M(a^2+2l^2-3a^2\cos2\theta)}{2l\kappa\sin^2\theta}\end{pmatrix}\frac{1}{R_c^3}+O(1/R_c^4),$$

$$\sigma_{ab}=\begin{pmatrix}\frac{\rho^2}{\Delta_\theta} & 0\\ 0 & \frac{((r^2+a^2)^2\Delta_\theta-a^2\Delta\sin^2\theta)}{\rho^2\Sigma^2}\end{pmatrix}. \tag{4.43}$$

We get a contribution of $iEP/2$ from the integration over $\varepsilon^{\rm ren}$ term and another contribution of $iEP/2$ from the integration over $s^{\rm ren}_{ab}$ term. The $j^{\rm ren}_\phi$ gives a contribution of $i\Omega_\infty PJ$. Together we have again

$$S_{\mathcal{B}}=iEP-i\Omega_\infty PJ. \tag{4.44}$$

Again using Eq. (4.40), the free energy takes the form

$$F_N=E-TS-\Omega J \tag{4.45}$$

where $\Omega=\Omega_H-\Omega_\infty$ is the potential relevant for the thermodynamics. So we end up getting the exact same expressions for F_N and F_D (in covariant and canonical approaches, separately).

The emergence of the canonical ensemble together with the Smarr formula implies the first law as well. This follows from the discussion in Ref. [24], so we will not repeat it.

V. ALTERNATIVE QUANTIZATIONS IN ADS

We would like to investigate whether these boundary conditions can define a consistent quantum gravity in AdS. If so, this will provide a set of boundary conditions that are different from the standard Dirichlet boundary conditions familiar from AdS/CFT. On the other hand, a skeptic could choose to think of our discussion as merely a class of well-defined boundary conditions/terms for *classical* gravity in AdS. However, the fact that these boundary conditions give rise to finite actions that lead to correct thermodynamical relations is suggestive to us of an underlying quantum theory: so let us try and explore to see whether we can take these boundary conditions seriously at the quantum level.[10] We will not prove in this paper (but see Ref. [28]) that our approach can be the starting point of a consistent quantum theory, but we will merely make some related observations.

From the boundary theory point of view, the translation from the metric-fixed to stress-tensor-fixed point of view is a Legendre transform that takes the boundary partition function to the boundary effective action.[11] This seems to us to be a perfectly natural and consistent operation as we discussed in Sec. II E, so we believe there should be a legitimate formulation of holography in which the correspondence is phrased in the language of the effective action and not in terms of the generating functional. Note that for this, we will have to move away from the standard Dirichlet formulation of holography where the boundary values of bulk fields are interpreted as sources.

The trouble is that it is well known that (for example) for scalars in a fixed AdS background, of the two modes (which we can call Dirichlet and Neumann) only the Dirichlet mode is typically normalizable [21,37]. The exception to this is when the mass of the scalar falls in the Breitenlohner-Freedman window, where a Legendre transform analogous to ours takes the Dirichlet scalar theory to the Neumann scalar theory, and both are well defined quantum mechanically [21]. When the scalar mass is not in this specific range, there is only one choice of acceptable normalizable mode and a unique quantization in a fixed AdS background.

To understand this better, let us note that the reason why we want normalizable modes is because we want them to be well-defined states in the Hilbert space of the putative quantum theory, with finite norm. This translates to a notion of finite energy: when the scalar mode has finite energy in the bulk of AdS, it can be well defined as a state in the Hilbert space of the quantum theory. This is what happens in the case of scalar quantum field theory in a fixed AdS background [21,37,38].

Now, let us consider the case when the background is not rigid and the metric is allowed to fluctuate. Let us start by considering scalar fields in such a setup. We note two things. One is that a dynamical background makes the notion of energy more subtle, and second the notion of mass of the scalar is ambiguous because (say) a term of the form

$$(m^2 + \lambda R)\phi^2 \tag{5.1}$$

where R is the curvature scalar of the background will look like a usual mass term in the rigid limit. So a nonminimal coupling can sometimes be difficult to distinguish. As it happens both these issues have been addressed in Ref. [25] (see also Refs. [39,40]) and it was found that once one deals with the appropriate notion of (canonical) energy both quantizations are admissible. We will take this as an encouraging fact: when dealing with the full gravity theory with appropriate counterterms etc. it is not necessarily only a Dirichlet boundary condition that can be well defined; the notion of canonical energy needs to take into account the full theory.

Indeed, a similar conclusion was arrived at by Compere and Marolf [14], who considered the possibility of not fixing the boundary metric, and instead considered simply integrating it over in the path integral. At the semiclassical level, the variational principle would then yield

$$\delta S_D^{\rm ren} = \text{Eqs. of motion} + \frac{1}{2}\int_{\partial M} d^d x \sqrt{-g_0}\, T^{ij}\delta g_{0_{ij}}, \tag{5.2}$$

where now there is no assumption that $\delta g_{0_{ij}} = 0$ because we are letting it fluctuate. This means that to ensure that the action is stationary, now we need the boundary (renormalized) stress tensor to vanish.[12] Remarkably, Compere and Marolf found that such boundary metric fluctuations are in fact normalizable with respect to the canonical (symplectic)

[10] C.K. thanks K. Skenderis for comments on (non)normalizable modes and the choice of quantizations in AdS.

[11] A similar approach for scalar fields was taken in Ref. [21]; the source and condensate are dual variables in the Legendre transform sense.

[12] The boundary stress-energy tensor that we have often used in our discussions in this paper is given by the relation

$$T^{\rm ren}_{ij}[\gamma] = -\frac{2}{\sqrt{-\gamma}}\frac{\delta S_D^{\rm ren}}{\delta\gamma^{ij}} \tag{5.3}$$

where the boundary is placed at $\rho = \epsilon$. This is related to the CFT stress tensor (which is the true renormalized stress tensor, and the one we are using in this section) through

$$\begin{aligned} T_{ij} &= \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{d/2-1}}T^{\rm ren}_{ij}[\gamma]\right) = \lim_{\epsilon\to 0}\left(-\frac{2}{\sqrt{-g(x,\epsilon)}}\frac{\delta S_D^{\rm ren}}{\delta g^{ij}}\right)\\ &= -\frac{2}{\sqrt{g_0}}\frac{\delta S_D^{\rm ren}}{\delta g_0^{ij}}. \end{aligned} \tag{5.4}$$

Here, g_0 is the leading term in the Fefferman-Graham expansion.

structure defined by the *full* renormalized Dirichlet action $S_D^{\rm ren}$. Furthermore they also showed that the symplectic structure is also conserved when the boundary condition $T_{ij} = 0$ holds. They further showed that if we couple the full renormalized bulk Dirichlet action above to a boundary action that is a functional of the boundary metric (i.e., the boundary is dynamical), so that the variation now becomes

$$\delta S_D^{\rm bndry} \equiv \delta(S_D + S_{\rm bndry}) = {\rm e.o.m} - \frac{1}{2}\int_{\partial\mathcal{M}} d^d x \sqrt{-g_0} T^{ij} \delta g_{0ij} + \int_{\partial\mathcal{M}} d^d x \frac{\delta S_{\rm bndry}}{\delta g_{0ij}} \delta g_{0ij} = {\rm e.o.m} - \frac{1}{2}\int_{\partial\mathcal{M}} d^d x \sqrt{g_0} \left(T^{ij} - \frac{2}{\sqrt{g_0}} \frac{\delta S_{\rm bndry}}{\delta g_{0ij}} \right) \delta g_{0ij} \tag{5.5}$$

then *again* the claims above hold, if instead of requiring $T^{ij} = 0$ we now require

$$T^{ij} - \frac{2}{\sqrt{g_0}} \frac{\delta S_{\rm bndry}}{\delta g_{0ij}} = 0. \tag{5.6}$$

With that aside, let us turn to our Neumann case. We will merely discuss some connections between our work and that of Compere and Marolf and leave it at that for now. We first note that the usual Dirichlet action plus a boundary term, after a Legendre transform of the kind we discussed, takes the form

$$S_N^{\rm bndry} \equiv S_D + S_{\rm bndry} + \int_{\partial\mathcal{M}} d^d x \frac{\sqrt{g_0}}{2} \left(T^{ij} - \frac{2}{\sqrt{g_0}} \frac{\delta S_{\rm bndry}}{\delta g_{0ij}} \right) g_{0ij}. \tag{5.7}$$

This has the variation

$$\delta S_N^{\rm bndry} = {\rm e.o.m} + \frac{1}{2}\int_{\partial\mathcal{M}} d^d x g_{0ij} \delta \left[\sqrt{g_0} \left(T^{ij} - \frac{2}{\sqrt{g_0}} \frac{\delta S_{\rm bndry}}{\delta g_{0ij}} \right) \right]. \tag{5.8}$$

Note that this is of the Neumann form, but now with boundary dynamics. It would be interesting to see if this leads to normalizable fluctuations, perhaps if one imposes the condition (5.6) that Compere and Marolf did. It is worth mentioning here that what the authors of Ref. [14] called the Neumann boundary condition is (as is often conventional in the gravity literature) the vanishing of T_{ij}. This is the gravitational analogue of starting with the standard *Dirichlet* action in particle mechanics, letting the coordinate q fluctuate at the boundary, but demanding that $\dot{q} = 0$ at the boundary so that the boundary piece dies anyway, so that the variational problem is well defined. A genuine Neumann boundary condition is less constraining: it merely says that the normal derivative/canonical conjugate is *fixed*, and not necessarily zero. This is what we do in this paper.

Of course to conclusively settle this question requires further work, but we suspect that when one takes into account the full dynamics of the system instead of a fixed AdS background, more boundary conditions than what are usually considered will lead to consistent quantum theories. It seems likely that one can discuss the normalizability via the symplectic structure in a covariant phase space approach, and we will report on work in this direction elsewhere.[13]

VI. MICROCANONICAL IN GRAVITY, MICROCANONICAL IN CFT, AND NEUMANN

The Neumann path integral that we have considered in this paper is related to the “microcanonical” path integral that was considered by Brown and York [30]. Their approach amounts to holding *some* of the components of the quasilocal (boundary) stress tensor density fixed, whereas our approach is in some sense more covariant: we hold the entire boundary stress tensor density fixed. We saw that this has a natural interpretation as a Neumann problem, and results in a very simple Neumann action that leads to various nice features, some of which we investigated in Refs. [7,20] as well as this paper.

The path integral of Ref. [30] was called a “microcanonical” functional integral. The motivation of Ref. [30] for this nomenclature was that in gravity, the total charges reduce to surface integrals over the boundary. In Ref. [30] this surface integral was not explicitly done, but we believe this surface integral actually needs to be done in order to get a true charge, and to make the path integral truly “microcanonical” from the gravity perspective.

We would like to emphasize however that even keeping the integrated charge (energy) fixed on the gravity side in the sense of Ref. [30] is not quite the same as holding the conformal field theory (CFT) energy fixed in AdS/CFT. This is because in Ref. [30] the boundary metric is allowed to fluctuate. In AdS/CFT however, in the microcanonical ensemble when we hold the CFT energy fixed, we also hold the metric fixed. If we have infinite resolution, there is no ensemble of states in the CFT satisfying both these conditions.

In AdS/CFT, the natural microcanonical object to hold fixed from the CFT perspective is the total CFT energy, which should be compared to a charge (the boundary stress tensor density is a current from the CFT perspective). In the thermodynamic limit, the microcanonical density of states is a Laplace transform of the canonical partition function [41]. The usual discussion of the Hawking-Page transition in AdS/CFT is in the context of the canonical ensemble, but by doing this Laplace transform we can move to the microcanonical ensemble as well. The resulting discussion is guaranteed to match with the discussion of AdS thermodynamics in the microcanonical ensemble done in

[13] Progress has been made in this direction after the first version of this paper appeared; it will be reported in Ref. [28].

the Hawking-Page paper [42],[14] because the corresponding canonical discussions match.

Our construction, as we have emphasized, is different from both Ref. [30] as well as the AdS/CFT discussion. Morally it is more similar to Ref. [30] because we also do not pin down the metric at the boundary. Our approach could be viewed as an alternate implementation of holography in AdS where the boundary metric is allowed to fluctuate. In a follow-up paper [28], further evidence will be provided (along the lines of the suspicions expressed in Sec. V) that these boundary conditions may be consistent boundary conditions for quantum gravity in AdS: we will find that in odd d the fluctuations are normalizable, and that in even d, normalizability of the bulk fluctuations is guaranteed when the dynamics of the boundary metric is controlled by conformal gravity. Another direction that is being explored is the possibility of doing a similar renormalized construction for flat space Neumann gravity along the lines of the Dirichlet case discussed by Mann and Marolf [43]. We have recently also constructed Robin boundary terms for gravity. Considering the fact that the Dirichlet boundary term [1,2] has had numerous applications since its inception more than 40 years ago, perhaps it is not surprising that the Neumann term [7] also leads to natural applications and generalizations.

ACKNOWLEDGMENTS

We thank Pallab Basu, Justin David, Jarah Evslin, Bob McNees, Shahin Sheikh-Jabbari, Kostas Skenderis and Sergey Solodukhin for questions, comments and/or discussions. C. K. thanks the organizers and attendees at conferences at Bangalore, Phitsanulok and Yerevan, for discussions and hospitality.

APPENDIX A: ASYMPTOTIC SOLUTION

The relation between the various g_i's (with $i < d$) in the Fefferman-Graham expansion is determined by solving Einstein's equation iteratively. This was worked out in detail in Ref. [23] and here we collect some useful results for completeness. The indices below are raised with the metric g_0.

The determinant of the induced metric on $\rho = \epsilon$ boundary can be expanded as follows:

$$\sqrt{-g} = \sqrt{-g_0}\bigg(1 + \frac{1}{2}\epsilon \mathrm{Tr}(g_0^{-1}g_2) + \frac{1}{8}\epsilon^2((\mathrm{Tr}(g_0^{-1}g_2))^2 - \mathrm{Tr}(g_0^{-1}g_2)^2) + O(\epsilon^3)\bigg). \tag{A1}$$

[14]This discussion is in the last section of their paper, and is not as well known as their canonical discussion. The only thing relevant for our purposes here is that they change ensembles via the aforementioned Laplace transform.

The leading coefficients g_n for $n \neq d$ are given by[15]

$$g_{2\,ij} = -\frac{1}{(d-2)}\left(R_{ij} - \frac{1}{2(d-1)}Rg_{0ij}\right), \tag{A2}$$

$$\begin{aligned} g_{4\,ij} = \frac{1}{(d-4)}\bigg(&\frac{1}{8(d-1)}D_iD_jR - \frac{1}{4(d-2)}D^kD_kR_{ij} \\ &+ \frac{1}{8(d-1)(d-2)}g_{0ij}D^kD_kR \\ &- \frac{1}{2(d-2)}R^{kl}R_{\mathrm{ikjl}} + \frac{(d-4)}{2(d-2)^2}R_i^kR_{kj} \\ &+ \frac{1}{(d-1)(d-2)^2}RR_{ij} + \frac{1}{4(d-2)^2}R^{kl}R_{kl}g_{0ij} \\ &- \frac{3d}{16(d-1)^2(d-2)^2}R^2g_{0ij}\bigg). \end{aligned} \tag{A3}$$

For $n = d$, one can obtain the trace and divergence of g_n as well as the coefficient of the logarithmic term h_d from Einstein's equation and we refer the reader to Appendix A of Ref. [23]. The on-shell g_2 is determined in terms of the induced metric γ as [23]

$$\begin{aligned} \mathrm{Tr}g_2 &= \frac{1}{2\epsilon(d-1)}\bigg(-R[\gamma] + \frac{1}{(d-2)}\bigg(R_{ij}[\gamma]R^{ij}[\gamma] - \frac{1}{2(d-1)}R^2[\gamma]\bigg) + O[R^3[\gamma]]\bigg), \\ \mathrm{Tr}g_2^2 &= \frac{1}{(d-2)^2\epsilon^2}\bigg(R_{ij}[\gamma]R^{ij}[\gamma] + \frac{4-3d}{4(d-1)^2}R^2[\gamma] + O[R^3[\gamma]]\bigg). \end{aligned} \tag{A4}$$

APPENDIX B: LEGENDRE TRANSFORM APPROACH

The Neumann action can be thought of as a boundary Legendre transform of the Dirichlet action. The Neumann and Dirichlet actions are related by [20]

$$S_N^{\mathrm{ren}} = S_D^{\mathrm{ren}} - \int_{\partial\mathcal{M}} d^{D-1}x\pi_{ab}^{\mathrm{ren}}\gamma^{ab} \tag{B1}$$

where $\pi_{ab}^{\mathrm{ren}} = \frac{\delta S_D^{\mathrm{ren}}}{\delta\gamma^{ab}}$. π_{ab}^{ren} is further related to the renormalized boundary stress tensor as

$$\pi_{ab}^{\mathrm{ren}} = -\frac{\sqrt{-\gamma}}{2}T_{ab}^{\mathrm{ren}}. \tag{B2}$$

So, given the renormalized action and the boundary stress tensor for the Dirichlet case, we can use the above relations

[15]Our convention for the Ricci tensor and Ricci scalar differ from Ref. [23] by a minus sign.

between the Dirichlet and Neumann action to obtain a renormalized action for the Neumann case. This serves as an independent check of the holographic renormalization of the Neumann case and we will go through each case ($D = 3, 4, 5$) separately here.

1. AdS_3

The renormalized Dirichlet action and stress tensor for AdS_3 are given by [22,23]

$$S_D^{\text{ren}} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^3x\sqrt{-g}(R-2\Lambda) + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^2x\sqrt{-\gamma}\Theta - \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^2x\sqrt{-\gamma} \tag{B3}$$

and

$$T_{ab}^{\text{ren}} = \frac{1}{\kappa}(\Theta_{ab} - \Theta\gamma_{ab} + \gamma_{ab}) \tag{B4}$$

where we have set $l = 1$. Using Eqs. (B1) and (B2) we immediately see that

$$S_N^{\text{ren}} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^3x\sqrt{-g}(R-2\Lambda) + \frac{1}{2\kappa}\int_{\partial\mathcal{M}} d^2x\sqrt{-\gamma}\Theta \tag{B5}$$

which matches with the renormalized Neumann action obtained by holographic renormalization.

2. AdS_4

In AdS_4, the renormalized Dirichlet action and stress tensor are [22,23] (for $l = 1$)

$$S_D^{\text{ren}} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R-2\Lambda) + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-\gamma}\Theta - \frac{2}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-\gamma}\left(1+\frac{^{(3)}R}{4}\right) \tag{B6}$$

and

$$T_{ab}^{\text{ren}} = \frac{1}{\kappa}(\Theta_{ab} - \Theta\gamma_{ab} + 2\gamma_{ab} - {}^{(3)}G_{ab}). \tag{B7}$$

Using Eqs. (B1) and (B2) we obtain

$$S_N^{\text{ren}} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R-2\Lambda) + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-\gamma}\left(1-\frac{^{(3)}R}{4}\right) \tag{B8}$$

which is in agreement with the renormalized Neumann action obtained by holographic renormalization.

3. AdS_5

For the case of AdS_5 the renormalized action and stress tensor are given by [22,23] (for $l = 1$)

$$S_D^{\text{ren}} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^5x\sqrt{-g}(R-2\Lambda) + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}\Theta - \frac{3}{\kappa}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}\left(1+\frac{^{(4)}R}{12}\right) \tag{B9}$$

and

$$T_{ab}^{\text{ren}} = \frac{1}{\kappa}\left(\Theta_{ab} - \Theta\gamma_{ab} + 3\gamma_{ab} - \frac{1}{2}{}^{(4)}G_{ab}\right). \tag{B10}$$

Using Eqs. (B1) and (B2) we once again obtain the renormalized Neumann action which matches with the one obtained by holographic renormalization

$$S_N^{\text{ren}} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^5x\sqrt{-g}(R-2\Lambda) - \frac{1}{2\kappa}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}\Theta + \frac{3}{\kappa}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}. \tag{B11}$$

APPENDIX C: GAUSS-CODAZZI-RICCI RELATIONS

Gauss-Codazzi relations help us express the spacetime curvature tensors in terms of the intrinsic and extrinsic curvatures of the embedding hypersurface. They can be summarized as follows:

$$\begin{aligned} R + 2R_{ab}u^a u^b &= {}^{(2)}R - \hat{k}_{ab}\hat{k}^{ab} + \hat{k}^2, \\ \sigma_{ab}u_c R^{bc} &= d_a\hat{k} - d_b\hat{k}^b_a, \\ \sigma_{ac}\sigma_{bd}R^{cd} &= -\frac{1}{N}\mathcal{L}_m\hat{k}_{ab} - \frac{1}{N}d_a d_b N + {}^{(2)}R_{ab} \\ &\quad + \hat{k}\hat{k}_{ab} - 2\hat{k}_{ac}\hat{k}^c_b \end{aligned} \tag{C1}$$

where $\mathcal{L}_m$ refers to the Lie derivative with respect to the vector $m^a = Nu^a$, d_a is the covariant derivative with respect to the metric σ_{ab} and $\hat{k}_{ab}$ is the extrinsic curvature of B embedded in $\mathcal{B}$.

The last of these relations does not arise as commonly as the first two; we refer the reader to Ref. [44]. We need all three of them in our simplifications of the ADM version of the renormalized actions.

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