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Scale versus conformal invariance from entanglement entropy

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For a generic conformal field theory (CFT) in four dimensions, the scale anomaly dictates that on conformally flat backgrounds, the universal part of entanglement entropy across a sphere $[\mathcal{C}_{\text{univ}}(S^2)]$ is positive. Based on this fact, we explore the consequences of assuming a positive sign for $\mathcal{C}_{\text{univ}}(S^2)$ on such backgrounds in a four-dimensional scale-invariant theory (SFT). In the absence of a dimension two scalar operator $\mathcal{O}_2$ in the spectrum of a SFT, we show that this assumption suggests that SFT is a CFT. In the presence of $\mathcal{O}_2$, we show that this assumption can fix the coefficient of the nonlinear coupling term $\int d^4x\sqrt{g}R\mathcal{O}_2$ to a conformal value.

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I. INTRODUCTION

The asymptotic structure of Poincaré invariant unitary quantum field theories in deep UV and IR is of great importance in physics. A deep understanding of this issue is achievable via the profound idea of Wilson [1]. According to this idea, the fixed points of a renormalization group (RG) are the dwellings of those asymptotics, and therefore, the asymptotic theories are scale invariant. Other new dwellings are the renormalization group limit cycles which also describe the scale-invariant field theories. Remarkably, with a few known exceptions, unitary scale-invariant theories (SFT's) always exhibit full conformal symmetry. A natural question is whether it is possible for a theory to be scale invariant but not conformal invariant? The converse question, i.e., whether a theory can be invariant under conformal transformations but not under scaling, is easy to answer. The commutator between the conserved generators of translations and conformal transformations gives the scaling generator together with the Lorentz ones. This means that Poincaré plus conformal invariance comprises scale invariance. The converse is still an open question since Poincaré and scaling generators form a closed algebra.

Recently, there were considerable efforts to answer this question. The task has been done in some spacetime dimensions, but the problem is still open for $D = 4$. Although some comprehensive arguments are available in 4D, they still suffer from a serious loophole. In this paper we study the problem of scale vs conformal invariance in 4D by making use of entanglement entropy. For a generic conformal field theory (CFT) in 4D, the scale anomaly dictates that on conformally flat backgrounds, the universal part of entanglement entropy across a sphere $[\mathcal{C}_{\text{univ}}(S^2)]$ is positive [2]. Based on this fact, we explore the consequences of assuming a positive sign for $\mathcal{C}_{\text{univ}}(S^2)$ on such backgrounds in a 4D SFT. In the absence of a dimension two scalar operator $\mathcal{O}_2$ in the spectrum of a SFT, we show that this assumption suggests that the SFT actually is a CFT. In the presence of $\mathcal{O}_2$, which is actually related to the loophole in previous studies, we show that this assumption fixes the coefficient of the nonlinear coupling term $\int d^4x\sqrt{g}R\mathcal{O}_2$ to a conformal value.

The paper is organized as follows. The first section is devoted to a comprehensive review on previous studies on the subject of scale vs conformal invariance by emphasizing on 4D. Since our work is highly based on using a scale anomaly in SFT's, we will dedicate some parts of the first section to this topic and its crucial rule in the subject of scale vs conformal invariance. Also in this section, the remaining problem in previous studies is mentioned. In Sec. II we study scale vs conformal invariance in 4D via entanglement entropy. Finally in the last section, we will discuss the possible appearance of a scale anomaly in other measures of entanglement and their rule in the problem of scale vs conformal invariance.

II. PREVIOUS ATTEMPTS ON SCALE VS CONFORMAL INVARIANCE

In $D = 2$, based on the argument of Zamolodchikov [3], Polchinski proved that any unitary SFT exhibits full conformal symmetry [4]. Polchinski assumed that a unitary 2D SFT has a well-defined energy-momentum tensor together with a discrete spectrum and finite energy-momentum two-point function. Later on, Riva and Cardy presented a model with scale but without conformal symmetry [5]. However, their model does not violate Polchinski's argument because it does not have reflection positivity, the Euclidean version of unitarity, and, more precisely, it does not have a discrete spectrum. An earlier model by Hull and Townsend [6], which seems to be in contradiction with the Polchinski proof, is not also a

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counterexample, because this model violates the assumption of having well-behaved energy-momentum two-point function. More recent proposed counterexamples also violate one of the assumptions of the theorem (unitarity, existence, and finiteness of correlators) [7,8].

For $D \geq 3$, the situation was unclear until 2011. Actually, all of the perturbative fixed points, which were introduced in the preexisting literature, belonged to two general categories. In the first category, the fixed points come from the RG flow in theories that do not have any candidate for virial current and, therefore, that fixed points were automatically conformal invariant [9,10]. In the second category, which is more interesting, although the studied theories have a nontrivial candidate for virial current, at the fixed points no virial current appears and therefore they also exhibit full conformal symmetry [4,11]. Consequently, a general conjecture seemed to be that the Zamolodchikov-Polchinski theorem is even true in $D \geq 3$, even though a proof has not been available. Interestingly, in 2011 it was demonstrated that this conjecture is false, at least in $D = 3$ and in $D \geq 5$ [12]. The counterexample is simply the free Maxwell theory. This scale-invariant field theory is unitary—it has a well-defined energy-momentum tensor and also has a discrete spectrum, but it is not a CFT.

Therefore, we remain with $D = 4$. First, it is shown that at all $4D$ perturbative fixed points, the scale symmetry is enhanced to the full conformal symmetry [13,14]. Indeed, the approach of [13] is based on the idea of Komargodski-Schwimmer's a-theorem, while [14] is based on the concept of the local Callan-Symanzik equation. The argument in [13] holds even for theories with gravitational anomalies. Furthermore, it is argued that perturbative scale-invariant trajectories correspond to rare RG flows, namely limit cycles with nonvanishing beta functions [15], also enjoy the benefit of conformal symmetry [18–21]. Moreover in [13] it was proposed that a scale anomaly can be used to understand the scale vs conformal invariance at a non-perturbative level. Anomalies are caused by quantum effects. At the classical level a general SFT has a local conserved scale current S^μ [22]

$$S^\mu = x^\nu T^\mu_\nu + V^\mu, \tag{1}$$

where $T_{\mu\nu}$ denotes the energy-momentum tensor and V^μ is the so-called "virial current." Conservation of scale current gives

$$0 = \partial_\mu S^\mu = T^\mu_\mu + \partial_\mu V^\mu, \tag{2}$$

which means that for scale-invariant theories $T^\mu_\mu = -\partial_\mu V^\mu$. Note that we have used the fact that the energy-momentum tensor is conserved. Obviously, if the virial current in a SFT is conserved, that SFT is actually a CFT. The less obvious case in which a unitary SFT would be a CFT is when the virial current is a total derivative, i.e.,

$$V_\mu = \partial_\mu L. \tag{3}$$

In such a case, one can find an improved energy-momentum tensor

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + \frac{1}{3}(\partial_\mu \partial_\nu - \eta_{\mu\nu}\Box)L, \tag{4}$$

which is conserved and traceless [4,23,24]. In the following, by SFT we mean a theory in which its virial current is neither conserved nor a total derivative. At the quantum level, in general, scale invariance may be broken by anomalies. The anomalies can be represented in terms of the Wess-Zumino action. In order to proceed, a convenient formalism is to introduce background fields $g_{\mu\nu}$ and C_μ as a source for $T_{\mu\nu}$ and V_μ, respectively. In this way,

$$e^{W[g_{\mu\nu},C_\mu]} = \int d[\varphi] e^{-S[\varphi; g_{\mu\nu},C_\mu]},$$
$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}, \qquad V^\mu = -\frac{1}{\sqrt{g}} \frac{\delta S}{\delta C_\mu}, \tag{5}$$

where W is the generating functional of connected graphs. Under the generalized Weyl transformation [25]

$$\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}, \qquad \delta_\sigma C_\mu = \partial_\mu \sigma, \tag{6}$$

we have

$$\delta_\sigma W = \int d^4x \left(-2\sigma g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} + \partial_\mu \sigma \frac{\delta}{\delta C_\mu}\right) W,$$
$$= \int d^4x \sqrt{g} \sigma \langle T^\mu_\mu + \nabla_\mu V^\mu \rangle. \tag{7}$$

If the SFT is nonanomalous, $\delta_\sigma W$ vanishes. But in the presence of an anomaly in general, we have

$$\delta_\sigma W = S_{\mathrm{WZ}}|_\sigma, \tag{8}$$

which results in

$$\int d^4x \sqrt{g} \sigma \langle T^\mu_\mu + \nabla_\mu V^\mu \rangle = S_{\mathrm{WZ}}|_\sigma. \tag{9}$$

Here, $S_{\mathrm{WZ}}|_\sigma$ denotes those terms in a Wess-Zumino action (S_{WZ}), which are linear in σ. The most general parity even Wess-Zumino action involving the metric and the gauge field C_μ for a 4D SFT is given by [13,26]

$$S_{\mathrm{WZ}}[g_{\mu\nu}, C_\mu; \sigma] = \int d^4x\sqrt{g}\Big\{-a\Big[\sigma E_4 + 4\Big(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\Big) \times \partial_\mu\sigma\partial_\nu\sigma - 4(\partial\sigma)^2\Box\sigma + 2(\partial\sigma)^4\Big] + c\sigma W^2 - e\sigma\Sigma^2 + f\sigma C_{\mu\nu}C^{\mu\nu}\Big\}, \quad (10)$$

where E_4 and W^2 are the Euler density and square of the Weyl tensor, respectively, and

$$\Sigma = \frac{1}{6}R + \nabla_\mu C^\mu - C_\mu C^\mu, \quad C_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu. \quad (11)$$

The coefficients a and c are the standard conformal anomaly coefficients of a CFT, while the e and f terms appear only in a SFT [27]. It should be noted that in the presence of a dimension two scalar operator $\mathcal{O}_2$, the term $\xi\int d^4x\sqrt{g}\Sigma\mathcal{O}_2$ can be added to the action, which only shifts the anomaly coefficient e [26,28].

According to Eqs. (9) and (10), under the global scale transformations we have

$$\int d^4x\sqrt{g}\sigma\langle T^\mu_\mu + \nabla_\mu V^\mu\rangle|_{C_\mu=0} = \int d^4x\sqrt{g}\sigma(-aE_4 + cW^2 - \tilde{e}R^2), \quad (12)$$

where the normalized $\tilde{e} \equiv \frac{e}{36}$ is introduced. The e anomaly plays a crucial role in the problem of scale vs conformal invariance at a nonperturbative level which can be understood as follows. From (12), the two-point function of the trace of energy-momentum tensor in a 4D flat anomalous scale-invariant theory is given by [26,29]

$$\langle T(q)T(-q)\rangle = -\tilde{e}q^4\log\frac{q^2}{\mu^2} + B(\mu)q^4, \quad (13)$$

where μ is an arbitrary renormalization scale and $B(\mu)$ is a scheme-dependent constant. It is shown that unitarity imposes $\tilde{e} \geq 0$ [26,31]. Note that the Fourier transformation of the q^4 term in (13) is a derivative of the delta function, so if $\tilde{e} = 0$ we have

$$\langle T(x)T(0)\rangle = 0, \quad x \neq 0. \quad (14)$$

This means that in a unitary theory, T must be equal to zero as an operator identity, and the scale-invariant theory becomes fully conformal. It should be noted that to have a CFT in the presence of $\mathcal{O}_2$, $\tilde{e}$ is not necessarily zero and should satisfy another condition [31]. When this condition holds, one may improve T such that the new T vanishes. Based on these observations, it was argued that the structure of a special anomalous 3-point function in any SFT is not compatible with operator product expansions (OPEs); this implies that the e term must vanish, and thus all unitary SFTs are CFTs [26]. Later on, the authors of [31] pointed out a subtlety in the relation between OPEs and the large momentum limit which invalidates this argument. While the OPE controls the leading nonlocal contribution in the large momentum limit, there are semilocal contributions which dominate over the OPE contribution in the relevant case, and therefore, the statement in [26] is false. After that, based on the proof of the a-theorem and using the concept of dilaton scattering amplitudes, it is argued that unitary SFTs must be either CFTs, or the trace of the energy-momentum tensor behaves like a generalized free field [23]. Moreover, it is shown that if no scalar operator of dimension precisely 2 appears in the spectrum of a SFT, in which its energy-momentum tensor is a generalized free field, that theory would be conformal [24]. In the presence of a scalar operator with dimension precisely 2, which can mix with T, one can show that there is at least one improvement such that an improved T is not a generalized free field [24]. Thus the only loophole that remains in the proof of [24] is the case where the energy-momentum tensor is a generalized free field and the scalar operator with dimension precisely 2 exists in the spectrum [32].

In the next section we explore some consequences of assuming a positive sign for $\mathcal{C}_{\mathrm{univ}}(S^2)$ in the subject of scale vs conformal invariance, specially in the case where a dimension two scalar operator exists in the spectrum of a SFT.

III. ENTANGLEMENT ENTROPY AND SCALE VS CONFORMAL

The properties of nonlocal quantities are important as the correlation functions of local operators in a given quantum field theory. In particular, they are important for the understanding of quantum phase structures. One of the important nonlocal physical quantities is the Wilson loop operator in gauge theories, which is a very useful order parameter for the understanding of the confinement [34]. Quantum entanglement (QE) is also a momentous nonlocal quantity in more generic QFTs. QE has made an increasingly dominant impression on the understanding of quantum complex systems in a diverse set of areas including condensed matter physics [35–39], quantum information theory [40–42], and quantum gravity [43–50]. One of the measures of QE is entanglement entropy (EE). Considering a pure state of a relativistic SFT defined on a $3+1$ dimensional manifold $\mathcal{M}$, EE is defined by tracing out those modes which reside outside an entangling region Υ. This entangling region is a submanifold of $\mathcal{M}$ at a fixed time. The result of the trace-out action is a mixed state ρ_Υ. In order to calculate EE, one should first obtain the $\mathrm{Tr}_\Upsilon(\rho^n_\Upsilon)$ and find the Rényi entropy

$$S_n(\rho_\Upsilon) = \frac{1}{1-n}\log\,\mathrm{Tr}_\Upsilon(\rho^n_\Upsilon), \quad (15)$$

where n is a positive integer. Upon analytically continuing n to positive real values, one can take the limit $n \to 1$ to obtain the entanglement, or von Neumann entropy as

$$S_{\rm EE} = \lim_{n\to 1} S_n = -\partial_n \log {\rm Tr}_\Upsilon(\rho_\Upsilon^n)|_{n=1}. \qquad (16)$$

Furthermore, the ${\rm Tr}_\Upsilon(\rho_\Upsilon^n)$ can be computed from the partition function Z_n on a n-sheeted $3+1$ dimensional manifold $\mathcal{M}_n$ as

$$\log {\rm Tr}_\Upsilon(\rho_\Upsilon^n) = \log Z_n - n \log Z_1. \qquad (17)$$

Thus Eq. (16) becomes

$$S_{\rm EE} = -\partial_n (\log Z_n - n \log Z_1)|_{n=1}. \qquad (18)$$

For the closed connected surface Υ, we can define a length scale s. Therefore by using Eq. (6) together with (17) and (7) we have

$$s\frac{d}{ds}\log {\rm Tr}_\Upsilon(\rho_\Upsilon^n) = \int_{\mathcal{M}_n} d^4x\sqrt{g}\langle T^\mu_\mu + \nabla_\mu V^\mu\rangle|_{C_\mu=0} - n\int_{\mathcal{M}_1} d^4x\sqrt{g}\langle T^\mu_\mu + \nabla_\mu V^\mu\rangle|_{C_\mu=0}. \qquad (19)$$

The above result together with (18) and (12) gives

$$s\frac{d}{ds}S_{\rm EE} = -\partial_n \int_{\mathcal{M}_n} d^4x\sqrt{g}(-aE_4 + cW^2 - \tilde{e}R^2)|_{n=1} + \int_{\mathcal{M}_1} d^4x\sqrt{g}(-aE_4 + cW^2 - \tilde{e}R^2). \qquad (20)$$

The n-sheeted $3+1$ dimensional manifold $\mathcal{M}_n$, in general, contains conical singularities. The procedure of calculating the integral of metric curvatures on manifolds with conical singularities has been developed in [51,52]. According to that procedure, we have

$$\int_{\mathcal{M}_n} d^4x\sqrt{g}E_4 = n\int_{\mathcal{M}_1} d^4x\sqrt{g}E_4 + 8\pi(1-n) \times \int_{\partial\Upsilon} d^2\chi\sqrt{\gamma}R[\gamma] + \mathcal{O}(1-n)^2$$

$$\int_{\mathcal{M}_n} d^4x\sqrt{g}W^2 = n\int_{\mathcal{M}_1} d^4x\sqrt{g}W^2 + 8\pi(1-n) \times \int_{\partial\Upsilon} d^2\chi\sqrt{\gamma}K[g;t,s;\mathcal{K}^\alpha_{ij}] + \mathcal{O}(1-n)^2,$$

$$\int_{\mathcal{M}_n} d^4x\sqrt{g}R^2 = n\int_{\mathcal{M}_1} d^4x\sqrt{g}R^2 + 8\pi(1-n) \times \int_{\partial\Upsilon} d^2\chi\sqrt{\gamma}R[g] + \mathcal{O}(1-n)^2, \qquad (21)$$

where

$$K[g;t,s;\mathcal{K}^\alpha_{ij}] = 2W_{\mu\nu\alpha\beta}t^\mu s^\nu t^\alpha s^\beta - [\mathcal{K}^\alpha_{ij}\mathcal{K}^{\alpha ij} - \frac{1}{2}(\mathcal{K}^{\alpha i}_i)^2], \qquad (22)$$

and g is the full 4D metric. Furthermore, γ_{ij} and $\mathcal{K}^\alpha_{ij}$ are the intrinsic metric and the extrinsic curvature of $\partial\Upsilon$, $\alpha = \{t, s\}$ indexing the two normal directions (one timelike t^μ and one spacelike s^μ) and the first term on the right-hand side of (22) is nothing but the pullback of the Weyl tensor onto $\partial\Upsilon$. Using the relations (21) in (20) one arrives at [53]

$$s\frac{d}{ds}S_{\rm EE} = -8\pi\int_{\partial\Upsilon} d^2\chi\sqrt{\gamma}(aR[\gamma] - cK[g;t,s;\mathcal{K}^\alpha_{ij}] + \tilde{e}R[g]). \qquad (23)$$

The right-hand side of (23), in the absence of the e term, is indeed the Graham-Witten anomaly [56] for a two-dimensional submanifold $\partial\Upsilon$ on the D-dimensional CFT [57]. The holographic realization of these anomalies comes from studying the Einstein spaces in the bulk, which are asymptotically locally AdS manifolds (AlAdS). The former statement means that in the presence of the e term, the right-hand side of (23) could be considered as (generalized) Graham-Witten anomalies for a two-dimensional submanifold on the D-dimensional SFT. To check this proposition one could redo the machinery of Graham and Witten for non-AlAdS manifolds, such as geometries in the foliation preserving diffeomorphic theory of gravity [58].

The point which should be stressed here is that $S_{\rm EE}$ is a UV divergent quantity in a continuum QFT. It has a universal part ($\mathcal{C}_{\rm univ}$), which is defined as its cutoff-independent term, which contains nontrivial physical information, including central charges and RG monotones [59–62]. Furthermore, $s\frac{d}{ds}S_{\rm EE}$ is equal to the minus of the $\mathcal{C}_{\rm univ}$ [63]. In many respects, these universal terms are the natural counterparts of quantum-mechanical entropies, which suggest that, in QFT, the $\mathcal{C}_{\rm univ}$ is also positive definite. Indeed, for spherical entangling surfaces ($\partial\Upsilon = S^2$) in the vacuum state of CFTs in flat (conformally flat) spacetime, this appears to be true [2,60–62]. Note that one can always pick complex enough entangling surfaces to violate this positivity [2]. In this paper, we would especially like to study the effect of an e anomaly on the sign of $\mathcal{C}_{\rm univ}(S^2)$. We take a conformally flat metric, $g_{\mu\nu} = e^{-2\tau}\eta_{\mu\nu}$, as a background metric. Because the $K[g;t,s;\mathcal{K}^\alpha_{ij}]$ is a Weyl invariant, it does not contribute to $\mathcal{C}_{\rm univ}(S^2)$. Moreover by noting that

$$\int_{S^2} d^2\chi\sqrt{\gamma}R[g]|_{g=e^{-2\tau}\eta} = 6\int_{S^2} d^2\chi\sqrt{\gamma_\eta}[\Box\tau - (\partial\tau)^2], \qquad (24)$$

from (23) we have

$$\mathcal{C}_{\rm univ}(S^2) = 16\pi\left(a + 3\tilde{e}\int_{S^2} d^2\chi\sqrt{\gamma_\eta}[\Box\tau - (\partial\tau)^2]\right). \quad (25)$$

Remember that in a unitary SFT, $\tilde{e} \geq 0$. By assuming $\tilde{e} > 0$, one can check that for any positive value of a, there exists a function τ for which the $\mathcal{C}_{\rm univ}(S^2)$ becomes negative. On the other hand, for a generic CFT in 4D, the scale anomaly dictates that on conformally flat backgrounds, $\mathcal{C}_{\rm univ}(S^2)$ is positive [2]. By assuming that $\mathcal{C}_{\rm univ}(S^2)$ on such backgrounds is also positive for a SFT [64], Eq. (25) implies that $\tilde{e} = 0$. Thus, in the absence of a dimension two scalar operator $\mathcal{O}_2$ in the spectrum of a SFT, we have shown that the positivity of $\mathcal{C}_{\rm univ}(S^2)$ suggests that a SFT is a CFT.

Furthermore, as we mentioned in the previous section, the only loophole in the proof of [24] is related to the case where the trace of an energy-momentum tensor is a generalized free field and a scalar operator with dimension of precisely 2 exists in the spectrum. Also we noted that, in the presence of $\mathcal{O}_2$, one can add the term $\xi\int d^4x\sqrt{g}R\mathcal{O}_2$ to the action in order to change the trace of an energy-momentum tensor. This means that the universal part of EE can be changed by adding this nonlinear coupling term. To be more precise, this nonlinear term just shifts the e anomaly coefficient [26] in Eq. (25)

$$\mathcal{C}_{\rm univ}(S^2) = 16\pi(a + 3(\tilde{e} - \alpha\xi)\int_{S^2} d^2\chi\sqrt{\gamma_\eta} \times [\Box\tau - (\partial\tau)^2]), \quad (26)$$

where α is a positive number. For example, for a free scalar theory, the universal part of EE is calculated by using a heat Kernel method [52] which leads to $\tilde{e} = \frac{1}{72}$ and $\alpha = \frac{1}{12}$. Interestingly, the positivity of $\mathcal{C}_{\rm univ}(S^2)$ fixes the coefficient of the nonlinear coupling term to $\xi = \frac{\tilde{e}}{\alpha}$, where for the free scalar theory it becomes $\xi = \frac{1}{6}$. This value for ξ is exactly the one to have a conformal scalar theory. This means that in free scalar theory, the positivity of $\mathcal{C}_{\rm univ}(S^2)$ suggests that the theory can be improved to a CFT.

IV. DISCUSSION

In the previous section, we have shown that the existence of an e anomaly can affect the sign of $\mathcal{C}_{\rm univ}(S^2)$, which plays a crucial role in the subject of scale vs conformal invariance in $D = 4$. For a generic CFT in four dimensions, the scale anomaly dictates that the $\mathcal{C}_{\rm univ}(S^2)$ on conformally flat backgrounds is positive. Based on this fact, we have explored the consequences of assuming a positive sign for $\mathcal{C}_{\rm univ}(S^2)$ on such backgrounds in a four-dimensional SFT. In the absence of a dimension two scalar operator $\mathcal{O}_2$ in the spectrum of a SFT, we have shown that this assumption suggests that SFT is a CFT. In the presence of $\mathcal{O}_2$ in a SFT, we have shown that this assumption fixes the coefficient of the nonlinear coupling term $\int d^4x\sqrt{g}R\mathcal{O}_2$ to a conformal value.

The e anomaly may have an effect on strong subadditivity (SSA) inequalities. SSA inequalities state that, given a tripartite quantum system A, B, C and a joint density matrix $\rho(ABC)$, the EEs of the subsystems obey the following inequalities:

$$S_{\rm EE}(AB) + S_{\rm EE}(BC) - S_{\rm EE}(ABC) - S_{\rm EE}(B) \geq 0,$$
$$S_{\rm EE}(AB) + S_{\rm EE}(BC) - S_{\rm EE}(A) - S_{\rm EE}(C) \geq 0. \quad (27)$$

SSA is a general theorem that depends only on basic facts about Hilbert spaces and the definition of the von Neumann entropy [45]. It is obeyed as long as the bulk spacetime satisfies the null energy condition (NEC) [65,66]. In general, it is believed that the NEC is related to unitarity [60]. Therefore, if in the presence of an e anomaly SSA inequalities are violated, the theory is nonunitary and therefore any unitary 4D SFT is a CFT.

The e anomaly can also affect other measures of QE. For a mixed state, the EE is no longer a good measure of entanglement since it mixes quantum and classical correlations. An interesting computable measurement of entanglement for the mixed states is the logarithmic negativity (LN) [67–69], which gives an upper bound on distillable entanglement in quantum mechanics, and is thus strictly greater than the EE. It is argued that the universal part of LN is also related to the scale anomalies, and for CFTs, it is positive definite across spherical entangling surfaces [2,70–72]. Therefore a natural question would be what happens to the sign of the universal part of LN in the presence of an e anomaly? To answer this question one should calculate Rényi entropies in SFTs. This might be done using the method of [73].

The e anomaly may also appear in nonlocal measures of quantum phase transitions (QPT). One of these nonlocal measures is EE. In the vicinity of QPTs, EE obeys a scaling behavior [74–76] and its universal properties have been investigated in a family of models [74,75]. Many other studies of different measures of QPTs have been presented recently. For example QPTs are characterized in terms of the overlap (fidelity) function between two ground states obtained for two close values of external parameters [77–79]. At the critical point, fidelity shows a peak. This overlap suggests that fidelity may capture some information about finite size scaling and universality classes. Interestingly, the holographic counterpart of the fidelity is proposed very recently in [80,81]. For sure, studying the effect of an e anomaly on critical exponents and comparing them with simulations may help us to have a better understanding of scale vs conformal invariance.

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