PHYSICAL REVIEW D **94,** 125005 (2016)

Scheme-independent series expansions at an infrared zero of the beta function in asymptotically free gauge theories

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(Received 1 October 2016; published 2 December 2016)

We consider an asymptotically free vectorial gauge theory, with gauge group G and N_f fermions in a representation R of G, having an infrared (IR) zero in the beta function at $\alpha_{\rm IR}$. We present general formulas for scheme-independent series expansions of quantities, evaluated at $\alpha_{\rm IR}$, as powers of an N_f-dependent expansion parameter, Δ_f. First, we apply these to calculate the derivative $d\beta/d\alpha$ evaluated at $\alpha_{\rm IR}$, denoted $\beta'_{\rm IR}$, which is equivalent to the anomalous dimension of the ${\rm Tr}(F_{\mu\nu}F^{\mu\nu})$ operator, to order Δ_f^4 for general G and R, and to order Δ_f^5 for $G = {\rm SU}(3)$ and fermions in the fundamental representation. Second, we calculate the scheme-independent expansions of the anomalous dimension of the flavor-nonsinglet and flavor-singlet bilinear fermion antisymmetric Dirac tensor operators up to order Δ_f^3. The results are compared with rigorous upper bounds on anomalous dimensions of operators in conformally invariant theories. Our other results include an analysis of the limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c fixed, calculation and analysis of Padé approximants, and comparison with conventional higher-loop calculations of $\beta'_{\rm IR}$ and anomalous dimensions as power series in α.

DOI: 10.1103/PhysRevD.94.125005

I. INTRODUCTION

The evolution of an asymptotically free gauge theory from the ultraviolet (UV) to the infrared is of fundamental importance. The evolution of the running gauge coupling $g = g(\mu)$, as a function of the Euclidean momentum scale, μ, is described by the renormalization-group (RG) beta function [1], $\beta_g = dg/dt$ or equivalently,

$$\beta = \frac{d\alpha}{dt} = \frac{g}{2\pi}\beta_g, \tag{1.1}$$

where $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $dt = d\ln\mu$ (the argument μ will often be suppressed in the notation). Here we consider an asymptotically free (AF) vectorial gauge theory with non-Abelian, Yang-Mills gauge group G and N_f copies (flavors) of fermions ψ_j, $j = 1, \ldots, N_f$ transforming according to the representation R of G. We take the fermions to be massless, since a massive fermion with mass m_0 would be integrated out of the effective field theory at scales $\mu < m_0$ [2] and hence would not affect the infrared limit $\mu \to 0$ that we study here.

In an asymptotically free theory with sufficiently large fermion content, the beta function has an infrared zero at $\alpha_{\rm IR}$ that controls the UV to IR evolution. Here we consider vectorial theories of this type. As the scale μ decreases from large values in the UV to small values in the IR, $\alpha(\mu)$ approaches $\alpha_{\rm IR}$ as $\mu \to 0$. The properties of the theory at this IR zero of the beta function are of considerable interest. If this IR zero of the beta function occurs at sufficiently weak coupling so that the gauge interaction does not produce any spontaneous chiral symmetry breaking (SχSB), then it is an exact IR fixed point (IRFP) of the renormalization group. The theory thus exhibits scale invariance with anomalous dimensions for various (gauge-invariant) operators. In this infrared limit, the theory is in a chirally symmetric, deconfined, non-Abelian Coulomb phase (NACP). If, on the other hand, as μ decreases and $\alpha(\mu)$ increases toward $\alpha_{\rm IR}$, there is a scale $\mu = \Lambda$ at which $\alpha(\mu)$ exceeds a critical value, denoted α_{cr}, then the gauge interaction produces a nonzero chiral condensate, with associated spontaneous chiral symmetry breaking and dynamical mass generation for the fermions. These fermions are thus integrated out of the low-energy effective field theory that is operative for $\mu < \Lambda$. In this case, $\alpha_{\rm IR}$ is only an approximate IRFP. We define $N_{f,cr}$ to be the critical value of N_f such that if $N_f > N_{f,cr}$, then the (asymptotically free) theory does not undergo spontaneous chiral symmetry breaking. At the two-loop (2ℓ) level, $\alpha_{{\rm IR},2\ell} = -4\pi b_1/b_2$, where b_ℓ denotes the ℓ-loop coefficient in the beta function [see Eqs. (2.1) and (2.5) below], and since b_1 [3] and b_2 [4] are independent of the scheme used for regularization and renormalization of the theory [5], it follows that $\alpha_{{\rm IR},2\ell}$ is also scheme-independent.

Physical quantities evaluated at an infrared fixed point of the renormalization group at $\alpha = \alpha_{\rm IR}$ are of basic interest. Since these are physical, their exact values must be scheme-independent. In conventional computations of these quantities, first, one expresses them as series expansions in powers of α, calculated to n-loop order; second, one computes the IR zero of the beta function, denoted $\alpha_{\mathrm{IR},n}$, to the same n-loop order; and third, one sets $\alpha = \alpha_{\mathrm{IR},n}$ in the series expansion for the given quantity to obtain its value at the IR zero of the beta function to this n-loop order. However, these conventional series expansions in powers of α, calculated to a finite order, are scheme-dependent beyond the leading one or two terms. Specifically, the terms in the beta function are scheme-dependent at loop order $\ell \geq 3$ and the terms in an anomalous dimension are scheme-dependent at loop order $\ell \geq 2$. Indeed, as is well known, the presence of scheme-dependence in higher-order perturbative calculations is a general property in quantum field theory.

Clearly, it would be very valuable to have a calculational framework in which these physical quantities evaluated at $\alpha = \alpha_{\mathrm{IR}}$ are expressed as a series expansion such that at every order in this expansion the result is scheme-independent. A key point that was noted early on [3,4,6] is that α_{IR} becomes small as the number N_f of fermions increases toward the value $N_{f,b1z}$ [given below in Eq. (2.4)] at which the one-loop term in the beta function, b_1, passes through zero. At the two-loop level, $\alpha_{\mathrm{IR}} \propto \Delta_f$, where

$$\Delta_f = N_{f,b1z} - N_f. \tag{1.2}$$

Indeed, in a theory with $G = \mathrm{SU}(N_c)$ and fermions in the fundamental representation, in the limit $N_c \to \infty$ and $N_f \to \infty$ with N_f/N_c fixed, α_{IR} can be made arbitrarily small. Hence, one can envision reliable perturbative calculations of series expansions for physical quantities at this IRFP [4,6] and, in particular, series expansions of these quantities in powers of Δ_f for reasonably small Δ_f [7]. Because Δ_f is obviously scheme-independent, it follows that this perturbative series expansion in powers of Δ_f is scheme-independent. Some early work on this was reported in [7,8]. Recently, in [9], a procedure for calculating the coefficients of this scheme-independent expansion was given for the anomalous dimension of the (gauge-invariant) fermion bilinear at the IR zero of the beta function, and the coefficients in this expansion were calculated up to order Δ_f^3 in a vectorial asymptotically free gauge theory with gauge group G and N_f fermions in a representation R. This work also presented an analogous calculation for a theory with $\mathcal{N} = 1$ supersymmetry to order Δ_f^2. The results were then evaluated in the case of $\mathrm{SU}(N_c)$ with fermions in the fundamental (F) representation, $R = F$, with Young tableau $\square$. In [10], for $G = \mathrm{SU}(3)$ and $R = F$, we calculated the n-loop value of the squared coupling, $\alpha_{\mathrm{IR},n\ell}$ and the resultant value of $\gamma_{\bar\psi\psi}$ to five-loop order, and in [11] we calculated the scheme-independent expansion of $\gamma_{\bar\psi\psi}$ for the representations R is theory to order Δ_f^4, using five-loop inputs, and performed an extrapolation to infinite order in Δ_f to estimate the exact value of $\gamma_{\bar\psi\psi}$ as a function of N_f. The improvement in the knowledge of the anomalous dimension $\gamma_{\bar\psi\psi}$ obtained from the scheme-independent series expansions in [9,11] is valuable not only for general field-theoretic purposes, but also because theories with large anomalous dimensions of fermion bilinears may be relevant for ultraviolet completions of the Standard Model. Indeed, there has been considerable interest in theories that might produce large $\gamma_{\bar\psi\psi} \sim O(1)$ associated with an IR zero of the beta function and resultant quasi-conformal behavior [12]. In [11] we also compared our results with recent lattice measurements of $\gamma_{\bar\psi\psi}$.

In this paper we report a number of new results on scheme-independent series expansions in powers of Δ_f. As noted, we consider an asymptotically free vectorial gauge theory with gauge group G and N_f fermions in the representation R. First, we present general formulas for coefficients in the scheme-independent expansion in powers of Δ_f of an arbitrary (gauge-invariant) physical quantity evaluated at α_{IR}. We calculate the scheme-independent expansion of the derivative of the beta function, $\beta' = d\beta/d\alpha$, evaluated at α_{IR}, denoted β'_{IR}, to order Δ_f^4. As a consequence of the trace anomaly relation, in a theory with massless fermions, β'_{IR} is equivalent to $\gamma_{F^2,\mathrm{IR}}$, the anomalous dimension, evaluated at α_{IR}, of the operator $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, where $F^a_{\mu\nu}$ is the non-Abelian Yang Mills field-strength tensor. For the special case where the gauge group is SU(3) and the fermions are in the fundamental (triplet) representation, we compute this expansion to order Δ_f^5. This SU(3) theory corresponds to quantum chromodynamics (QCD) with N_f massless quarks. For general G and R, we calculate the scheme-independent expansion coefficients to order Δ_f^3 for the anomalous dimension, evaluated at α_{IR}, of the flavor-nonsinglet and flavor-singlet fermion bilinear Dirac tensor operators. Since the Δ_f expansion starts at the upper end of the non-Abelian Coulomb phase (NACP) at $\Delta_f = 0$, i.e., $N_f = N_{f,b1z}$, and extends downward in N_f with increasing Δ_f, we focus mainly on the infrared behavior in the NACP. We show that our scheme-independent calculations of the anomalous dimensions of $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ and fermion bilinear operators in the non-Abelian Coulomb phase obey respective rigorous upper bounds for conformally invariant theories. As part of our analysis, we compare results for various quantities calculated via the scheme-independent expansion with results calculated via a conventional higher-loop scheme-dependent expansion. Further, for the case with $G = \mathrm{SU}(N_c)$ and fermions in the fundamental representation, we discuss the limit $N_c \to \infty$ and $N_f \to \infty$ with N_f/N_c fixed and finite. From ratios of scheme-independent expansion coefficients for β'_{IR}, $\gamma_{\bar\psi\psi,\mathrm{IR}}$ and the anomalous dimension of the fermion bilinear antisymmetric Dirac tensor operator, we show, in agreement with, and extending [9], that the scheme-independent Δ_f expansion should be reasonably accurate in the non-Abelian Coulomb phase. As with our earlier work, the present study is motivated by the value of the new results for a basic understanding of the renormalization-group evolution of asymptotically free gauge theories, and also may be relevant to ultraviolet completions of the Standard Model.

The paper is organized as follows. Some relevant background and methods are discussed in Sec. II. In Sec. III we present explicit formulas for the calculations of certain coefficients [a_n and k_n in Eqs. (3.1) and (3.3)] that are needed for the rest of our work. General scheme-independent expansion formulas for anomalous dimensions of operators are given in Sec. IV. In this section we also discuss rigorous upper bounds on anomalous dimensions in a conformally invariant theory and their application here. We give our new results on scheme-independent calculations of $\beta'_{\rm IR}$ in Sec. V. In Sec. VI we extend the analysis of the scheme-independent expansion of the anomalous dimension for the $m = 0$ fermion bilinear previously studied in [9] and [11] with several new results. These include calculations for the limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c fixed and analyses of Padé approximants, with comparison to scheme-dependent higher-loop conventional calculations. Section VII presents scheme-independent calculations of the anomalous dimension for the fermion bilinear (flavor-nonsinglet and flavor-singlet) antisymmetric rank-2 Dirac tensor operator. Our conclusions are given in Sec. VIII and some auxiliary formulas are listed in Appendix.

II. BACKGROUND AND METHODS

The beta function of this theory has the series expansion

$$\beta = -2\alpha \sum_{\ell=1}^{\infty} b_\ell a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \alpha^\ell, \tag{2.1}$$

where

$$a = \frac{g^2}{16\pi^2} = \frac{\alpha}{4\pi}, \tag{2.2}$$

b_ℓ is the ℓ-loop coefficient, $\bar{b}_\ell = b_\ell/(4\pi)^\ell$, and we extract a minus sign for convenience, so $b_1 > 0$ for asymptotic freedom. For analysis of an IR zero of β, it is convenient to extract the α^2 factor that gives rise to the UV zero at $\alpha = 0$ and define a reduced (r) beta function

$$\beta_r = \frac{\beta}{(-\frac{\alpha^2 b_1}{2\pi})} = 1 + \frac{1}{b_1}\sum_{\ell=2}^{\infty} b_\ell a^{\ell-1}. \tag{2.3}$$

The n-loop ($n\ell$) beta function, denoted $\beta_{n\ell}$ and reduced beta function, denoted $\beta_{r,n\ell}$ are obtained from the respective Eqs. (2.1) and (2.3) by changing the upper limit on the ℓ-loop summation from ∞ to n. As noted above, b_1 and b_2 are scheme-independent (SI), while the b_ℓ with $\ell \geq 3$ are scheme-dependent (SD) [5]. For a general gauge group G and fermion representation R, the coefficients b_1 and b_2 were calculated in [3] and [4], and b_3 and b_4 were calculated in [13] and [14] (and checked in [15]) in the commonly used mass-independent $\overline{\rm MS}$ scheme [16]. Recently, for $G = {\rm SU}(3)$ and $R = F$, b_5 was calculated in [17]. For reference and to show our normalizations explicitly, b_1 and b_2 are listed in Appendix. As N_f increases, b_1 decreases through positive values and vanishes with sign reversal at $N_f = N_{f,b1z}$, where

$$N_{f,b1z} = \frac{11C_A}{4T_f} \tag{2.4}$$

(the subscript $b1z$ means "b_1 zero"), where C_A and T_f are group-theoretic invariants [18,19]. The asymptotic freedom condition therefore implies the upper bound $N_f < N_{f,b1z}$. We denote the interval $0 \leq N_f < N_{f,b1z}$ as I_{AF}.

For N_f close to, but less than, $N_{f,b1z}$, $b_2 < 0$, so the two-loop beta function has an IR zero, at the value

$$\alpha_{{\rm IR},2\ell} = -\frac{\bar{b}_1}{\bar{b}_2} = -\frac{4\pi b_1}{b_2}. \tag{2.5}$$

In general, the n-loop beta function has a double UV zero at $\alpha = 0$ and $n-1$ zeros away from the origin. Among the latter, the smallest (real, positive) zero, if such a zero occurs, is the physical IR zero, denoted $\alpha_{{\rm IR},n\ell}$. As N_f decreases from $N_{f,b1z}$, b_2 passes through zero to positive values as N_f passes through the value

$$N_{f,b2z} = \frac{17C_A^2}{2T_f(5C_A + 3C_f)}. \tag{2.6}$$

Hence, with N_f formally extended from nonnegative integers to nonnegative real numbers [19], $\beta_{2\ell}$ has an IR zero (IRZ) for N_f in the interval

$$I_{\rm IRZ}\colon\ N_{f,b2z} < N_f < N_{f,b1z}. \tag{2.7}$$

We denote this interval as $I_{\rm IRZ}$.

As N_f decreases in this interval, $\alpha_{{\rm IR},2\ell}$ increases toward strong coupling. Hence, to study the IR zero for N_f toward the middle and lower part of $I_{\rm IRZ}$ with reasonable accuracy, one requires higher-loop calculations. These were performed in [20–27] for $\alpha_{{\rm IR},n\ell}$ and for the anomalous dimension of the fermion bilinear operator. Clearly, a perturbative calculation of the IR zero of $\beta_{n\ell}$ is only reliable if the resultant $\alpha_{{\rm IR},n\ell}$ is not excessively large. Moreover, since the b_ℓ with $\ell \geq 3$ are scheme-dependent, it is also incumbent upon one to ascertain the degree of sensitivity of the value obtained for $\alpha_{{\rm IR},n\ell}$ for $n \geq 3$ to the scheme used for the calculation. This task was carried out in [28–31]. One way to do this is to perform the calculation of $\alpha_{\mathrm{IR},n\ell}$ in one scheme, say $\overline{\mathrm{MS}}$, apply a scheme transformation to obtain the value of $\alpha_{\mathrm{IR},n\ell}$ in another scheme, and compare how close the two values are. As we discussed in [28,29], an acceptable scheme transformation function must satisfy a set of conditions, and although these are automatically satisfied in the local vicinity of the origin, $\alpha = 0$ (as in optimized schemes for perturbative QCD calculations), they are not automatically satisfied, and indeed, are quite restrictive conditions, when one applies the scheme transformation at an IR zero away from the origin. Anomalous dimensions of composite fermion operators for $G = \mathrm{SU}(3)$ have been calculated in [32].

The one-loop coefficient b_1 is a polynomial of degree 1 in N_f and the higher-loop coefficients b_ℓ with $\ell \geq 2$ are polynomials of degree $\ell - 1$ in N_f. Let us define

$$b_\ell^{(0)} = b_\ell|_{N_f = N_{f,b1z}} \tag{2.8}$$

and, for $r \geq 1$,

$$b_\ell^{(r)} = \frac{d^r b_\ell}{(dN_f)^r}|_{N_f=N_{f,b1z}} = (-1)^r \frac{d^r b_\ell}{(d\Delta_f)^r}\bigg|_{\Delta_f=0}. \tag{2.9}$$

Then one has the scheme-independent results

$$b_1^{(0)} = 0 \tag{2.10}$$

(which is equivalent to the definition of $N_{f,b1z}$),

$$b_1^{(1)} = \frac{4T_f}{3}, \tag{2.11}$$

$$b_2^{(0)} = -C_A(7C_A + 11C_f) \equiv -C_A D, \tag{2.12}$$

where

$$D = 7C_A + 11C_f, \tag{2.13}$$

and

$$b_2^{(1)} = -\frac{4}{3}(5C_A + 3C_f)T_f. \tag{2.14}$$

It is convenient to introduce the definition (2.13), since powers of D occur in the denominators of the scheme-independent expansion coefficients of anomalous dimensions of bilinear fermion Dirac tensor operators and of $d\beta/d\alpha$ evaluated at the IR zero of the beta function.

Thus, one has the finite Taylor series expansions

$$b_1 = b_1^{(1)}(N_f - N_{f,b1z}) = -b_1^{(1)}\Delta_f \tag{2.15}$$

and, for $\ell \geq 2$,

$$b_\ell = \sum_{r=0}^{\ell-1} \frac{1}{r!} b_\ell^{(r)} (N_f - N_{f,b1z})^r = \sum_{r=0}^{\ell-1} \frac{(-1)^r}{r!} b_\ell^{(r)} \Delta_f^r. \tag{2.16}$$

We write Eqs. (2.15) and (2.16) in a unified manner as

$$b_\ell = \sum_{r=0}^{r_{\max}(\ell)} \frac{(-1)^r}{r!} b_\ell^{(r)} \Delta_f^r, \tag{2.17}$$

where $r_{\max}(1) = 1$ and $r_{\max}(\ell) = \ell - 1$ if $\ell \geq 2$.

It will also be useful to recall some basic properties of the theory regarding global flavor symmetries. Because the N_f fermions are massless, the Lagrangian is invariant under the classical global chiral flavor (fl) symmetry $G_{fl,cl} = \mathrm{U}(N_f)_L \otimes \mathrm{U}(N_f)_R$, or equivalently,

$$G_{fl,cl} = \mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R \otimes \mathrm{U}(1)_V \otimes \mathrm{U}(1)_A \tag{2.18}$$

(where V and A denote vector and axial-vector). The $\mathrm{U}(1)_V$ represents fermion number, which is conserved by the bilinear operators that we consider. The $\mathrm{U}(1)_A$ symmetry is broken by instantons, so the actual nonanomalous global flavor symmetry is

$$G_{fl} = \mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R \otimes \mathrm{U}(1)_V. \tag{2.19}$$

This G_{fl} symmetry is respected in the (deconfined) non-Abelian Coulomb phase, since there is no spontaneous chiral symmetry in this phase. As noted before, we focus on this phase in the present work, since the (scheme-independent) Δ_f expansion starts from the upper end of the interval I_{IRZ} in this phase where $\alpha_{\mathrm{IR}} \to 0$ as $\Delta_f \to 0$. In contrast, in the phase with confinement and spontaneous chiral symmetry breaking, the gauge interaction produces a bilinear fermion condensate, which can be written as $\sum_{j=1}^{N_f} \bar{\psi}_j \psi_j$, and this breaks G_{fl} to $\mathrm{SU}(N_f)_V \otimes \mathrm{U}(1)_V$.

III. CALCULATION OF THE SERIES EXPANSION COEFFICIENTS k_n AND a_n

We know that the exact α_{IR} (and also the n-loop approximation to this exact α_{IR}) vanishes (linearly) as a function of Δ_f and that it is analytic at $\Delta_f = 0$, so we can expand it, or equivalently, $a_{\mathrm{IR}} = \alpha_{\mathrm{IR}}/(4\pi)$, as a series expansion in this variable, Δ_f. We write

$$a_{\mathrm{IR}} = \sum_{j=1}^{\infty} a_j \Delta_f^j. \tag{3.1}$$

[Note that a_j as defined here is equal to $a_j/2$ in terms of the a_j in Eq. (8) of [9].]

One calculates the coefficients a_j in two steps. First, one evaluates β_r in Eq. (2.3) at $\alpha = \alpha_{IR}$, where it vanishes. Since the prefactor $-8\pi a_{IR}^2$ in Eq. (2.1) is nonzero in general (although it does vanish at $\Delta_f = 0$), it follows that

$$\sum_{\ell=1}^{\infty} b_\ell (a_{IR})^{\ell-1} = 0. \tag{3.2}$$

One then substitutes the finite Taylor series expansions for b_ℓ, and a_{IR}, Eqs. (2.15), (2.17), and (3.1), in Eq. (3.2) and thereby obtains the equation

$$\beta_r|_{\alpha=\alpha_{IR}} = 0 = \sum_{\ell=1}^{\infty}\left[\left(\sum_{r=0}^{r_{\max}(\ell)} b_\ell^{(r)}\Delta_f^r\right)\left(\sum_{j=1}^{\infty} a_j\Delta_f^j\right)^\ell\right]$$
$$= \sum_{n=1}^{\infty} k_n \Delta_f^n. \tag{3.3}$$

The results for the first three k_n were given in [9] and are

$$k_1 = a_1 b_2^{(0)} - b_1^{(1)}, \tag{3.4}$$

$$k_2 = a_2 b_2^{(0)} + a_1^2 b_3^{(0)} - a_1 b_2^{(1)}, \tag{3.5}$$

and

$$k_3 = a_3 b_2^{(0)} + 2a_1 a_2 b_3^{(0)} + a_1^3 b_4^{(0)} - a_2 b_2^{(1)} - a_1^2 b_3^{(1)}. \tag{3.6}$$

From Eq. (3.3), it follows that the coefficient a_n occurs linearly in the expression for k_n, in the single term $a_n b_2^{(0)}$ [9]. To further show the structural forms of the k_n, we give k_4 and k_5 here:

$$k_4 = a_4 b_2^{(0)} + (a_2^2 + 2a_1 a_3) b_3^{(0)} + 3a_1^2 a_2 b_4^{(0)} + a_1^4 b_5^{(0)}$$
$$- a_3 b_2^{(1)} - 2a_1 a_2 b_3^{(1)} - a_1^3 b_4^{(1)} + \frac{1}{2} a_1^2 b_3^{(2)} \tag{3.7}$$

$$k_5 = a_5 b_2^{(0)} + 2(a_1 a_4 + a_2 a_3) b_3^{(0)} + 3a_1(a_2^2 + a_1 a_3) b_4^{(0)}$$
$$+ 4a_1^3 a_2 b_5^{(0)} + a_1^5 b_6^{(0)} - a_4 b_2^{(1)} - (a_2^2 + 2a_1 a_3) b_3^{(1)}$$
$$- 3a_1^2 a_2 b_4^{(1)} - a_1^4 b_5^{(1)} + a_1 a_2 b_3^{(2)} + \frac{1}{2} a_1^3 b_4^{(2)}. \tag{3.8}$$

In addition to the property that k_n contains a term $a_n b_2^{(0)}$, we remark on two other general properties of the k_n: (i) k_n contains a term $a_1^n b_{n+1}^{(0)}$ [which coincides with the term $a_n b_2^{(0)}$ if $n = 1$] and (ii) if $n \ge 2$, then k_n contains a term $-a_{n-1} b_2^{(1)}$.

Next, one observes that in Eq. (3.3), since Δ_f is variable, this implies that the coefficients k_n of each power Δ_f^n must vanish individually. One can solve the equations $k_n = 0$ for the a_n. The solutions are unique because of the property that a_n occurs linearly in k_n. The solutions for the a_n with with $1 \le n \le 3$ were given in [9]. Thus, the equation $k_1 = 0$ yields

$$a_1 = \frac{b_1^{(1)}}{b_2^{(0)}}. \tag{3.9}$$

One then substitutes this into the equation $k_2 = 0$ and solves for a_2, obtaining

$$a_2 = \frac{b_1^{(1)}}{(b_2^{(0)})^3}(b_2^{(0)} b_2^{(1)} - b_1^{(1)} b_3^{(0)}). \tag{3.10}$$

One then proceeds iteratively in the manner, substituting the solutions for the a_k with $1 \le k \le n-1$ in the equation $k_n = 0$ and solving for a_n. For a_3, this yields

$$a_3 = \frac{b_1^{(1)}}{(b_2^{(0)})^5}[(b_2^{(0)} b_2^{(1)})^2 - 3b_1^{(1)} b_2^{(0)} b_2^{(1)} b_3^{(0)} + 2(b_1^{(1)} b_3^{(0)})^2$$
$$+ b_1^{(1)} (b_2^{(0)})^2 b_3^{(1)} - (b_1^{(1)})^2 b_2^{(0)} b_4^{(0)}]. \tag{3.11}$$

In general, a_n depends on the b_ℓ coefficients for $1 \le \ell \le n+1$. The a_n with $1 \le n \le 3$ were all the coefficients of this type that were needed in [9] since the b_ℓ have only been computed for a general gauge group G and fermion representation R up to $\ell = 4$ loop order. These a_n have a factorized structure with a prefactor

$$a_n \propto \frac{b_1^{(1)}}{(b_2^{(0)})^{2n-1}}. \tag{3.12}$$

In [11] we also calculated and presented the result for a_4 for the specific case $G = \mathrm{SU}(3)$ and fermion representation $R = F$, since we were using the recent calculation of the five-loop coefficient b_5 for this case in [17]. Here we give the general result for a_4 for arbitrary gauge group G and fermion representation R:

$$a_4 = \frac{b_1^{(1)}}{(b_2^{(0)})^7}[(b_2^{(0)} b_2^{(1)})^3 - \frac{1}{2} b_1^{(1)} (b_2^{(0)})^4 b_3^{(2)}$$
$$+ (b_1^{(1)})^2 (b_2^{(0)})^3 b_4^{(1)} - 4(b_1^{(1)} b_2^{(0)})^2 (b_2^{(1)} b_4^{(0)} + b_3^{(0)} b_3^{(1)})$$
$$+ 3b_1^{(1)} (b_2^{(0)})^2 b_2^{(1)} (b_2^{(0)} b_3^{(1)} - 2b_2^{(1)} b_3^{(0)})$$
$$+ 10(b_1^{(1)})^2 b_2^{(0)} b_2^{(1)} (b_3^{(0)})^2$$
$$+ 5(b_1^{(1)})^3 b_3^{(0)} (b_2^{(0)} b_4^{(0)} - (b_3^{(0)})^2)$$
$$- (b_1^{(1)})^3 (b_2^{(0)})^2 b_5^{(0)}]. \tag{3.13}$$

In the same manner, we have calculated a_5 by substituting our solutions for the a_k with $1 \le k \le 4$ in Eq. (3.8), and so forth for higher a_k.

IV. SCHEME-INDEPENDENT SERIES EXPANSION FOR ANOMALOUS DIMENSIONS AT α_{IR}

Let us consider a (gauge-invariant) operator $\mathcal{O}$. Because of the interactions, the full scaling dimension of this operator, denoted $D_{\mathcal{O}}$, differs from its free-field value, $D_{\mathcal{O},\mathrm{free}}$:

$$D_{\mathcal{O}} = D_{\mathcal{O},\mathrm{free}} - \gamma_{\mathcal{O}}, \tag{4.1}$$

where $\gamma_{\mathcal{O}}$ is the anomalous dimension of the operator [33]. Since $\gamma_{\mathcal{O}}$ arises from the gauge interaction, it can be expressed as a power series about $a = 0$:

$$\gamma_{\mathcal{O}} = \sum_{\ell=1}^{\infty} c_{\mathcal{O},\ell}\, a^{\ell}, \tag{4.2}$$

where $c_{\mathcal{O},\ell}$ is the ℓ-loop coefficient.

The exact anomalous dimension $\gamma_{\mathcal{O}}$ evaluated at a zero of the exact beta function, denoted $\gamma_{\mathcal{O},\mathrm{IR}}$, is a physical quantity and hence is scheme-independent. This was shown formally for the fermion bilinear operator $\mathcal{O} = \bar\psi\psi$ in [5], and the proof given there can be straightforwardly extended to other (gauge-invariant) operators $\mathcal{O}$. However, this scheme independence is not preserved in a finite-order perturbative calculation, owing to the scheme dependence of the b_ℓ for $\ell \geq 3$ and of the $c_{\mathcal{O},\ell}$ for $\ell \geq 2$.

As mentioned above, a method for calculating $\gamma_{\bar\psi\psi,\mathrm{IR}}$ as a perturbative series expansion in powers of Δ_f was presented in [9], with the important property that at each order of the expansion the resulting approximation to $\gamma_{\bar\psi\psi,\mathrm{IR}}$ is scheme-independent. We can calculate a scheme-independent series expansion in powers of Δ_f for the anomalous dimension $\gamma_{\mathcal{O}}$ of a general (gauge-invariant) operator $\mathcal{O}$, evaluated at α_{IR} by taking the series (4.2) and inserting the expansions of $c_{\mathcal{O},\ell}$ and a_{IR} as functions of Δ_f. An advantage of this type of series expansion is that since Δ_f is scheme-independent, so is the expansion for $\gamma_{\mathcal{O}}$, in contrast to the expression of $\gamma_{\mathcal{O}}$ as a series in powers of $\alpha_{\mathrm{IR},n\ell}$.

We proceed to give a generalization of the results of [9] for the anomalous dimension of an arbitrary (gauge-invariant) operator $\mathcal{O}$ in an asymptotically free gauge theory with gauge group G and N_f fermions in the representation R, evaluated at α_{IR}. We denote this anomalous dimension as $\gamma_{\mathcal{O},\mathrm{IR}}$. Specifically, we present a general method for calculating a series expansion of $\gamma_{\mathcal{O},\mathrm{IR}}$ in powers of Δ_f.

We begin with the series expansion (4.2) and substitute the series expansions for the $c_{\mathcal{O},\ell}$ and for a_{IR}. Let

$$c^{(0)}_{\mathcal{O},\ell} = c_{\mathcal{O},\ell}|_{N_f=N_{f,b1z}} \tag{4.3}$$

and, for $r \geq 1$,

$$c^{(r)}_{\mathcal{O},\ell} = \frac{d^r c_{\mathcal{O},\ell}}{(dN_f)^r}\bigg|_{N_f=N_{f,b1z}} = (-1)^r \frac{d^r c_{\mathcal{O},\ell}}{(d\Delta_f)^r}\bigg|_{\Delta_f=0} \tag{4.4}$$

Then

$$\gamma_{\mathcal{O},\mathrm{IR}} = \sum_{\ell=1}^{\infty}\left[\left(\sum_r c^{(r)}_{\mathcal{O},\ell}\Delta_f^r\right)\left(\sum_{j=1}^{\infty} a_j \Delta_f^j\right)^{\ell}\right] = \sum_{n=1}^{\infty}\kappa_{\mathcal{O},n}\Delta_f^n. \tag{4.5}$$

We denote the value of $\gamma_{\mathcal{O},\mathrm{IR}}$ obtained from this series calculated to order $O(\Delta_f^p)$, i.e., from the last line of Eq. (5.7) with the upper limit of the summand changed from ∞ to p, as $\gamma_{\mathcal{O},\mathrm{IR},\Delta_f^p}$.

We calculate

$$\kappa_{\mathcal{O},1} = a_1 c^{(0)}_{\mathcal{O},1}, \tag{4.6}$$

$$\kappa_{\mathcal{O},2} = a_2 c^{(0)}_{\mathcal{O},1} + a_1^2 c^{(0)}_{\mathcal{O},2}, \tag{4.7}$$

$$\kappa_{\mathcal{O},3} = a_3 c^{(0)}_{\mathcal{O},1} + 2a_1 a_2 c^{(0)}_{\mathcal{O},2} + a_1^3 c^{(0)}_{\mathcal{O},3} + a_1^2 c^{(1)}_{\mathcal{O},2}, \tag{4.8}$$

$$\kappa_{\mathcal{O},4} = a_4 c^{(0)}_{\mathcal{O},1} + (2a_1a_3 + a_2^2)c^{(0)}_{\mathcal{O},2} + 3a_1^2 a_2 c^{(0)}_{\mathcal{O},3} + a_1^4 c^{(0)}_{\mathcal{O},4} + 2a_1a_2 c^{(1)}_{\mathcal{O},2} + a_1^3 c^{(1)}_{\mathcal{O},3}, \tag{4.9}$$

etc. for $\kappa_{\mathcal{O},n}$ with $n \geq 5$. To calculate $\kappa_{\mathcal{O},n}$, one needs to know the a_j and c_j for $1 \leq j \leq n$. These $\kappa_{\mathcal{O},n}$ have the following general properties: (i) $\kappa_{\mathcal{O},n}$ contains the term $a_n c^{(0)}_{\mathcal{O},1}$ and (ii) $\kappa_{\mathcal{O},n}$ contains the term $a_1^n c^{(0)}_{\mathcal{O},n}$ (which coincides with (i) if $n = 1$).

A relevant question concerns the range of applicability of the scheme-independent series expansion (4.5). We address this question here. As noted above, our analysis in this paper is focused on the non-Abelian Coulomb phase, since there is no spontaneous symmetry breaking in this phase, and hence a zero of the beta function is an exact IR fixed point of the renormalization group. This means that the theory at this fixed point is scale-invariant. A number of studies have concluded that in this case of an exact IRFP in this asymptotically free gauge theory, scale invariance implies the larger symmetry of conformal invariance [34,35].

We will use several methods to assess the range of validity of the (scheme-independent) small-Δ_f expansion. A general comment is that the properties of the theory change qualitatively as N_f decreases through the value $N_{f,cr}$ and spontaneous chiral symmetry breaking occurs and the fermions gain dynamical masses. The (chirally symmetric) non-Abelian Coulomb phase with $N_{f,cr} < N_f < N_{f,b1z}$ is clearly qualitatively different from the confined phase with spontaneous chiral symmetry breaking at smaller N_f below $N_{f,cr}$. Therefore, one does not, in general, expect the small-Δ_f series expansion to hold below $N_{f,cr}$. Estimating the range of applicability of this expansion is thus connected with estimating the value of $N_{f,cr}$.

For this purpose, as in our previous work [9,22,24], we can apply a rigorous upper bound on the anomalous dimension of an operator from the unitarity of a conformal field theory. If the approximate calculation of the anomalous dimension of a given quantity at a fixed value of Δ_f, computed up to order Δ_f^p, yields a value that exceeds this upper bound, then we can infer that the calculation is not applicable at this value of Δ_f or equivalently, N_f. In particular, this can give information on the extent of the non-Abelian Coulomb phase and the value of $N_{f,cr}$. This bound is applicable whether or not the coefficients $\kappa_{\mathcal{O},n}$ are all of the same sign, but it is most useful if these coefficients do have the same sign, since in this case for a fixed Δ_f the anomalous dimension is a monotonic function of the order to which the small-Δ_f series expansion is calculated.

A second method that we shall use to estimate the range of applicability of the series expansions in powers of Δ_f is the ratio test. If a function $f(z)$ has a Taylor series $f(z) = \sum_{n=1}^{\infty} s_n z^n$, then the ratio test states that the series is (absolutely) convergent if $|z| < z_0$, where

$$z_0 = \lim_{n\to\infty} \frac{|s_n|}{|s_{n+1}|}. \tag{4.10}$$

Our application of the ratio test here is only intended to give a rough estimate of this range of applicability of the Δ_f series expansion since (i) we do not assume that the Δ_f expansion is a Taylor series expansion, and (ii) with only a few terms in the series for a given quantity, we can compute only a few ratios of adjacent coefficients.

Finally, a third method that we shall use is to calculate $[p,q]$ Padé approximants to the Δ_f series expansions. As rational functions of Δ_f, the approximants with $q \ge 1$ have poles, and the nearest poles to the origin give one estimate of the range of validity of the expansions.

A. Upper bound on anomalous dimensions

We now state and apply the upper bound on the anomalous dimension of an operator in a theory with scale invariance or conformal invariance. Recall that a (finite-dimensional) representation of the Lorentz group is specified by the set (j_1, j_2), where j_1 and j_2 take on nonnegative integral or half-integral values [36]. A Lorentz scalar operator thus transforms as (0,0), a Lorentz vector as $(1/2, 1/2)$, an antisymmetric tensor like the field-strength tensor $F^a_{\mu\nu}$ as $(1,0) \oplus (0,1)$, etc. Then the requirement of unitarity in a scale-invariant theory (in four spacetime dimensions) requires that the full dimension $D_{\mathcal{O}}$ of an operator (other than the identity) must satisfy the lower bound [35]

$$D_{\mathcal{O}} \ge j_1 + j_2 + 1. \tag{4.11}$$

With the definition (4.1), this is equivalent to the upper bound on the anomalous dimension

$$\gamma_{\mathcal{O}} \le D_{\mathcal{O},\mathrm{free}} - (j_1 + j_2 + 1). \tag{4.12}$$

The case $(j_1, j_2) = (0,0)$ includes the Lorentz scalar operators $F^a_{\mu\nu}F^{a\mu\nu}$, and the flavor-nonsinglet and flavor-singlet fermion bilinear operators $\bar\psi T_b \psi$ and $\bar\psi\psi$, where here T_b is an element of the Lie algebra of the global flavor symmetry group $\mathrm{SU}(N_f)$. Hence, first, since $D_{F^2,\mathrm{free}} = 4$, it follows from (4.12) that the anomalous dimension of the $F^a_{\mu\nu}F^{a\mu\nu}$, evaluated at α_{IR}, must satisfy

$$\gamma_{F^2,\mathrm{IR}} \le 3. \tag{4.13}$$

Second, (4.12) implies that the (equal) anomalous dimensions of the flavor-nonsinglet and flavor-singlet fermion bilinear operators $\bar\psi T_b\psi$ and $\bar\psi\psi$ evaluated at α_{IR}, denoted $\gamma_{\bar\psi\psi,\mathrm{IR}}$, must satisfy

$$\gamma_{\bar\psi\psi,\mathrm{IR}} \le 2. \tag{4.14}$$

The flavor-nonsinglet and flavor-singlet fermion bilinear antisymmetric rank-2 Dirac tensor operators proportional to $\bar\psi T_b \sigma_{\mu\nu}\psi$ and $\bar\psi\sigma_{\mu\nu}\psi$ to be analyzed below correspond to the case $(j_1, j_2) = (1,0) \oplus (0,1)$ (as is clear from the fact that they can couple to the non-Abelian field-strength tensor to form a Lorentz scalar). Hence, with $j_1 + j_2 = 1$ for $(j_1, j_2) = (1,0)$ or (0,1), the bound (4.12) implies that the (equal) anomalous dimensions of these operators evaluated at α_{IR}, denoted $\gamma_{T,\mathrm{IR}}$, must satisfy

$$\gamma_{T,\mathrm{IR}} \le 1. \tag{4.15}$$

We have applied the upper bound (4.14) in our previous calculations of $\gamma_{\bar\psi\psi,\mathrm{IR},n\ell}$ at the n-loop level, up to $n = 4$ loops [9–11,22,26,27]. We have also applied a corresponding upper bound in [9,24,27] on the anomalous dimension of the (gauge-invariant) bilinear chiral superfield operator $\Phi\tilde\Phi$ in a vectorial asymptotically free gauge theory with gauge group G, $\mathcal{N} = 1$ supersymmetry, and N_f pairs of chiral superfields Φ_j and $\tilde\Phi_j$, $1 \le j \le N_f$, transforming according to the representations R and $\bar R$ of G [24,27]. A theory of particular interest is the case $R = F$; here, $N_{f,b1z} = 3N_c$ and the lower end of the conformal phase is known, namely $N_{f,cr} = (3/2)N_c$ [37,38] (which is integral and hence physical if N_c is even). This theory corresponds to supersymmetric QCD with massless matter fields, and is often denoted SQCD. In this case, the upper bound is $\gamma_{\bar\psi\psi} \le 1$, and this is saturated at the lower end of the non-Abelian Coulomb phase. The scheme-independent expansion in [9] exhibited excellent agreement with this exact result.

V. SCHEME-INDEPENDENT CALCULATION OF $\beta'_{\rm IR}$

A. Calculation to order Δ_f^4 for general G and R

An important property of an asymptotically free theory at an IR zero of the beta function (IRFP of the renormalization group) is the derivative of this beta function evaluated at $\alpha = \alpha_{\rm IR}$,

$$\beta'_{\rm IR} = \frac{d\beta}{d\alpha}\bigg|_{\alpha=\alpha_{\rm IR}}. \tag{5.1}$$

This is scheme-independent, as was proved in [5] [39]. In a theory with massless fermions, as considered here, the trace of the energy-momentum tensor, T^μ_μ, satisfies the relation [40]

$$T^\mu_\mu = \frac{\beta}{4\alpha} F^a_{\mu\nu} F^{a\mu\nu}, \tag{5.2}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g c^{abc} A^b_\mu A^c_\nu$ is the gluon field strength tensor [41]. Since the energy-momentum tensor is conserved, its anomalous dimension is zero, and its full dimension is equal to its free-field dimension, 4. Consequently, the full scaling dimension of the rescaled operator $F^a_{\mu\nu}F^{a\mu\nu}$, denoted D_{F^2}, satisfies

$$D_{F^2} = 4 + \beta' - \frac{2\beta}{\alpha}, \tag{5.3}$$

where we use the shorthand notation $F^2 \equiv F^a_{\mu\nu}F^{a\mu\nu}$ [42,43]. We denote the anomalous dimension of the operator $F^a_{\mu\nu}F^{a\mu\nu}$ as γ_{F^2} and its evaluation at $\alpha_{\rm IR}$ as $\gamma_{F^2,{\rm IR}}$. From Eq. (5.3), it follows that at a zero of the beta function away from the origin, in particular, the IR zero of an asymptotically free gauge theory of interest here at $\alpha = \alpha_{\rm IR}$, the derivative $\beta'_{\rm IR}$ is equivalent to the anomalous dimension [33] of the operator $F^a_{\mu\nu}F^{a\mu\nu}$:

$$\beta'_{\rm IR} = -\gamma_{F^2,{\rm IR}}. \tag{5.4}$$

From Eq. (2.1), one obtains the conventional series expansion for $\beta'_{\rm IR}$ in powers of α, or equivalently, a:

$$\beta'_{\rm IR} = -2\sum_{\ell=1}^\infty (\ell+1) b_\ell a_{\rm IR}^\ell. \tag{5.5}$$

We denote $\beta'_{{\rm IR},n\ell}$ as the n-loop truncation of this infinite series. The two-loop value is scheme-independent [26]:

$$\beta'_{{\rm IR},2\ell} = -\frac{2b_1^2}{b_2} = \frac{(11C_A - 4T_fN_f)^2}{3[2(5C_A+3C_f)T_fN_f - 17C_A^2]}, \tag{5.6}$$

which is positive for $N_f \in I_{\rm IRZ}$. However, at the level of $n \ge 3$ loops, the quantity $\beta'_{{\rm IR},n\ell}$ is scheme-dependent. This quantity was calculated up to the four-loop level in [26,27], using b_3 and b_4 computed in the $\overline{\rm MS}$ scheme from [13,14] (for SU(3), see also the four-loop study [44]).

Here we calculate a scheme-independent expansion of $\beta'_{\rm IR}$ in powers of Δ_f to order Δ_f^4 for general G and R and to the five-loop level, i.e., order Δ_f^5, for SU(3). For general G and R, we substitute the expansions of b_ℓ and $a_{\rm IR}$, as series in Δ_f, in Eq. (5.5) to obtain

$$\beta'_{\rm IR} = -2\sum_{\ell=1}^\infty (\ell+1)\bigg[\bigg(\sum_{r=0}^{r_{\rm max}(\ell)} b_\ell^{(r)}\Delta_f^r\bigg)\bigg(\sum_{j=1}^\infty a_j\Delta_f^j\bigg)^\ell\bigg] = \sum_{n=1}^\infty d_n \Delta_f^n. \tag{5.7}$$

We denote the value of $\beta'_{\rm IR}$ obtained from this series calculated to order Δ_f^p as $\beta'_{{\rm IR},\Delta_f^p}$. The calculation of d_n contains explicit dependence on the b_ℓ for $1 \le \ell \le n$ and on the a_j for $1 \le j \le n-1$; since a_j depends on b_ℓ for $1 \le \ell \le j+1$, it follows that the calculation of d_n requires knowledge of b_ℓ for $1 \le \ell \le n$. Since the b_ℓ have been calculated for general gauge group G and fermion representation R up to four-loop level, we can thus calculate explicit expressions for the d_n up to $n = 4$. For our calculation, in addition to the scheme-independent results for b_1 and b_2 [3,4], we have used the expressions for b_3 and b_4 calculated in the $\overline{\rm MS}$ scheme in [13,14]. However, we stress that it does not matter which scheme we use for b_3 and b_4, because the resulting series expansion for $\beta'_{\rm IR}$ in powers of Δ_f is scheme-independent.

Substituting the $b_\ell^{(r)}$ and a_j into these equations, we find the following results. First,

$$d_1 = 0, \tag{5.8}$$

so that $\beta'_{\rm IR}$ vanishes quadratically with Δ_f as $\Delta_f \to 0$, i.e., as $N_f \to N_{f,b1z}$. For $n \ge 2$, with the denominator factor $D = 7C_A + 11C_f$ as defined in Eq. (2.13), we calculate

$$d_2 = \frac{2^5 T_f^2}{3^2 C_A D}, \tag{5.9}$$

$$d_3 = \frac{2^7 T_f^3(5C_A + 3C_f)}{3^3 C_A^2 D^2}, \tag{5.10}$$

and

$$
\begin{aligned}
d_4 = -\frac{2^3 T_f^2}{3^6 C_A^4 D^5}\Big[& C_A^5 T_f^2(-412335+1241856\zeta_3) + C_A^4 T_f^2 C_f(-310800+2661120\zeta_3) \\
& + C_A^3 T_f^2 C_f^2(-217848-836352\zeta_3) + C_A^3 \frac{d_R^{abcd} d_R^{abcd}}{d_A}(-2385152+5203968\zeta_3) + C_A^2 T_f^2 C_f^3(-2855424-3066624\zeta_3) \\
& + C_A^2 T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(630784-6150144\zeta_3) + C_A^2 C_f \frac{d_R^{abcd} d_R^{abcd}}{d_A}(-3748096+8177664\zeta_3) \\
& + 191664 C_A T_f^2 C_f^4 + C_A T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A}(-35840+946176\zeta_3) + C_A T_f C_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(991232-9664512\zeta_3) \\
& + T_f^2 C_f \frac{d_A^{abcd} d_A^{abcd}}{d_A}(-56320+1486848\zeta_3)\Big]. \qquad (5.11)
\end{aligned}
$$

Here,

$$\zeta_s = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (5.12)$$

is the Riemann zeta function, the quantities C_A, C_f, T_f are group invariants, and the contractions $d_A^{abcd} d_A^{abcd}$, $d_R^{abcd} d_A^{abcd}$, $d_R^{abcd} d_R^{abcd}$ are additional group-theoretic quantities given in [14], and d_A is the dimension of the adjoint representation of G. These calculations thus determine the quantity $\beta'_{\rm IR}$ to order Δ_f^4 for an arbitrary gauge group G and fermion representation R. We have also calculated d_5, but the expression is sufficiently lengthy that we do not include it here; however, we shall use it below.

We note a general result on the signs of the first two nonzero coefficients in the scheme-independent expansion for $\beta'_{\rm IR}$:

$$d_n > 0 \quad \text{for } n = 2, 3 \quad \text{and arbitrary } G, R. \qquad (5.13)$$

TABLE I. Signs of expansion coefficients discussed in the text for gauge group $G = {\rm SU}(N_c)$ and fermion representation $R = F$ (fundamental) and $R = adj$ (adjoint). Several results on signs actually apply more generally for arbitrary G and R; see text for details. For $G = {\rm SU}(3)$, we have also calculated $d_{5,F}$ in Eq. (5.20) and find that it is negative. The entry for $\kappa_{4,F}$ applies for $G = {\rm SU}(3)$ [see Eq. (6.14)], as calculated in [11], and this is indicated by the $(*)$. The entry NA means "not available," i.e. the coefficient has not yet been calculated.

n	$d_{n,F}$	$d_{n,adj}$	$\kappa_{n,F}$	$\kappa_{n,adj}$	$\kappa_{T,n,F}$	$\kappa_{T,n,adj}$
1	0	0	+	+	−	−
2	+	+	+	+	−	−
3	+	+	+	+	+	+
4	−	+	+(*)	NA	NA	NA

These positivity results are clear from Eqs. (5.9) and (5.10). In contrast, there are terms of both signs in the large square bracket in the expression for d_4, Eq. (5.11); for example, in the large square bracket in Eq. (5.11), the coefficients of the $C_A^5 T_f^2$ and $C_A^4 T_f^2 C_f$ terms are positive while the coefficient of the $C_A^3 T_f^2 C_f^2$ term is negative, etc. Indeed, we will show below in Eqs. (5.16) and (5.61) that for $G = {\rm SU}(N_c)$, d_4 is negative if $R = F$ and positive if $R = adj$. A summary of the sign results for these coefficients and others is given in Table I for the case where $G = {\rm SU}(N_c)$.

In Table II we list the (scheme-independent) values that we calculate for $\beta'_{{\rm IR},\Delta_f^p}$ with $2 \le p \le 4$ for the illustrative gauge groups $G = {\rm SU}(2)$, SU(3), and SU(4), as functions of N_f in the respective intervals $I_{\rm IRZ}$ given in Eq. (2.7). For comparison, we list the n-loop values of $\beta'_{{\rm IR},n\ell}$ with the $2 \le n \le 4$, where $\beta'_{{\rm IR},3\ell}$ and $\beta'_{{\rm IR},4\ell}$ are computed in the $\overline{\rm MS}$ scheme. Values that exceed $\beta'_{\rm IR} = 3$ are marked as such. In the case of SU(3), we also include our calculation of $\beta'_{{\rm IR},\Delta_f^5}$.

B. Evaluation for $G = {\rm SU}(N_c)$ and $R = F$

We proceed to evaluate our general formulas for the d_n coefficients for a case of particular interest, namely that in which the gauge group is $G = {\rm SU}(N_c)$ with N_f fermions in the fundamental representation, $R = F$. In addition to Eq. (5.8), our general results (5.9)–(5.11) yield

$$d_{2,{\rm SU}(N_c),F} = \frac{2^4}{3^2(25N_c^2 - 11)}, \qquad (5.14)$$

$$d_{3,{\rm SU}(N_c),F} = \frac{2^5(13N_c^2 - 3)}{3^3 N_c (25N_c^2 - 11)^2}, \qquad (5.15)$$

and

TABLE II. Scheme-independent values of $\beta'_{\mathrm{IR},\Delta_f^p}$ with $2 \le p \le 4$ for $G = \mathrm{SU}(2)$, SU(3), and SU(4), as functions of N_f in the respective intervals I_{IRZ} given in Eq. (2.7) with (2.4) and (2.6). For comparison, we list the n-loop values of $\beta'_{\mathrm{IR},n\ell}$ with $2 \le n \le 4$, where $\beta'_{\mathrm{IR},3\ell}$ and $\beta'_{\mathrm{IR},4\ell}$ are computed in the $\overline{\mathrm{MS}}$ scheme. Values that exceed the upper bound (4.13) are marked as such. In the case of SU(3), we also include our calculation of $\beta'_{\mathrm{IR},\Delta_f^5}$. The notation ae-n means $a \times 10^{-n}$. The notation – means that the entry has not been calculated.

N_c	N_f	$\beta'_{\mathrm{IR},2\ell}$	$\beta'_{\mathrm{IR},3\ell,\overline{\mathrm{MS}}}$	$\beta'_{\mathrm{IR},4\ell,\overline{\mathrm{MS}}}$	$\beta'_{\mathrm{IR},\Delta_f^2}$	$\beta'_{\mathrm{IR},\Delta_f^3}$	$\beta'_{\mathrm{IR},\Delta_f^4}$	$\beta'_{\mathrm{IR},\Delta_f^5}$
2	6	>3	1.620	0.975	0.499	0.957	0.734	–
2	7	1.202	0.728	0.677	0.320	0.554	0.463	–
2	8	0.400	0.318	0.300	0.180	0.279	0.250	–
2	9	0.126	0.115	0.110	0.0799	0.109	0.1035	–
2	10	0.0245	0.0239	0.0235	0.0200	0.0236	0.0233	–
3	9	>3	1.475	1.464	0.467	0.882	0.7355	0.602
3	10	1.523	0.872	0.853	0.351	0.621	0.538	0.473
3	11	0.720	0.517	0.498	0.251	0.415	0.3725	0.344
3	12	0.360	0.2955	0.282	0.168	0.258	0.239	0.228
3	13	0.174	0.1556	0.149	0.102	0.144	0.137	0.134
3	14	0.0737	0.0699	0.678	0.0519	0.0673	0.0655	0.0649
3	15	0.0227	0.0223	0.0220	0.0187	0.0220	0.0218	0.0217
3	16	2.21e-3	2.20e-3	2.20e-3	2.08e-3	2.20e-3	2.20e-3	2.20e-3
4	11	>3	2.189	2.189	0.553	1.087	0.898	–
4	12	>3	1.430	1.429	0.457	0.858	0.729	–
4	13	1.767	0.965	0.955	0.370	0.663	0.578	–
4	14	0.984	0.655	0.639	0.292	0.498	0.445	–
4	15	0.581	0.440	0.424	0.224	0.362	0.331	–
4	16	0.348	0.288	0.276	0.1645	0.251	0.234	–
4	17	0.204	0.180	0.1725	0.114	0.164	0.156	–
4	18	0.113	0.105	0.101	0.0731	0.0988	0.0955	–
4	19	0.0558	0.0536	0.0522	0.0411	0.0520	0.0509	–
4	20	0.0222	0.0218	0.0215	0.0183	0.0215	0.0213	–
4	21	5.01e-3	4.99e-3	4.96e-3	4.57e-3	4.97e-3	4.96e-3	–

$$d_{4,\mathrm{SU}(N_c),F} = -\frac{2^4}{3^5 N_c^2 (25N_c^2 - 11)^5} \times [N_c^8(-366782 + 660000\zeta_3) + N_c^6(865400 - 765600\zeta_3) + N_c^4(-1599316 + 2241888\zeta_3) + N_c^2(571516 - 894432\zeta_3) + 3993]. \quad (5.16)$$

As is evident, the coefficients $d_{2,\mathrm{SU}(N_c),F}$ and $d_{3,\mathrm{SU}(N_c),F}$ are positive-definite for all physical values of N_c. We find that $d_{4,\mathrm{SU}(N_c),F}$ is negative-definite for all physical values of $N_c \ge 2$.

C. Calculation to $O(\Delta_f^5)$ for $G = \mathrm{SU}(3)$ and $R = F$

For the special case where the gauge group is $G = \mathrm{SU}(3)$ and the N_f fermions are in the fundamental representation, $R = F$, we can make use of the recent calculation of b_5 in the $\overline{\mathrm{MS}}$ scheme in [17] to carry out the scheme-independent calculation of β'_{IR} to one order higher than for general G and R, namely to order Δ_f^5. We first give the special cases of our results in Eqs. (5.8)–(5.11) for this theory. In addition to $d_{1,\mathrm{SU}(3),F} = 0$, we find

$$d_{2,\mathrm{SU}(3),F} = \frac{8}{3^2 \cdot 107} = 0.830737 \times 10^{-2}, \quad (5.17)$$

$$d_{3,\mathrm{SU}(3),F} = \frac{304}{3^3 \cdot (107)^2} = 0.983427 \times 10^{-3}, \quad (5.18)$$

and

$$d_{4,\mathrm{SU}(3),F} = \frac{633325687}{2 \cdot 3^6 \cdot (107)^5} - \frac{682880}{3^4 \cdot (107)^4}\zeta_3 = -(0.463417 \times 10^{-4}). \quad (5.19)$$

For $d_{5,\mathrm{SU}(3),F}$ we calculate

$$d_{5,\mathrm{SU}(3),F} = -\frac{66670528901419}{2 \cdot 3^9 \cdot (107)^7} - \frac{122882810048}{3^8 \cdot (107)^6}\zeta_3 + \frac{196275200}{3^6 \cdot (107)^5}\zeta_5 = -(0.564349 \times 10^{-5}). \quad (5.20)$$

In these equations we have indicated the simple factorizations of the denominators that were already evident in the general analytic expressions (5.8)–(5.11).

The numerators do not, in general, have such simple systematic factorizations; for example, in $d_{4,\mathrm{SU}(3),F}$, the number $633325687 = 227 \cdot 311 \cdot 8971$, etc. We will also use this factorization format, indicating the factorizations of the denominators, in later equations. Substituting these coefficients into Eq. (5.7), we have, to $O(\Delta_f^5)$,

$$\beta'_{\mathrm{IR}} = \Delta_f^2[(0.830737\times 10^{-2}) + (0.983427\times 10^{-3})\Delta_f - (0.463417\times 10^{-4})\Delta_f^2 - (0.564349\times 10^{-5})\Delta_f^3], \quad (5.21)$$

to the indicated floating-point accuracy.

In Fig. 1 we plot the values of β'_{IR}, calculated to order Δ_f^p with $2 \le p \le 5$. In the general calculations of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ as a series in powers of Δ_f to maximal power $p = 3$ (i.e., order Δ_f^3) in [9] and, for $G = \mathrm{SU}(3)$ and $R = F$, to maximal power $p = 4$ in [11], it was found that, for a fixed value of N_f, or equivalently, Δ_f, in the interval I_{IRZ}, these anomalous dimensions increased monotonically as a function of p. This feature motivated our extrapolation to $p = \infty$ in [9] to obtain estimates for the exact $\gamma_{\bar\psi\psi,\mathrm{IR}}$. In contrast, here we find that, for a fixed value of N_f, or equivalently, Δ_f, in I_{IRZ}, as a consequence of the fact that different coefficients d_n do not all have the same sign, $\beta'_{\mathrm{IR},\Delta_f^p}$ is not a monotonic function of p. Because of this nonmonotonicity, we do not attempt to extrapolate our series to $p = \infty$. Lattice measurements of $\gamma_{F^2,\mathrm{IR}}$ or β'_{IR} would be useful here (see also [44]). In particular, for $G = \mathrm{SU}(3)$ and fermions in the fundamental representation, the lattice measurements of $\gamma_{F^2,\mathrm{IR}}$ could be compared with our scheme-independent calculation of β'_{IR} to order Δ_f^5, similar to the comparison of our scheme-independent calculation of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ to order Δ_f^4 (which also used five-loop inputs [45]) with lattice results that we carried out in [11].

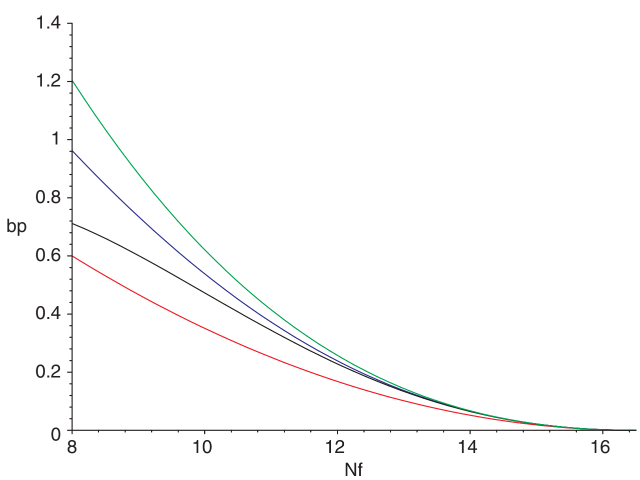

FIG. 1. Plot of $\beta'_{\mathrm{IR},\Delta_f^p}$ for $2 \le p \le 5$ as a function of N_f for the SU(3) theory with N_f fermions in the fundamental representation. From bottom to top, the curves (with colors online) refer to $\beta'_{\mathrm{IR},\Delta_f^2}$ (red), $\beta'_{\mathrm{IR},\Delta_f^5}$ (black) $\beta'_{\mathrm{IR},\Delta_f^4}$ (blue), $\beta'_{\mathrm{IR},\Delta_f^3}$ (green).

To get a rough estimate of the range of accuracy and applicability of the series expansion for β'_{IR}, we can compute ratios of coefficients, as discussed in connection with Eq. (4.10). Thus, we have

$$\frac{d_{2,\mathrm{SU}(3),F}}{d_{3,\mathrm{SU}(3),F}} = 8.447, \quad (5.22)$$

$$\frac{d_{3,\mathrm{SU}(3),F}}{|d_{4,\mathrm{SU}(3),F}|} = 21.221, \quad (5.23)$$

and

$$\frac{|d_{4,\mathrm{SU}(3),F}|}{|d_{5,\mathrm{SU}(3),F}|} = 8.2115. \quad (5.24)$$

Since $N_{f,b1z} = 16.5$ and $N_{f,b2z} = 153/19 = 8.053$ in this SU(3) theory, the maximal value of Δ_f in the interval I_{IRZ} is

$$(\Delta_f)_{\max} = \frac{321}{38} = 8.447 \quad \text{for SU(3)}, \quad N_f \in I_{\mathrm{IRZ}}. \quad (5.25)$$

Therefore, these ratios suggest that the small-Δ_f expansion may be reasonably reliable in most of this interval, I_{IRZ} and the associated non-Abelian Coulomb phase.

D. Calculation in the LNN limit and comparison with conventional calculation

For theories having gauge the group $G = \mathrm{SU}(N_c)$ with N_f fermions in the fundamental representation of this group, i.e., $R = F$, it is of interest to consider the limit

$$N_c \to \infty, \qquad N_f \to \infty$$
$$\text{with} \quad r \equiv \frac{N_f}{N_c} \ \text{fixed and finite}$$
$$\text{and} \quad \xi(\mu) \equiv \alpha(\mu) N_c \ \text{is a finite function of } \mu. \quad (5.26)$$

We will use the symbol $\lim_{LNN}$ for this limit, where "LNN" stands for "large N_c and N_f" [with the constraints in Eq. (5.26) imposed]. In this LNN ('t Hooft-Veneziano) limit we define the quantities

$$x = \lim_{LNN} \frac{g^2 N_c}{16\pi^2} = \frac{\xi}{4\pi}, \quad (5.27)$$

$$r_{b1z} = \lim_{LNN} \frac{N_{f,b1z}}{N_c}, \quad (5.28)$$

and

$$r_{b2z} = \lim_{LNN} \frac{N_{f,b2z}}{N_c}, \tag{5.29}$$

with values

$$r_{b1z} = \frac{11}{2} = 5.5 \tag{5.30}$$

and

$$r_{b2z} = \frac{34}{13} = 2.615 \tag{5.31}$$

(to the indicated floating-point accuracy). With $I_{\rm IRZ}$ being $N_{f,b2z} < N_f < N_{f,b1z}$, the corresponding interval in the ratio r is

$$I_{{\rm IRZ},r}: \ \frac{34}{13} < r < \frac{11}{2}, \quad \text{i.e.}, 2.615 < r < 5.5. \tag{5.32}$$

We define the scaled scheme-independent expansion parameter for the LNN limit

$$\Delta_r \equiv \frac{\Delta_f}{N_c} = r_{b1z} - r = \frac{11}{2} - r \tag{5.33}$$

and

$$r_c = \lim_{LNN} \frac{N_{f,cr}}{N_c}. \tag{5.34}$$

After these preliminaries, we now proceed to calculate the scheme-independent expansion of $\beta'_{\rm IR}$ in this LNN limit (5.26), and compare with the conventional calculation of this quantity. The beta function that is finite in this LNN limit is

$$\beta_\xi = \frac{d\xi}{dt} = \lim_{LNN} \beta N_c, \tag{5.35}$$

where $\xi = \lim_{LNN} \alpha N_c$ was defined in Eq. (5.26). This has the series expansion

$$\beta_\xi \equiv \frac{d\xi}{dt} = -8\pi x \sum_{\ell=1}^{\infty} \hat{b}_\ell x^\ell, \tag{5.36}$$

where

$$\hat{b}_\ell = \lim_{LNN} \frac{b_\ell}{N_c^\ell}. \tag{5.37}$$

The $\hat{b}_\ell$ are listed for reference in Appendix.

Since the derivative $d\beta_\xi/d\xi$ satisfies the relation

$$\frac{d\beta_\xi}{d\xi} = \frac{d\beta}{d\alpha} \equiv \beta', \tag{5.38}$$

it follows that β' is finite in the LNN limit (5.26). In terms of the variable x defined in Eq. (5.27), we have

$$\beta' = -2 \sum_{\ell=1}^{\infty} (\ell+1) \hat{b}_\ell x^\ell. \tag{5.39}$$

Because $\beta'_{\rm IR}$ is scheme-invariant and is finite in the LNN limit, one is motivated to calculate the LNN limit of the scheme-independent expansion (5.7). For this purpose, in addition to the rescaled quantities Δ_r defined in Eq. (5.33), we define the rescaled coefficient

$$\hat{d}_n = N_c^n d_n, \tag{5.40}$$

which is finite in the LNN limit. Then each term

$$\lim_{LNN} d_n \Delta_f^n = (N_c^n d_n) \left(\frac{\Delta_f}{N_c}\right)^n = \hat{d}_n \Delta_r^n \tag{5.41}$$

is finite in this limit. Thus, writing $\lim_{LNN} \beta'_{\rm IR}$ as $\beta'_{{\rm IR},LNN}$, we have

$$\beta'_{{\rm IR},LNN} = \sum_{n=1}^{\infty} d_n \Delta_f^n = \sum_{n=1}^{\infty} \hat{d}_n \Delta_r^n. \tag{5.42}$$

We denote the value of $\beta'_{{\rm IR},LNN}$ obtained from this series calculated to order $O(\Delta_f^p)$ as $\beta'_{{\rm IR},LNN,\Delta_f^p}$.

From Eqs. (5.8)–(5.11), we find that the approach to the LNN limits for $\hat{d}_n$ involves correction terms that vanish like $1/N_c^2$. This is the same property that was found in [26,27] and, in the same way, it means that the approach to the LNN limit for finite N_c and N_f with fixed $r = N_f/N_c$ is rather rapid, as discussed in [27]. We calculate $\hat{d}_1 = 0$ and

$$\hat{d}_2 = \frac{2^4}{3^2 \cdot 5^2} = 0.0711111, \tag{5.43}$$

$$\hat{d}_3 = \frac{416}{3^3 \cdot 5^4} = 2.465185 \times 10^{-2}, \tag{5.44}$$

and

$$\hat{d}_4 = \frac{5868512}{3^5 \cdot 5^{10}} - \frac{5632}{3^4 \cdot 5^6} \zeta_3 = -(2.876137 \times 10^{-3}). \tag{5.45}$$

Thus, numerically

$$\begin{aligned} \beta'_{{\rm IR},LNN} &= \gamma_{F^2,{\rm IR}} \\ &= \Delta_r^2 [0.07111 + (2.4652 \times 10^{-2}) \Delta_r \\ &\quad - (2.8761 \times 10^{-3}) \Delta_r^2]. \end{aligned} \tag{5.46}$$

We may again calculate ratios of successive magnitudes of these coefficients to get a rough estimate of the range over which the small-Δ_r expansion is reliable in this LNN limit. We find

$$\frac{\hat{d}_2}{\hat{d}_3} = 2.885 \tag{5.47}$$

and

$$\frac{\hat{d}_3}{|\hat{d}_4|} = 8.571. \tag{5.48}$$

For $r \in I_{\mathrm{IRZ},r}$, the maximal value of Δ_r is

$$(\Delta_r)_{\max} = \frac{75}{26} = 2.885 \quad \text{for } r \in I_{\mathrm{IRZ},r}. \tag{5.49}$$

Therefore, these LNN ratios suggest, in agreement with our analysis for SU(3) and $R = F$, that the small-Δ_r expansion may be reasonably reliable over much of the interval $I_{\mathrm{IRZ},r}$.

It is useful to compare these scheme-independent calculations of $\beta'_{\mathrm{IR},LNN}$ with the results of conventional n-loop calculations, denoted $\beta'_{\mathrm{IR},n\ell,LNN}$. These derivatives are computed from the n-loop truncation of the series in Eq. (5.39). As a special case of our remark below Eq. (5.5), we note that in calculating the n-loop truncation of the series (5.39) at the IR zero of the beta function, for $n \geq 3$, one uses the property that

$$\sum_{\ell=1}^{n} \hat{b}_\ell x_{\mathrm{IR},n\ell}^{\ell-1} = 0, \tag{5.50}$$

to eliminate the highest-loop term $\hat{b}_n x_{\mathrm{IR}}^{n-1}$, expressing it as $\hat{b}_n x_{\mathrm{IR}}^{n-1} = -\sum_{\ell=1}^{n-2} \hat{b}_\ell x_{\mathrm{IR},n\ell}^{\ell-1}$. The two-loop result for x_{ir} is

$$x_{\mathrm{IR},2\ell} = \frac{11-2r}{13r-34} \quad \text{for } r \in I_{\mathrm{IRZ},r}. \tag{5.51}$$

The resultant two-loop for β'_{IR} is

$$\beta'_{\mathrm{IR},2\ell} = \frac{2(11-2r)^2}{3(13r-34)}. \tag{5.52}$$

Both $x_{\mathrm{IR},2\ell}$ and $\beta'_{\mathrm{IR},2\ell}$ are scheme-independent. However, the higher-loop expressions for these quantities at loop level $n \geq 3$ do not preserve the scheme-independence of the exact β'_{IR}. Let us define the polynomials (see Eqs. (3.9) and (2.26) in [27])

$$C_{3\ell} = -52450 + 41070r - 7779r^2 + 448r^3 \tag{5.53}$$

and

$$D_{3\ell} = -2857 + 1709r - 112r^2, \tag{5.54}$$

both of which are positive for $r \in I_{\mathrm{IRZ},r}$. The three-loop value of the IR zero of the beta function in the LNN limit, computed in the $\overline{\mathrm{MS}}$ scheme, is [27]

$$x_{\mathrm{IR},3\ell} = \frac{3[-3(13r-34) + \sqrt{C_{3\ell}}\,]}{D_{3\ell}}. \tag{5.55}$$

We calculate the three-loop result for β'_{IR}, or equivalently the anomalous dimension of $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, in the LNN limit, again in the $\overline{\mathrm{MS}}$ scheme, to be

$$\begin{aligned}\beta'_{\mathrm{IR},3\ell} = {} & \frac{2[-3(13r-34) + \sqrt{C_{3\ell}}\,]}{D_{3\ell}^2} \\ & \times [-52450 + 41070r - 7779r^2 + 448r^3 \\ & - 3(13r-34)\sqrt{C_{3\ell}}\,].\end{aligned} \tag{5.56}$$

We compute the four-loop result $\beta'_{\mathrm{IR},4\ell}$ in this scheme in a similar manner. In Table III we list the numerical values of these conventional n-loop calculations in comparison with our scheme-independent results calculated to $O(\Delta_f^p)$ for $2 \leq n \leq 4$ and $1 \leq p \leq 3$. We see that, especially for r values in the upper part of the interval $I_{\mathrm{IRZ},r}$, the results are rather close, and, furthermore, that, as expected, for a given r, the higher the loop level n and the truncation order p in the respective calculations of $\beta'_{\mathrm{IR},n\ell}$ in the $\overline{\mathrm{MS}}$ scheme and the scheme-independent $\beta'_{\mathrm{IR},\Delta_f^p}$, the better the agreement between these two results. All of the entries shown in Table III have $\beta'_{\mathrm{IR}} < 3$ except for the two-loop values $\beta'_{\mathrm{IR},2\ell}$ for $r = 3.0$ and $r = 2.8$ which are 3.333 and 8.100, respectively.

E. Calculation of the d_n to $O(\Delta_f^4)$ for $G = \mathrm{SU}(N_c)$ and $R = adj$

It is worthwhile to compare our results obtained for $G = \mathrm{SU}(N_c)$ with N_f fermions in the fundamental representation to the case in which the fermions are in the adjoint representation, denoted as adj for short. In this case, the general expressions for $N_{f,b1z}$ and $N_{f,b2z}$ are

$$N_{f,b1z} = \frac{11}{4} = 2.75 \quad \text{for } R = adj \tag{5.57}$$

and

$$N_{f,b2z} = \frac{17}{16} = 1.0625 \quad \text{for } R = adj, \tag{5.58}$$

so the interval I_{IRZ} only contains the single integer value $N_f = 2$.

For this theory, our general expressions (5.9) and (5.10) reduce to pure numbers, independent of N_c:

TABLE III. Scheme-independent values of $\beta'_{\mathrm{IR},\Delta_r^p}$ for $2 \le p \le 4$ in the LNN limit (5.26) as functions of $r = 5.5 - \Delta_r$. For comparison, we also list the n-loop values $\beta'_{\mathrm{IR},n\ell}$ with $2 \le n \le 4$, where $\beta'_{\mathrm{IR},3\ell}$ and $\beta'_{\mathrm{IR},4\ell}$ are computed in the $\overline{\mathrm{MS}}$ scheme [and values that exceed the upper bound (4.13) are marked as such]. The notation ae-n means $a \times 10^{-n}$.

r	$\beta'_{\mathrm{IR},2\ell}$	$\beta'_{\mathrm{IR},3\ell,\overline{\mathrm{MS}}}$	$\beta'_{\mathrm{IR},4\ell,\overline{\mathrm{MS}}}$	$\beta'_{\mathrm{IR},\Delta_r^2}$	$\beta'_{\mathrm{IR},\Delta_r^3}$	$\beta'_{\mathrm{IR},\Delta_r^4}$
2.8	>3	1.918	1.949	0.518	1.004	0.851
3.0	>3	1.376	1.523	0.444	0.830	0.717
3.2	1.856	1.006	1.100	0.376	0.676	0.596
3.4	1.153	0.7395	0.72985	0.314	0.542	0.486
3.6	0.752	0.542	0.528	0.257	0.426	0.388
3.8	0.500	0.393	0.378	0.2055	0.327	0.303
4.0	0.333	0.279	0.267	0.160	0.243	0.229
4.2	0.219	0.193	0.185	0.120	0.174	0.166
4.4	0.139	0.128	0.122	0.0860	0.119	0.115
4.6	0.0837	0.0792	0.0766	0.0576	0.0756	0.0737
4.8	0.0460	0.0445	0.0435	0.0348	0.0433	0.0426
5.0	0.0215	0.0212	0.0208	0.0178	0.0209	0.0207
5.2	0.714e-2	0.710e-2	0.706e-2	0.640e-2	0.707e-2	0.704e-2
5.4	0.737e-3	0.736e-3	0.7356e-3	0.7111e-3	0.7358e-3	0.7355-3

$$d_{2,\mathrm{SU}(N_c),adj} = \frac{2^4}{3^4} = 0.19753, \tag{5.59}$$

$$d_{3,\mathrm{SU}(N_c),adj} = \frac{2^8}{3^7} = 0.11706. \tag{5.60}$$

For d_4 we calculate

$$d_{4,\mathrm{SU}(N_c),adj} = \frac{46871 N_c^2 + 85248}{2^2 \cdot 3^{12} N_c^2}. \tag{5.61}$$

This coefficient $d_{4,\mathrm{SU}(N_c),adj}$ is manifestly positive and has the large-N_c limit

$$\lim_{N_c\to\infty} d_{4,\mathrm{SU}(N_c),adj} = \frac{46871}{2^2 \cdot 3^{12}} = 0.022049. \tag{5.62}$$

In contrast to our results for the $d_{n,\mathrm{SU}(N_c),F}$, here all of the coefficients $d_{n,\mathrm{SU}(N_c),adj}$ that we have calculated, for $1 \le n \le 4$, are positive. These signs are recorded in Table I.

With these coefficients, one can again compute ratios to obtain a crude idea of the region over which the small-Δ_f series expansion is reliable. We have

$$\frac{d_{2,\mathrm{SU}(N_c),adj}}{d_{3,\mathrm{SU}(N_c),adj}} = 1.687 \tag{5.63}$$

and, taking the large-N_c limit for simplicity,

$$\lim_{N_c\to\infty} \frac{d_{3,\mathrm{SU}(N_c),adj}}{d_{4,\mathrm{SU}(N_c),adj}} = 5.309. \tag{5.64}$$

These ratios are consistent with the inference that the small-Δ_f expansion may again be reasonably accurate in the interval I_{IRZ} and for the corresponding value $N_f = 2$ in this theory.

VI. ANALYSIS OF SCHEME-INDEPENDENT EXPANSION COEFFICIENTS FOR $\gamma_{\bar\psi\psi,\mathrm{IR}}$

A. Review of calculation to $O(\Delta_f^3)$ for general G and R

We consider the (gauge-invariant) flavor-nonsinglet (fns) and flavor-singlet (fs) bilinear fermion operators

$$J_{0,fns} = \sum_{j,k=1}^{N_f} \bar\psi_j (T_b)_{jk} \psi_k, \tag{6.1}$$

where here T_b with $b = 1, \ldots, N_f^2 - 1$ is an generator of the global flavor group $\mathrm{SU}(N_f)$, and

$$J_{0,fs} = \sum_{j=1}^{N_f} \bar\psi_j \psi_j. \tag{6.2}$$

We will often suppress the flavor indices and write these simply as $\bar\psi T_b \psi$ and $\bar\psi\psi$. These have the same anomalous dimension (e.g., [46]), which we denote as $\gamma_{\bar\psi\psi}$. (Thus, one may simply consider the operator $\bar\psi_j\psi_j$ with no sum on j, but here we shall refer to $J_{0,fns}$ and $J_{0,fs}$.) The operator $J_{0,fns}$ has the chiral decomposition $\bar\psi T_b \psi = \bar\psi_L T_b \psi_R + \bar\psi_R T_b \psi_L$. Hence, in the non-Abelian Coulomb phase where the flavor symmetry is (2.19), one may regard the T_b in the term $\bar\psi_L T_b \psi_R$ acting to the right as an element of $\mathrm{SU}(N_f)_R$ and acting to the left as an element of $\mathrm{SU}(N_f)_L$.

The usual series expansion of $\gamma_{\bar\psi\psi}$ in powers of α, or equivalently, a, is

$$\gamma_{\bar{\psi}\psi} = \sum_{\ell=1}^{\infty} c_\ell a^\ell, \tag{6.3}$$

where c_ℓ is the ℓ-loop coefficient. For general G and R the coefficients c_ℓ have been calculated up to $\ell = 4$ loop level [47] (earlier work includes [48]) and for the special case $G = \mathrm{SU}(3)$ and $R = F$, c_5 has been calculated [49]. The scheme-independent expansion of $\gamma_{\bar{\psi}\psi}$ can be written as

$$\gamma_{\bar{\psi}\psi,\mathrm{IR}} = \sum_{n=1}^{\infty} \kappa_n \Delta_f^n. \tag{6.4}$$

We denote the truncation of this sum to maximal power $n = p$ as $\gamma_{\bar{\psi}\psi,\mathrm{IR},\Delta_f^p}$. For a general asymptotically free vectorial gauge theory with gauge group G and N_f fermions in an arbitrary representation R, the coefficients κ_n were given in [9] up to order $n = 3$, yielding the expansion of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ to order Δ_f^3. For reference, we display the κ_n coefficients from Ref. [9] [with the denominator factor D given in Eq. (2.13)]:

$$\kappa_1 = \frac{8T_f C_f}{C_A D}, \tag{6.5}$$

$$\kappa_2 = \frac{4T_f^2 C_f (5C_A + 88C_f)(7C_A + 4C_f)}{3C_A^2 D^3}, \tag{6.6}$$

and

$$\begin{aligned}\kappa_3 = \frac{4T_f C_f}{3^4 C_A^4 D^5}\Bigg[&-55419 T_f^2 C_A^5 + 432012 T_f^2 C_A^4 C_f + 5632 T_f^2 C_f \frac{d_A^{abcd} d_A^{abcd}}{d_A}(-5 + 132\zeta_3)\\
&+ 16 C_A^3\left(122043 T_f^2 C_f^2 + 6776 \frac{d_R^{abcd} d_R^{abcd}}{d_A}(-11 + 24\zeta_3)\right)\\
&+ 704 C_A^2\left(1521 T_f^2 C_f^3 + 112 T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(4 - 39\zeta_3) + 242 C_f \frac{d_R^{abcd} d_R^{abcd}}{d_A}(-11 + 24\zeta_3)\right)\\
&+ 32 T_f C_A\left(53361 T_f C_f^4 - 3872 C_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(-4 + 39\zeta_3) + 112 T_f \frac{d_A^{abcd} d_A^{abcd}}{d_A}(-5 + 132\zeta_3)\right)\Bigg].\end{aligned} \tag{6.7}$$

B. Evaluation of κ_n for $G = \mathrm{SU}(N_c)$ and $R = F$

For the case where the N_f fermions are in the representation $R = F$, these results (6.5)–(6.7) from [9] take the following forms:

$$\kappa_{1,\mathrm{SU}(N_c),F} = \frac{4(N_c^2 - 1)}{N_c(25N_c^2 - 11)}, \tag{6.8}$$

$$\kappa_{2,\mathrm{SU}(N_c),F} = \frac{4(N_c^2 - 1)(9N_c^2 - 2)(49N_c^2 - 44)}{3N_c^2(25N_c^2 - 11)^3}, \tag{6.9}$$

and

$$\begin{aligned}\kappa_{3,\mathrm{SU}(N_c),F} = \frac{8(N_c^2 - 1)}{3^3 N_c^3 (25N_c^2 - 11)^5}&[274243 N_c^8 - 455426 N_c^6 - 114080 N_c^4 + 47344 N_c^2 + 35574\\
&- 4224 N_c^2 (4N_c^2 - 11)(25N_c^2 - 11)\zeta_3].\end{aligned} \tag{6.10}$$

We find that these coefficients $\kappa_{n,\mathrm{SU}(N_c),F}$ with $1 \le n \le 3$ are positive-definite for all physical $N_c \ge 2$. This is obvious for $n = 1$, 2, and an examination of the polynomial in square brackets in Eq. (6.10), of degree 8 in N_f, proves the result for $n = 3$.

C. Calculation of κ_n coefficients to $O(\Delta_f^4)$ for $G = \mathrm{SU}(3)$ and $R = F$

For comparison with the κ_n with other values of N_c, we recall our calculation of the κ_n to order $n = 4$, i.e., to order $O(\Delta_f^4)$ in [11]. We found

$$\kappa_{\mathrm{SU}(3),F,1} = \frac{16}{3\cdot 107} = 4.9844\times 10^{-2}, \tag{6.11}$$

$$\kappa_{\mathrm{SU}(3),F,2} = \frac{125452}{(3\cdot 107)^3} = 3.7928\times 10^{-3}, \tag{6.12}$$

$$\kappa_{\mathrm{SU}(3),F,3} = \frac{972349306}{(3\cdot 107)^5} - \frac{140800}{3^3\cdot(107)^4}\zeta_3 = 2.3747\times 10^{-4}, \tag{6.13}$$

$$\kappa_{\mathrm{SU}(3),F,4} = \frac{33906710751871}{2^2(3\cdot 107)^7} - \frac{1684980608}{3^5\cdot(107)^6}\zeta_3 + \frac{59840000}{(3\cdot 107)^5}\zeta_5 = 3.6789\times 10^{-5}. \tag{6.14}$$

In Ref. [9] the ratio test was applied to the first three coefficients, $\kappa_{\mathrm{SU}(3),F,n}$, $n = 1,\ 2,\ 3$ and the excellent convergence was noted. Here, using our calculation of $\kappa_{\mathrm{SU}(3),F,4}$ in [11], we calculate the next ratio, $\kappa_{\mathrm{SU}(3),F,3}/\kappa_{\mathrm{SU}(3),F,4}$. We have

$$\frac{\kappa_{\mathrm{SU}(3),F,1}}{\kappa_{\mathrm{SU}(3),F,2}} = 13.142, \tag{6.15}$$

$$\frac{\kappa_{\mathrm{SU}(3),F,2}}{\kappa_{\mathrm{SU}(3),F,3}} = 15.972, \tag{6.16}$$

and

$$\frac{\kappa_{\mathrm{SU}(3),F,3}}{\kappa_{\mathrm{SU}(3),F,4}} = 6.455. \tag{6.17}$$

Since the maximal value of Δ_f in the interval I_{IRZ} is 8.447 [see Eq. (5.25)], these ratios suggest, as noted in [9] and in agreement with our earlier calculation of coefficient ratios for β'_{IR}, that the small-Δ_f expansion may be reasonably reliable over much of the interval I_{IRZ}.

The positivity of the $\kappa_{\mathrm{SU}(3),F,n}$ for $1 \le n \le 3$ is in agreement with our more general positivity results given above, and, as we noted in [11], we also found that $\kappa_{\mathrm{SU}(3),F,4}$ is positive. These signs are recorded in Table I. The positivity of all of these coefficients played an important role in our analysis in [11] because it meant that for a given value of N_f, or equivalently, Δ_f, the value of $\gamma_{\bar\psi\psi}$ calculated to $O(\Delta_f^n)$, denoted $\gamma_{\bar\psi\psi,\Delta_f^n}$, is a monotonically increasing function of n over the full range $1 \le n \le 4$ that we calculated. We then conjectured that this positivity would be true for all n, i.e., we conjectured that $\kappa_n > 0$ for all $n \ge 1$. Assuming the validity of this conjecture, we then computed the extrapolation to $n \to \infty$ for an exact $\gamma_{\bar\psi\psi,\mathrm{IR}}$ in the SU(3) theory with $R = F$. A generalization of our conjecture in [11] that is motivated by our present results is that, in the notation of Eqs. (6.11)–(6.14), $\kappa_{n,\mathrm{SU}(N_c),F} > 0$ for all $n \ge 1$ and all $N_c \ge 2$. Importantly, in [11] we noted that, if this monotonicity property holds, then, combining it with the upper bound $\gamma_{\bar\psi\psi,\mathrm{IR}} < 2$, one would infer that if γ_{IR} saturates its upper bound (4.14) as N_f decreases and passes through the value $N_{f,cr}$ at the lower end of the non-Abelian Coulomb phase, it would follow from our extrapolated values of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ that $8 < N_{f,cr} < 9$. Here one must mention the caveat that it is not known if, in fact, γ_{IR} saturates its upper bound in this way as $N_f \searrow N_{f,cr}$. Indeed, the nature of the transition as N_f decreases through $N_{f,cr}$ has not been definitely established. Analyses via Schwinger-Dyson equations suggested that, as $N_f \nearrow N_{f,cr}$ from within the phase with confinement and chiral symmetry breaking, the fermion condensate $\langle\bar\psi\psi\rangle$ could vanish with an essential zero [50]. Some insight into this may be derived from the known results in SQCD. In SQCD, as noted above, the upper bound is $\gamma_{\bar\psi\psi,\mathrm{IR}} < 1$ and is saturated at the lower end of the non-Abelian Coulomb phase [37,38].

In the case $G = \mathrm{SU}(3)$ and $R = F$, one of the major values of the five-loop calculation of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ in [10] and the scheme-independent calculations of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ to order Δ_f^3 in [9] and to order Δ_f^4 in [11], with the additional analysis here, is the comparison of these results with fully nonperturbative lattice measurements of this anomalous dimension [51]. [Since our discussion here is on the operator $\bar\psi\psi$ and the gauge group SU(3), when there is no danger of confusion, we omit these subscripts in the $\bar\psi\psi$ anomalous dimension.] A number of lattice groups have obtained data and carried out analyses of these data for the SU(3) theory with $N_f = 12$ fermions with $R = F$. These groups have reported the following values: $\gamma_{\mathrm{IR}} = 0.414 \pm 0.016$ [52], $\gamma_{\mathrm{IR}} \sim 0.35$ [53], $\gamma_{\mathrm{IR}} \simeq 0.4$ [54], $\gamma_{\mathrm{IR}} = 0.27(3)$ [55], $\gamma_{\mathrm{IR}} \simeq 0.25$ [56], $\gamma_{\mathrm{IR}} = 0.235(46)$ [57], and $0.2 \lesssim \gamma \lesssim 0.4$ [58]. (For comparative discussions of these different results and estimates of overall uncertainties, the reader is advised to consult the reviews in [51] and the original papers [52–54,57–59].) As we noted in [11], our value $\gamma_{\mathrm{IR},\Delta^4} = 0.338$ and our extrapolated $\gamma_{\mathrm{IR}} = 0.40$ are consistent with this range of lattice measurements, taking into account the different methods of lattice data analysis used, and are somewhat higher than the five-loop value $\gamma_{\mathrm{IR},5\ell} = 0.255$ from the conventional α series that we obtained in [10]. The $\gamma_{\mathrm{IR},5\ell} = 0.255$ value in [10] is in very good agreement with the measured values of γ_{IR} reported in [55,57,58,60–62].

There have also been lattice studies of the SU(3) theory with $N_f = 10$ [60] and $N_f = 8$ [51,61,62]. For the SU(3) theory with $N_f = 10$ fermions, our scheme-independent calculation presented in [11] and discussed further here gives $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^4} = 0.615$ and our extrapolation to infinite order in the Δ_f expansion yields $\gamma_{\bar\psi\psi,\mathrm{IR}} = 0.95(6)$, consistent with estimates that $\gamma_{\bar\psi\psi,\mathrm{IR}} \sim 1$ from lattice studies [51,60]. In the SU(3) theory (with N_f fermions in the representation $R = F$), the lower end of I_{IRZ} occurs at $N_{f,b2z} = 8.047$, but one may still formally consider the results of the small-Δ_f expansion evaluated at $N_f = 8$. In this case we obtain $\gamma_{\mathrm{IR},\Delta_f^p} = 0.424$, 0.698, 0.844, 1.04 for $1 \le p \le 4$. These are again consistent with the rough estimates $\gamma_{\bar\psi\psi,\mathrm{IR}} \sim 1$ from lattice studies [51,61,62]. There is not yet a consensus on the value of $N_{f,cr}$ from lattice studies [51]. In this context, one should keep in mind that for $N_f < N_{f,cr}$, there is spontaneous chiral symmetry breaking, so the IR zero of the beta function is only approximate, since the theory flows away from this value as the fermions gain dynamical mass and are integrated out, leaving a pure gluonic low-energy effective field theory. For such a theory, the quantity extracted from either continuum or lattice analyses as $\gamma_{\bar\psi\psi,\mathrm{IR}}$ is only an effective anomalous dimension that describes the renormalization-group behavior as the theory is flowing near to the approximate zero of the beta function.

D. Evaluation of $\kappa_{n,\mathrm{SU}(N_c),R}$ to $O(\Delta_f^3)$ for $R = adj$

In the case $R = adj$, the general results in [9] reduce as follows:

$$\kappa_{1,\mathrm{SU}(N_c),adj} = \frac{4}{3^2} = 0.4444, \tag{6.18}$$

$$\kappa_{2,\mathrm{SU}(N_c),adj} = \frac{341}{2\cdot 3^6} = 0.23388, \tag{6.19}$$

$$\kappa_{3,\mathrm{SU}(N_c),adj} = \frac{61873N_c^2 - 42624}{2^3\cdot 3^{10}N_c^2}. \tag{6.20}$$

This is positive for all physical N_c and has the large-N_c limit

$$\lim_{N_c\to\infty} \kappa_{3,\mathrm{SU}(N_c),adj} = \frac{61873}{2^3\cdot 3^{10}} = 0.130978. \tag{6.21}$$

The positive signs of these $\kappa_{n,\mathrm{SU}(N_c),adj}$ coefficients are recorded in Table I.

E. Comparison of scheme-independent calculation of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ with conventional calculations

It is of considerable interest to compare the results obtained in [9] for the scheme-independent expansion of $\gamma_{\bar\psi\psi,\mathrm{IR}}$ to order $O(\Delta_f^3)$ (using calculations of the b_n to $n = 4$ loop order and c_n to $n = 3$ loop order) with results obtained previously with the conventional calculation of the n-loop $\gamma_{\bar\psi\psi,\mathrm{IR},n\ell}$ in powers of the n-loop $\alpha_{\mathrm{IR},n\ell}$ in [22] (using calculations of the b_n and c_n up to $n = 4$ loop order). Here and below, for specific calculations we take the gauge group to be $\mathrm{SU}(N_c)$ with various values of N_c. For notational brevity, in this section we will often leave the subscript $\bar\psi\psi$ implicit on these and other quantities and thus write $\gamma_{\mathrm{IR}} \equiv \gamma_{\bar\psi\psi,\mathrm{IR}}$, $\gamma_{\mathrm{IR},n\ell} \equiv \gamma_{\bar\psi\psi,\mathrm{IR},n\ell}$, $\kappa_n \equiv \kappa_{\bar\psi\psi,n}$, etc. in this and the next section. Since $\gamma_{\mathrm{IR},n\ell}$ is scheme-dependent beyond the lowest order, one must choose a scheme for this comparison. Here we choose the widely used $\overline{\mathrm{MS}}$ scheme, for which b_3 and b_4 and c_n for $2 \le n \le 4$ were calculated for a general gauge group G and fermion representation R [13–15] [47]. In the special case of $G = \mathrm{SU}(3)$ and $R = F$, using the recent calculations of the five-loop coefficients b_5 and c_5 in the $\overline{\mathrm{MS}}$ scheme, we computed $\gamma_{\mathrm{IR},n\ell}$ up to $n = 5$ loop level [10] in this $\overline{\mathrm{MS}}$ scheme and performed a scheme-independent calculation up to order Δ_f^4 [11]. For this special case we compared the results obtained via these two different approaches. Here we carry out a similar comparison for other $\mathrm{SU}(N_c)$ theories. The scheme-independent expansion of γ_{IR} has the form (6.4). We denote the value of γ_{IR} obtained from this series calculated to order $O(\Delta_f^p)$ as $\gamma_{\mathrm{IR},\Delta_f^p}$

As discussed above, our discussion is restricted to the interval I_{IRZ} of values of N_f, given in Eq. (2.7), for which the (scheme-independent) two-loop beta function has an IR zero. Using the results for the lower and upper ends of this interval, $N_{f,b2z}$ and $N_{f,b1z}$ from Eqs. (2.4) and (2.6), one has, for $(N_{f,b1z}, N_{f,b2z})$, the respective values (5.55,11), (8.05,16.5), and (10.61,22) for $N_c = 2, 3, 4$ [19], and hence the physical intervals I_{IRZ} with integral N_f: $6 \le N_f \le 10$ for SU(2), $9 \le N_f \le 16$ for SU(3), and $11 \le N_f \le 21$ for SU(4). Our results for these three illustrative values of N_c are listed in Table IV. For the special case $N_c = 3$, we have carried these calculations one order higher, namely to five-loop level and to order Δ_f^4 in [10,11].

Since the calculation of κ_n and the resultant $\gamma_{\mathrm{IR},\Delta_f^n}$ uses information from the $(n+1)$-loop beta function from (2.1) and the n-loop expansion of γ in (4.2), it is natural to compare the (SI) $\gamma_{\mathrm{IR},\Delta_f^n}$ with the (SD) $\gamma_{\mathrm{IR},n'\ell}$ for $n' = n$ and $n' = n + 1$. Since $\gamma_{\mathrm{IR},\Delta_f^n}$ includes n-loop information about $\gamma_{\mathrm{IR},n\ell}$, one would expect the closest agreement between $\gamma_{\mathrm{IR},\Delta_f^n}$ and $\gamma_{\mathrm{IR},n\ell}$, and our results confirm this expectation. In the upper and middle part of the interval I_{IRZ} for a given N_c, we find that $\gamma_{\mathrm{IR},\Delta_f^n}$ is slightly larger than $\gamma_{\mathrm{IR},3\ell}$, with the difference increasing as N_f decreases below $N_{f,b1z}$, i.e., as Δ_f increases.

We recall the upper bound (4.14) that applies at an IRFP in the non-Abelian Coulomb phase, based on the scale invariance and inferred conformal invariance in this phase. The bound (4.14) also applies, for a different reason, in the

TABLE IV. Values of the anomalous dimension $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p}$ calculated to order $p = 1, 2, 3$, for $G = \mathrm{SU}(N_c)$ and $R = F$, as functions of N_c and N_f. To save space, we omit the subscript $\bar\psi\psi$, writing $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_f^p} \equiv \gamma_{\mathrm{IR},\Delta_f^p}$. For comparison, we also include the (scheme-independent) $\gamma_{\mathrm{IR},2\ell}$ and $\gamma_{\mathrm{IR},n\ell,\overline{\mathrm{MS}}}$, $n = 3, 4$. $\gamma_{\mathrm{IR},4\ell,\overline{\mathrm{MS}}}$. Values that exceed the bound $\gamma_{\bar\psi\psi,\mathrm{IR}} < 2$ in Eq. (4.14) are marked as such; in these cases, the $\gamma_{\mathrm{IR},n\ell,\overline{\mathrm{MS}}}$ are unlisted (u).

N_c	N_f	$\gamma_{\mathrm{IR},2\ell}$	$\gamma_{\mathrm{IR},3\ell,\overline{\mathrm{MS}}}$	$\gamma_{\mathrm{IR},4\ell,\overline{\mathrm{MS}}}$	$\gamma_{\mathrm{IR},\Delta_f}$	$\gamma_{\mathrm{IR},\Delta_f^2}$	$\gamma_{\mathrm{IR},\Delta_f^3}$
2	6	>2	u	u	0.337	0.520	0.596
2	7	>2	u	u	0.270	0.387	0.426
2	8	0.752	0.272	0.204	0.202	0.268	0.285
2	9	0.275	0.161	0.157	0.135	0.164	0.169
2	10	0.0910	0.0738	0.0748	0.0674	0.07475	0.07535
3	9	>2	u	u	0.374	0.587	0.687
3	10	>2	u	u	0.324	0.484	0.549
3	11	1.61	0.439	0.250	0.274	0.389	0.428
3	12	0.773	0.312	0.253	0.224	0.301	0.323
3	13	0.404	0.220	0.210	0.174	0.221	0.231
3	14	0.212	0.146	0.147	0.125	0.148	0.152
3	15	0.0997	0.0826	0.0836	0.0748	0.0833	0.0841
3	16	0.0272	0.0258	0.0259	0.0249	0.0259	0.0259
4	11	>2	u	u	0.424	0.694	0.844
4	12	>2	u	u	0.386	0.609	0.721
4	13	>2	u	u	0.347	0.528	0.610
4	14	>2	u	u	0.308	0.451	0.509
4	15	1.32	0.420	0.281	0.270	0.379	0.418
4	16	0.778	0.325	0.269	0.231	0.312	0.336
4	17	0.481	0.251	0.234	0.193	0.249	0.263
4	18	0.301	0.189	0.187	0.154	0.190	0.197
4	19	0.183	0.134	0.136	0.116	0.136	0.139
4	20	0.102	0.0854	0.0865	0.0771	0.0860	0.086
4	21	0.0440	0.0407	0.0409	0.0386	0.0408	0.0409

phase with confinement and spontaneous chiral symmetry breaking; in that phase it is a consequence of the physical requirement that the momentum-dependent dynamically generated effective fermion mass

$$m(k) \sim \Lambda \left(\frac{\Lambda}{k}\right)^{2-\gamma_{\mathrm{IR}}} \tag{6.22}$$

must satisfy the constraint $\lim_{k\to\infty} m(k) = 0$, where k is the Euclidean momentum. In the upper and middle parts of the interval I_{IRZ} in the NACP, the values of $\gamma_{\mathrm{IR},n\ell}$ calculated in the conventional series expansion in powers of $\alpha_{\mathrm{IR},n\ell}$ obey this upper bound. However, for a given N_c, toward the lower end of the respective intervals I_{IRZ}, the IR coupling $\alpha_{\mathrm{IR},n\ell}$ become too large for the perturbative calculations to be applicable, and some resultant values of the anomalous dimensions exceed the bound (4.14). This occurs for the scheme-independent two-loop values $\gamma\mathrm{IR}, 2\ell$ for $N_f = 6, 7$ if $N_c = 2$; for $N_f = 9, 10$ if $N_c = 3$, and for $11 \le N_f \le 14$ if $N_f = 4$. In these cases, since it is not clear that the higher-order values $\gamma_{\mathrm{IR},n\ell}$ are reliable, we leave them unlisted (u), as we did in [22].

From these calculations and the entries in Table IV, one of the important advances achieved by the scheme-independent Δ_f expansion is evident, namely that the values of $\gamma_{\mathrm{IR},\Delta_f^p}$ with $1 \le p \le 3$ (and, for SU(3) also $p = 4$ in [11]) that we calculate via this method obey the upper bound (4.14) throughout all of the interval I_{IRZ} and associated non-Abelian Coulomb phase, in contrast with some of the values calculated via the conventional loop expansion toward the lower end of I_{IRZ}. In general, for all of the N_c values considered, our results for $\gamma_{\mathrm{IR},\Delta_f^p}$ here satisfy the upper bound (4.14) and hence are consistent with the conclusion that the Δ_f expansion is reasonably reliable throughout the interval I_{IRZ} and non-Abelian Coulomb phase. We regard this, together with the scheme-independence itself, as being a major advantage of the Δ_f expansion.

F. LNN limit for $\gamma_{\bar\psi\psi,\mathrm{IR}}$

Here we consider theories with $G = \mathrm{SU}(N_c)$ and N_f copies of fermions in the representation $R = F$ in the LNN limit ([27]). We recall that in this LNN limit, the interval I_{IRZ} is given by Eq. (5.32) and the scaled Δ_r is defined by Eq. (5.33). We define rescaled coefficients $\hat\kappa_n$

$$\hat\kappa_n \equiv \lim_{N_c\to\infty} N_c^n \kappa_n \tag{6.23}$$

that are finite in this LNN limit. The anomalous dimension $\gamma_{\bar\psi\psi,\mathrm{IR}}$ is also finite in this limit and is given by

$$\lim_{LNN} \gamma_{\bar\psi\psi,\mathrm{IR}} = \sum_{n=1}^{\infty} \kappa_n \Delta_f^n = \sum_{n=1}^{\infty} \hat\kappa_n \Delta_r^n. \tag{6.24}$$

From (5.32), it follows that as r decreases from r_{b1z} to r_{b2z}, Δ_r increases from 0 to the its maximal value

$$(\Delta_r)_{\max} = \frac{75}{26} = 2.8846 \quad \text{for } r \in I_{\mathrm{IRZ},r}. \tag{6.25}$$

From the results for κ_n, $n = 1, 2, 3$ in [9] or the special cases given above for $G = \mathrm{SU}(N_c)$ and $R = F$ in Eqs. (6.8)–(6.10), we find

$$\hat\kappa_1 = \frac{4}{25} = 0.1600, \tag{6.26}$$

$$\hat\kappa_2 = \frac{588}{5^6} = 0.037632, \tag{6.27}$$

and

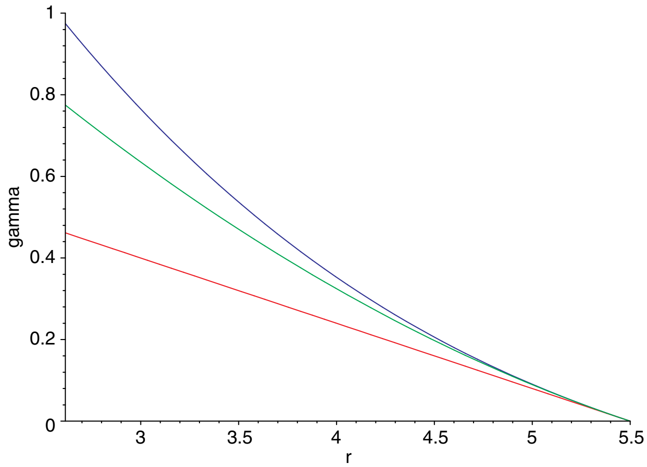

FIG. 2. Plot of $\gamma_{\bar{\psi}\psi,\mathrm{IR},\Delta_r^p}$ for $1 \le p \le 3$ as a function of $r \in I_{\mathrm{IRZ},r}$ in the LNN limit (5.26). From bottom to top, the curves (with colors online) refer to $\gamma_{\bar{\psi}\psi,\mathrm{IR},\Delta_r}$ (red), $\gamma_{\bar{\psi}\psi,\mathrm{IR},\Delta_r^2}$ (green) $\gamma_{\bar{\psi}\psi,\mathrm{IR},\Delta_r^3}$ (blue).

$$\hat{\kappa}_3 = \frac{2193944}{3^3 \cdot 5^{10}} = 0.83207 \times 10^{-2}, \qquad (6.28)$$

where, as above, we indicate the factorization of the denominators. Numerically, to order $O(\Delta_r^3)$,

$$\lim_{LNN} \gamma_{\bar{\psi}\psi,\mathrm{IR}} = \Delta_r[0.160000 + 0.037632\Delta_r + 0.0083207\Delta_r^2 + O(\Delta_f^3)]. \qquad (6.29)$$

We plot the value of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ calculated to order Δ_r^p, denoted $\gamma_{\bar{\psi}\psi,\mathrm{IR},\Delta_r^p}$, for $1 \le p \le 3$, as a function of $r \in I_{\mathrm{IRZ},r}$ in Fig. 2. As a consequence of the positivity of the $\hat{\kappa}_p$ in Eqs. (6.26)–(6.28), for a fixed r, $\gamma_{\bar{\psi}\psi,\mathrm{IR},\Delta_r^p}$ is a monotonically increasing function of the order of calculation, p. Interestingly, as r decreases toward the lower end of the interval $I_{\mathrm{IRZ},r}$ at $r = r_{b2z} = 34/13 = 2.6154$, the value of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ calculated to the highest order in this LNN limit, namely $O(\Delta_r^3)$ is slightly less than 1. This is similar to the behavior that was found for the specific cases of SU(2) and SU(3) gauge groups and $R = F$ in [9] and for SU(3) with $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ calculated to the next order, $O(\Delta_r^4)$ in [11].

As discussed above, our calculations of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ via the Δ_f expansion, both for specific values of N_c and in the LNN limit, have yielded results satisfying the upper bound (4.14) throughout the interval I_{IRZ}. These results support the conclusion that the small-Δ_f series expansion is reliable throughout this interval I_{IRZ} and associated non-Abelian Coulomb phase. It is also worthwhile to obtain an estimate of the range of applicability of the small-Δ_f series expansion via a different method, the aforementioned ratio test. From the coefficients $\hat{\kappa}_n$ that we have calculated with $1 \le n \le 3$, we compute the ratios

$$\frac{\hat{\kappa}_1}{\hat{\kappa}_2} = 4.252 \qquad (6.30)$$

and

$$\frac{\hat{\kappa}_2}{\hat{\kappa}_3} = 4.523. \qquad (6.31)$$

Recalling that the maximal value of Δ_r in the interval $I_{\mathrm{IRZ},r}$ is 2.885 [Eq. (5.49)], these ratios are again consistent with the inference that the small-Δ_r series expansion may be reasonably accurate in this interval I_{IRZ}. Since r has a maximal value of 5.5 in this LNN limit, the above ratios also suggest that one could not reliably apply the small Δ_r expansion down to small r (see also [63]). This is in agreement with the fact that the properties of theory change qualitatively as r decreases below r_c in Eq. (5.34); in particular, there is spontaneous chiral symmetry breaking at small r

G. Analysis with Padé approximants

To get further insight into the behavior of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$, we shall calculate and analyze Padé approximants (PAs) [64]. For this purpose, we shall use a reduced function normalized to unity at $\Delta_f = 0$, namely

$$\bar{\gamma}_{\bar{\psi}\psi,\mathrm{IR}} = \frac{\gamma_{\bar{\psi}\psi,\mathrm{IR}}}{\kappa_1 \Delta_r} = 1 + \frac{1}{\kappa_1}\sum_{n=2}^{\infty} \kappa_n \Delta_r^{n-1}. \qquad (6.32)$$

The calculation of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ to order Δ_r^3 yields $\bar{\gamma}_{\bar{\psi}\psi,\mathrm{IR}}$ to order Δ_r^2. In turn, from this we can compute three PAs: $[2,0]_{\bar{\gamma}_{\bar{\psi}\psi,\mathrm{IR}}}$, $[1,1]_{\bar{\gamma}_{\bar{\psi}\psi,\mathrm{IR}}}$, and $[0,2]_{\bar{\gamma}_{\bar{\psi}\psi,\mathrm{IR}}}$. Since the [2,0] PA is just $\bar{\gamma}_{\bar{\psi}\psi,\mathrm{IR}}$ itself, to order Δ_r^2, we focus on the [1,1] and [0,2] PAs. We calculate

$$[1,1]_{\bar{\gamma}_{\bar{\psi}\psi,\mathrm{IR}}} = \frac{1 + \frac{34957}{2480625}\Delta_r}{1 - \frac{548486}{2480625}\Delta_r} \qquad (6.33)$$

and

$$[0,2]_{\bar{\gamma}_{\bar{\psi}\psi,\mathrm{IR}}} = \frac{1}{1 - \frac{147}{625}\Delta_r + \frac{34957}{10546875}\Delta_r^2}. \qquad (6.34)$$

The [1,1] PA has no physical zero and a pole at

$$(\Delta_r)_{\mathrm{pole},[1,1]_{\bar{\gamma}_{\bar{\psi}\psi,\mathrm{IR}}}} = \frac{2480625}{548486} = 4.523. \qquad (6.35)$$

Since this value is well beyond the maximum value of Δ_r for $r \in I_{\mathrm{IRZ},r}$, namely 2.885, it follows that the [1,1] PA is finite for all $r \in I_{\mathrm{IRZ},r}$.

The [0,2] PA obviously has no zero, and has two poles, at

TABLE V. Values of $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_r^3}$, $[1,1]_{\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_r^3}}$, and $[1,1]_{\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_r^3}}$, together with $\gamma_{\bar\psi\psi,\mathrm{IR},n\ell}$ with $n=2$, 3, 4 from Table V of [27] for comparison, as a function of r for $r \in I_{\mathrm{IRZ},r}$ and satisfying $\gamma_{\mathrm{IR}} < 2$. Here, $\Delta_r = 5.5 - r$, as in Eq. (5.33). To save space, we omit the subscript $\bar\psi\psi$ below. Values that exceed the bound $\gamma_{\bar\psi\psi,\mathrm{IR}} < 2$ from conformal invariance [see Eq. (4.14)] are marked as such.

r	$\gamma_{\mathrm{IR},2\ell}$	$\gamma_{\mathrm{IR},3\ell}$	$\gamma_{\mathrm{IR},4\ell}$	$\gamma_{\mathrm{IR},\Delta_r^3}$	$[1,1]_{\gamma_{\mathrm{IR},\Delta_r^3}}$	$[0,2]_{\gamma_{\mathrm{IR},\Delta_r^3}}$
2.8	>2	1.708	0.1902	0.8701	1.1127	1.1102
3.0	>2	1.165	0.2254	0.7652	0.9259	0.9244
3.2	>2	0.8540	0.2637	0.6683	0.7731	0.7722
3.4	>2	0.6563	0.2933	0.5790	0.6458	0.6453
3.6	1.853	0.5201	0.3083	0.4969	0.5383	0.5380
3.8	1.178	0.4197	0.3061	0.4216	0.4463	0.4461
4.0	0.7847	0.3414	0.2877	0.3528	0.3667	0.3666
4.2	0.5366	0.2771	0.2566	0.2899	0.2973	0.2972
4.4	0.3707	0.2221	0.2173	0.2326	0.2362	0.23615
4.6	0.2543	0.1735	0.1745	0.1805	0.18205	0.18205
4.8	0.1696	0.1294	0.1313	0.1333	0.1338	0.1338
5.0	0.1057	0.08886	0.08999	0.09045	0.09058	0.09058
5.2	0.05620	0.05123	0.05156	0.05161	0.05163	0.05163
5.4	0.01682	0.01637	0.01638	0.01638	0.01638	0.01638

$$(\Delta_r)_{\mathrm{poles},[0,2]_{\bar\gamma_{\bar\psi\psi,\mathrm{IR}}}} = \frac{1875}{69914}(1323 \pm 17\sqrt{4605}) = 4.5425, 66.420. \tag{6.36}$$

The first of these, at $\Delta_r = 4.5425$, is well beyond $(\Delta_r)_{\max} = 2.885$ so that the [0,2] PA is finite for all $r \in I_{\mathrm{IRZ},r}$, and the second is also irrelevant, since it corresponds to the value, $r = 72$, far beyond the AF interval, $r \in [0, 34/13]$. The irrelevance of these poles in the Padé approximants is in agreement with the conclusion that we have reached from our other methods that the small-Δ_f expansion is reasonably reliable throughout the interval I_{IRZ} and related non-Abelian Coulomb phase. In Table V we list our results for $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_r^3}$, $[1,1]_{\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_r^3}}$, and $[0,2]_{\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_r^3}}$, together with $\gamma_{\bar\psi\psi,\mathrm{IR},n\ell}$ with $n=2$, 3, 4 from [27] for comparison.

We find that if r is in the upper part of the interval $I_{\mathrm{IRZ},r}$, then there is excellent agreement between our higher-loop calculations of $\gamma_{\bar\psi\psi,\mathrm{IR},3\ell,\overline{\mathrm{MS}}}$ and $\gamma_{\bar\psi\psi,\mathrm{IR},4\ell,\overline{\mathrm{MS}}}$ from [22] and the present calculations of $\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_r^3}$, $[1,1]_{\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_r^3}}$, and $[0,2]_{\gamma_{\bar\psi\psi,\mathrm{IR},\Delta_r^3}}$. As r decreases in this interval $I_{\mathrm{IRZ},r}$, the values of the anomalous dimension calculated in the various different ways begin to exhibit small deviations from each other, and, as expected, these deviations become larger as r descends toward the lower end of the interval $I_{\mathrm{IRZ},r}$.

VII. SCHEME-INDEPENDENT CALCULATION OF ANOMALOUS DIMENSION $\gamma_{T,\mathrm{IR}}$ TO $O(\Delta_f^3)$

A. Calculation for general G and R

In this section we present a scheme-independent calculation of the anomalous dimension of the (gauge-invariant) bilinear fermion antisymmetric rank-2 Dirac tensor operators evaluated at α_{IR}. The flavor-nonsinglet and flavor-singlet tensor operators of this type are

$$J_{2,fns} = \bar\psi T_b \sigma_{\mu\nu}\psi \tag{7.1}$$

and

$$J_{2,fs} = \bar\psi \sigma_{\mu\nu}\psi, \tag{7.2}$$

where, as defined before, T_b, $b = 1, \dots, N_f^2 - 1$, is a generator of algebra of $\mathrm{SU}(N_f)$, and

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \tag{7.3}$$

is the usual antisymmetric rank-2 Dirac tensor. As was true of the operators $J_{0,fns}$ and $J_{0,fs}$, the anomalous dimensions of $J_{2,fns}$ and $J_{2,fs}$ are equal (e.g., [46]), so we will denote both with the single symbol γ_T (T for tensor) and the evaluation at α_{IR} as $\gamma_{T,\mathrm{IR}}$. The usual power series expansion for γ_T in powers of a is

$$\gamma_T = \sum_{\ell=1} c_{T,\ell} a^\ell. \tag{7.4}$$

The $c_{T,\ell}$ have been calculated up to $\ell = 3$ loop order in [46,65]. We write the scheme-independent expansion of this anomalous dimension as

$$\gamma_{T,\mathrm{IR}} = \sum_{n=1}^{\infty} \kappa_{T,n}\Delta_f^n \tag{7.5}$$

and denote the truncation of this series at maximal power $n = p$ as $\gamma_{T,\mathrm{IR},\Delta_f^p}$.

For general gauge group G and fermion representation R, using the three-loop results from [46,65] together with the four-loop beta function coefficients b_ℓ with $1 \le \ell \le 4$ [3,4,13,14], we calculate the following coefficients in the scheme-independent expansion of $\gamma_{T,\mathrm{IR}}$

$$\kappa_{T,1} = -\frac{8C_f T_f}{3C_A D}, \tag{7.6}$$

$$\kappa_{T,2} = -\frac{4C_f T_f^2(259C_A^2 + 428C_A C_f - 528C_f^2)}{9C_A^2 D^3}, \tag{7.7}$$

$$\kappa_{T,3} = \frac{4C_f T_f}{3^5 C_A^4 D^5}\bigg[3C_A T_f^2\{C_A^4(-11319+188160\zeta_3)+C_A^3 C_f(-337204+64512\zeta_3)+C_A^2 C_f^2(83616-890112\zeta_3) + C_A C_f^3(1385472-354816\zeta_3)+C_f^4(-212960+743424\zeta_3)\} - 512T_f^2 D(-5+132\zeta_3)\frac{d_A^{abcd}d_A^{abcd}}{d_A} - 15488C_A^2 D(-11+24\zeta_3)\frac{d_R^{abcd}d_R^{abcd}}{d_A}+11264C_A T_f D(-4+39\zeta_3)\frac{d_R^{abcd}d_A^{abcd}}{d_A}\bigg]. \tag{7.8}$$

We note that

$$\kappa_{T,1} = -\frac{1}{3}\kappa_1. \tag{7.9}$$

B. Evaluation for $G = \mathrm{SU}(N_c)$ and $R = F$

As we did with the κ_n coefficients, we exhibit the reduction of these general formulas for the gauge group $G = \mathrm{SU}(N_c)$ with N_f fermions in the representation $R = F$. In accordance with Eq. (7.9), we obtain

$$\kappa_{T,1,\mathrm{SU}(N_c),F} = -\frac{4(N_c^2-1)}{3N_c(25N_c^2-11)}. \tag{7.10}$$

Further,

$$\kappa_{T,2,\mathrm{SU}(N_c),F} = -\frac{4(N_c^2-1)(341N_c^4+50N_c^2-132)}{3^2 N_c^2(25N_c^2-11)^3} \tag{7.11}$$

and

$$\kappa_{T,3,\mathrm{SU}(N_c),F} = \frac{8(N_c^2-1)}{3^4 N_c^3(25N_c^2-11)^5}\times[23057N_c^8-557686N_c^6+1084692N_c^4-354200N_c^2-13310 + 192(25N_c^2-11)(163N_c^4-225N_c^2-22)\zeta_3]. \tag{7.12}$$

The coefficient $\kappa_{T,1,\mathrm{SU}(N_c),F}$ is manifestly negative for all $N_c \ge 2$, and this is also true of $\kappa_{T,2,\mathrm{SU}(N_c),F}$, while we find that $\kappa_{T,3,\mathrm{SU}(N_c),F}$ is positive for all $N_c \ge 2$.

C. LNN l for $\gamma_{T,\mathrm{IR}}$

Here we evaluate the $\kappa_{T,n}$ and $\gamma_{T,\mathrm{IR}}$ in the LNN limit. The rescaled quantities that are finite in this limit are the analogues of those that we defined and studied for $\gamma_{\bar\psi\psi,\mathrm{IR}}$ in Sec. VI F. We calculate

$$\hat\kappa_{T,n} = \lim_{N_c\to\infty} N_c^n \kappa_{T,n} \tag{7.13}$$

have the values

$$\hat\kappa_{T,1} = -\frac{4}{3\cdot 5^2} = -0.053333, \tag{7.14}$$

$$\hat\kappa_{T,2} = -\frac{1364}{3^2\cdot 5^6} = -(0.969956\times 10^{-2}), \tag{7.15}$$

and

$$\hat\kappa_{T,3} = \frac{184456}{3^4\cdot 5^{10}} = 2.3319\times 10^{-4}. \tag{7.16}$$

Hence, to third order in the rescaled quantity Δ_r defined in Eq. (5.33), we have the following scheme-independent expansion for $\gamma_{T,\mathrm{IR}}$ in the LNN limit:

$$\lim_{LNN}\gamma_{T,\mathrm{IR}} = \Delta_r[-0.053333-(0.96996\times 10^{-2})\Delta_r + (2.3319\times 10^{-4})\Delta_r^2 + O(\Delta_f^3)]. \tag{7.17}$$

In Fig. 3 we plot $\gamma_{T,\mathrm{IR},\Delta_r^p}$ for $1 \le p \le 3$ as a function of r in the interval $I_{\mathrm{IRZ},r}$. As a consequence of the fact that both $\hat\kappa_{T,1}$ and $\hat\kappa_{T,2}$ are negative, for a fixed value of r, $\gamma_{T,\mathrm{IR},\Delta_p^2}$ is negative and larger in magnitude than $\gamma_{T,\mathrm{IR},\Delta_p^2}$. Although

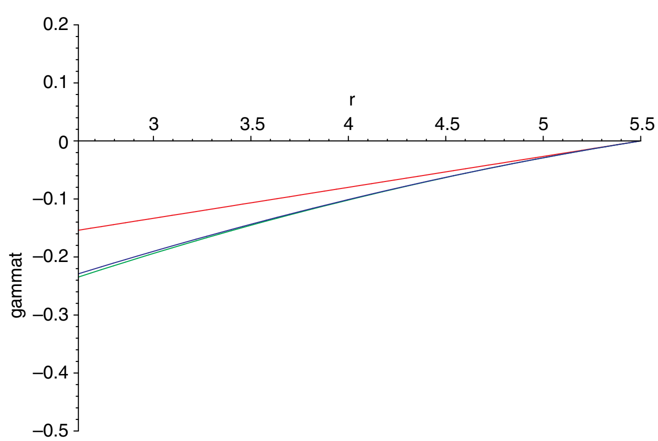

FIG. 3. Plot of $\gamma_{T,\mathrm{IR},\Delta_r^p}$ for $1 \le p \le 3$ as a function of $r \in I_{\mathrm{IRZ},r}$ in the LNN limit (5.26). From bottom to top, the curves (with colors online) refer to $\gamma_{T,\mathrm{IR},\Delta_r}$ (red), $\gamma_{T,\mathrm{IR},\Delta_r^2}$ (green),, $\gamma_{T,\mathrm{IR},\Delta_r^3}$ (blue).

$\hat{\kappa}_{T,3}$ is positive, it is sufficiently small that for a given r, the value of r, $\gamma_{T,\mathrm{IR},\Delta_p^3}$ is close to the value of r, $\gamma_{T,\mathrm{IR},\Delta_p^2}$.

D. Calculation of $\gamma_{T,\mathrm{IR}}$ to $O(\Delta_f^3)$ for $G=\mathrm{SU}(3)$ and $R=F$

As another interesting comparison, we evaluate our general expressions for the $\kappa_{T,n}$ in the special case where the gauge group is $G=\mathrm{SU}(3)$ and the fermion representation is $R=F$. We find

$$\kappa_{T,\mathrm{SU}(3),F,1} = -\frac{16}{3^2\cdot 107} = -(1.6615\times 10^{-2}), \tag{7.18}$$

$$\kappa_{T,\mathrm{SU}(3),F,2} = -\frac{37252}{(3\cdot 107)^3} = -(1.12625\times 10^{-3}), \tag{7.19}$$

and

$$\kappa_{T,\mathrm{SU}(3),F,3} = -\frac{341234350}{3^7\cdot (107)^5} + \frac{2855936}{3^6\cdot (107)^4}\zeta_3 = 2.480155\times 10^{-5}. \tag{7.20}$$

Thus, the leading two terms in the Δ_f expansion for J_2 are negative, with the coefficient of Δ_f^3 being positive but smaller in magnitude. These results may be contrasted to those obtained in [9] for $\kappa_n \equiv \kappa_{\bar{\psi}\psi,n}$ with $1\le n\le 3$ and in [11] for $n=4$ for this SU(3) theory with $R=F$, which are listed above in (6.11)–(6.14). We have computed ratios of the magnitudes of successive coefficients as before and again infer that the small-Δ_f expansion can be reliable in the interval I_{IRZ}.

E. Evaluation for $R=adj$

For $G=\mathrm{SU}(N_c)$ and $R=adj$, our general results above reduce to

$$\kappa_{T,1,\mathrm{SU}(N_c),adj} = -\frac{4}{3^3} = -0.05333, \tag{7.21}$$

$$\kappa_{T,2,\mathrm{SU}(N_c),adj} = -\frac{53}{2\cdot 3^7} = -(1.2117\times 10^{-2}), \tag{7.22}$$

and

$$\kappa_{T,3,\mathrm{SU}(N_c),adj} = \frac{N_c^2(34799-9216\zeta_3)+42624}{2^3\cdot 3^{11}N_c^2}. \tag{7.23}$$

This is positive for all physical N_c and has the large-N_c limit

$$\lim_{N_c\to\infty}\kappa_{T,3,\mathrm{SU}(N_c),adj} = \frac{34799-9216\zeta_3}{2^3\cdot 3^{11}} = 0.0167381. \tag{7.24}$$

Thus, the signs of the first three coefficients $\kappa_{T,n,\mathrm{SU}(N_c),adj}$ are the same as those of the coefficients $\kappa_{T,n,\mathrm{SU}(N_c),F}$. These are summarized in Table I.

VIII. CONCLUSIONS

In conclusion, in this paper we have presented a number of new results on scheme-independent calculations of various quantities in an asymptotically free vectorial gauge theory having an IR zero of the beta function. We consider a theory with a (non-Abelian) gauge group G and N_f fermions in a representation R of G. First, we have calculated the derivative β'_{IR}, equivalent to $\gamma_{F^2,\mathrm{IR}}$, to order Δ_f^4 for general G and R, and have given explicit results for $G=\mathrm{SU}(N_f)$ and fermions in the fundamental and adjoint representations. For the case $G=\mathrm{SU}(3)$ and fermions in the fundamental representation, we have also calculated β'_{IR} to the next higher order, Δ_f^5. It would be useful to have lattice measurements of $\gamma_{F^2,\mathrm{IR}}$, which, in the case of SU(3), could be compared with our calculation of this anomalous dimension. Second, we have given more details on the scheme-independent analysis of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ studied earlier in [9] and [11], including explicit analytic results for $G=\mathrm{SU}(N_c)$ with fermions in the fundamental and adjoint representations. In the former case, we have also investigated the LNN limit (5.26), calculated Padé approximants, and compared with results from the conventional higher-loop calculation of this anomalous dimension. Our results are useful for comparisons with lattice measurements of $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ and for the fundamental question of the value of $N_{f,cr}$ and whether $\gamma_{\bar{\psi}\psi,\mathrm{IR}}$ saturates its upper bound at the lower end of the conformal non-Abelian Coulomb phase. Moreover, the type of theory considered here may be relevant for ultraviolet extensions of the Standard Model. Third, we have presented a scheme-independent calculation to order Δ_f^3 of the anomalous dimension $\gamma_{T,\mathrm{IR}}$ of the (flavor-nonsinglet and flavor-singlet) bilinear fermion antisymmetric rank-2 Dirac tensor operators. We have shown that our scheme-independent calculations of the anomalous dimensions of $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ and various fermion bilinear operators in the non-Abelian Coulomb phase obey respective rigorous upper bounds for conformally invariant theories. This, together with other inputs including Padé approximants indicates that the series expansions in powers of Δ_f should be reasonably accurate throughout the non-Abelian Coulomb phase. We believe that the results presented here show the value of scheme-independent expansions of quantities evaluated at an infrared zero of the beta function in gauge theories.

ACKNOWLEDGMENTS

This research was supported in part by the Danish National Research Foundation Grant No. DNRF90 to

CP3-Origins at SDU (T. A. R.) and by the U.S. NSF Grant No. NSF-PHY-16-1620628 (R. S.)

APPENDIX: SERIES COEFFICIENTS FOR β_ξ AND $\gamma_{\bar{\psi}\psi}$ IN THE LNN LIMIT

For reference, we list here the rescaled series coefficients for β_ξ and $\gamma_{\bar{\psi}\psi}$ in the LNN limit (5.26). First, we recall that [3]

$$b_1 = \frac{1}{3}(11C_A - 4T_f N_f) \tag{A1}$$

and [4]

$$b_2 = \frac{1}{3}[34C_A^2 - 4(5C_A + 3C_f)T_f N_f], \tag{A2}$$

where C_A, C_f, and T_f are group invariants [18]. It follows that in the LNN limit the $\hat{b}_\ell$ with $\ell = 1, 2$ are

$$\hat{b}_1 = \frac{1}{3}(11 - 2r) \tag{A3}$$

and

$$\hat{b}_2 = \frac{1}{3}(34 - 13r). \tag{A4}$$

The coefficients b_3 and b_4 have been calculated in the $\overline{\mathrm{MS}}$ scheme [13,14]. With these inputs, one obtains [27]

$$\hat{b}_3 = \frac{1}{54}(2857 - 1709r + 112r^2) \tag{A5}$$

and

$$\hat{b}_4 = \frac{150473}{486} - \left(\frac{485513}{1944}\right)r + \left(\frac{8654}{243}\right)r^2 + \left(\frac{130}{243}\right)r^3 + \frac{4}{9}(11 - 5r + 21r^2)\zeta_3. \tag{A6}$$

For the coefficients $\hat{c}_\ell$ in Eq. (6.24), one has ([47] and references therein)

$$\hat{c}_1 = 3, \tag{A7}$$

$$\hat{c}_2 = \frac{203}{12} - \frac{5}{3}r, \tag{A8}$$

$$\hat{c}_3 = \frac{11413}{108} - \left(\frac{1177}{54} + 12\zeta_3\right)r - \frac{35}{27}r^2, \tag{A9}$$

and

$$\hat{c}_4 = \frac{460151}{576} - \frac{23816}{81}r + \frac{899}{162}r^2 - \frac{83}{81}r^3 + \left(\frac{1157}{9} - \frac{889}{3}r + 20r^2 + \frac{16}{9}r^3\right)\zeta_3 + r(66 - 12r)\zeta_4 + (-220 + 160r)\zeta_5. \tag{A10}$$

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[65] We note the following misprint in [44]: in the last line of Eq. (7) of that paper, the term $-144T_F^2C_F^2$ should read $-144T_F^2N_F^2$. We thank J. A. Gracey for confirming this (private communication).