Spatially inhomogeneous and irrotational geometries admitting intrinsic conformal symmetries

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"Diagonal" spatially inhomogeneous (SI) models are introduced under the assumption of the existence of (proper) intrinsic symmetries and can be seen, in some sense, as complementary to the Szekeres models. The structure of this class of spacetimes can be regarded as a generalization of the (twist-free) locally rotationally symmetric geometries without any global isometry containing, however, these models as special cases. We consider geometries where a six-dimensional algebra $\mathcal{IC}$ of intrinsic conformal vector fields (ICVFs) exists that acts on a two-dimensional (pseudo)-Riemannian manifold. Its members $\mathbf{X}_\alpha$—constituted of three intrinsic Killing vector fields and three *proper* and *gradient* ICVFs—and the specific form of the gravitational field are given explicitly. An interesting consequence, in contrast with the Szekeres models, is the immediate existence of *conserved quantities along null geodesics*. We check computationally that the magnetic part H_{ab} of the Weyl tensor vanishes, whereas the shear σ_{ab} and the electric part E_{ab} share a common eigenframe irrespective of the fluid interpretation of the models. A side result is the fact that the spacetimes are foliated by a set of *conformally flat* three-dimensional *timelike* slices when the anisotropy of the *flux-free* fluid is described only in terms of the three *principal inhomogeneous "pressures"* p_α, or equivalently when the Ricci tensor shares the same basis of eigenvectors with σ_{ab} and E_{ab}. The conformal flatness also indicates that it is highly possible that a *ten-dimensional algebra* of ICVFs Ξ that acts on the three-dimensional timelike slices exists, enriching in that way the set of conserved quantities admitted by the SI models found in the present paper.

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I. INTRODUCTION

An inspection of the Einstein field equations (EFEs)

$$G^a{}_b \equiv R^a{}_b - \frac{1}{2}R\delta^a{}_b = T^a{}_b \qquad (1)$$

reveals the rich and strong correlation between the geometry of spacetime and the dynamics. The latter is primarily encoded in a realistic[1] energy-momentum tensor T_{ab}. However, even if we assume that the spacetime does not contain any dynamical fields, $g_{ab}(x^c)$ itself becomes a dynamical variable, showing the complexity that arises from this duality. It is thus evident that any intention to simplify $g_{ab}(x^c)$ with some kind of symmetry must take into account the fusion between the gravitational field and the spacetime geometry.

On the other hand, observable quantities necessitate the existence of a unit timelike vector field u^a representing an average velocity [1] and its kinematical quantities θ (volume expansion scalar), σ_{ab} (anisotropic expansion trace-free tensor), ω_{ba} (the congruence's twist tensor), and $\dot{u}^a$ (nongeodesic indication 1-form) describe the distortion of the integral curves of u^a as measured in the rest space of a *comoving* observer,

$$\theta \equiv u_{a;b}h^{ab}, \qquad \sigma_{ab} \equiv u_{(c;d)}\left(h^c_a h^d_b - \frac{1}{3}h^{cd}h_{ab}\right),$$
$$\dot{u}_a \equiv u_{a;b}u^b, \qquad \omega_{ba} \equiv u_{[c;d]}h^c_a h^d_b, \qquad (2)$$

where $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor normal to u^a. In the generic case there are no *a priori* reasons to impose special features on the timelike congruence and only the interplay of physics (plus observations) and geometry with the inclusion of appropriate boundary data (at spatial or null past/future infinity) should enforce the need for such characteristics.

The third constituent element in this "arena" is the presence of a matter fluid which is described in terms of the geometry and the kinematics as

$$T^a{}_b = \rho u^a u_b + p h^a{}_b + q^a u_b + u^a q_b + \pi^a{}_b, \qquad (3)$$

where ρ and p are the energy density and the isotropic pressure, respectively, q^a is the direction of the momentum flow, and π_{ab} is the anisotropic and trace-free pressure tensor,

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[1] Realistic implies that the dynamical portions of T_{ab} must be derived from a set of well-established phenomenological laws and not by hand.

$$\rho \equiv T_{ab}u^a u^b \qquad p \equiv \frac{1}{3}T_{ab}h^{ab},$$
$$q_a \equiv -h_a^c T_{cd}u^d, \qquad \pi_{ab} \equiv \left(h_a^c h_b^d - \frac{1}{3}h^{cd}h_{ab}\right)T_{cd}. \quad (4)$$

Each of the above dynamical components has (and must have) a phenomenologically sound meaning [2] that can be justified from observations at some acceptable cosmological scale. It should be noticed that the choice of the observer is not unique and can be chosen either comoving u^a or noncomoving $\tilde{u}^a(\neq u^a)$, in which case the interpretation for each one should be completely different, leading to the notion of *tilted* models [3].

Spatially inhomogeneous (SI) models [4] have contributed significantly to our understanding of structure formation and the effect of local density and pressure fluctuations in the accelerated phase of the Universe. It is clear that they represent not an alternative to the linearized version of the perturbed Friedmann-Lemaître-Robertson-Walker models, but rather *exact perturbation solutions* within a homogeneous and isotropic background. Although to date a quite generic SI model without special characteristics (in the sense that will become transparent in the next sections) has not been found, the known exact SI solutions can serve as toy models in various directions [5].

Szekeres' solution [6] was the first SI model without any (global) isometry and, as such, is well fitted along the aforementioned research lines. From a geometrical and kinematical point of view, it admits a tetrad of unit vector fields $\{u^a, x^a, y^a, z^a\}$ that are hypersurface orthogonal and any pair $\{u^a, x^a\}$, $\{u^a, y^a\}$, $\{u^a, z^a\}$ is surface forming, which implies that

$$y^k L_{\mathbf{u}} x_k = z^k L_{\mathbf{u}} x_k = 0,$$
$$x^k L_{\mathbf{u}} y_k = z^k L_{\mathbf{u}} y_k = 0,$$
$$x^k L_{\mathbf{u}} z_k = y^k L_{\mathbf{u}} z_k. \quad (5)$$

In addition, the unit timelike vector field u^a is geodesic, consistent with a dust fluid content (Refs. [7,8] provide a generalization of the Szekeres spacetime with $p \neq 0$), which results in the Szekeres family of quasisymmetric models [9,10],

$$ds^2 = -dt^2 + S^2\left\{\frac{[(\ln S/E)']^2}{\epsilon + F}dr^2 + \frac{dy^2 + dz^2}{V^2\{1 + \frac{k}{4}[(y-Y)^2 + (z-Z)^2]\}^2}\right\}, \quad (6)$$

$$ds^2 = -dt^2 + S^2\left\{\frac{[(\ln S/E)']^2}{F}dr^2 + \frac{4(dy^2 + dz^2)}{[(y-Y)^2 + (z-Z)^2]^2}\right\}, \quad (7)$$

where $k = \epsilon/V^2$, and $Y(r)$, $Z(r)$, $V(r)$, and $F(r)$ are arbitrary functions of the radial coordinate. An important property of these models is the vanishing of the magnetic part of the Weyl tensor,

$$\frac{1}{2}\eta_{ac}{}^{ij}C_{ijbd}u^c u^d \equiv H_{ab} = 0, \quad (8)$$

which implies that gravitational radiation cannot propagate [11,12] within this class of models. Essentially, Eq. (8) is true for the general diagonal metric ($u^a = C^{-1}\delta_t^a$)

$$ds^2 = g_{ab}dx^a dx^b = A^2dx^2 + B^2dz^2 - C^2dt^2 + D^2dy^2, \quad (9)$$

and therefore it can be seen entirely as an "artifact" of the specific geometrical character of the tetrad $\{u^a, x^a, y^a, z^a\}$ irrespective of further dynamical restrictions. Spacetimes that satisfy Eq. (8) are usually referred as purely "electrical" and a lot of work has been done regarding the dynamical structure and the existence of perfect fluid models (see, e.g., Ref. [13] and references cited therein) with vanishing H_{ab}. The analysis is focused mainly on perfect fluids with a barotropic equation of state $p = p(\rho)$ or rotational dust (geodesic) models.

The key feature of the family (6) or (7) is the *conformal flatness* of the three-dimensional slices $t = \text{const}$ [9] which, geometrically, could be reminiscent of the constant curvature of the two-dimensional hypersurfaces $t, r = \text{const}$ and the subsequent existence of a *six-dimensional algebra* of intrinsic conformal vector fields (ICVFs) $\mathbf{X}$ satisfying [10]

$$p_a^c p_b^d \mathcal{L}_{\mathbf{X}} p_{cd} = 2\phi(\mathbf{X})p_{ab}, \quad (10)$$

where $p_{ab} = h_{ab} - x_a x_b$ is the projection tensor normal to the pair $\{u^a, x^a\}$ and, given the structure of Eq. (6) or Eq. (7), represents the induced metric of the two-dimensional manifold $\mathbf{u}\wedge\mathbf{x} = \mathbf{0}$.

The notion of intrinsic symmetries was introduced in Ref. [14–17], but their covariant form was not. In order to investigate the implications of the existence of geometric symmetries in general relativity we must take into account the holonomy group structure of the spacetime manifold together with the associated local diffeomorphisms [18]. Furthermore, it is necessary to reformulate the necessary and sufficient (integrability) conditions (coming from the existence of the symmetry) in a covariant way and study their consequences in the kinematics and dynamics of the corresponding model. The fact that Szekeres models admit (proper) ICVFs that act on two-dimensional (and possibly three-dimensional) submanifolds shows that ICVFs could be more relevant and impose far less restrictions than the full CVF models, which are very rare [4].

The purpose of the present paper is to extend the investigation of the existence of ICVFs to spacetimes with the metric (9), thus providing a kind of *geometrical* classification with respect to the intrinsic conformal algebra *without assuming* any matter content, and thus providing a much richer diversity of possible physically sound models than those that have been reported so far [13]. In particular, in Sec. II we assume that a *six-dimensional algebra* of ICVFs exists that acts on the timelike distribution $\mathbf{x}\wedge\mathbf{z}=\mathbf{0}$, which implies that the latter has constant curvature and the resulting spacetimes can be referred to as quasisymmetric. We give the explicit form of the ICVFs and the associated spacetime metrics and show computationally that the magnetic part H_{ab} of the Weyl tensor vanishes, whereas the shear σ_{ab} and the electric part $E_{ab}=C_{acbd}u^c u^d$ share a common eigenframe irrespective of the fluid interpretation of the models. Furthermore, nontilted perfect fluids (where, in general, p and ρ do not satisfy a barotropic equation of state) cannot be excluded at once since the H-divergence constraint is trivially satisfied. Two interesting results then arise: in contrast with the Szekeres models, there exist *infinite conserved quantities* along null geodesics. Furthermore, the hypersurfaces $x=\text{const}$ are *conformally flat* when the fluid is flux free ($q^a=0$) and its anisotropy is described only in terms of the three *principal inhomogeneous "pressures"* p_α or, equivalently, when the Einstein tensor $G^a{}_b$ is "diagonal." One should expect the existence of a *ten-dimensional algebra* of ICVFs Ξ of the $\mathbf{x}_\perp$ distribution that satisfies

$$\hat{h}^c_a\hat{h}^d_b\mathcal{L}_\Xi\hat{h}_{cd}=2\phi(\Xi)\hat{h}_{ab}, \tag{11}$$

where $\hat{h}_{ab}=g_{ab}-x_a x_b$ is regarded as the induced metric of $\mathbf{x}_\perp$. In Sec. III, for completeness, we also give the six-dimensional algebra of ICVFs that act on the $\mathbf{x}\wedge\mathbf{u}=\mathbf{0}$ spacelike distribution when $\dot{u}^a\neq 0=x^a_{;b}x^b$. As expected, the x slices are also conformally flat provided that $T^a{}_b=\text{diag}(\rho,p_1,p_2,p_3)$. Section IV includes our conclusions and further areas of research.

Throughout this paper, the following conventions are used: the spacetime manifold is endowed with a Lorentzian metric of signature $(-,+,+,+)$, spacetime indices are denoted by lower case latin letters $a,b,\ldots=0,\ 1,\ 2,\ 3$, spatial frame indices are denoted by lower case greek letters $\alpha,\beta,\ldots=1,\ 2,\ 3$, and we use geometrized units such that $8\pi G=1=c$.

II. SPATIALLY INHOMOGENEOUS AND IRROTATIONAL MODELS OF TYPE II

We consider a spacetime geometry where a unit timelike vector field u^a is twist free ($\omega_{ab}=0$) but *nongeodesic* ($\dot{u}^a\neq 0$). We make the assumption that there exist three independent spacelike unit vector fields $\{\mathbf{x},\mathbf{y},\mathbf{z}\}$, normal to u^a, and each of these is hypersurface orthogonal

$$x_{[a}x_{b;c]}=y_{[a}y_{b;c]}=z_{[a}z_{b;c]}=0. \tag{12}$$

The unit spacelike vector field x^a is taken to be geodesic, i.e., $(x_a)^*\equiv x_{a;b}x^b=0$ and the pairs $\{u^a,x^a\}$, $\{u^a,y^a\}$, and $\{u^a,z^a\}$ are surface forming, satisfying Eq. (5).

Under these conditions, the most general metric adapted to the geodesic coordinates of x^a has the following form:

$$ds^2=g_{ab}dx^a dx^b=dx^2+B^2dz^2-C^2dt^2+D^2dy^2, \tag{13}$$

where the functions $B(t,x,y,z)$, $C(t,x,y,z)$, and $D(t,x,y,z)$ depend on all four coordinates. It follows from Eq. (13) that the magnetic part of the Weyl tensor with respect to u^a vanishes ($H_{ab}=0$) and, in general, the Petrov type is I, that is, $E_{ab}=\text{diag}(0,E_1,E_2,E_3)$.

Essentially, the induced metric of the distribution $\mathbf{x}\wedge\mathbf{z}=\mathbf{0}$ is represented by the second-order symmetric tensor $p_{ab}\equiv g_{ab}-x_a x_b-z_a z_b$, where $p^k_a x_k=0=p^k_a z_k$. We assume that there exists a six-dimensional algebra $\mathcal{IC}(\mathbf{X}_A)$ $(A=1,\ldots,6)$ of ICVFs that act on a two-dimensional pseudo-Riemannian manifold that obeys

$$p^c_a p^d_b\mathcal{L}_\mathbf{X}g_{cd}=p^c_a p^d_b\mathcal{L}_\mathbf{X}p_{cd}\equiv\bar{\nabla}_{(b}X_{a)}=2\phi(\mathbf{X})p_{ab}, \tag{14}$$

where $\phi(\mathbf{X}_A)$ are the conformal factors of the vectors $\mathbf{X}_A$ that are lying and acting on the submanifold $\mathbf{x}\wedge\mathbf{z}=\mathbf{0}$, and $\bar{\nabla}_a$ represents a well-defined covariant derivative

$$\bar{\nabla}_c p_{ab}=p^k_c p^i_a p^j_b\nabla_k p_{ij}=0 \tag{15}$$

for any tensorial quantity

$$\bar{\nabla}_c\Pi^a_b\equiv p^k_c p^a_i p^k_b\Pi^i_{j;k}.$$

From the inspection of Eq. (14) it follows that $C=D$, and the general solution shows that $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ are *intrinsic Killing vector fields* and $\mathbf{X}_4$, $\mathbf{X}_5$, $\mathbf{X}_6$ are *proper* and *gradient* ICVFs, i.e., their associated bivectors vanish identically ($\bar{\nabla}_{[b}X_{a]}=0$),

$$\mathbf{X}_1=\mathbf{M}_{yt}=(y-Y)\partial_t+(t-T)\partial_y, \tag{16}$$

$$\mathbf{X}_2=\left\{\frac{k}{4}[(y-Y)^2+(t-T)^2]-1\right\}\partial_t+\frac{k}{2}(y-Y)(t-T)\partial_y, \tag{17}$$

$$\mathbf{X}_3=\frac{k}{2}(y-Y)(t-T)\partial_t+\left\{1+\frac{k}{4}[(y-Y)^2+(t-T)^2]\right\}\partial_y, \tag{18}$$

$$\mathbf{X}_4=\mathbf{H}=(t-T)\partial_t+(y-Y)\partial_y, \tag{19}$$

$$\mathbf{X}_5=\left\{\frac{k}{4}[(t-T)^2+(y-Y)^2]+1\right\}\partial_t+\frac{k}{2}(t-T)(y-Y)\partial_y, \tag{20}$$

$$\mathbf{X}_6=\frac{k}{2}(t-T)(y-Y)\partial_t+\left\{\frac{k}{4}[(y-Y)^2+(t-T)^2]-1\right\}\partial_y, \tag{21}$$

with the associated conformal factors

$$\phi(\mathbf{X}_1)=\phi(\mathbf{X}_2)=\phi(\mathbf{X}_3)=0, \tag{22}$$

$$\phi(\mathbf{X}_4)=\left\{1-\frac{k}{4}[(y-Y)^2-(t-T)^2]\right\}N, \tag{23}$$

$$\phi(\mathbf{X}_5)=kN(t-T),$$
$$\phi(\mathbf{X}_6)=kN(y-Y). \tag{24}$$

Consequently, the two-dimensional manifold $\mathbf{x}\wedge\mathbf{z}=\mathbf{0}$ has (locally) *constant curvature* and the metric (13) takes the form

$$ds^2=dx^2+B^2dz^2+\frac{S^2}{V^2}\frac{-dt^2+dy^2}{\{1+\frac{\epsilon}{4V^2}[(y-Y)^2-(t-T)^2]\}^2}, \tag{25}$$

where $S(x,z)$, $Y(z)$, $T(z)$, and $V(z)$ are arbitrary functions of their arguments and $\epsilon=\pm1$ ($\neq 0$) corresponds to the constant curvature of the hypersurfaces $x,z=$ const.

If we define the function $E(t,y,z)$ according to ($k=\epsilon/V^2$)

$$E(t,y,z)=V\left\{1+\frac{k}{4}[(y-Y)^2-(t-T)^2]\right\}, \tag{26}$$

then

$$N(t,y,z)=\frac{1}{E(t,y,z)}, \tag{27}$$

and the metric becomes

$$ds^2=dx^2+B^2dz^2+\frac{S^2}{E^2}(-dt^2+dy^2). \tag{28}$$

The case where the distribution $\mathbf{x}\wedge\mathbf{z}=\mathbf{0}$ has *zero curvature* is treated similarly. The ICVFs are

$$\mathbf{X}_1=\mathbf{M}_{yt}=(y-Y)\partial_t+(t-T)\partial_y, \tag{29}$$

$$\mathbf{X}_2=[(y-Y)^2+(t-T)^2]\partial_t+2(y-Y)(t-T)\partial_y, \tag{30}$$

$$\mathbf{X}_3=2(y-Y)(t-T)\partial_t+[(y-Y)^2+(t-T)^2]\partial_y, \tag{31}$$

$$\mathbf{X}_4=\mathbf{H}=(t-T)\partial_t+(y-Y)\partial_y, \tag{32}$$

$$\mathbf{X}_5=\partial_t, \qquad \mathbf{X}_6=\partial_y, \tag{33}$$

with the conformal factors

$$\phi(\mathbf{X}_1)=\phi(\mathbf{X}_2)=\phi(\mathbf{X}_3)=0, \tag{34}$$

$$\phi(\mathbf{X}_4)=-1, \tag{35}$$

$$\phi(\mathbf{X}_5)=\frac{2(t-T)}{(y-Y)^2-(t-T)^2}, \tag{36}$$

$$\phi(\mathbf{X}_6)=\frac{2(y-Y)}{(t-T)^2-(y-Y)^2}, \tag{37}$$

and the metric function $E(t,y,z)$ is given by

$$E(t,y,z)=\frac{1}{N(t,y,z)}=\frac{1}{4}[(y-Y)^2-(t-T)^2]. \tag{38}$$

A potential application of the IC algebra $\mathcal{IC}(\mathbf{X}_A)$ found in the present section could be the existence of conserved currents and quantities. For example, consider a null geodesic vector field l^a *lying* in the two-dimensional manifold $\mathbf{x}\wedge\mathbf{z}=\mathbf{0}$ and the quantities nt$Q_A=l^aX_{(A)a}$. It is easy to see that Q_A are conserved *along* the null geodesics since

$$[Q_{(A)}]_{;a}l^a=l^b{}_{;a}X_{(A)b}l^a+l^al^bX_{(A)b;a}=0. \tag{39}$$

For the metric (28) a null geodesic vector field is $l^a=f(u^a+y^a)=fn^a$, where $f(x^a)$ satisfies $(f_{;k}n^k)n^a=-fn^a{}_{;k}n^k$ [we note that $n^a=u^a+y^a$ is not geodesic for a generic form of Eq. (28)].

In the search for fluid solutions we usually start by analyzing the structure of the constraints of the EFEs (1). The "temporal" constraints $G^0{}_\alpha=0$ for the metric (28) reduce to

$$SB_{,tx}-B_{,t}S_{,x}=0, \tag{40}$$

$$B_{,y}E_t+B_{,t}E_{,y}+EB_{,ty}=0, \tag{41}$$

$$BS(EE_{,zt}-E_{,t}E_{,z})+EB_{,t}(ES_{,z}-SE_{,z})=0, \tag{42}$$

whereas the "spatial" constraints $G^\alpha{}_\beta=0$ have the forms

$$B_{,y}S_{,x}-SB_{,yx}=0, \tag{43}$$

$$B(ES_{,zx}-S_{,x}E_{,z})+B_{,x}(SE_{,z}-ES_{,z})=0, \tag{44}$$

$$BS(EE_{,zy}-E_{,y}E_{,z})+EB_{,y}(ES_{,z}-SE_{,z})=0, \tag{45}$$

where a “,” denotes partial differentiation with respect to the corresponding coordinate.

The *general solution* of the above set of coupled differential equations is

$$B = \frac{S[\ln(S/E)]_{,z}}{\sqrt{\epsilon + F(z)}}, \tag{46}$$

where $F(z)$ is an arbitrary function and $E(t, y, z)$ is given in Eq. (26) or Eq. (38).

It should be emphasized that the existence of the $\mathcal{IC}(\mathbf{X}_A)$ intrinsic conformal algebra is a *direct consequence* of the general solution (46), (26), or (38), and therefore in order to determine the exact form of $\mathbf{X}_A$ we could simply apply the methodology of Ref. [10], thus avoiding Eq. (14). Furthermore, we can verify that the Petrov type is D, i.e., the eigenvalues of the electric part of the Weyl tensor $E_1 = E_3$ (in contrast with the Szekeres models where $E_2 = E_3$).

The EFEs (1) then become

$$G^a{}_b = T^a{}_b = \mathrm{diag}(\rho, p_1, p_2, p_3), \tag{47}$$

i.e., the Ricci tensor $R^a{}_b$ *shares the same basis of eigenvectors* with σ_{ab} and E_{ab}.

The *directional* and *inhomogeneous* “pressures” p_α are not necessarily equal and the fluid is, in general, anisotropic for the comoving observers $u^a = (E/S)\delta^a{}_t$. In order to show whether a specific perfect fluid solution exists (i.e., $p_1 = p_2 = p_3$), one must monitor the integrability conditions, i.e., the consistent evolution of the *nontrivial* constraints. We can prove, however, that the H-divergence constraint is trivially satisfied. We observe computationally that the three mutually orthogonal and unit spacelike vector fields $\{x^a, y^a, z^a\}$ areeigenvectorsof $E_{ab} = \mathrm{diag}(0, E_1, E_2, E_1)$ and $\sigma_{ab} = \mathrm{diag}(0, \sigma_1, \sigma_2, \sigma_3)$. Because H_{ab} vanishes identically for the metric (28), the further requirement $p_1 = p_2 = p_3$ gives $\pi_{ab} = 0$ and the H-divergence equation [1]

$$\epsilon^{\alpha\beta\gamma}\sigma_{\beta\delta}E^\delta{}_\gamma = 0$$

implies that the shear σ_{ab} and the electric part E_{ab} tensors commute, i.e., they must share a common eigenframe (which they actually do).

An important consequence of the solution (26) or (38) and Eq. (46) is that the Cotton-York tensor [19,20]

$$C_{abc} = 2\left(R_{a[b} - \frac{1}{4}Rg_{a[b}\right)_{;c]} \tag{48}$$

vanishes, i.e., the hypersurfaces $x = \mathrm{const}$ are *conformally flat*. Therefore, in complete analogy with the two-dimensional case, one should expect the existence of a *ten-dimensional algebra* of ICVFs Ξ of the $\mathbf{x}_\perp$ distribution that satisfies

$$\hat{h}_a^c\hat{h}_b^d\mathcal{L}_\Xi\hat{h}_{cd} = 2\phi(\Xi)\hat{h}_{ab}, \tag{49}$$

where $\hat{h}_{ab} = g_{ab} - x_a x_b$ is regarded as the induced metric of $\mathbf{x}_\perp$.

We note that by relaxing the flux-free restrictions [Eq. (40)–(42)], exact perfect-fluid models could exist for noncomoving (tilted) observers $\tilde{u}^a$, similar to the case of spatially homogeneous tilted perfect models (e.g., Refs. [21–23]) which necessitates the presence of *nonzero vorticity* [24]. This could also be possible for the Szekeres geometries, i.e., when $\tilde{u}^a$ are *comoving with the (perfect) fluid*, in which case the u^a observers will interpret it as imperfect.[2] Again, if such a solution exists, it must be proven that it evolves consistently along u^a that “see” an anisotropic and nonzero flux matter fluid.

III. SPATIALLY INHOMOGENEOUS AND IRROTATIONAL MODELS OF TYPE III

We are interested in the case where the induced metric of the distribution $\mathbf{x}\wedge\mathbf{u} = \mathbf{0}$, represented by the second-order symmetric tensor $p_{ab} \equiv g_{ab} - x_a x_b + u_a u_b$ (where $p_a^k x_k = 0 = p_a^k u_k$), admits the six-dimensional algebra $\mathcal{IC}(\mathbf{X}_A)$ $(A = 1, \ldots, 6)$ of ICVFs:

$$p_a^c p_b^d \mathcal{L}_{\mathbf{X}} p_{cd} = 2\phi(\mathbf{X}) p_{ab}, \tag{50}$$

$$\mathbf{X}_1 = \mathbf{M}_{yz}, \tag{51}$$

$$\mathbf{X}_2 = \left\{1 + \frac{k}{4}[(y - Y)^2 - (z - Z)^2]\right\}\partial_y + \frac{k}{2}(y - Y)(z - Z)\partial_z, \tag{52}$$

$$\mathbf{X}_3 = \frac{k}{2}(y - Y)(z - Z)\partial_y + \left\{1 + \frac{k}{4}[(z - Z)^2 - (y - Y)^2]\right\}\partial_z, \tag{53}$$

$$\mathbf{X}_4 = \mathbf{H} = (y - Y)\partial_y + (z - Z)\partial_z, \tag{54}$$

$$\mathbf{X}_5 = \left\{\frac{k}{4}[(y - Y)^2 - (z - Z)^2] - 1\right\}\partial_y + \frac{k}{2}(y - Y)(z - Z)\partial_z, \tag{55}$$

[2] In Ref. [25] the “environment” was completely different since the comoving interpretation remained that of a perfect fluid (i.e., the exact Szekeres model) and the tilted observers were derived from a Lorentz boost of u^a.

$$\mathbf{X}_6 = \frac{k}{2}(y-Y)(z-Z)\partial_y + \left\{\frac{k}{4}[(z-Z)^2-(y-Y)^2]-1\right\}\partial_z. \quad (56)$$

$\mathbf{X}_4$, $\mathbf{X}_5$, and $\mathbf{X}_6$ are *proper* and *gradient* ICVFs and the conformal factors are given by

$$\phi(\mathbf{X}_1) = \phi(\mathbf{X}_2) = \phi(\mathbf{X}_3) = 0, \quad (57)$$

$$\phi(\mathbf{X}_4) = \left\{1 - \frac{k}{4}[(y-Y)^2 + (z-Z)^2]\right\}N, \quad (58)$$

$$\phi(\mathbf{X}_5) = kN(y-Y), \qquad \phi(\mathbf{X}_6) = kN(z-Z). \quad (59)$$

The two-dimensional manifold $\mathbf{x}\wedge\mathbf{u} = \mathbf{0}$ is of *constant curvature* and the metric (13) is

$$ds^2 = dx^2 - C^2dt^2 + \frac{S^2}{V^2}\frac{dy^2+dz^2}{\{1+\frac{\epsilon}{4V^2}[(y-Y)^2+(z-Z)^2]\}^2}, \quad (60)$$

where $S(t,x)$ and $Y(t)$, $Z(t)$, and $V(t)$ are now arbitrary functions of t, and $\epsilon = \pm 1$ ($\neq 0$) corresponds to the constant curvature of the hypersurfaces $x, t = \text{const}$.

Similarly with type II, we define the function $E(t,y,z)$ according to ($k = \epsilon/V^2$)

$$E(t,y,z) = V\left\{1 + \frac{k}{4}[(y-Y)^2 + (z-Z)^2]\right\}, \quad (61)$$

with

$$N(t,y,z) = \frac{1}{E(t,y,z)}, \quad (62)$$

and the metric becomes

$$ds^2 = dx^2 - C^2dt^2 + \frac{S^2}{E^2}(dy^2+dz^2). \quad (63)$$

For completeness we give the corresponding expressions for the ICVFs and the metric for the case where the curvature of $\mathbf{x}\wedge\mathbf{u} = \mathbf{0}$ vanishes:

$$\mathbf{X}_1 = \mathbf{M}_{yz} = (z-Z)\partial_y - (y-Y)\partial_z, \quad (64)$$

$$\mathbf{X}_2 = [(y-Y)^2 - (z-Z)^2]\partial_y + 2(y-Y)(z-Z)\partial_z, \quad (65)$$

$$\mathbf{X}_3 = 2(y-Y)(z-Z)\partial_y + [(z-Z)^2 - (y-Y)^2]\partial_z, \quad (66)$$

$$\mathbf{X}_4 = \mathbf{H} = (y-Y)\partial_y + (z-Z)\partial_z, \quad (67)$$

$$\mathbf{X}_5 = \partial_y, \qquad \mathbf{X}_6 = \partial_z. \quad (68)$$

The conformal factors are

$$\phi(\mathbf{X}_1) = \phi(\mathbf{X}_2) = \phi(\mathbf{X}_3) = 0, \quad (69)$$

$$\phi(\mathbf{X}_4) = -1, \quad (70)$$

$$\phi(\mathbf{X}_5) = -\frac{2(y-Y)}{(y-Y)^2 + (z-Z)^2}, \quad (71)$$

$$\phi(\mathbf{X}_6) = -\frac{2(z-Z)}{(y-Y)^2 + (z-Z)^2}, \quad (72)$$

and the metric function $E(t,y,z)$ assumes the form

$$E(t,y,z) = \frac{1}{N(t,y,z)} = \frac{1}{4}[(y-Y)^2 + (z-Z)^2]. \quad (73)$$

In contrast with the previous case, the spacetime (63) does not allow the existence of conserved currents and quantities constructed from null vector fields "living" in $\mathbf{x}\wedge\mathbf{u} = \mathbf{0}$ due to the positive-definite character of the quasisymmetric two-dimensional metric p_{ab}. However, we can check for a flux-free solution which implies the "temporal" constraints ($G^0_\alpha = 0$)

$$C(ES_{,tx} - S_{,x}E_{,t}) + C_{,x}(SE_{,t} - ES_{,t}) = 0, \quad (74)$$

$$CS(EE_{,ty} - E_{,t}E_{,y}) + EC_{,y}(ES_{,t} - SE_{,t}) = 0, \quad (75)$$

$$CS(EE_{,zt} - E_{,t}E_{,z}) + EC_{,z}(ES_{,t} - SE_{,t}) = 0, \quad (76)$$

and the associated "spatial" constraints ($G^\alpha_\beta = 0$)

$$SC_{,yx} - C_{,y}S_{,x} = 0, \quad (77)$$

$$SC_{,zx} - C_{,z}S_{,x} = 0, \quad (78)$$

$$C_{,y}E_{,z} + C_{,z}E_{,y} + EC_{,zy} = 0. \quad (79)$$

We can verify that the *general solution* of Eqs. (74)–(79) is

$$C = \frac{S[\ln(S/E)]_{,t}}{\sqrt{\epsilon + F(t)}}, \quad (80)$$

where $F(t)$ is an arbitrary function and $E(t,y,z)$ is given in Eq. (61) or Eq. (73). In this type it also becomes evident that the existence of the $\mathcal{IC}(\mathbf{X}_A)$ intrinsic conformal algebra is a direct consequence of the general solution (80), (61), or (73).

Using the same arguments, the directional "pressures" p_α are, in general, not equal and the fluid is anisotropic. This, however, does not exclude *a priori* a perfect-fluid (nontilted) solution once the consistency of the integrability conditions is established. In addition, the existence of the general solution of Eqs. (74)–(79) is *equivalent* to the fact that the x slices are *conformally flat* and timelike, which indicates a *ten-dimensional algebra* of ICVFs Υ which will give rise to conserved quantities along null geodesics of the form $l^a = f_0 u^a + f_1 y^a + f_2 z^a$, where f_0, f_1, and f_2 are some functions satisfying the orthonormality condition $f_0^2 = f_1^2 + f_2^2$ and the geodesic assumption $(f_0 u^a + f_1 y^a + f_2 z^a)_{;b} l^b = 0$.

Summarizing the results of Secs. II and III regarding the conformal flatness of the $\mathbf{x}_\perp$ distribution in both types II and III, we can speculate that a spacetime with the metric (9) is foliated with conformally flat three-dimensional hypersurfaces iff a six-dimensional subalgebra of ICVFs exists that acts on two-dimensional submanifolds and the $\mathbf{u}_\perp$ *or* $\mathbf{x}_\perp$ distributions are almost $(1+2)$ decomposable (in the spirit of the arguments in Ref. [10]). Equivalently, the above holds true iff a six-dimensional subalgebra of ICVFs exists that acts on two-dimensional (pseudo)-Riemannian manifolds and the Ricci tensor shares a common basis of eigenvectors with shear σ_{ab} and the electric part E_{ab} of the Weyl tensor.

IV. CONCLUSIONS

It should be noticed that the existence of the six-dimensional algebra of ICVFs that act on two-dimensional manifolds is *independent* of the geodesic assumption of the unit spacelike vector field x^a, and the form of the metrics (28) and (63) is altered only by an arbitrary function in the g_{xx} component with a subsequent change in the dynamics. As such, the structure of the class of spacetimes presented in this paper can be regarded as a generalization of the (irrotational) locally rotationally symmetric geometries without any global isometry which, however, contain these models as special cases [26].

An interesting aspect of the analysis of the preceding sections is the existence of *infinite conserved quantities* along null geodesics originating from the ICVFs admitted by the two-dimensional submanifold (in type II) or the $\mathbf{x}_\perp$ submanifold (in types II and III). Therefore, it could be enlightening to determine the ten-dimensional algebra of ICVFs due to the emerged conformal flatness of the hypersurfaces $x = \text{const}$ when the fluid is flux free ($q^a = 0$) and its anisotropy is described only in terms of the three principal inhomogeneous "pressures" p_α (or equivalently when the Einstein tensor $G^a{}_b$ is diagonal).

As we have seen, a perfect fluid solution was not excluded *a priori*. In this direction, it would be interesting to allow the inclusion of a cosmological constant Λ similar to the case of the Szekeres models [27] or the Petrov type I silent universes [28] where exact solutions have been shown to exist. Furthermore, the models of type II [Eqs. (28) and (46), with Eqs. (26) and (38)] or type III [Eqs. (63) and (80), with Eqs. (61) and (73)] could also be relevant for studying the effect of *small anisotropic* and *inhomogeneous* "pressures" on the expansion dynamics—either as the relic of various physical sources [29] or as the result of backreaction terms of the density fluctuations [30,31]—provided that the use of purely phenomenological laws governing the appearance of "pressures" is consistent with the kinetic theory approach of the fluid thermodynamics.

Relaxing the flux-free restrictions [Eqs. (40)–(42) or Eqs. (74)–(76)] opens the possibility that exact tilted perfect-fluid solutions could be found for the spacetimes presented in this paper. Unlike the symmetric Lemaître-Tolman-Bondi subclass [32] (or the plane/hyperbolic analogues) where a tilted (twisted) perfect-fluid solution cannot exist [26] (due to the locally rotational symmetry), it is far from obvious that the *intrinsic locally rotational symmetry* induced from the ICVF could be strong enough to forbid a noncomoving perfect-fluid interpretation.

We emphasize that every attempt to assign a dynamical (vacuum or nonvacuum) interpretation to the spacetimes presented in this paper must take into account the induced (nonsymmetry) integrability conditions. This can be done by examining whether a suitable set of initial data evolves consistently, which is equivalent to demanding that the constraints (spatial divergence and curl equations encoded in the set of the initial data) are consistent with the evolution equations, and hence that they are preserved identically along the timelike congruence u^a *without* imposing new geometrical, kinematical, or dynamical restrictions [11,12]. Therefore, it is necessary to covariantly formulate the necessary and sufficient conditions coming from the existence of the symmetry, and study their consequences in the dynamics. We believe that all of the above points are physically sound and require further investigation.

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