Soliton mechanics

Sharmila Gunasekaran,[*] Uzair Hussain,[†] and Hari K. Kunduri[‡]

Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland and Labrador, A1C 4P5 Canada

(Received 7 October 2016; published 20 December 2016)

The domain of outer communication of five-dimensional asymptotically flat stationary spacetimes may possess nontrivial 2-cycles (bubbles). Spacetimes containing such 2-cycles can have nonzero energy, angular momenta, and charge even in the absence of horizons. A mass variation formula has been established for spacetimes containing bubbles and possibly a black hole horizon. This "first law of black hole and soliton mechanics" contains new intensive and extensive quantities associated with each 2-cycle. We consider examples of such spacetimes for which we explicitly calculate these quantities and show how regularity is essential for the formulas relating them to hold. We also derive new explicit expressions for the angular momenta and charge for spacetimes containing solitons purely in terms of fluxes supporting the bubbles.

DOI: 10.1103/PhysRevD.94.124029

I. INTRODUCTION

A striking feature of the Einstein-Maxwell theory in four dimensions is the absence of globally stationary, asymptotically flat solutions with nonzero energy—that is, there are "no solitons without horizons" [1]. This property is closely linked to uniqueness theorems for black holes, and indeed it fails to hold in the Einstein–Yang-Mills theory for which "hairy" black holes exist (see, e.g., [2]). In five and higher dimensions, however, nontrivial topology in the spacetime can support the existence of such horizonless solitons even in Einstein-Maxwell supergravity theories. For an asymptotically flat solution, the topological censorship theorem [3] asserts that the domain of outer communication of a spacetime must be simply connected. In four dimensions, that is sufficient to ensure the absence of any cycles in the exterior. In five dimensions, simple connectedness is a weaker constraint, and in particular does not exclude the possibility of 2-cycles ("bubbles"). Physically, these cycles are supported by magnetic flux supplied by Maxwell fields and contribute to both the energy and angular momenta of the spacetime.

In this article we will focus on five-dimensional asymptotically flat stationary spacetimes with two commuting rotational Killing fields, possibly containing a single black hole. In this case it has been shown that the topology of the domain of outer communication is $\mathbb{R}\times\Sigma$, where[1]

$$\Sigma \cong (\mathbb{R}^4 \# n(S^2\times S^2)\# n'(\pm\mathbb{CP}^2))\backslash B, \qquad (1)$$

for some n, $n'\in\mathbb{N}_0$ and B is the black hole region, where the horizon $H=\partial B$ must topologically be one of S^3, $S^1\times S^2$, or $L(p,q)$ [4–7]. The integers n, n' determine the 2-cycle structure of Σ.

In the absence of black holes, soliton spacetimes with 2-cycles supported by flux are known to exist, with a large number of supersymmetric (see the review [8]) and non-supersymmetric examples [9–11]. The largest known family of solutions to our knowledge of these two types appeared in [12] and [13], respectively. These spacetimes carry positive energy. The relationship between the mass of these spacetimes and their fluxes is expressed in a Smarr-type formula, as observed for BPS solitons in supergravity theories by Gibbons and Warner [14]. Subsequently, it was shown that under stationary, $U(1)^2$-invariant variations satisfying the linearized field equations, variations of the mass and magnetic fluxes for general soliton spacetimes are governed by a "first law" formula [15] [see (11) below].

Furthermore, one can derive a generalized mass and mass variation formula for $\mathbb{R}\times U(1)^2$-invariant spacetimes containing a black hole with an arbitrary number of 2-cycles in the exterior region. Similar to the soliton case it was found that on top of the familiar terms for a black hole, extra terms due to the bubbles are present. However, unlike the pure soliton case, these additional terms are most naturally expressed in terms of variations of an intensive quantity (a potential), as opposed to an extensive quantity (a flux). For Einstein-Maxwell theory, possibly with a Chern-Simons term, the mass formula is [15]

$$M=\frac{3\kappa A_H}{16\pi}+\frac{3}{2}\Omega_i J_i+\Phi_H Q+\frac{1}{2}\sum_{[C]}\mathcal{Q}[C]\Phi[C]+\frac{1}{2}\sum_{[D]}\mathcal{Q}[D]\Phi[D] \qquad (2)$$

[*] sdgg82@mun.ca
[†] uh1681@mun.ca
[‡] hkkunduri@mun.ca

[1] In fact, the statement regarding Σ is still true if only one rotational Killing field is assumed, although then there are more possibilities for the horizon topology [4].

and the first law of black hole mechanics is

$$\delta M = \frac{\kappa \delta A_H}{8\pi} + \Omega_i \delta J_i + \Phi_H \delta Q + \sum_{[C]} \mathcal{Q}[C]\delta\Phi[C] + \sum_{[D]} \mathcal{Q}[D]\delta\Phi[D]. \tag{3}$$

In the above $[C]$ is a basis for the second homology of Σ, $[D]$ are certain disk topology surfaces which extend from the horizon, Φ are magnetic potentials, and $\mathcal{Q}$ are certain "electric" fluxes defined on these surfaces which we will define precisely below. This shows that nontrivial spacetime topology plays an important role in black hole thermodynamics, thus providing further motivation to study such objects beyond the obvious implications for black hole nonuniqueness [16].

It should be noted that most explicitly known examples of soliton spacetimes are supersymmetric, in which case the mass variation formula simply follows from the Bogomol'nyi-Prasad-Sommerfield (BPS) relation. The same is true for the supersymmetric solution describing a rotating black hole with a soliton in the exterior region [16]. Indeed, quite generally for BPS black hole solutions one can show that the additional terms arising in (2) and (3) vanish identically. This is analogous to the fact that for BPS black holes in these theories, the surface gravity and angular velocities also vanish identically. For nonsupersymmetric solutions describing black holes with exterior bubbles, however, these terms would generically contribute. Examples of such solutions are not explicitly known, although there seems to be no obstruction to their existence, even in the vacuum.

The purpose of this paper is to apply the formalism developed in [15] to explicitly compute the various potentials and fluxes appearing above for some known spacetimes with nontrivial Σ. In so doing we will verify the first variation formula above. We will also derive some new relations that show how the angular momenta and total electric charge of a spacetime may arise solely from the presence of flux through the 2-cycles. Finally, we will reexamine the singly rotating dipole black ring [17]. The solution is characterized by a local dipole "charge" resulting from magnetic flux through the S^2 of the ring horizon. The first law for black rings derived in [18] contains additional terms due to the dipole charge, and we show how this is recovered using the general formalism of [15]. This will use in a crucial way the disk topology region that lies in the domain of outer communication of the black ring.

II. FIRST LAW FOR BLACK HOLES AND SOLITONS IN SUPERGRAVITY

The mass and mass variation formulas for asymptotically flat, stationary spacetimes invariant under two commuting rotational symmetries have been established for a general five-dimensional theory of gravity coupled to an arbitrary set of Maxwell fields and uncharged scalars. We will be concerned with specific soliton and black hole solutions to five-dimensional minimal supergravity, whose bosonic action is (setting Newton's constant $G_5 = 1$)

$$S = \frac{1}{16\pi}\int_{\mathcal{M}} \left(\star R - 2F \wedge \star F - \frac{8}{3\sqrt{3}} F \wedge F \wedge A \right). \tag{4}$$

Here $F = \mathrm{d}A$, and A is a locally defined gauge potential. The existence of a nontrivial second homology H_2 implies that F is closed but not exact. The theory can be recovered from the general theory considered in [15] upon setting $I = 1$, $g_{IJ} = 2$, and $C_{IJK} = 16/\sqrt{3}$. We will follow this convention throughout when appealing to the construction of potentials and fluxes used in [15]. The equations of motion are

$$R_{ab} = \frac{4}{3} F_{ac}F^c_b + \frac{1}{3} G_{acd}G_b^{cd}, \qquad \mathrm{d} \star F + \frac{2}{\sqrt{3}} F \wedge F = 0, \tag{5}$$

where $G = \star F$. The central observation of [14] was that the nontriviality of the second homology H_2 makes it more natural to work with G rather than the gauge potential A which cannot be globally defined.

Let ξ be the stationary Killing field normalized so that $|\xi|^2 \to -1$ at spatial infinity (in the case of a spacetime containing a black hole, ξ is instead identified with the Killing field which is the null generator of the event horizon). Using the fact that F is closed and invariant under this action, we have a globally defined potential Φ_ξ defined by

$$\mathrm{d}\Phi_\xi \equiv i_\xi F \tag{6}$$

and the requirement $\Phi_\xi \to 0$ at spatial infinity. From the Maxwell equation one may define a closed 2-form

$$\Theta = 2 i_\xi G - \frac{8}{\sqrt{3}} F \Phi_\xi. \tag{7}$$

If, in addition to being stationary, the spacetime is invariant under a $U(1)^2$ isometry generated by the Killing fields $m_i = (m_1, m_2)$ (normalized to have 2π-periodic orbits), we also have globally defined magnetic potentials

$$\mathrm{d}\Phi_i = i_{m_i} F, \tag{8}$$

and we also fix the freedom by requiring these vanish at an asymptotically flat end. Together (ξ, m_i) generate an $\mathbb{R} \times U(1)^2$ action acting as isometries on $(\mathcal{M}, g, F)$. Using these potentials one can finally deduce the existence of globally defined potentials U_i

$$\mathrm{d}U_i = i_{m_i}\Theta + \frac{8}{\sqrt{3}}\mathrm{d}\Phi_i \Phi^H_\xi, \tag{9}$$

which are again fixed by requiring they vanish at the asymptotically flat end. Here Φ^H_ξ is the pullback of Φ_ξ to the horizon if a black hole is present in the spacetime; for a pure soliton spacetime this term is ignored. The potentials and fluxes defined above can be thought of as functions on a 2D (two-dimensional) orbit space $\mathcal{B} \cong \Sigma/U(1)^2$ [5]. The rank of the matrix $\lambda_{ij} = m_i \cdot m_j$ divides the space into two-dimensional interior points, one-dimensional boundary segments ($\partial\mathcal{B}$) called rods, and zero-dimensional points that lie on "corners" where the segments intersect. A black hole is represented by a compact rod $I_H \cong H/U(1)^2$ where the timelike Killing field goes null. There are two non-compact semi-infinite rods corresponding to the two asymptotic axes of rotation extending out to spatial infinity. The rest of $\partial\mathcal{B}$ contains finite rods I_i where an integer linear combination $v^i m_i$, $v^i \in \mathbb{Z}$ of the rotational Killing fields vanishes. These orbit space data thus encode the action of the isometry group and determine the full spacetime topology up to diffeomorphism [5]. In particular, finite rods represent two-dimensional submanifolds which may have the topology of either S^2 or a closed disk D if the corresponding rod is adjacent to I_H. We will discuss below specific examples of spacetimes containing such 2-cycles and disks.

For purely soliton spacetimes (i.e., without black holes), the Smarr formula and mass variation reduce to [15]

$$M = \frac{1}{2}\sum_{[C]} \Psi[C] q[C], \tag{10}$$

$$\delta M = \sum_{[C]} \Psi[C] \delta q[C], \tag{11}$$

where

$$q[C] = \frac{1}{4\pi}\int_C F \quad \text{and} \quad \Psi[C] = \pi v^i U_i \tag{12}$$

represent the magnetic flux and magnetic potential associated with each element of $[C]$. Note that in (11) the extensive variable $q[C]$ appears naturally in the first law in contrast to (3).

Before discussing specific examples, we would like to present new Smarr-type formulas for the angular momenta and electric charge for purely soliton spacetimes as a sum over fluxes through the 2-cycles. These are useful as they demonstrate how a spacetime can possess such conserved charges in the absence of horizons.

First, consider the angular momenta J_i associated with the rotational Killing field m_i defined by the Komar integrals

$$J[m_i] = \frac{1}{16\pi}\int_{S^3_\infty} \star \mathrm{d}m_i. \tag{13}$$

The Maxwell equation and Killing property of the m_i imply the existence of two closed (though not necessarily exact) 2-forms Υ_i defined by

$$\Upsilon_i \equiv 2 i_{m_i} G - \frac{8}{\sqrt{3}} F \Phi_i. \tag{14}$$

Cartan's formula immediately implies the existence of global potential functions χ_{ij} satisfying $\mathrm{d}\chi_{ij} = i_{m_i}\Upsilon_j$. Note that we can always choose the integration constant so that $\chi_{ij} = 0$ on an interval on which m_i vanishes for fixed j. Now using Stokes' theorem

$$\begin{aligned} J[m_i] &= \frac{1}{8\pi}\int_\Sigma \star\mathrm{Ric}(m_i) \\ &= \frac{1}{8\pi}\int_\Sigma \left(-\frac{1}{3}\right)\Upsilon_i \wedge F + \frac{4}{3}\mathrm{d}\star(F\Phi_i). \end{aligned} \tag{15}$$

The final term above may be shown to vanish by converting it to an integral over S^3_∞ where Φ_i vanishes. We can evaluate this integral over the orbit space $\mathcal{B}$, giving

$$J[m_i] = \frac{\pi}{6}\int_\mathcal{B} \eta^{jk}\mathrm{d}\chi_{ji} \wedge \mathrm{d}\Phi_k = \frac{\pi}{6}\int_\mathcal{B} \mathrm{d}[\eta^{jk}\chi_{ji} \wedge \mathrm{d}\Phi_k], \tag{16}$$

where η^{ij} is the antisymmetric symbol with $\eta^{12} = 1$. The final term can be converted to a boundary term on $\partial\mathcal{B}$, and using the fact that the potentials vanish on the semi-infinite rods $I_\pm$, we are left with

$$J[m_i] = \frac{\pi}{6}\sum_i \int_{I_i} \eta^{jk}\chi_{ji}\mathrm{d}\Phi_k. \tag{17}$$

This can be further simplified by using the fact that each rod is specified by a pair of integers v^i, so that $v^i m_i$ vanishes. By definition $v^i \mathrm{d}\Phi_i = 0$ on the rod, so that $\Phi[C] \equiv v^i \Phi_i$ is constant. By an $SL(2,\mathbb{Z})$ change of basis let us define a new basis $(\hat{m}_1, \hat{m}_2)$ for the $U(1)^2$ generators such that $\hat{m}_1 = v^i m_i$. The other Killing field $\hat{m}_2$ is non-vanishing on the rod except at the end points (these correspond to topologically S^2 submanifolds in the spacetime). Note that in the obvious notation, $\hat{\chi}_{1i}$, $\hat{\Phi}_1$ are constants on the rod. Using $SL(2,\mathbb{Z})$-invariance, $\eta^{jk}\chi_{ji}\mathrm{d}\Phi_k = \eta^{jk}\hat{\chi}_{ji}\mathrm{d}\hat{\Phi}_k$. Putting the above facts together we arrive at

$$J[m_i] = \frac{1}{3}\sum_{[C]} \chi_i[C] q[C], \tag{18}$$

where $q[C]$ are the magnetic fluxes associated with a given cycle C and $\chi_i[C] \equiv -\pi\hat{\chi}_{1i} = -\pi v^j \chi_{ji}$ is a constant associated with each cycle. It is natural to interpret the $\chi_i[C]$ as *magnetic angular momenta potentials* as they encode how the magnetic flux $q[C]$ contribute to the total angular momenta of the spacetime.

Now let us turn to an expression for the total electric charge Q, defined by

$$Q \equiv \frac{1}{4\pi}\int_{S^3_\infty} \star F = -\frac{1}{2\sqrt{3}\pi}\int_\Sigma F \wedge F. \tag{19}$$

It may appear counterintuitive that magnetic fluxes contribute to the electric charge, but it should be noted that the Maxwell equation in supergravity is self-sourced. We now proceed to evaluate this over the boundary of the orbit space. Using the definition of the magnetic potentials, we have

$$Q = \frac{\pi}{\sqrt{3}}\int_{\mathcal{B}} \eta^{ij} \mathrm{d}\Phi_i \wedge \mathrm{d}\Phi_j = \frac{\pi}{\sqrt{3}}\int_{\partial\mathcal{B}} \eta^{ij}\Phi_i \mathrm{d}\Phi_j. \tag{20}$$

We can now express this as a sum over the 2-cycles using the argument used above for the angular momenta. The result is

$$Q = -\frac{4\pi}{\sqrt{3}}\sum_{[C]} \Phi[C] q[C], \tag{21}$$

where $\Phi[C] = v^i\Phi_i$ are constant magnetic potentials associated with each 2-cycle with corresponding rod vector v^i.

III. EXAMPLES

A. Single soliton spacetime

Our first example is a charged, nonsupersymmetric gravitational soliton with spatial slices $\Sigma \cong \mathbb{R}^4 \# \mathbb{CP}^2$ which was concisely analyzed in [14] (see also [10] for a discussion of a generalization which is asymptotically AdS_5). In the following we will use a different parametrization which is convenient for our purposes. The equations of motion (5) admit the following local solution, invariant under an $\mathbb{R} \times SU(2) \times U(1)$ isometry:

$$\mathrm{d}s^2 = -\frac{r^2 W(r)}{4b(r)^2}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{W(r)} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + b(r)^2(\sigma_3 + f(r)\mathrm{d}t)^2, \tag{22}$$

$$F = \frac{\sqrt{3q}}{2}\mathrm{d}\left[\left(\frac{1}{r^2}\right)\left(\frac{j}{2}\sigma_3 - \mathrm{d}t\right)\right], \tag{23}$$

where σ_i are left-invariant 1-forms on $SU(2)$,

$$\sigma_1 = -\sin\psi \mathrm{d}\theta + \cos\psi \sin\theta \mathrm{d}\phi,$$
$$\sigma_2 = \cos\psi \mathrm{d}\theta + \sin\psi \sin\theta \mathrm{d}\phi,$$
$$\sigma_3 = \mathrm{d}\psi + \cos\theta \mathrm{d}\phi, \tag{24}$$

which satisfy $\mathrm{d}\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$ and $\psi \sim \psi + 4\pi$, $\phi \sim \phi + 2\pi$, $\theta \in [0, \pi]$ is required for asymptotic flatness. The functions appearing in the metric are given by

$$W(r) = 1 - \frac{2}{r^2}(p - q) + \frac{q^2 + 2pj^2}{r^4},$$
$$f(r) = -\frac{j}{2b(r)^2}\left(\frac{2p - q}{r^2} - \frac{q^2}{r^4}\right), \tag{25}$$

$$b(r)^2 = \frac{r^2}{4}\left(1 - \frac{j^2 q^2}{r^6} + \frac{2j^2 p}{r^4}\right), \tag{26}$$

where $p, q, j \in \mathbb{R}$. We will take $m_i = (\partial_{\hat\psi}, \partial_\phi)$, $\hat\psi = \psi/2$, to be our basis for the generators of the $U(1)^2$ action with 2π-periodic orbits.

The parameters (p, q, j) in the above local metric can be chosen to describe asymptotically flat, charged rotating black holes. However, we may obtain a regular soliton spacetime by requiring that the S^1 parametrized by the coordinate ψ degenerates smoothly at some $r = r_0$ in the spacetime, leaving an S^2 bolt, or bubble. We therefore require $g_{\psi\psi} = b(r)^2$ vanishes at r_0. Regularity of the spacetime metric imposes that $W(r_0) = 0$. The existence of a simultaneous root fixes

$$p = \frac{r_0^4(r_0^2 - j^2)}{2j^4}, \qquad q = -\frac{r_0^4}{j^2}. \tag{27}$$

In order for $\partial_{\hat\psi}$ to degenerate smoothly and avoid a conical singularity at $r = r_0$ requires $W'(r_0)(b^2(r_0))' = 1$, or equivalently

$$(1 - x)(2 + x)^2 = 1 \tag{28}$$

for $x = x_* = r_0^2/j^2$. This cubic has a unique positive solution at $x \approx 0.870385$, and in particular, $r_0^2 < j^2$.

With this inequality it is easy to check that $W(r)$, $b(r)^2 > 0$ for $r > r_0$ and the spacetime metric is globally regular. Further

$$g^{tt} = -\frac{4b(r)^2}{r^2 W(r)} < 0 \tag{29}$$

so the spacetime is stably causal, and in particular, the $t = \mathrm{const}$ hypersurfaces are Cauchy surfaces. It can be verified that $g_{tt} < 0$ everywhere, so $\partial/\partial t$ is globally timelike and, in particular, there are no ergoregions. However, if one uplifts the soliton to six dimensions, we expect it will suffer from the instability discussed in [19].

We thus obtain a 1-parameter family of $\mathbb{R} \times SU(2) \times U(1)$-invariant soliton spacetime.

The S^2 at $r = r_0$ has a round metric

$$\mathrm{d}s_2^2 = \frac{r_0^2}{4}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \tag{30}$$

and carries a magnetic flux

$$q[C] = \frac{1}{4\pi}\int_{S^2} F = \frac{\sqrt{3}r_0^2}{4j}. \tag{31}$$

It is straightforward to read off

$$\Phi_\xi = \frac{\sqrt{3}q}{2r^2}, \qquad \Phi_{\hat\psi} = -\frac{\sqrt{3}qj}{2r^2}, \qquad \Phi_\phi = -\frac{\sqrt{3}qj\cos\theta}{4r^2}. \tag{32}$$

A long but straightforward calculation yields, using (7) and (9),

$$\mathrm{d}U_{\hat\psi} = \left[\frac{2\sqrt{3}jq}{r^3} - \frac{4\sqrt{3}jq^2}{r^5}\right]\mathrm{d}r, \tag{33}$$

$$\mathrm{d}U_\phi = \left[-\frac{2\sqrt{3}jq^2\cos\theta}{r^5} + \frac{\sqrt{3}jq\cos\theta}{r^3}\right]\mathrm{d}r + \left[-\frac{\sqrt{3}jq^2\sin\theta}{2r^4} + \frac{\sqrt{3}jq\sin\theta}{2r^2}\right]\mathrm{d}\theta, \tag{34}$$

which leads to

$$U_{\hat\psi} = \frac{\sqrt{3}jq}{r^2}\left(\frac{q}{r^2} - 1\right), \qquad U_\phi = \frac{\sqrt{3}jq\cos\theta}{2r^2}\left(\frac{q}{r^2} - 1\right), \tag{35}$$

where the integration constants have been fixed so that the potentials vanish as $r \to \infty$.

On the S^2 "bolt" at $r = r_0$, the Killing field $\partial_{\hat\psi} = 2\partial_\psi$ degenerates smoothly. The interval structure of the orbit space is given below (see Fig. 1) in the basis of rotational Killing fields orthogonal at infinity $(\partial_{\phi_1}, \partial_{\phi_2})$ where $\partial_{\phi_1} = \partial_\psi - \partial_\phi$ and $\partial_{\phi_2} = \partial_\phi + \partial_\psi$. In this basis the two semi-infinite rods can manifestly be seen as axes of rotation with vanishing ∂_{ϕ_1} or ∂_{ϕ_2}. We now turn to the computation of the potentials associated with the soliton. First,

$$\Psi[C] = \pi U_{\hat\psi}(r_0) = \frac{\sqrt{3}\pi r_0^2(j^2 + r_0^2)}{j^3}. \tag{36}$$

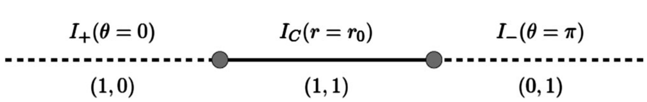

FIG. 1. Rod structure for single soliton spacetime in (ϕ_1, ϕ_2) basis.

We then find

$$\frac{\Psi[C]q[C]}{2} = \frac{3\pi}{8}\left(\frac{r_0}{j}\right)^4 (j^2 + r_0^2), \tag{37}$$

which is indeed the Arnowitt-Deser-Misner (ADM) mass of the spacetime, which can easily be read off from the expansion

$$g_{tt} = -1 + \frac{8M}{3\pi r^2} + O(r^{-4}). \tag{38}$$

Finally the first law of soliton mechanics asserts that

$$\mathrm{d}M = \Psi[C]\mathrm{d}q[C]. \tag{39}$$

In our explicit example,

$$\mathrm{d}M - \Psi[C]\mathrm{d}q[C] = \frac{3\pi r_0^5}{4j^5}(j\mathrm{d}r_0 - r_0\mathrm{d}j) \tag{40}$$

and the right hand side vanishes as a consequence of the regularity condition $r_0^2/j^2 = x_*$. We emphasize that the Smarr-type relation for the mass does not require regularity of the spacetime to hold, whereas the first law is, in fact, a finer probe of regularity. Finally one can explicitly check that the electric charge is indeed given by

$$Q = -\frac{4\pi}{\sqrt{3}}\Phi[C]q[C] = -\frac{\sqrt{3}\pi r_0^4}{2j^2}. \tag{41}$$

To compute the magnetic angular momentum potentials χ_{ij}, it is convenient to work in the $U(1)^2$ basis $(\partial_\psi, \partial_\phi)$ and then convert to the basis $(\partial_{\phi_1}, \partial_{\phi_2})$ which is orthogonal at the asymptotically flat end, in order to fix integration constants. A long but straightforward calculation yields

$$\begin{aligned}
\chi_{\psi\psi} &= -\frac{\sqrt{3}q^2j^2}{4r^4} + \frac{\sqrt{3}q}{4}, \\
\chi_{\phi\psi} &= \frac{\sqrt{3}q\cos\theta}{4}\left(1 - \frac{qj^2}{r^4}\right), \\
\chi_{\phi\phi} &= -\frac{\sqrt{3}q^2j^2\cos^2\theta}{4r^4} - \frac{\sqrt{3}q}{4}, \\
\chi_{\psi\phi} &= -\frac{\sqrt{3}q\cos\theta}{4}\left(1 + \frac{qj^2}{r^4}\right).
\end{aligned} \tag{42}$$

Since the 2-cycle is specified by the vanishing of $\partial_{\hat\psi}$, using the formula (18) we find

$$J_\psi = \frac{\pi r_0^6}{4j^3}, \qquad J_\phi = 0, \tag{43}$$

where in the second equality we observe that $\chi_{\psi\phi} = 0$ on C using (27). It is easy to check that these expressions agree with the standard ADM angular momenta computed from the asymptotic falloff of the metric. As expected, the $SU(2)\times U(1)$-invariant solution has equal angular momenta in orthogonal 2-planes, $J_1 = J_2 = J_\psi$. Note that $J_\psi \neq 0$ for the soliton; indeed, we have the constraint

$$J_\psi = -\frac{2Qq[C]}{3} = \frac{16\pi q[C]^3}{3\sqrt{3}}. \tag{44}$$

B. Double soliton spacetime

Our second example is a supersymmetric, asymptotically flat spacetime containing two nonhomologous 2-cycles. The spatial slices $\Sigma \cong \mathbb{R}^4\#(S^2\times S^2)$ where the connected sum with $\mathbb{R}^4$ corresponds to removing a point. The solution is originally given in the more general $U(1)^3$ five-dimensional supergravity [20]. We will quickly review this double soliton solution to the minimal supergravity theory (4) as this particular case does not seem to be reproduced explicitly in the literature. Note that it belongs to the general family of solutions with the Gibbons-Hawking base space first analyzed in detail in [21].

The spacetime metric takes the canonical form of a timelike fibration over a hyper-Kähler "base space"

$$\mathrm{d}s^2 = -f^2(\mathrm{d}t+\omega)^2 + f^{-1}\mathrm{d}s_B^2, \tag{45}$$

where $V = \partial/\partial t$ is the supersymmetric, timelike Killing vector field and $\mathrm{d}s_M^2$ is a hyper-Kähler base [21]. The solution has a Gibbons-Hawking hyper-Kähler base

$$\mathrm{d}s_M^2 = H^{-1}(\mathrm{d}\psi+\chi)^2 + H(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)), \tag{46}$$

where (r,θ,ϕ) are spherical coordinates on $\mathbb{R}^3$, the function H is harmonic on $\mathbb{R}^3$, and χ is a 1-form on $\mathbb{R}^3$ satisfying $\star_3\mathrm{d}\chi = \mathrm{d}H$.

The analysis of [21] shows a general technique for constructing solutions of the above form. Defining the following harmonic functions on $\mathbb{R}^3$ [20]:

$$H = \frac{1}{r} - \frac{1}{r_1} + \frac{1}{r_2}, \qquad K = \frac{k_0}{r} + \frac{k_1}{r_1} + \frac{k_2}{r_2}, \tag{47}$$

$$L = 1 + \frac{\ell_0}{r} + \frac{\ell_1}{r_1} + \frac{\ell_2}{r_2}, \qquad M = m + \frac{m_1}{r_1} + \frac{m_2}{r_2}, \tag{48}$$

with

$$r_1 = \sqrt{r^2 + a_1^2 - 2ra_1\cos\theta},$$
$$r_2 = \sqrt{r^2 + a_2^2 - 2ra_2\cos\theta}, \tag{49}$$

where we assume $0 < a_1 < a_2$, we arrive at a solution provided

$$f^{-1} = H^{-1}K^2 + L, \qquad \omega = \omega_\psi(\mathrm{d}\psi+\chi) + \hat{\omega}, \tag{50}$$

where

$$\omega_\psi = H^{-2}K^3 + \frac{3}{2}H^{-1}KL + M, \tag{51}$$

$$\star_3\mathrm{d}\hat{\omega} = H\mathrm{d}M - M\mathrm{d}H + \frac{3}{2}(K\mathrm{d}L - L\mathrm{d}K). \tag{52}$$

The Maxwell field is then

$$F = \frac{\sqrt{3}}{2}\mathrm{d}[f(\mathrm{d}t+\omega) - KH^{-1}(\mathrm{d}\psi + \chi_i\mathrm{d}x^i) - \xi_i\mathrm{d}x^i], \tag{53}$$

where the 1-form ξ satisfies $\star_3\mathrm{d}\xi = -\mathrm{d}K$. For the above choice of harmonic functions one finds

$$\chi = \left[\cos\theta - \frac{r\cos\theta - a_1}{r_1} + \frac{r\cos\theta - a_2}{r_2}\right]\mathrm{d}\phi, \tag{54}$$

and

$$\xi = -\left[k_0\cos\theta + \frac{k_1(r\cos\theta - a_1)}{r_1} + \frac{k_2(r\cos\theta - a_2)}{r_2}\right]\mathrm{d}\phi, \tag{55}$$

where we have absorbed the integration constant in χ by suitably shifting ψ. One may also integrate explicitly for $\hat{\omega} = \hat{\omega}_\phi\mathrm{d}\phi$.

For a suitable choice of constants this solution is asymptotically flat provided $\Delta\psi = 4\pi$, $\Delta\phi = 2\pi$, and $0 \le \theta \le \pi$. In particular, setting $r = \rho^2/4$ and sending $\rho \to \infty$ one finds

$$\mathrm{d}s_M^2 \sim \mathrm{d}\rho^2 + \frac{\rho^2}{4}[(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi)^2)^2\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2] \tag{56}$$

with $O(\rho^{-2})$ corrections in the associated Cartesian chart. Finally, choosing

$$m = -\frac{3}{2}(k_0 + k_1 + k_2) \tag{57}$$

and suitably fixing the integration constant in $\hat{\omega}_\phi$, we find $f = 1 + \mathcal{O}(\rho^{-2})$, $\omega_\psi = \mathcal{O}(\rho^{-2})$, and $\hat{\omega}_\phi = \mathcal{O}(\rho^{-2})$. Thus the spacetime is asymptotically Minkowski $\mathbb{R}^{1,4}$.

The free parameters characterizing these local "three-center" solutions may be chosen so that globally the spacetime describes a two-soliton spacetime (see, e.g., [14]). It is clear that the spacetime metric is regular apart from possible singularities at the "centers" which lie at the points $\mathbf{x_0} = (0,0,0)$, $\mathbf{x_1} = (0,0,a_1)$, and $\mathbf{x_2} = (0,0,a_2)$ in the usual Cartesian coordinates on the ambient $\mathbb{R}^3$ on the base space. To ensure that the spacetime metric degenerates smoothly at these points, it is sufficient to first require that the base space be smooth. It can be shown that this is, in fact, the case without any further restriction of parameters (the base space metric approaches, up to an overall sign, the Euclidean metric near the origin of $\mathbb{R}^4$). Note that on the base space, ∂_ψ degenerates smoothly at the centers.

Next to ensure that the spacetime metric is well behaved and has the correct signature, we must have $f \neq 0$ ($f = 0$ would correspond to an event horizon). Equivalently we must ensure f^{-1} does not diverge, which fixes

$$\ell_2 = -k_2^2, \qquad \ell_1 = k_1^2, \qquad \ell_0 = -k_0^2. \tag{58}$$

Further, since ∂_ψ degenerates on the base, near the centers we have

$$|\partial_\psi|^2 = -f^2 \omega_\psi^2 \leq 0, \tag{59}$$

which immediately implies that ω_ψ must *vanish* at these points. It turns out generically ω_ψ actually has simple poles at these points. Removing these requires

$$m_1 = \frac{k_1^3}{2}, \qquad m_2 = \frac{k_2^3}{2}, \qquad k_0 = 0. \tag{60}$$

Actually imposing that $\omega_\psi = 0$ leads to the so-called "bubble equations"

$$a_2 k_1^3 + a_1 k_2^3 - 3 a_1 a_2 (k_1 + k_2) = 0, \tag{61}$$

$$a_1 (k_1 + k_2)^3 + (a_2 - a_1)(k_1^3 - 3 a_1 (2k_1 + k_2)) = 0, \tag{62}$$

$$a_2 (k_1 + k_2)^3 - (a_2 - a_1)(k_2^3 + 3 a_2 k_1) = 0, \tag{63}$$

which correspond to the enforcing regularity at $r = 0, r = a_1$, and $r = a_2$, respectively. This leaves a one-parameter family of 2-soliton spacetimes parametrized by (a_1, a_2, k_1, k_2) subject to the three regularity constraints. An analysis of the geometry shows that the spacetime is stably causal ($g^{tt} \leq 0$) [14].

Let us now consider the boundary structure of the orbit space $\mathcal{B} = \Sigma / U(1)^2$, which determines the topology of the spacetime. There is a semi-infinite rod I_+ corresponding to one of the axes of symmetry in the asymptotically flat region. The appropriately normalized Killing field which vanishes on this rod is $v_+ = \partial_\psi - \partial_\phi$. In terms of the spherical coordinates on the ambient $\mathbb{R}^3$ associated with the Gibbons-Hawking space, $I_+ = \{r > a_2, \theta = 0\}$. Next, there is a finite rod $I_{C_2} = \{a_1 < r < a_2, \theta = 0\}$ with associated vanishing Killing field $v_2 = -(\partial_\phi + \partial_\psi)$. Note that the Killing field ∂_ψ is nonvanishing on C_2 and degenerates smoothly at the end points $r = a_1, a_2$ implying that C_2 is a topologically S^2-submanifold in the spacetime. The second bubble corresponds to the interval $I_{C_1} = \{0 < r < a_1, \theta = 0\}$ with associated Killing field $v_1 = -\partial_\phi + \partial_\psi$. The Killing field ∂_ψ is again nonvanishing on this interval and degenerates smoothly at the end points $r = 0, r = a_1$. Finally, there is a second semi-infinite rod $I_- = \{r > 0, \theta = \pi\}$ with associated Killing field $v_- = \partial_\phi + \partial_\psi$.

The rod structure (see Fig. 2 below) is most naturally expressed in terms of the basis of Killing fields $m_1 = v_+, m_2 = v_-$ which have 2π periodic orbits,

$$v_+ = (1, 0), \qquad v_2 = (0, -1),$$
$$v_1 = (1, 0), \qquad v_- = (0, 1), \tag{64}$$

from which it is easy to check that the compatibility condition $|\det(v_i^T v_{i+1}^T)| = 1$ is satisfied for adjacent rods. We now turn to a computation of the various intensive and extensive quantities appearing in the first law. The magnetic fluxes through the bubbles C_1, C_2 are found to be

$$q[C_2] = \frac{1}{4\pi} \int_{S_2^2} F = -\frac{\sqrt{3}}{2}(k_1 + k_2),$$
$$q[C_1] = \frac{1}{4\pi} \int_{S_1^2} F = \frac{\sqrt{3}}{2} k_1. \tag{65}$$

The computation of the "electric" potentials U_i requires some more work. For a general supersymmetric solution in the timelike class, one can derive the relation

$$i_\xi \star F = \frac{\sqrt{3}}{2} f^2 \star_4 \mathrm{d}\omega - \frac{f G^+}{\sqrt{3}}, \tag{66}$$

where $\star_4$ is the Hodge dual taken with respect to the base space and $G^+ = \frac{f}{2}(\mathrm{d}\omega + \star_4 \mathrm{d}\omega)$ is a self-dual 2-form. Using this and the general form of the Maxwell field leads to the simple expression

$$\Theta = \sqrt{3}\mathrm{d}(f^2(\mathrm{d}t + \omega)) - 4F \tag{67}$$

from which it is manifest that Θ is closed, though not exact, as expected. We then have

$$U_\psi = -\sqrt{3} f^2 \omega_\psi + 4 A_\psi + 2\sqrt{3}(k_1 + k_2), \tag{68}$$

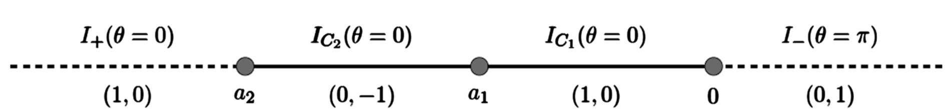

FIG. 2. Rod structure for double soliton spacetime in the (ϕ_1, ϕ_2) basis. Here, $\partial_{\phi_1} = \partial_\psi - \partial_\phi$ and $\partial_{\phi_2} = \partial_\phi + \partial_\psi$.

$$U_\phi = -\sqrt{3}f^2\omega_\phi + 4A_\phi, \quad (69)$$

where A_ψ, A_ϕ are the components of the gauge field and integration constants have been chosen so that U_i vanish at spatial infinity. As discussed above, $v^i_{C_2}U_i$ and $v^i_{C_1}U_i$ must be constant on the 2-cycles C_2 and C_1, respectively. In order to demonstrate this, one must make use of the regularity constraints (61). We find

$$\Psi[C_2] = \pi U_{C_2} \equiv -\pi(U_\psi + U_\phi)|_{I_{C_2}} = -4\sqrt{3}k_1, \quad (70)$$

$$\Psi[C_1] = \pi U_{C_1} \equiv \pi(U_\psi - U_\phi)|_{I_{C_1}} = 4\pi\sqrt{3}(k_1 + k_2). \quad (71)$$

Using this we can indeed verify that

$$\frac{1}{2}\sum_C \Psi[C]q[C] = 6\pi k_1(k_1 + k_2) = M. \quad (72)$$

The first law

$$\delta M = \Psi[C_1]\delta q[C_1] + \Psi[C_2]\delta q[C_2] \quad (73)$$

can then be verified explicitly [we emphasize this is independent from (72)]. Note that it is straightforward to check that the magnetic potentials are

$$\Phi[C_1] = -\sqrt{3}(k_1 + k_2) = -\frac{1}{4\pi}\Psi[C_1],$$
$$\Phi[C_2] = \sqrt{3}k_1 = -\frac{1}{4\pi}\Psi[C_2], \quad (74)$$

and inserting these into (41) for the total electric charge expressed as a sum over the basis of 2-cycles, one recovers the usual BPS relation $M = \sqrt{3}Q/2$. The variational formula (73) is surprising as it represents a genuine "first law" for BPS geometries, whereas for BPS black holes, the first law trivially follows from the BPS condition (i.e., $\delta M = \sqrt{3}\delta Q/2$).

The calculation of angular momenta from the general formula (18) is less straightforward. The difficulty arises from the complexity of the solution, and although it is possible to show that $\mathrm{d}\chi_{ij} = 0$, obtaining the integrated potentials in closed form has proved difficult. However, it should be noted that the asymptotic conditions $v^i_+\chi_{ij} = 0$ on I_+ and $v^i_-\chi_{ij}$ on I_-, as well as the evaluation of $\chi_i[C]$ on each cycle, only require knowledge of χ_{ij} on the "axes" $\theta = 0, \pi$. Hence we need only integrate for $\chi_{ij}(r, 0)$ and $\chi_{ij}(r, \pi)$ on each segment on the axis (i.e., $I_\pm$, I_{C_i}). Since the χ_{ij} must be continuous functions of r along the axes across the rod points at $r = a_2, r = a_1$, and $r = 0$, the integration constants arising from integrating separately over each segment are determined completely by the asymptotic conditions. Carrying this out carefully one finds

$$\chi_\phi[C_2] = 2\sqrt{3}k_1(k_1 + 2k_2),$$
$$\chi_\phi[C_1] = -2\sqrt{3}(k_2^2 - k_1^2) \quad (75)$$

and

$$\chi_\psi[C_2] = -2\sqrt{3}k_1(3k_1 + 2k_2),$$
$$\chi_\psi[C_1] = 2\sqrt{3}(3k_1^2 + 4k_1k_2 + k_2^2), \quad (76)$$

where we have used the regularity constraints (61) to significantly simplify these expressions. Using the expressions for the fluxes (65) we obtain the angular momenta

$$J_\psi = 3\pi k_1(k_1 + k_2)(2k_1 + k_2),$$
$$J_\phi = -3\pi k_1k_2(k_1 + k_2), \quad (77)$$

which do, in fact, agree with the standard ADM angular momenta provided that (61) is used to simplify the latter.

Using the above expressions for the charges (J_ψ, J_ϕ, Q) and fluxes $q[C_i]$, we can derive

$$J_\psi = \frac{Q}{2}(q[C_1] - q[C_2]) = \frac{8\pi}{\sqrt{3}}q[C_1]q[C_2](q[C_2] - q[C_1]), \quad (78)$$

$$J_\phi = \frac{Q}{2}(q[C_2] + q[C_1]) = -\frac{8\pi}{\sqrt{3}}q[C_1]q[C_2](q[C_2] + q[C_1]). \quad (79)$$

The angular momenta about the $\psi-$ and $\phi-$ directions thus is a measure of the difference and sum of the magnetic fluxes out of the two bubbles.

C. Dipole black ring

As a last example, we consider asymptotically flat dipole black rings [17] where the horizon topology is $S^1 \times S^2$ and $\Sigma \cong \mathbb{R}^4\#(S^2 \times D^2)$ [22,23]. The rings are a solution to five-dimensional Einstein-Maxwell theory (and also the minimal supergravity theory because the Chern-Simons term is of no consequence to the solutions). For convenience to match with the conventions used in [17], in this section we take $g_{IJ} = 1/2$ in the general formalism of [15]. The metric is given by

$$ds^2 = -\frac{F(y)}{F(x)}\left(\frac{H(x)}{H(y)}\right)\left(\mathrm{d}t + C(\nu,\lambda)\frac{1+y}{F(y)}\mathrm{d}\psi\right)^2 + \frac{R^2}{(x-y)^2}F(x)(H(x)H(y)^2)\left[-\frac{G(y)}{F(y)H(y)^3}\mathrm{d}\psi^2 - \frac{\mathrm{d}y^2}{G(y)} + \frac{\mathrm{d}x^2}{G(x)} + \frac{G(x)}{F(x)H(x)^3}\mathrm{d}\varphi^2\right] \quad (80)$$

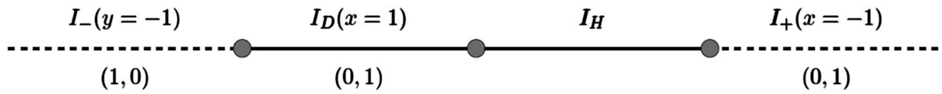

FIG. 3. Rod structure for dipole ring.

with the gauge potential,

$$A_\varphi = \sqrt{3}C(\nu,-\mu)R\frac{1+x}{H(x)}. \tag{81}$$

The functions in the metric are defined as follows:

$$F(\xi) = 1+\lambda\xi, \qquad G(\xi) = (1-\xi^2)(1+\nu\xi), \qquad H(\xi) = 1-\mu\xi$$
$$\text{with} \quad 0<\nu\le\lambda<1, 0\le\mu<1, \quad \text{and} \quad C(\alpha,\beta) = \sqrt{\beta(\beta-\alpha)\frac{1+\beta}{1-\beta}}, \tag{82}$$

where α and β are any two of the parameters μ, ν, and λ.

The following relations remove conical singularities at $y=-1$, $x=-1$, and $x=+1$:

$$\Delta\psi = \Delta\varphi = 2\pi\frac{(1+\mu)^{3/2}\sqrt{1-\lambda}}{1-\nu}, \qquad \frac{1-\lambda}{1+\lambda}\left(\frac{1+\mu}{1-\mu}\right)^3 = \left(\frac{1-\nu}{1+\nu}\right)^2. \tag{83}$$

Thermodynamic quantities for (80) were calculated in [17]. Here, we specifically focus on rederiving the extra terms that contribute to the mass using the results in [15]. These extra terms arise from disk topology surfaces denoted by D that meet the horizon. The fluxes and potentials evaluated on these surfaces may equivalently be evaluated on any other surface that is homologous to D with the same boundary as D. Studying the rod structure of the solution reveals a disk topology surface at $x=1$ (see Fig. 3 above). The disk D is parametrized by (y,ψ) at constant t, ϕ, and $x=1$. The flux $\mathcal{Q}[D]$ is given by

$$\mathcal{Q}[D] = \int_{[D]}\Theta = -\frac{\sqrt{3}\pi(\mu+1)R\sqrt{\mu(1-\lambda)(1-\mu)}}{4\sqrt{(\mu+\nu)}}. \tag{84}$$

(For usual Einstein-Maxwell theory $g_{IJ}=\frac{1}{2}$ and $C_{IJK}=0$.) ∂_φ vanishes at $x=1$. $(v^1,v^2)=(0,1)$ in the $(\hat\partial_\psi,\hat\partial_\varphi)$ basis, where the Killing fields are normalized to have 2π periodic orbits,

$$\Phi[D] = v^i\Phi_i = -\frac{2\sqrt{3}(1+\mu)R\sqrt{\mu(1-\lambda)(\mu+\nu)}}{\sqrt{(1-\mu)}(1-\nu)}. \tag{85}$$

It is easily checked that the potential $\Phi[D]=-2\mathcal{D}$ and flux $\mathcal{Q}[D]=-\frac{1}{2}\hat\Phi$ where $\mathcal{D}$ is the local dipole charge and $\hat\Phi$ is the magnetic potential introduced[2] in [17]. Therefore, we see that the Smarr relation and first law given in [17]

$$M = \frac{3}{16\pi}\kappa A_H + \frac{3}{2}\Omega_H J + \frac{1}{2}\mathcal{D}\hat\Phi,$$
$$\delta M = \frac{\kappa\delta A_H}{8\pi} + \Omega_H\delta J + \hat\Phi\delta\mathcal{D} \tag{86}$$

match precisely with the derived expressions in (2) and (3). An important point to emphasize is that, although the local dipole charge $\mathcal{D}$ arises as a *flux* integral of F over the S^2 of the black ring [17], in our formalism it arises as the constant value of Φ evaluated on the equipotential disk surface D which ends on the horizon. Hence, although it seems counterintuitive that variations of an "intensive" variable such as $\Phi[D]$ appear in the general first law, we see that at least in the present case, it is more naturally interpreted as an extensive variable (the dipole charge). Indeed, if one looks at the falloff of the gauge field A at the asymptotically flat region [24], this quantity can be interpreted as producing a dipole contribution. The fact that $\Phi[D]$ captures, in an invariant way, the dipole charge has also been observed in the context of black lenses [25–27]. In the case of black lenses, there is, in fact, no natural 2-cycle in the spacetime on which to define a dipole charge as there is for a ring [26].

[2]The quantities $\mathcal{D}$ and $\hat\Phi$ are referred to as $\mathcal{Q}$ and Φ, respectively, in the notation of [17]. We are using different symbols to avoid confusion with the notation of [15].

IV. DISCUSSION

We have explicitly computed the additional terms in the Smarr relation and first law arising from nontrivial spacetime topology in three different geometries, two describing solitons and another describing a black ring. For purely soliton spacetimes, we have complemented the results in [15] with a Smarr type formula for J and Q. These expressions also demonstrate the presence of conserved charges in the absence of a horizon. We have seen that spacetime regularity is crucial for the first law to be satisfied for all examples.

A conjectured relation [28] between dynamical and thermodynamic instability has been established by Hollands and Wald [29]. They have shown that the black p-brane spacetime $M \times \mathbb{T}^p$ associated with a thermodynamically unstable black hole M is itself dynamically unstable. This result, of course, applies to spacetimes with horizons only and does not pertain to the soliton spacetimes considered here. Very recently, the linear stability of supersymmetric soliton geometries has been investigated [30] (see also [31] for a rigorous analysis of the scalar wave equation). In particular, the authors of [30] have produced evidence that these solutions suffer from a nonlinear instability associated with the slow decay of linear waves. It would be interesting if a connection could be found between these studies of dynamical instability and an analogue of thermodynamic instability using the laws of soliton mechanics discussed in this work.

ACKNOWLEDGMENTS

H. K. K. is supported by an NSERC Discovery Grant No. 418537-2012. This research was supported in part by Perimeter Institute for Theoretical Physics. The research at Perimeter Institute is supported by the Government of Canada and by the Province of Ontario. We thank James Lucietti for a number of useful suggestions and for comments on the draft.

[1] G. W. Gibbons, Supergravity vacua and solitons, in *Duality and Supersymmetric Theories*, edited by P. West, Publications of the Newton Institute (Cambridge University Press, Cambridge, UK, 1999), pp. 267–296; https://inspirehep.net/record/458559/files/arXiv:1110.0918.pdf.

[2] A. Ashtekar, A. Corichi, and D. Sudarsky, Hairy black holes, horizon mass and solitons, Classical Quantum Gravity **18**, 919 (2001).

[3] J. L. Friedman, K. Schleich, and D. M. Witt, Topological Censorship, Phys. Rev. Lett. **71**, 1486 (1993); Erratum, Phys. Rev. Lett. **75**, 1872(E) (1995).

[4] S. Hollands, J. Holland, and A. Ishibashi, Further restrictions on the topology of stationary black holes in five dimensions, Ann. Henri. Poincaré **12**, 279 (2011).

[5] S. Hollands and S. Yazadjiev, Uniqueness theorem for 5-dimensional black holes with two axial Killing fields, Commun. Math. Phys. **283**, 749 (2008).

[6] S. Hollands and S. Yazadjiev, A uniqueness theorem for stationary Kaluza-Klein black holes, Commun. Math. Phys. **302**, 631 (2011).

[7] S. Hollands and A. Ishibashi, Black hole uniqueness theorems in higher dimensional spacetimes, Classical Quantum Gravity **29**, 163001 (2012).

[8] I. Bena and N. P. Warner, Black holes, black rings and their microstates, Lect. Notes Phys. **755**, 1 (2008).

[9] I. Bena, S. Giusto, C. Ruef, and N. P. Warner, A (running) bolt for new reasons, J. High Energy Phys. 11 (2009) 089.

[10] G. Compere, K. Copsey, S. de Buyl, and R. B. Mann, Solitons in five dimensional minimal supergravity: Local charge, exotic ergoregions, and violations of the BPS bound, J. High Energy Phys. 12 (2009) 047.

[11] N. Bobev and C. Ruef, The nuts and bolts of Einstein-Maxwell solutions, J. High Energy Phys. 01 (2010) 124.

[12] I. Bena, S. Giusto, E. J. Martinec, R. Russo, M. Shigemori, D. Turton, and N. P. Warner, Smooth Horizonless Geometries Deep Inside the Black-Hole Regime, Phys. Rev. Lett. **117**, 201601 (2016).

[13] I. Bena, G. Bossard, S. Katmadas, and D. Turton, Non-BPS multi-bubble microstate geometries, J. High Energy Phys. 02 (2016) 073.

[14] G. W. Gibbons and N. P. Warner, Global structure of five-dimensional fuzzballs, Classical Quantum Gravity **31**, 025016 (2014).

[15] H. K. Kunduri and J. Lucietti, The first law of soliton and black hole mechanics in five dimensions, Classical Quantum Gravity **31**, 032001 (2014).

[16] H. K. Kunduri and J. Lucietti, Black hole non-uniqueness via spacetime topology in five dimensions, J. High Energy Phys. 10 (2014) 082.

[17] R. Emparan, Rotating circular strings, and infinite non-uniqueness of black rings, J. High Energy Phys. 03 (2004) 064.

[18] K. Copsey and G. T. Horowitz, The role of dipole charges in black hole thermodynamics, Phys. Rev. D **73**, 024015 (2006).

[19] V. Cardoso, O. J. C. Dias, J. L. Hovdebo, and R. C. Myers, Instability of non-supersymmetric smooth geometries, Phys. Rev. D **73**, 064031 (2006).

[20] I. Bena and N. P. Warner, Bubbling supertubes and foaming black holes, Phys. Rev. D **74**, 066001 (2006).

[21] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis, and H. S. Reall, All supersymmetric solutions of minimal supergravity in five- dimensions, Classical Quantum Gravity **20**, 4587 (2003).

[22] A. Alaee, H. K. Kunduri, and E. M. Pedroza, Notes on maximal slices of five-dimensional black holes, Classical Quantum Gravity **31**, 055004 (2014).
[23] L. Andersson, M. Dahl, G. J. Galloway, and D. Pollack, On the geometry and topology of initial data sets with horizons, arXiv:1508.01896.
[24] R. Emparan and H. S. Reall, Black rings, Classical Quantum Gravity **23**, R169 (2006).
[25] H. K. Kunduri and J. Lucietti, Supersymmetric Black Holes with Lens-Space Topology, Phys. Rev. Lett. **113**, 211101 (2014).
[26] H. K. Kunduri and J. Lucietti, Black lenses in string theory, Phys. Rev. D **94**, 064007 (2016).
[27] S. Tomizawa and M. Nozawa, Supersymmetric black lenses in five dimensions, Phys. Rev. D **94**, 044037 (2016).
[28] S. S. Gubser and I. Mitra, The evolution of unstable black holes in anti-de Sitter space, J. High Energy Phys. 08 (2001) 018.
[29] S. Hollands and R. M. Wald, Stability of black holes and black branes, Commun. Math. Phys. **321**, 629 (2013).
[30] F. C. Eperon, H. S. Reall, and J. E. Santos, Instability of supersymmetric microstate geometries, J. High Energy Phys. 10 (2016) 031.
[31] J. Keir, Wave propagation on microstate geometries, arXiv:1609.01733.