PHYSICAL REVIEW D **94,** 124027 (2016)

Construction of regular black holes in general relativity

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(Received 25 October 2016; published 20 December 2016)

We present a general procedure for constructing exact black hole solutions with electric or magnetic charges in general relativity coupled to a nonlinear electrodynamics. We obtain a variety of two-parameter family spherically symmetric black hole solutions. In particular, the singularity at the center of the space-time can be canceled in the parameter space and the black hole solutions become regular everywhere in space-time. We study the global properties of the solutions and derive the first law of thermodynamics. We also generalize the procedure to include a cosmological constant and construct regular black hole solutions that are asymptotic to anti–de Sitter space-time.

DOI: 10.1103/PhysRevD.94.124027

I. INTRODUCTION

The celebrated singularity theorems proved by Penrose and Hawking [1] claim that under some circumstances the existence of singularities is inevitable in general relativity. This is in accordance with the observation that the first known exact black hole solutions in general relativity have a singularity inside the event horizon. However, it is widely believed that the singularities are nonphysical objects that are created by classical theories of gravity and they do not exist in nature. In fact, the quantum arguments given by Sakharov [2] and Gliner [3] suggest that the space-time singularities could be avoided for matter sources with a de Sitter core at the center of the space-time. Based on this idea, Bardeen proposed the first static spherically symmetric regular black hole solution [4]. Other regular black hole models were also proposed later [5–14]. It is easily shown that all these regular black hole models violate the strong energy condition[1] and hence can break the singularity theorems.

It was established by Ayón-Beato and García [15–19] that the regular black hole models can be interpreted as the gravitational field of a nonlinear electric or magnetic monopole. Thus, the physical source of the regular black holes could be a nonlinear electromagnetic field. This is also ensured by other authors in the literature [20]. Recently, it was shown in [21] that some regular black hole solutions can be constructed in $f(T)$ gravity coupled to a nonlinear electrodynamics.

In this paper, motivated by the ideas of Ayón-Beato and García, we study whether there exists a general procedure for constructing regular black hole solutions in general relativity coupled to a nonlinear electrodynamics. We find that the answer is yes. In fact, we can construct many static spherically symmetric black hole solutions with two independent integration constants. The regular black holes emerge as some degenerated solutions in the parameter space. We study the thermodynamic properties of the solutions and derive the first law, Smarr formula, and entropy product formulas, respectively. We also find that the procedure can be straightforwardly generalized to include a cosmological constant and construct black hole solutions that are asymptotic to anti–de Sitter space-time.

The paper is organized as follows. In Sec. II, we study Einstein gravity coupled to a nonlinear electrodynamics and discuss the geometric conditions for regular black holes. In Sec. III, we construct a variety of magnetically charged black hole solutions in the gravity model. In Sec. IV, we demonstrate the procedure for constructing electrically charged solutions in the gravity model. In Sec. V, we study the thermodynamic properties of the solutions above and derive the first law of thermodynamics. In Sec. VI, we generalize the procedure to gravity theories with a cosmological constant and construct AdS black hole solutions. We conclude this paper in Sec. VII.

II. EINSTEIN GRAVITY COUPLED TO A NONLINEAR ELECTRODYNAMICS

We consider Einstein gravity coupled to a nonlinear electromagnetic field of the type

$$I = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}(R - \mathcal{L}(\mathcal{F})), \qquad (1)$$

where $F = dA$ is the field strength of the vector field, $\mathcal{F} \equiv F_{\mu\nu}F^{\mu\nu}$, and the Lagrangian density $\mathcal{L}$ is a function of $\mathcal{F}$. The covariant equations of motion are

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[1]The rotating regular black hole models also violate the weak energy condition.

$$G_{\mu\nu} = T_{\mu\nu}, \qquad \nabla_\mu(\mathcal{L}_\mathcal{F} F^{\mu\nu}) = 0, \tag{2}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor and $\mathcal{L}_\mathcal{F} = \frac{\partial\mathcal{L}}{\partial\mathcal{F}}$. The energy momentum tensor is

$$T_{\mu\nu} = 2\left(\mathcal{L}_\mathcal{F} F^2_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{L}\right). \tag{3}$$

In this paper, we consider the static spherically symmetric black hole solutions with nonlinear electric/magnetic charges. The most general ansatz is given by

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2,$$
$$A = a(r)dt + Q_m \cos\theta d\phi, \tag{4}$$

where $f = f(r)$, $d\Omega^2 = d\theta^2 + \sin\theta^2 d\phi^2$ denotes the metric of a unit 2-sphere, and Q_m is the total magnetic charge defined by

$$Q_m = \frac{1}{4\pi}\int F. \tag{5}$$

Note that in above ansatz $-g_{tt} = g^{rr} = f$ is consistent with the Einstein equations of motion. This will be shown later in detail. It turns out that the construction of analytical black hole solutions with dyonic charges is a problem of great difficulty. The situation becomes much simpler for the single charged case, namely, when $a(r) = 0$ or $Q_m = 0$. Hence, in the following sections we will explicitly show how to construct exact black hole solutions with either magnetic or electric charges.

Since the main motivation of this paper is to construct regular black holes in this gravity model, it is instructive to first discuss what kind of a metric is regular at the origin of the space-time. For this purpose, we parametrize the metric function as

$$f = 1 - \frac{2m(r)}{r}, \tag{6}$$

where the constant mass of the Schwarzschild black hole is replaced by a mass distribution function $m(r)$. To govern the existence of an event horizon, we shall require the mass function to be positive definite, namely, $m(r) > 0$ when $r > 0$. To exclude the space-time singularity at the origin, we consider a smooth function $m(r)$ that is at least three times differentiable and approaches zero sufficiently fast in the limit $r \to 0$: $m(r), m'(r), m''(r)$ vanish but the third-order derivative $m'''(r)$ is finite (zero or nonzero) at the origin $r = 0$. Then to ensure the space-time regularity, a sufficient condition $m(r)/r^3$ is finite in the limit $r \to 0$ because the curvature polynomials involve at most second-order derivatives of the metric. To be concrete, we present some low-lying curvature polynomials as follows:

$$R = \frac{4m'}{r^2} + \frac{2m''}{r}, \qquad R_{\mu\nu}R^{\mu\nu} = \frac{8m'^2}{r^4} + \frac{2m''^2}{r^2},$$
$$R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} = \frac{48m^2}{r^6} - \frac{16m}{r^3}\left(\frac{4m'}{r^2} - \frac{m''}{r}\right) + 4\left(\frac{8m'^2}{r^4} - \frac{4m'm''}{r^3} + \frac{m''^2}{r^2}\right). \tag{7}$$

It is clear that if $m(r)/r^3$ is finite, and hence $m'(r)/r^2$, $m''(r)/r$ are also finite in the limit $r \to 0$, all of these polynomials will be finite constants at the origin.[2] Thus, from a purely mathematical point of view there exists a variety of candidates for regular black holes in nature except for those with a de Sitter core [namely, $m(r)/r^3$ is a finite but nonzero constant] at the center of the space-time. The existence of such regular black hole solutions cannot be ruled out before we have a better understanding of the theory of quantum gravity.

III. ASYMPTOTICALLY FLAT BLACK HOLES WITH MAGNETIC CHARGES

In this section, we will explicitly demonstrate the construction procedure of exact black hole solutions with magnetic charges. In this case, the general ansatz is given by (4) with $a(r) = 0$. It turns out that the nonlinear Maxwell equations are automatically satisfied. For Einstein equations, we find that there are only two independent equations, given by

$$0 = \frac{f'}{r} + \frac{f-1}{r^2} + \frac{1}{2}\mathcal{L}, \tag{8}$$

$$0 = f'' + \frac{2f'}{r} + \mathcal{L} - \frac{4Q_m^4}{r^4}\mathcal{L}_\mathcal{F}, \tag{9}$$

where a prime denotes the derivative with respect to the radial coordinate. One can first solve the Lagrangian density $\mathcal{L}$ as a function of r,

$$\mathcal{L} = -2\left(\frac{f'}{r} + \frac{f-1}{r^2}\right), \tag{10}$$

and then substitute it into the second equation. We find that the latter is automatically satisfied for any given metric function f. Hence, the metric ansatz (4) is indeed most general for static spherically symmetric solutions with magnetic charges. Under the parametrization (6), the Lagrangian density simplifies to

$$\mathcal{L} = \frac{4m'(r)}{r^2}. \tag{11}$$

[2]Of course, one should further verify that the metric behaves regularly everywhere in the space-time when an exact solution satisfying these conditions is successfully constructed.

In addition, the square of the field strength $\mathcal{F}$ is

$$\mathcal{F} = \frac{2Q_m^2}{r^4}. \tag{12}$$

Thus, one can freely choose a mass function $m(r)$ that is interesting in physics and then solve the Lagrangian density analytically as a function of $\mathcal{F}$. This completes the construction of static solutions with magnetic charges. However, there is a potential shortcoming in this procedure. The magnetic charge Q_m and the integration constants from the metric function f may in general appear in the derived Lagrangian density as well. This means that the solution has no free parameters because all the constants in the solution are the coupling constants of the corresponding theory. As such a solution is less interesting in physics, we shall focus on constructing the solution with at least one free integration constant.

To check the consistency of the above procedure, let us discuss two simple examples. The first is when $m(r) = \text{const}$. The metric is a Schwarzschild black hole, which is the solution of vacuum Einstein equations while Eq. (11) implies $\mathcal{L} = 0$, as expected. The second example is

$$m(r) = M - \frac{Q_m^2}{2r}. \tag{13}$$

The metric is a magnetically charged Reissner-Nordström black hole, which is the solution of Einstein-Maxwell theories. On the other hand, from Eqs. (11)–(12), we find $\mathcal{L} = \mathcal{F}$, as expected.

Using the procedure demonstrated above, we can easily construct a lot of exact black hole solutions with magnetic charges in the gravity model. In the following, we will present three different classes of solutions, which include the well-known regular black hole models such as the Bardeen black hole [4] and the Hayward black hole [9].

A. Case 1: Bardeen class

The first class solution that we present is valid for a Lagrangian density,

$$\mathcal{L} = \frac{4\mu}{\alpha}\frac{(\alpha\mathcal{F})^{5/4}}{(1+\sqrt{\alpha\mathcal{F}})^{1+\mu/2}}, \tag{14}$$

where $\mu > 0$ is a dimensionless constant and $\alpha > 0$ has the dimension of length squared. In the weak field limit, the vector field behaves as $\mathcal{L} \sim \alpha^{1/4}\mathcal{F}^{5/4}$, which is slightly stronger than a Maxwell field. The general two-parameter family black hole solution is

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2,$$
$$A = Q_m \cos\theta d\phi,$$
$$f = 1 - \frac{2M}{r} - \frac{2\alpha^{-1}q^3 r^{\mu-1}}{(r^2+q^2)^{\mu/2}}, \tag{15}$$

where q is a free integration constant that is related to the magnetic charge

$$Q_m = \frac{q^2}{\sqrt{2\alpha}}. \tag{16}$$

For the same physical charge, the parameter q can be either positive or negative. However, for our purpose we shall require $q > 0$ (throughout this paper) because the solution with $M = 0$ will no longer be a black hole in the $q < 0$ case.[3] Note that the solution reduces to a Schwarzschild black hole in the neutral limit. For $M = 0$, $\mu = 3$, the solution is the Bardeen black hole [4], which was first constructed in [18]. For later convenience, we refer to the parameter M as the "Schwarzschild mass." The Arnowitt-Deser-Misner (ADM) mass of the black hole can be read off from the asymptotic behavior of the metric functions

$$f = 1 - \frac{2(M+\alpha^{-1}q^3)}{r} + \cdots. \tag{17}$$

We have

$$M_{\text{ADM}} = M + M_{\text{em}}, \qquad M_{\text{em}} = \alpha^{-1}q^3. \tag{18}$$

It is worth pointing out that the ADM mass has two copies of contributions, one is the Schwarzschild mass which describes the condensate of the massless graviton from its nonlinear self-interactions and the other is a charged term which is associated with the nonlinear interactions between the graviton and the photon. The latter contribution is impossible for a Maxwell field or a Born-Infeld field. Since the M term introduces an unavoidable space-time singularity, we focus on discussing the degenerate case with zero Schwarzschild mass. The metric function $f(r)$ for various μ is depicted in Fig. 1. It is clear that for $\mu \geq 1$, $f(r)$ approaches a finite constant in the limit $r \to 0$. In fact, near the origin, the metric function behaves as

$$f = 1 - 2\alpha^{-1}q^{3-\mu}r^{\mu-1} + \cdots. \tag{19}$$

As emphasized earlier, to exclude the space-time singularity the mass function of the solution should satisfy the condition $m(r)/r^3 \sim$ const in the limit $r \to 0$. This selects a special class of solution that has $\mu \geq 3$. Calculating the low-lying curvature polynomials, we find

$$R = \text{regular term} \times r^{\mu-3},$$
$$R_{\mu\nu}R^{\mu\nu} = \text{regular term} \times r^{2\mu-6},$$
$$R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} = \text{regular term} \times r^{2\mu-6}, \tag{20}$$

[3] For some of the solutions in this section such as (24), there exists an additional singularity at $r = -q > 0$ when $q < 0$, which can be covered by an event horizon even for $M = 0$. However, in this case the graviton mode becomes ghostlike. Thus we always require $q > 0$ in this paper.

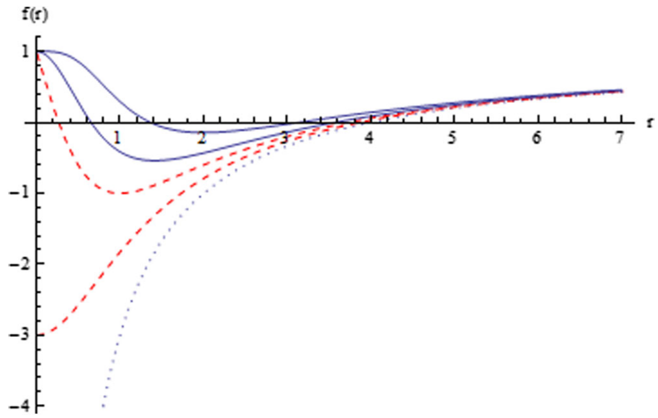

FIG. 1. The metric function $f(r)$ for a Bardeen class solution with zero Schwarzschild mass. Along the vertical axis, the value of μ decreases from top to bottom. For solid lines $\mu = 5, 3$ and for dashed lines $\mu = 2, 1$. Some parameters have been set as $\alpha = 1/2$, $q = 1$. The dotted line corresponds to a Schwarzschild black hole with $M = 2$.

where "regular term" denotes the terms that have a regular limit at the origin. Therefore, for $\mu \geq 3$ the singularity at the origin is indeed canceled. Finally, we remark that for generic μ, the solution violates the strong energy condition while the weak energy condition is still preserved.

B. Case 2: Hayward class

The second class of solution that we present is valid for a Lagrangian density:

$$\mathcal{L} = \frac{4\mu}{\alpha} \frac{(\alpha\mathcal{F})^{\frac{\mu+3}{4}}}{(1 + (\alpha\mathcal{F})^{\frac{\mu}{4}})^2}. \tag{21}$$

In the weak field limit, the vector field behaves as $\mathcal{L} \sim \alpha^{\frac{\mu-1}{4}} \mathcal{F}^{\frac{\mu+3}{4}}$. It could be either stronger ($\mu > 1$) or weaker ($0 < \mu < 1$) than a Maxwell field. A critical case occurs when $\mu = 1$, at which the nonlinear electrodynamics reduces to a Maxwell field in the weak field limit. The general static spherically symmetric solution reads

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2, \\ A &= Q_m \cos\theta d\phi, \\ f &= 1 - \frac{2M}{r} - \frac{2\alpha^{-1} q^3 r^{\mu-1}}{r^\mu + q^\mu}, \end{aligned} \tag{22}$$

where the magnetic charge and the ADM mass are still given by (16) and (18), respectively. For $M = 0$, $\mu = 3$, the solution is the Hayward black hole [9], which has been constructed in [22]. For the solution with zero Schwarzschild mass, the behaviors of the metric function f and the low-lying curvature polynomials are still given by (19) and (20), respectively. Thus, the regular black hole solution has $\mu \geq 3$ as well.

C. Case 3: A new class

Perhaps the most interesting theories that admit regular black hole solutions are such that the vector field approaches a Maxwell field in the weak field limit. We find that such theories indeed exist:

$$\mathcal{L} = \frac{4\mu}{\alpha} \frac{\alpha\mathcal{F}}{(1 + (\alpha\mathcal{F})^{1/4})^{\mu+1}}. \tag{23}$$

The black hole solution reads

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2, \\ A &= Q_m \cos\theta d\phi, \\ f &= 1 - \frac{2M}{r} - \frac{2\alpha^{-1} q^3 r^{\mu-1}}{(r + q)^\mu}, \end{aligned} \tag{24}$$

where the magnetic charge and ADM mass are still given by (16) and (18), respectively. When $M = 0$, the metric function f and the curvature polynomials behave as (19) and (20) near the origin. Thus, the regular black hole solution also has $\mu \geq 3$.

D. Generic case

For a generic mass function $m(r)$ of the type

$$m(r) = M + \frac{\alpha^{-1} q^3 r^\mu}{(r^\nu + q^\nu)^{\mu/\nu}}, \tag{25}$$

the Lagrangian density of the nonlinear electromagnetic field turns out to be

$$\mathcal{L} = \frac{4\mu}{\alpha} \frac{(\alpha\mathcal{F})^{\frac{\nu+3}{4}}}{(1 + (\alpha\mathcal{F})^{\frac{\nu}{4}})^{\frac{\mu+\nu}{\nu}}}, \tag{26}$$

where an extra dimensionless parameter ν is introduced. The Lagrangian density reduces to (14), (21), and (23) when $\nu = 2, \mu, 1$, respectively. For later convenience, we also write down the corresponding black hole solution as follows:

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2, \\ A &= Q_m \cos\theta d\phi, \\ f &= 1 - \frac{2M}{r} - \frac{2\alpha^{-1} q^3 r^{\mu-1}}{(r^\nu + q^\nu)^{\mu/\nu}}. \end{aligned} \tag{27}$$

Following the analysis above, the regular black hole solution has $M = 0$ and $\mu \geq 3$. Of course, one can also consider constructing black hole solutions with more general mass functions if the difficulty of solving the Lagrangian density analytically can be overcome. [One example is the first exact regular black hole solutions presented by Ayón-Beato and García in [15]. In this paper, we only consider the black hole solutions with the mass function (25) because the solutions can have one independent integration constant. For more general mass functions, this condition will no longer hold, but one can still construct those solutions using our procedure.]

IV. ASYMPTOTICALLY FLAT BLACK HOLES WITH ELECTRIC CHARGES

For electrically charged black hole solutions, the general ansatz is given by (4) with $Q_m = 0$. In this case, we find that there are three independent equations,

$$\begin{aligned} 0 &= \frac{a''}{a'} + \frac{2}{r} + \frac{\mathcal{L}'_{\mathcal{F}}}{\mathcal{L}_{\mathcal{F}}}, \\ 0 &= f'' + \frac{2f'}{r} + \mathcal{L}, \\ 0 &= \frac{f'}{r} + \frac{f-1}{r^2} + \frac{1}{2}\mathcal{L} + 2a'^2\mathcal{L}_{\mathcal{F}}, \end{aligned} \tag{28}$$

where the first is the equation of the vector field which can be solved as

$$\mathcal{L}_{\mathcal{F}} = \frac{Q_e}{r^2 a'}. \tag{29}$$

Here Q_e is the electric charge carried by the black hole

$$Q_e = \frac{1}{4\pi}\int \mathcal{L}_{\mathcal{F}}{}^*F. \tag{30}$$

The Lagrangian density can be solved from the second equation:

$$\mathcal{L} = -f'' - \frac{2f'}{r}. \tag{31}$$

Substituting the above results into the last equation, one finds

$$0 = f'' - \frac{2(f-1)}{r^2} - \frac{4Q_e a'}{r^2}. \tag{32}$$

This is the equation that one should solve to obtain the electric field for a given metric function. To check the consistency of the procedure, we calculate $\mathcal{L}_{\mathcal{F}}$ from its definition $\mathcal{L}_{\mathcal{F}} = \partial\mathcal{L}/\partial\mathcal{F} = \mathcal{L}'/\mathcal{F}', \mathcal{F} = -2a'^2$ and use Eqs. (31) and (32). We find that the result exactly coincides with (29).

Under the parametrization (6), the Lagrangian density simplifies to

$$\mathcal{L} = \frac{2m''}{r}. \tag{33}$$

Equation (32) can be analytically solved as

$$a = \frac{1}{2Q_e}(3m - rm') + c, \tag{34}$$

where c is an integration constant associated with the gauge choice. In the following, we shall choose the gauge $a(\infty) = 0$. This completes the construction of black hole solutions with nonlinear electric charges. One can first choose a physically interesting mass function and then obtain the corresponding gauge potential and the Lagrangian density as a function of r from above two equations. The remaining problem is how to express the Lagrangian density explicitly as a function of the field strength squared $\mathcal{F}$. In general, this is very difficult because $\mathcal{F} = -2a'^2$ has a rather complicated expression.[4] Nevertheless, Eq. (29) allows us to rewrite the Lagrangian density at least as a function of P, where $P = \mathcal{F}(\mathcal{L}_{\mathcal{F}})^2$, namely, $\mathcal{L} = \mathcal{L}(P)$. In fact, in this case it may be more appropriate to describe the system by means of a Legendre transformation [15,23]:

$$\mathcal{H} = \mathcal{F}\mathcal{L}_{\mathcal{F}} - \mathcal{L}. \tag{35}$$

It is easy to show that $\mathcal{H}$ is naturally a function of P, $d\mathcal{H} = (\mathcal{L}_{\mathcal{F}})^{-1}d(\mathcal{F}(\mathcal{L}_{\mathcal{F}})^2) = \mathcal{H}_P dP$, and the Lagrangian density can be derived as $\mathcal{L} = 2P\mathcal{H}_P - \mathcal{H}$. It should be emphasized that the original $\mathcal{L}(\mathcal{F})$ formalism may not be appropriate any longer in this case[5] because one will end with a multivalued $\mathcal{L}(\mathcal{F})$, which has different branches for a well-defined single one $\mathcal{H}(P)$.

For the generic mass function (25), the electrically charged black hole solution reads

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2, \\ f &= 1 - \frac{2M}{r} - \frac{2\alpha^{-1}q^3 r^{\mu-1}}{(r^\nu + q^\nu)^{\mu/\nu}}, \\ A &= \frac{q}{\sqrt{2\alpha}}\left(\left(3 - (\mu-3)\left(\frac{q}{r}\right)^\nu\right)\right. \\ &\quad \times \left.\left(1 + \left(\frac{q}{r}\right)^\nu\right)^{-\frac{\mu+\nu}{\nu}} - 3\right)dt, \end{aligned} \tag{36}$$

where the electric charge is given by

$$Q_e = \frac{q^2}{\sqrt{2\alpha}}. \tag{37}$$

[4]The situation becomes much simpler for a Maxwell and Born-Infeld field.

[5]We are grateful to E. Ayón-Beato for this point.

The corresponding Lagrangian density can be solved as

$$\mathcal{L} = \frac{2\mu}{\alpha} z^{\mu-3}(1+z^\nu)^{-\frac{\mu+2\nu}{\nu}}(\mu - 1 - (\nu+1)z^\nu),$$
$$z = \frac{q}{r} = \frac{1}{(-\alpha P)^{1/4}}. \tag{38}$$

Unlike the magnetically charged case, the field strength of the nonlinear electromagnetic field can behave regularly for some cases. We find

$$\mathcal{F} = -2a'^2 = -\frac{1}{2Q_e^2}(2m' - rm'')^2,$$
$$= -\alpha^{-1}\mu^2 q^{2\nu+2} r^{2\mu-2}((\nu+3)r^\nu - (\mu-3)q^\nu)^2 (r^\nu + q^\nu)^{-\frac{2\mu+4\nu}{\nu}}. \tag{39}$$

It is easy to see that when $\mu \geq 1$, the electric field has a regular limit at the origin of space-time. Thus, in the subset of the parameter space $M = 0, \mu \geq 3$, we have regular black holes with regular electric fields.

To end this section, we point out that the construction will become much more complicated for dyonic regular black hole solutions (4) because the field strength squared becomes $\mathcal{F} = \frac{2Q_m^2}{r^4} - 2a'^2$, while $\mathcal{L}_\mathcal{F}$ still takes the form (29). Thus, in this case it is a problem of great difficulty to solve the Lagrangian density $\mathcal{L}$ as a function of $\mathcal{F}$ or P for a given mass function. Perhaps one can start the story from a given Lagrangian density such as (26) and then try to solve the electric field as well as the metric functions analytically or half-analytically. We leave this as a future direction for research.

V. THE FIRST LAW OF THERMODYNAMICS

A. Derivation of the first law

For asymptotically flat black holes with nonlinear electric/magnetic charges, the first law was derived in [24] using a covariant approach. It was shown that the standard first law,

$$dM_{\text{ADM}} = TdS + \Phi dQ_e + \Psi dQ_m, \tag{40}$$

was satisfied. Here T, S are the Hawking temperature and entropy

$$T = \frac{\kappa}{2\pi}, \qquad S = \frac{1}{4}A, \tag{41}$$

where κ is the surface gravity and A is the area of the event horizon. The physical charges Q_e, Q_m and the conjugate potentials Φ, Ψ are defined by

$$Q_e = \frac{1}{4\pi}\int_{\Sigma_2} \mathcal{L}_\mathcal{F} {}^*F,$$
$$\Phi = A_t(\infty) - A_t(r_0),$$
$$Q_m = \frac{1}{4\pi}\int_{\Sigma_2} F,$$
$$\Psi = \tilde{A}_t(r_0) - \tilde{A}_t(\infty),$$
$$\tilde{F} = d\tilde{A} = \mathcal{L}_\mathcal{F} {}^*F. \tag{42}$$

Note that the definitions for the electric charge and magnetic potential are properly generalized[6] and they coincide with the conventional ones for a linear Maxwell field (for more details, we suggest that interested readers refer to [24]). Furthermore, if α is taken as a thermodynamic variable, the first law generalized in the extended phase space reads[7]

$$dM_{\text{ADM}} = TdS + \Phi dQ_e + \Psi dQ_m + \Pi d\alpha, \tag{43}$$

where Π is a new quantity conjugate to α. It is defined by

$$\Pi = \frac{1}{4}\int_{r_0}^\infty \mathrm{d}r\sqrt{-g}\frac{\partial \mathcal{L}}{\partial \alpha}. \tag{44}$$

Note that Π has the dimension of energy. Then the scaling dimensional argument[8] implies that the Smarr formula is

$$M_{\text{ADM}} = 2TS + \Phi Q_e + \Psi Q_m + 2\Pi\alpha. \tag{45}$$

[6]To benefit readers, we shall briefly explain how we arrive at the definitions in (42) for a nonlinear electrodynamics. As usual the equations of motion and the Bianchi identities can be expressed as $d\tilde{F} = 0, dF = 0$. Then the physical charges can be defined by integrating the lhs of the equations over any closed-2 surface enclosing the charges. The electric and magnetic field vectors can be defined by $E_\mu = F_{\mu\nu}\xi^\nu, B_\mu = -\tilde{F}_{\mu\nu}\xi^\nu$, where ξ is a Killing vector that is null on the black hole event horizon (here our discussions are valid for generally stationary and axisymmetric black hole solutions). Using the equations of motion and Bianchi identities, one can show that $\nabla_{[\mu}E_{\nu]} = 0 = \nabla_{[\mu}B_{\nu]}$ due to time-translational and rotational symmetries. Hence the electric/magnetic field vectors can be written as $E_\mu = \partial_\mu\Phi, B_\mu = \partial_\mu\Psi$, which in fact defines the electric/magnetic potentials covariantly. Moreover, it is easily shown that $\Phi = A_\mu\xi^\mu$ and $\Psi = -\tilde{A}_\mu\xi^\mu$, up to a gauge choice. In static space-times, this gives the definitions in (42).

[7]It was shown by Zhang and Gao [25] that the additional terms in the first law for asymptotically flat black holes can be derived using the covariant approach in [24].

[8]Euler's theorem implies that for any given function $g(x_i)$ such that $\mu^\delta g(x_i) = g(\mu^{\delta_i}x_i)$, one has $\delta g(x_i) = \delta_i x_i \frac{\partial g}{\partial x_i}$. Here δ, δ_i denote the scaling dimensions of the function $g(x_i)$ and the variables x_i, respectively. For our first law (43), we have $[M_{\text{ADM}}] = L, [S] = L^2, [Q_e] = L, [Q_m] = L, [\alpha] = L^2$, implying that the Smarr formula is (45).

It is worth pointing out that the existence of the new conjugate (Π, α) is essential to govern the Smarr formula. However, the definition of the conjugate is not unique. One can redefine a new quantity $\tilde{\alpha} \sim \alpha^z$ and its conjugate as $\tilde{\Pi} d\tilde{\alpha} = \Pi d\alpha$. Then the Smarr formula (45) holds with the term $2\Pi\alpha$ replaced by $2z\tilde{\Pi}\,\tilde{\alpha}$. An interesting question is the physical interpretation of the new pair of conjugates, which however remains open and deserves further studies.

In the following, we shall test the first law (43) and the Smarr formula (45) for the exact solutions that we construct previously. First, for the magnetically charged solutions (27), the various thermodynamic quantities are given by

$$\begin{aligned} M_{\rm ADM} &= M + \alpha^{-1} q^3, \\ S &= \pi r_0^2, \\ T &= \frac{1}{4\pi r_0}(1 - 2\mu\alpha^{-1} q^4 r_0^{\mu-1}(r_0+q)^{-\mu-1}), \\ Q_m &= \frac{q^2}{\sqrt{2\alpha}}, \\ \Psi &= -\frac{q}{\sqrt{2\alpha}}\left(\left(3 - (\mu-3)\frac{q}{r_0}\right)\left(1+\frac{q}{r_0}\right)^{-\mu-1} - 3\right), \\ \Pi &= \frac{q^3}{4\alpha^2}\left(\left(1 + (\mu+1)\frac{q}{r_0}\right)\left(1+\frac{q}{r_0}\right)^{-\mu-1} - 1\right). \end{aligned} \tag{46}$$

It follows that the first law (43) and the Smarr formula (45) with vanishing electric charge hold straightforwardly.

For the electrically charged solutions (36), we have

$$\begin{aligned} M_{\rm ADM} &= M + \alpha^{-1} q^3, \\ S &= \pi r_0^2, \\ T &= \frac{1}{4\pi r_0}\left(1 - 2\mu\alpha^{-1} q^{\nu+3} r_0^{\mu-1}(r_0^\nu + q^\nu)^{-\frac{\mu+\nu}{\nu}}\right), \\ Q_e &= \frac{q^2}{\sqrt{2\alpha}}, \\ \Phi &= -\frac{q}{\sqrt{2\alpha}}\left((3 - (\mu-3)\left(\frac{q}{r_0}\right)^\nu\right) \\ &\quad \times \left(1+\left(\frac{q}{r_0}\right)^\nu\right)^{-\frac{\mu+\nu}{\nu}} - 3\right), \\ \Pi &= \frac{q^3}{4\alpha^2}\left(\left(1 + (\mu+1)\left(\frac{q}{r_0}\right)^\nu\right) \right. \\ &\quad \left. \times \left(1+\left(\frac{q}{r_0}\right)^\nu\right)^{-\frac{\mu+\nu}{\nu}} - 1\right). \end{aligned} \tag{47}$$

It is straightforward to verify that the first law (43) and the Smarr formula (45) with vanishing magnetic charge are indeed satisfied.

B. Entropy product formulas

Let us discuss the entropy product formulas for the solutions (27) and (36). For simplicity, we focus on the three special classes of solutions listed in Sec. III with some low-lying $\mu = 1, 2, 3$. For vanishing Schwarzschild mass, the maximal number of the horizons defined by the roots (both real and imaginary) of the equation $f(r) = 0$ is exactly equal to μ for all these solutions. For instance, for Bardeen class solutions (15), there is only one horizon $r_0 = \sqrt{4M_{\rm em}^2 - q^2}$ for $\mu = 1$ and two horizons $r_\pm = M_{\rm em} \pm \sqrt{M_{\rm em}^2 - q^2}$ for $\mu = 2$. In both cases, the reality of the horizons provides a lower bound for the physical charges: $Q > \mu\sqrt{\alpha/8}$. For $\mu = 3$, the equation $f(r) = 0$ is equivalent to a cubic equation of r^2,

$$0 = (r^2 + q^2)^3 - 4M_{\rm em}^2 r^4, \tag{48}$$

and hence there are six roots in total, which occur in pairs, with r^2 taking the same value. Here we follow [26] and view $\tilde{r} = r^2$ as the radial variable and consider only three roots. We find that the horizons radii have lengthy expressions that are not instructive to give. Nevertheless, the entropy product formula turns out to be very simple. We find

$$\begin{aligned} \mu &= 1, \qquad S = \pi(4M_{\rm em}^2 - \sqrt{2\alpha}Q), \\ \mu &= 2, \qquad \prod_{i=1}^{2} S_i = 2\alpha\pi^2 Q^2, \\ \mu &= 3, \qquad \prod_{i=1}^{3} S_i = -(2\alpha)^{3/2}\pi^3 Q^3, \end{aligned} \tag{49}$$

where Q collectively denotes the electric/magnetic charges. For $\mu = 4$, we can also derive the product $\prod_{i=1}^{4} S_i$, which is a rather involved function of $(M_{\rm em}, \alpha^{1/2}Q)$. For the Hayward class (22) and new class solutions (24), we obtain the same entropy product formulas:

$$\begin{aligned} \mu &= 1, \qquad S = \pi(2M_{\rm em}^2 - (2\alpha)^{1/4}Q^{1/2})^2, \\ \mu &= 2, \qquad \prod_{i=1}^{2} S_i = 2\alpha\pi^2 Q^2, \\ \mu &= 3, \qquad \prod_{i=1}^{3} S_i = (2\alpha)^{3/2}\pi^3 Q^3. \end{aligned} \tag{50}$$

Note that for all the solutions above, the entropy product formulas for $\mu = 2, 3$ are independent of $M_{\rm em}$.

For the solutions with nonzero Schwarzschild mass, we can also derive the product formulas of the entropies for $\mu = 1, 2$ cases. We find

$$\mu = 1, \qquad \prod_{i=1}^{2} S_i = 4(2\alpha)^{1/2}\pi^2 M^2 Q,$$
$$\mu = 2, \qquad \prod_{i=1}^{3} S_i = 8\alpha\pi^3 M^2 Q^2, \tag{51}$$

which intriguingly depend on the product of the Schwarzschild mass squared and the physical charges. Note that the $\mu = 1$ case of the Bardeen class solution has four horizons and hence is not included in the above results. The entropy product of this case and of the $\mu = 3$ case of all these solutions is in general a rather involved function of $(M, M_{\rm em}, \alpha^{1/2}Q)$.

VI. ASYMPTOTICALLY ANTI–DE SITTER BLACK HOLES

The charged AdS black holes play an important role in the application of the AdS/CFT correspondence. In this section, we would like to construct the AdS black hole solutions with nonlinear electric/magnetic charges. For this purpose, we include a cosmological constant in the action, namely,

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-g}(R + 6\ell^{-2} - \mathcal{L}(\mathcal{F})), \tag{52}$$

where ℓ is the AdS radius. The covariant equations of motion are still given by (2)–(3) but the Einstein tensor includes the cosmological constant $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}(R + 6\ell^{-2})g_{\mu\nu}$. We find that for maximally symmetric solutions with electric/magnetic charges, the procedure established in Secs. II and III still works well (of course, some of the equations involve new terms associated with the cosmological constant). Here we shall not repeat those details. The final results are for the same Lagrangian density $\mathcal{L}(\mathcal{F})$; the asymptotically flat black hole solutions obtained in Secs. II and III can be straightforwardly generalized to (A)dS black hole solutions with spherical/hyperbolic/toric topologies. For the magnetic case (26), the solution reads

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_k^2,$$
$$A = Q_m x dy,$$
$$f = r^2/\ell^2 + k - \frac{2M}{r} - \frac{2\alpha^{-1}q^3 r^{\mu-1}}{(r^\nu + q^\nu)^{\mu/\nu}}, \tag{53}$$

where $d\Omega_k^2 = dx^2/(1 - kx^2) + (1 - kx^2)dy^2$ denotes the metric of the two-dimensional sphere/hyperboloid/torus with constant curvature $k = 1, -1, 0$.

For the electrically charged case (38), the solution reads

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_k^2,$$
$$f = r^2/\ell^2 + k - \frac{2M}{r} - \frac{2\alpha^{-1}q^3 r^{\mu-1}}{(r^\nu + q^\nu)^{\mu/\nu}},$$
$$A = \frac{q}{\sqrt{2\alpha}}\left(\left(3 - (\mu - 3)\left(\frac{q}{r}\right)^\nu\right)\right.$$
$$\left.\times\left(1 + \left(\frac{q}{r}\right)^\nu\right)^{-\frac{\mu+\nu}{\nu}} - 3\right)dt. \tag{54}$$

Treating the cosmological constant as well as the parameter α as a thermodynamic variable [27,28], we argue that the first law in the extended phase space reads

$$dM_{\rm AMD} = TdS + \Phi dQ_e + \Psi dQ_m + \Pi d\alpha + V d\tilde{P}, \tag{55}$$

where $M_{\rm AMD}$ is the AMD mass [29,30] of AdS black holes and the conjugates $(\tilde{P}, V)$ are defined by [27,28]

$$\tilde{P} = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi\ell^2}, \qquad V = \frac{4\pi r_0^3}{3}. \tag{56}$$

The Smarr formula is

$$M_{\rm ADM} = 2TS + \Phi Q_e + \Psi Q_m + 2\Pi\alpha - 2V\tilde{P}. \tag{57}$$

To test the first law and Smarr formula for the above solutions, we first notice that the temperature of the solutions has an additional dependence on the cosmological constant as well as the topological parameter. We find

$$\text{magnetic solution: } T = \frac{1}{4\pi r_0}(3r_0^2\ell^{-2} + k - 2\mu\alpha^{-1}q^4 r_0^{\mu-1}$$
$$\times (r_0 + q)^{-\mu-1}),$$
$$\text{electric solution: } T = \frac{1}{4\pi r_0}\left(3r_0^2\ell^{-2} + k - 2\mu\alpha^{-1}\right.$$
$$\left.\times q^{\nu+3} r_0^{\mu-1}(r_0^\nu + q^\nu)^{-\frac{\mu+\nu}{\nu}}\right). \tag{58}$$

The mass and other thermodynamic quantities exactly coincide with those of (46) and (47), respectively. It follows that the above first law and Smarr formula are indeed satisfied for these solutions.

To end this section, we point out that Eq. (58) in fact gives the equation of state $\tilde{P} = \tilde{P}(T, V)$ for the black hole systems in the extended phase space. Then one can follow [22] and discuss the critical phenomena of these solutions.

VII. CONCLUSION

In this paper, we provide a generic procedure for constructing exact black hole solutions with electric or magnetic charges in general relativity coupled to a nonlinear electrodynamics. The Lagrangian density of the nonlinear electromagnetic field is proposed to be a function of the field strength squared, $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$. For general static spherically symmetric solutions, we find that the equations of motion allow us to choose an appropriate metric and then solve the gauge potential and the Lagrangian density of the nonlinear electromagnetic field. This is a simple but powerful procedure for constructing known regular black hole models in the literature.

We first construct magnetically charged solutions in the gravity model. We obtain a large class of solutions and derive the corresponding Lagrangian density of the nonlinear electromagnetic field analytically. The black hole solutions contain two free parameters and reduce to the Schwarzschild black hole in the neutral limit. In particular, in a subset of the parameter space the singularity at the origin of the space-time is canceled and the black holes become regular everywhere in the space-time. We also establish the procedure for constructing electrically charged solutions in the gravity model. We find that in this case, all the regular black holes have a regular electric field as well.

We then study the global properties of the above solutions. We derive the first law and the Smarr formula. For some of the solutions, we also derive the entropy product formulas and obtain many interesting results.

Finally, we generalize the construction procedure for gravity theories with a cosmological constant. We find that the above asymptotically flat black hole solutions (including the regular black holes) can be straightforwardly generalized to the maximally symmetric counterparts that are asymptotic to anti–de Sitter space-time.

ACKNOWLEDGMENTS

We are grateful to Sijie Gao and E. Ayon-Beato for valuable correspondence and discussions. Z. Y. Fan is supported in part by National Science Foundations of China Grants No. 11275010, No. 11335012, and No. 11325522. X. Wang is supported in part by National Science Foundations of China Grants No. 11235003 and No. 11375026.

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