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Quantum cosmology and the evolution of inflationary spectra

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We illustrate how it is possible to calculate the quantum gravitational effects on the spectra of primordial scalar/tensor perturbations starting from the canonical, Wheeler-De Witt, approach to quantum cosmology. The composite matter-gravity system is analyzed through a Born-Oppenheimer approach in which gravitation is associated with the heavy degrees of freedom and matter (here represented by a scalar field) with the light ones. Once the independent degrees of freedom are identified, the system is canonically quantized and a semiclassical approximation is used for the scale factor. The differential equation governing the dynamics of the primordial spectra with their quantum-gravitational corrections is then obtained and is applied to diverse inflationary evolutions. Finally, the analytical results are compared to observations through a Monte Carlo Markov chain technique and an estimate of the free parameters of our approach is finally presented and the results obtained are compared with previous ones.

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I. INTRODUCTION

The paradigm of inflation [1] has led to a beautiful connection between microscopic and macroscopic scales. This occurs since inflation acts as a "magnifying glass" insofar as microscopic quantum fluctuations at the beginning of time, when the Universe was very small, evolve into inhomogeneous structures [2]. Thus the observed structure of the present-day Universe is related to the very early time quantum dynamics. As a consequence, the former can be used to test the primordial dynamics and, in particular, the possible effects of quantum gravity at early times corresponding to a very small universe. The reason for this is that because of the huge value of the Planck mass quantum gravity effects are otherwise suppressed (of course one can also hope to observe quantum gravitational effects in the presence of very strong gravitational fields, for example in the proximity of black holes).

Composite systems which involve two mass (or time) scales such as molecules are amenable to treatment by a Born-Oppenheimer (BO) approach [3]. For molecules this is possible because of the different nuclear and electron masses; this allows one to suitably factorize the wave function of the composite system leading, in a first approximation, to a separate description of the motion of the nuclei and the electrons. In particular it is found that the former are influenced by the mean Hamiltonian of the latter and the latter (electrons) follow the former adiabatically (in the quantum mechanical sense). Similarly, for the matter gravity system as a consequence of the fact that gravity is characterized by the Planck mass, which is much greater than the usual matter mass, the heavy degrees of freedom are associated with gravitation and the light ones with matter [4]. As a consequence, to lowest order, gravitation will be driven by the mean matter Hamiltonian and matter will follow gravity adiabatically. As mentioned above, we shall quantize the composite system; by this we mean that we shall perform the canonical quantization of Einstein gravity and matter leading to the Wheeler-DeWitt (WDW) equation [5]. This is what we mean by quantum gravity; it is quite distinct to the introduction of so-called trans-Planckian effects (loosely referred to as quantum gravity) through *ad hoc* modifications of the dispersion relation [6] and/or the initial conditions [7]. Further, the equations we shall obtain after the BO decomposition, will be exact, in the sense that they also include nonadiabatic effects. The above approach has been previously illustrated in a mini-superspace model with the aim of studying the semi-classical emergence of time [4], which is otherwise absent in the quantum system. Conditions were found for the usual (unitary) time evolution of quantum matter (Schwinger-Tomonaga or Schrödinger) to emerge; essentially these are that nonadiabatic transitions (fluctuations) be negligible or that the Universe be sufficiently far from the Planck scale. In a series of papers [8] we have generalized the approach to nonhomogeneous cosmology in order to obtain corrections to the usual power spectrum of cosmological fluctuations produced during inflation. These corrections, which essentially amount to the inclusion of the effect of the nonadiabatic transitions, affect the infrared part of the spectrum and lead to an amplification or a suppression depending on the background evolution. More

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interestingly, they depend on the wave number k and scale as k^{-3}, in both the scalar and the tensor sectors, when background evolution is close to de Sitter. That non-adiabatic effects affect the infrared part of the spectrum, which is associated with large scales, is not surprising, since it is this part of the spectrum which exits the horizon in the early stages of inflation and is exposed to high energy and curvature effects for a longer time.

The latest Planck mission results [9] provide the most accurate constraints available currently to inflationary dynamics [1]. So far the slow roll (SR) mechanism has been confirmed to be a paradigm capable of reproducing the observed spectrum of cosmological fluctuations and the correct tensor to scalar ratio [2]. Since the inflationary period is the cosmological era describing the transition from the quantum gravitational scale down to the hot big bang scale, it may, somewhere, exhibit related peculiar features which could be associated with quantum gravity effects. Quite interestingly a loss of power, with respect to the expected flatness for the spectrum of cosmological perturbations, can be extrapolated from the data at large scales [10]. Since, as mentioned above, it is for such scales that quantum gravity effects due to nonadiabaticity may appear, this has motivated us to estimate such effects. Unfortunately, such a feature (evident already in the WMAP results) exhibits large errors due to cosmic variance. Nonetheless we feel that it is worth comparing our detailed analytical predictions for the quantum gravity effects with Planck data through a Monte Carlo Markov chain (MCMC) based method.

The paper is organized as follows. In Sec. II the basic equations are reviewed, and the canonical quantization method and the subsequent BO decomposition are illustrated. In the Sec. III we calculate the master equation governing the dynamics of the two-point function of the quantum fluctuations when the quantum gravitational effects are taken into account and the vacuum prescription for these fluctuations is briefly discussed. In Sec. IV we review the basic relations for de Sitter, power-law and slow-roll inflation and the quantum corrections to the primordial spectra are explicitly calculated for these three distinct cases. In Sec. V we illustrate how our analytical predictions are compared to observations and we comment on our results. Finally in Sec. VI we draw the conclusions.

II. BASIC EQUATIONS

The inflaton-gravity system is described by the following action

$$S = \int d\eta d^3x\sqrt{-g}\left[-\frac{{\rm M_P}^2}{2}R + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] \qquad (1)$$

where ${\rm M_P} = (8\pi G)^{-1/2}$ is the reduced Planck mass. The above action can be decomposed into a homogeneous part plus fluctuations around it. The fluctuations of the metric $\delta g_{\mu\nu}(\vec{x},\eta)$ are defined by

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu} \qquad (2)$$

where $g^{(0)}_{\mu\nu} = {\rm diag}[a(\eta)^2(1,-1,-1,-1)]$ and η is the conformal time. Only the scalar and the tensor fluctuations "survive" the inflationary expansion: $\delta g = \delta g^{(S)} + \delta g^{(T)}$. The scalar fluctuations of the metric can be defined as follows

$$\delta g_{\mu\nu} = a(\eta)^2\begin{pmatrix} 2A(\vec{x},\eta) & -\partial_i B(\vec{x},\eta) \\ -\partial_i B(\vec{x},\eta) & 2\delta_{ij}\psi(\vec{x},\eta) - D_{ij}E(\vec{x},\eta)\end{pmatrix} \qquad (3)$$

with $D_{ij} \equiv \partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2$. These four degrees of freedom (d.o.f.) mix with the inflaton fluctuation $\delta\phi(\vec{x},\eta)$, defined by $\phi(\vec{x},\eta) \equiv \phi_0(\eta) + \delta\phi(\vec{x},\eta)$. The scalar perturbations, defined in (3), are gauge dependent. One can either rewrite them in terms of just two Bardeen's potentials [11] or fix the gauge and set two of them to zero. Finally, on using the equations of motion, the scalar sector can be collectively described by a single field $v(\vec{x},\eta)$ which, in the uniform curvature gauge, is given by $v(\vec{x},\eta) = a(\eta)\delta\phi(\vec{x},\eta)$. Its Fourier transform, v_k, can then be decomposed into two parts: $v_{1,k} \equiv {\rm Re}(v_k)$ and $v_{2,k} \equiv {\rm Im}(v_k)$.

The tensor fluctuations are gauge invariant perturbations of the metric and are defined by

$$ds^2 = a(\eta)^2[d\eta^2 - (\delta_{ij} + h_{ij})dx^i dx^j] \qquad (4)$$

with $\partial^i h_{ij} = \delta^{ij}h_{ij} = 0$. For each direction of propagation of the perturbation k^i, the above conditions on h_{ij}, with the requirement $g_{\mu\nu} = g_{\nu\mu}$, give seven independent constraint equations for the components of the tensor perturbations, leading to only two remaining polarization physical degrees of freedom $h^{(+)}$ and $h^{(\times)}$. Then, on defining $v^{(\lambda)}_{1,k} \equiv \frac{a{\rm M_P}}{\sqrt{2}}{\rm Re}({\rm h_k})$ and $v^{(\lambda)}_{2,k} \equiv \frac{a{\rm M_P}}{\sqrt{2}}{\rm Im}({\rm h_k})$, one can describe the tensor perturbations in a manner similar to the scalar perturbations.

In what follows we shall illustrate in detail a point which is often glossed over: namely the fact that on working in a flat 3-space and considering both homogeneous and inhomogeneous quantities one must introduce an unspecified length L. Indeed the effective action of the homogeneous inflaton-gravity system plus the inhomogeneous perturbations finally is [12]

$$S = \int d\eta \Bigg\{ L^3 \left[-\frac{\tilde{\mathrm{M}}_{\mathrm{P}}^2}{2} a'^2 + \frac{a^2}{2} (\phi_0'^2 - 2V(\phi_0) a^2) \right] + \frac{1}{2} \sum_{i=1,2} \sum_{k \neq 0}^{\infty} \left[v_{i,k}'(\eta)^2 + \left(-k^2 + \frac{z''}{z} \right) v_{i,k}(\eta)^2 \right] + \frac{1}{2} \sum_{i=1,2} \sum_{\lambda=+,\times} \sum_{k \neq 0}^{\infty} \left[\left(\frac{v_{i,k}^{(\lambda)}}{d\eta} \right)^2 + \left(-k^2 + \frac{a''}{a} \right) (v_{i,k}^{(\lambda)})^2 \right] \Bigg\} \tag{5}$$

where $\tilde{\mathrm{M}}_{\mathrm{P}} = \sqrt{6}\mathrm{M}_{\mathrm{P}}$, $z \equiv \phi_0'/H$, $H = a'/a^2$ is the Hubble parameter and $L^3 \equiv \int d^3x$. Let us note that the action for the perturbations has been conveniently simplified by means of the homogeneous dynamics.

The interval ds has dimension of a length l and one generally may either take $[a] = l$ and $[dx] = [d\eta] = l^0$ or $[a] = l^0$ and $[dx] = [d\eta] = l$. Correspondingly one then has $[L] = l^0$ or $[L] = l$. One can eliminate the factor L^3 by replacing $a \to a/L$, $\eta \to \eta L$, $v \to \sqrt{L} v$ and $k \to k/L$. Such a redefinition is equivalent to setting $L = 1$ in the above action (5) (then implicitly assuming the convention $[a(\eta)] = l$ and $[dx] = [d\eta] = l^0$) and then proceeding with its quantization. Such a choice, although limited to the homogeneous part, has been previously illustrated [13]. Henceforth we shall use this latter simplifying choice. Only at the end, in order to compare our results with observations, we shall restore all quantities to their original definition and the dependence on L will become explicit. Let us finally note that the fact that L is infinite does not create a problem. As usual, the transition from the Fourier integral with respect to the wave number to the Fourier series eliminates the corresponding divergence.

In more detail the action (5) has been obtained from the usual Einstein action with a minimally coupled scalar field $\phi(\vec{x}, \eta)$ and a potential $V(\phi)$. The Einstein action is evaluated for a general metric, including the scalar metric perturbations (3) and the tensor perturbations (4). Concerning the scalar sector, the scalar field is expanded as $\phi(\vec{x}, \eta) = \phi_0(\eta) + \delta\phi(\vec{x}, \eta)$ and terms up to the second order in the perturbations are kept in the total action [12]. The first order contributions can be eliminated by using the equations of motion for the homogeneous parts. Variation with respect to the remaining scalar metric perturbations and $\delta\phi$ lead to three equations which can be used to eliminate the former in terms of the scalar field perturbation. One then has the Friedmann equations with the backreaction part coming from the scalar field and an equation of motion just for $\delta\phi$. The fact that the equations of motion are used in the derivation of the action implies that, under quantization, off-shell fluctuations are ignored. In addition, the fact that the perturbations have been described by the field v after adopting the uniform curvature gauge leaves open the issue of the quantum fluctuations of the gauge degrees of freedom, which are here neglected. Concerning the tensor sector the derivation is straightforward as the tensor perturbations are gauge invariant and, just as for the scalar sector, one is left with the second order contributions in the action.

Once the total action for the matter-gravity system is cast into the form (5), all the dynamical quantities (fields) are expressed through an infinite "tower" of homogeneous variables $v_{i,k}$. Such an effective description has a simplifying role in the quantization procedure, which we shall illustrate in detail in the next section.

A. Canonical quantization

The dynamics of each d.o.f. describing the perturbations, is formally analogous to that of a homogeneous scalar field with a time dependent mass. In order to illustrate the quantization procedure and the subsequent Born-Oppenheimer decomposition in detail, without losing generality, we single out the homogenous part and one real scalar field for the perturbations in (5):

$$S = \int d\eta \Bigg\{ \left[-\frac{\tilde{\mathrm{M}}_{\mathrm{P}}^2}{2} a'^2 + \frac{a^2}{2} (\phi_0'^2 - 2V(\phi_0) a^2) \right] + \frac{1}{2} \sum_{k \neq 0}^{\infty} [v_k'(\eta)^2 - \omega_k^2 v_k(\eta)^2] \Bigg\} \equiv \int d\eta \mathcal{L}_{\mathrm{tot}} \tag{6}$$

where $\omega_k^2 = k^2 + m^2(\eta)$ is time dependent and L has been set equal to 1. Let us note that $m^2(\eta)$ depends on the homogeneous quantities $a(\eta)$, $\phi_0(\eta)$ and their derivatives. The action describing the evolution of the cosmological perturbations is derived by substituting the homogenous, leading order, solutions into the perturbed Lagrangian. Such a derivation does not affect the perturbations but may have consequences on the quantization. Let us remember that in obtaining the reduced action (6) we have at most kept terms to quadratic order in the field and metric perturbations (v_k). Therefore, since quantum fluctuations around z''/z occur already multiplied by small field perturbations, we choose just to retain for it its classical homogeneous value. Thus our choice is to consider $m^2(\eta)$ as a generic function of time and consequently specify it at the end of the quantization procedure.

One can rewrite the above action in terms of an arbitrary time parameter τ with $N(\tau) d\tau = a(\eta) d\eta$, where $N(\tau)$ is the lapse function. The action (6) then becomes

$$S = \int d\tau \frac{N}{a} \Bigg\{ \left[-\frac{\tilde{\mathrm{M}}_{\mathrm{P}}^2}{2} \frac{a^2 \dot{a}^2}{N^2} + \frac{a^4}{2} \left(\frac{\dot{\phi}_0^2}{N^2} - 2V(\phi_0) \right) \right] + \frac{1}{2} \sum_{k \neq 0}^{\infty} \left[\frac{a^2 \dot{v}_k(\eta)^2}{N^2} - \omega_k^2 v_k(\eta)^2 \right] \Bigg\} \equiv \int d\tau \tilde{\mathcal{L}}_{\mathrm{tot}} \tag{7}$$

where the dot indicates the derivative with respect to τ. The lapse function plays the role of a Lagrange multiplier in the action. The variation of the action with respect to N leads to the following equation of motion:

$$0 = \frac{\delta \tilde{\mathcal{L}}_{\rm tot}}{\delta N} = \frac{\tilde{\rm M}_{\rm P}^2}{2}\frac{a\dot{a}^2}{N^2} - \frac{a^3\dot{\phi}_0^2}{2N^2} - a^3 V - \sum_{k\neq 0}^{\infty}\left[\frac{a\dot{v}_k^2}{2N^2} + \frac{\omega_k^2 v_k^2}{2a}\right] \tag{8}$$

having the form of a constraint equation. The system Hamiltonian is

$$\mathcal{H} = -\frac{N\pi_a^2}{2a\tilde{\rm M}_{\rm P}^2} + \frac{N\pi_\phi^2}{2a^3} + a^3 NV + \sum_{k\neq 0}^{\infty}\left[\frac{N\pi_k^2}{2a} + \frac{N\omega_k^2}{2a}v_k^2\right] \tag{9}$$

where

$$\pi_N = 0, \quad \pi_a = -\frac{\tilde{\rm M}_{\rm P}^2 a\dot{a}}{N}, \quad \pi_\phi = \frac{a^3\dot{\phi}_0}{N}, \quad \pi_k = \frac{a\dot{v}_k}{N}. \tag{10}$$

and is proportional to the above constraint (8):

$$0 = \frac{\delta\tilde{\mathcal{L}}_{\rm tot}}{\delta N} = \frac{\mathcal{H}}{N} \tag{11}$$

which is then called the "Hamiltonian constraint." It is a very particular energy conservation constraint which equates the system's total energy to zero. At the quantum level, when the degrees of freedom are canonically quantized, it plays the role of a time independent Schroedinger equation.

A convenient canonical quantization of the action (5) (which in particular implies the choice of a Hamiltonian for the field v) leads to the following Wheeler-De Witt equation [5] for the wave function of the Universe (matter plus gravity)

$$\left\{\frac{1}{2\tilde{\rm M}_{\rm P}^2}\frac{\partial^2}{\partial a^2} - \frac{1}{2a^2}\frac{\partial^2}{\partial\phi_0^2} + Va^4 + \sum_{k\neq 0}^{\infty}\left[-\frac{1}{2}\frac{\partial^2}{\partial v_k^2} + \frac{\omega_k^2}{2}v_k^2\right]\right\}\Psi(a,\phi_0,\{v_k\}) = 0. \tag{12}$$

Let us note that the time dependent mass in ω_k^2 is $m^2(\eta) = -\frac{z''}{z}$ for each mode of the scalar perturbation and $m^2(\eta) = -\frac{a''}{a}$ for each mode of the tensor perturbation, where $z(\eta)$, $a(\eta)$ are classical expressions.

B. Born-Oppeneheimer decomposition

Equation (12) can be written in the compact form

$$\left[\frac{1}{2\tilde{\rm M}_{\rm P}^2}\frac{\partial^2}{\partial a^2} + \hat{H}_0^{(M)} + \sum_k \hat{H}_k^{(M)}\right]\Psi(a,\phi_0,\{v_k\}) \equiv \left[\frac{1}{2\tilde{\rm M}_{\rm P}^2}\frac{\partial^2}{\partial a^2} + \hat{H}^{(M)}\right]\Psi(a,\phi_0,\{v_k\}) = 0 \tag{13}$$

where

$$\hat{H}_0^{(M)} = -\frac{1}{2a^2}\frac{\partial^2}{\partial\phi_0^2} + Va^4, \tag{14}$$

$$\hat{H}_k^{(M)} = -\frac{1}{2}\frac{\partial^2}{\partial v_k^2} + \frac{\omega_k^2}{2}v_k^2 \tag{15}$$

and is formally similar to a time independent Schroedinger equation, except for the sign in front of the kinetic term for the scale factor. Finding the general solution of the WDW equation, even when the perturbations are set to zero, is a very complicated task due to the interaction between matter and gravity.

A set of approximate solutions can be found within a BO approach. The BO approximation was originally introduced in order to simplify the Schroedinger equation of complex atoms and molecules [3].

It consists in factorizing the wave function of the Universe into a product

$$\Psi(a,\phi_0,\{v_k\}) = \psi(a)\chi(a,\phi_0,\{v_k\}) \tag{16}$$

where $\psi(a)$ is the wave function for the homogeneous gravitational sector and $\chi(a,\phi_0,\{v_k\})$ is that for matter (homogeneous plus perturbations). A similar decomposition for atoms consists in factorizing the atomic wave function $\Psi_A(r,R)$ into a nuclear wave function $\psi_N(R)$ and the electrons' wave functions $\chi_e(r,R)$, where r and R are the d.o.f. of electrons and nuclei respectively. The matter wave function in Eq. (16) can be further factorized as

$$\chi(a,\phi_0,\{v_k\}) = \chi_0(a,\phi_0)\prod_{k\neq 0}^{\infty}\chi_k(\eta,v_k) = \prod_{k=0}^{\infty}\chi_k. \tag{17}$$

Let us note that the wave function of each mode v_k depends parametrically on the conformal time η and, in the semiclassical limit, the evolution of the scale factor $a = a(\eta)$ fixes η as a function of a. This time is in fact state dependent, since it depends on the specific semiclassical state ψ for the scale factor. Its introduction implies that one neglects the quantum fluctuations of this state around its semiclassical peak, as will be discussed later in this section. The above factorization leads to the following set of partial differential equations, which are equivalent to the WDW equation:

$$\left[\frac{1}{2\tilde{\rm M}_{\rm P}^2}\frac{\partial^2}{\partial a^2} + \langle\hat{H}^{(M)}\rangle\right]\tilde{\psi} = -\frac{1}{2\tilde{\rm M}_{\rm P}^2}\left\langle\frac{\partial^2}{\partial a^2}\right\rangle\tilde{\psi} \tag{18}$$

which is the equation for the gravitational wave function and

$$\tilde{\psi}^*\tilde{\psi}[\hat{H}^{(M)} - \langle\hat{H}^{(M)}\rangle]\tilde{\chi} + \frac{1}{\tilde{\mathrm{M}}_{\mathrm{P}}^2}\left(\tilde{\psi}^*\frac{\partial}{\partial a}\tilde{\psi}\right)\frac{\partial}{\partial a}\tilde{\chi} = \frac{1}{2\tilde{\mathrm{M}}_{\mathrm{P}}^2}\tilde{\psi}^*\tilde{\psi}\left[\left\langle\frac{\partial^2}{\partial a^2}\right\rangle - \frac{\partial^2}{\partial a^2}\right]\tilde{\chi} \tag{19}$$

which is the equation for matter, where

$$\psi = e^{-i\int^a \mathcal{A}da'}\tilde{\psi}, \qquad \chi = e^{i\int^a \mathcal{A}da'}\tilde{\chi}, \qquad \mathcal{A} = -i\langle\chi|\frac{\partial}{\partial a}|\chi\rangle \tag{20}$$

with $v_0 = \phi_0$, $\langle\hat{O}\rangle = \langle\tilde{\chi}|\hat{O}|\tilde{\chi}\rangle$ and each mode is individually normalized by $\langle\chi_k|\chi_k\rangle = \int dv_k \chi_k^*\chi_k = 1$.

The right-hand side of Eqs. (18) and (19) are associated with nonadiabatic quantum effects. They are generally neglected in the leading order to the BO approximation.

On multiplying both sides by $\hat{P}_k = \prod_{j\neq k}\langle\tilde{\chi}_j|$ Eq. (19) can be split into a set of equations, each governing the dynamics of a single mode k of the matter field. One is then led to

$$\tilde{\psi}^*\tilde{\psi}[\hat{H}_k^{(M)} - \langle\tilde{\chi}_k|\hat{H}_k^{(M)}|\tilde{\chi}_k\rangle]\tilde{\chi}_k + \frac{1}{\tilde{\mathrm{M}}_{\mathrm{P}}^2}\left(\tilde{\psi}^*\frac{\partial\tilde{\psi}}{\partial a}\right) \times\frac{\partial\tilde{\chi}_k}{\partial a} = \frac{1}{2\tilde{\mathrm{M}}_{\mathrm{P}}^2}\tilde{\psi}^*\tilde{\psi}\left[\langle\tilde{\chi}_k|\frac{\partial^2}{\partial a^2}|\tilde{\chi}_k\rangle - \frac{\partial^2}{\partial a^2}\right]\tilde{\chi}_k. \tag{21}$$

We may now perform the semiclassical limit for the gravitational wave function $\psi(a)$ by setting

$$\tilde{\psi}(a) \sim (\tilde{\mathrm{M}}_{\mathrm{P}}^2 a')^{-1/2}\exp\left(-i\int^a \tilde{\mathrm{M}}_{\mathrm{P}}^2 a' da\right) \tag{22}$$

obtaining the Friedmann equation

$$-\frac{\tilde{\mathrm{M}}_{\mathrm{P}}^2}{2}a'^2 + \sum_{k=0}^{\infty}\langle\hat{H}_k^{(M)}\rangle = 0 \tag{23}$$

for Eq. (18), to the leading order. In such a way the BO decomposition of the wave function of the Universe is uniquely determined and a and η are related.

Of course we have assumed that a classical limit exists which implies that $|\tilde{\psi}|^2$ is strongly peaked on the classical trajectory $a(\eta)$. For a molecule this will correspond to considering the motion of the nuclei to be quasiclassical while that of the electrons is quantum mechanical. Clearly, the semiclassical limit is an addition to the Born-Oppenheimer factorization and is necessary for time to emerge. In the semiclassical limit, in a path integral representation for the wave function, neighboring paths will tend to yield cancelling contributions on account of the rapid variation of the phase associated with the exponential of the (effective) action. An exception to this rule occurs at stationary points of the exponent and the associated paths are related to classical trajectories. It is clear that for this limit to exist the fluctuations about the solutions to the classical equations of motions must be small and the integral over them finite. In general this leads to a constraint on the effective potential associated with the fluctuations; should this not be satisfied one could have, for example, fluctuations which increase exponentially in time which, of course, signal an instability. Obviously, adopting this semiclassical approximation involves ignoring all such quantum fluctuations, which will be eliminated from the subsequent analysis.

Now, on defining $|\chi_k\rangle_s \equiv e^{-i\int^\eta\langle\tilde{\chi}_k|\hat{H}_k^{(M)}|\tilde{\chi}_k\rangle d\eta'}|\tilde{\chi}_k\rangle$, Eq. (21) becomes

$$\begin{aligned} i\partial_\eta|\chi_k\rangle_s - \hat{H}_k^{(M)}|\chi_k\rangle_s &= \frac{\exp[-i\int^\eta\langle\tilde{\chi}_k|\hat{H}_k^{(M)}|\tilde{\chi}_k\rangle d\eta']}{2\tilde{\mathrm{M}}_{\mathrm{P}}^2} \\ &\quad\times\left[\partial_a^2 - \frac{a''}{(a')^2}\partial_a - \langle\tilde{\chi}_k|\left(\partial_a^2 - \frac{a''}{(a')^2}\partial_a\right)|\tilde{\chi}_k\rangle\right]|\tilde{\chi}_k\rangle \\ &\equiv \epsilon[\hat{\Omega}_k - \langle\hat{\Omega}_k\rangle_s]|\chi_k\rangle_s \end{aligned} \tag{24}$$

where $\langle\hat{O}\rangle_s \equiv {}_s\langle\chi_k|\hat{O}|\chi_k\rangle_s$ and $\epsilon \equiv \frac{1}{2\tilde{\mathrm{M}}_{\mathrm{P}}^2}$.

In the derivation of Eq. (24), we have treated the expectation value of the Hamiltonian of the perturbations as a c-number. Also, we have included contributions to $\mathcal{O}(\tilde{\mathrm{M}}_{\mathrm{P}}^{-2})$ (different expansions have been previously examined and compared for the homogeneous case [14]). The operator $\hat{\Omega}_k$ has the following form:

$$\hat{\Omega}_k = \frac{1}{a'^2}\frac{d^2}{d\eta^2} + \left[2i\frac{\langle\hat{H}_k^{(M)}\rangle_s}{a'^2} - 2\frac{a''}{a'^3}\right]\frac{d}{d\eta}. \tag{25}$$

The operator on the right-hand side of Eq. (24) has a nonlinear structure, since it depends on χ_s and χ_s^* through multiplicative factors of the form $\langle\hat{O}\rangle_s$. We immediately note that, in the absence of the right-hand side, Eq. (24) becomes the usual matter evolution equation (Schrödinger or Schwinger-Tomonaga). The terms on the right-hand side describe the nonadiabatic effects of quantum-gravitational origin.

III. TWO-POINT FUNCTION

We are interested in the observable features of the spectrum of the scalar/tensor fluctuations generated during inflation. Such features can be extracted from the two-point function

$$p(\eta) \equiv {}_s\langle 0|\hat{v}^2|0\rangle_s = \langle\hat{v}^2\rangle_0 \tag{26}$$

at late times (for the modes well outside the horizon). In (26) the vacuum state $|0\rangle_s$ satisfies the full equation (24) and, according to standard prescriptions, reduces to the Bunch-Davies (BD) vacuum [15] in the short wavelength regime (more general assumptions may be considered as well). Let us note that $p(\eta)$ also depends on k but, in order to keep notation compact, we decided to omit any explicit reference to it.

A. Unperturbed dynamics

Before tackling the problem of evaluating the evolution of $p(\eta)$ by taking into account the full dynamics given by (24), in this section we shall briefly review the basic formalism for the unperturbed dynamics.

For each k mode, on neglecting the quantum gravitational effects, Eq. (24) takes the form of a time dependent Schrödinger equation for a harmonic oscillator with time dependent frequency

$$\hat{H}_k^{(M)} = \frac{\hat{\pi}_k^2}{2} + \frac{\omega_k^2}{2}\hat{v}_k^2 \tag{27}$$

where $\omega_k = \omega_k(\eta)$. The subscript k and the label (M) will henceforth be omitted. The following consideration will be valid for both scalar and tensor perturbations.

At the classical level, v and π satisfy the Hamiltonian equations leading to the homogeneous classical Klein-Gordon equation (equation of a harmonic oscillator with a time dependent frequency):

$$v'' + \omega^2 v = 0. \tag{28}$$

At the quantum level, the solutions of the time dependent Schroedinger equation can be found by introducing a linear invariant operator $\hat{I}$, satisfying the differential equation

$$i\frac{d}{d\eta}\hat{I} + [\hat{I}, \hat{H}] = 0 \tag{29}$$

and building up a complete set of states from the invariant vacuum state $|\text{vac}\rangle$, defined by $\hat{I}|\text{vac}\rangle = 0$, and then iteratively applying $\hat{I}^\dagger$ to the vacuum. A linear invariant satisfying (29) is given by

$$I = i[\varphi^*\hat{\pi} - (\varphi^*)'\hat{v}] \tag{30}$$

where φ^* satisfies the classical equation of motion (28). The commutator satisfies $[\hat{I}, \hat{I}^\dagger] = 1$, provided the Wronskian condition

$$i[\varphi^*\varphi' - (\varphi^*)'\varphi] = 1 \tag{31}$$

holds. Then, in the coordinate representation, the properly normalized invariant vacuum is

$$\langle v|\text{vac}\rangle = \left[\frac{1}{2\pi(\varphi^*\varphi)}\right]^{1/4} \exp\left[\frac{i}{2}\frac{(\varphi^*)'}{\varphi^*}v^2\right] \tag{32}$$

and a suitable phase is needed in order for $|\text{vac}\rangle$ to satisfy the Schroedinger equation. One easily finds

$$|0\rangle_s = \exp\left[-\frac{i}{4}\int^\eta \frac{d\eta'}{\varphi^*\varphi}\right]|\text{vac}\rangle. \tag{33}$$

Let us note that the Wronskian condition, (31), does not fix the invariant vacuum in a unique way. In general, different linearly independent combinations of solutions of Eq. (28), satisfying the Wronskian condition, are allowed. The BD prescription is only one of the possible choices. Consequently the expression (32) is a more general vacuum state satisfying the unperturbed quantum dynamics.

The linear invariants may be alternatively defined in terms of the so-called Pinney variable. In particular $\hat{I}$ can be written as

$$\hat{I} = \frac{e^{i\Theta}}{\sqrt{2}}\left[\left(\frac{1}{\rho} - i\rho'\right)\hat{v} + i\rho\hat{\pi}\right] \tag{34}$$

where ρ is the Pinney variable, a real function satisfying the following nonlinear differential equation (the so-called Ermakov–Pinney equation [16])

$$\rho'' + \omega^2\rho = \frac{1}{\rho^3} \tag{35}$$

with $\Theta = \int^\eta \frac{d\eta'}{\rho^2}$. In terms of ρ the commutator $[\hat{I}, \hat{I}^\dagger] = 1$ is now trivially satisfied. The Pinney variable is related to the solution φ of the classical field equation (28) by

$$\rho = \sqrt{2\varphi^*\varphi}. \tag{36}$$

Hence, it is proportional to its modulus. In the coordinate representation, the properly normalized vacuum, expressed in terms of the Pinney variable, is

$$\langle v|0\rangle_s = \frac{1}{(\pi\rho^2)^{1/4}}\exp\left[-\frac{i}{2}\int^\eta \frac{d\eta'}{\rho^2} - \frac{v^2}{2}\left(\frac{1}{\rho^2} - i\frac{\rho'}{\rho}\right)\right]. \tag{37}$$

Let us finally note that the two-point function is given by

$$p(\eta) = \varphi^*\varphi = \frac{\rho^2}{2}. \tag{38}$$

B. Perturbed evolution

When quantum gravitational effects are taken into account, one must solve the integrodifferential equation (24), which is an extremely difficult task.

Instead of trying to solve (24) and then calculating the power spectrum, one can find the differential equation for the spectrum p, by iteratively differentiating the two-point function and using the canonical commutation relations. On taking $|\chi_k\rangle_s = |0\rangle_s$ in Eq. (24) (we are omitting the subscript k) one obtains the evolution equation for the vacuum

$$0 = i\frac{d}{d\eta}|0\rangle_s - \hat{H}|0\rangle_s - \left[(2i\langle\hat{H}\rangle_0 g(\eta) + g'(\eta))\right.$$
$$\left.\times\left(\frac{d}{d\eta} - \left\langle\frac{d}{d\eta}\right\rangle_0\right) + g(\eta)\left(\frac{d^2}{d\eta^2} - \left\langle\frac{d^2}{d\eta^2}\right\rangle_0\right)\right]|0\rangle_s \tag{39}$$

with $\langle\hat{O}\rangle_0 \equiv {}_s\langle 0|\hat{O}|0\rangle_s$ and $g(\eta) = \frac{1}{2\tilde{M}_P^2 a'^2}$. The evolution of the two-point function can be now calculated by differentiating (26) with regard to η and using (39). The first derivative of p with regard to the conformal time is

$$i\frac{dp}{d\eta} = \langle[\hat{v}^2, \hat{H}]\rangle_0 - \langle\hat{v}^2\rangle_0 F(\eta) + G_{\hat{v}^2}(\eta) \tag{40}$$

where

$$F(\eta) = (2ig\langle\hat{H}\rangle_0 + g')\langle\partial_\eta\rangle_0 + g\langle\partial_\eta^2\rangle_0 - \text{c.c.}, \tag{41}$$

$$G_{\hat{v}^2}(\eta) = (2ig\langle\hat{H}\rangle_0 + g')\langle\hat{v}^2\partial_\eta\rangle_0 + g\langle\hat{v}^2\partial_\eta^2\rangle_0 - \text{c.c} \tag{42}$$

Let us note that g is a real function and F and $G_{\hat{v}^2}$ are then purely imaginary functions of η by construction. The subscript $\hat{v}^2$ in (42) indicates that the function G depends on η and on the operator $\hat{v}^2$. The commutator in the expression (40) is $[\hat{v}^2, \hat{H}] = i\{\hat{v}, \hat{\pi}\}$ where the curly brackets denote an anticommutator. In a more compact form Eq. (40) can then be written as

$$\frac{d\langle\hat{v}^2\rangle_0}{d\eta} = \langle\{\hat{v}, \hat{\pi}\}\rangle_0 - iR(\hat{v}^2) \tag{43}$$

where R contains the quantum gravitational effects and is defined as $R(\hat{O}) = -\langle\hat{O}\rangle_0 F(\eta) + G_{\hat{O}}(\eta)$. The above expression can be differentiated once more with regard to η and takes the following form

$$\frac{d^2\langle\hat{v}^2\rangle_0}{d\eta^2} = \frac{d\langle\{\hat{v}, \hat{\pi}\}\rangle_0}{d\eta} - i\frac{dR(\hat{v}^2)}{d\eta}. \tag{44}$$

and, in analogy with (40)

$$\frac{d\langle\{\hat{v}, \hat{\pi}\}\rangle_0}{d\eta} = -i\langle[\{\hat{v}, \hat{\pi}\}, \hat{H}]\rangle_0 - iR(\{\hat{v}, \hat{\pi}\}). \tag{45}$$

The commutator in the expression above becomes $[\{\hat{v}, \hat{\pi}\}, \hat{H}] = 2i(\hat{\pi}^2 - \omega^2\hat{v}^2)$ and (44) can be then rewritten as

$$\frac{d^2\langle\hat{v}^2\rangle_0}{d\eta^2} = 2(\langle\hat{\pi}^2\rangle_0 - \omega^2\langle\hat{v}^2\rangle_0) - iR(\{\hat{v}, \hat{\pi}\}) - i\frac{dR(\hat{v}^2)}{d\eta}. \tag{46}$$

On then calculating the derivative of Eq. (46) we finally obtain

$$\frac{d^3\langle\hat{v}^2\rangle_0}{d\eta^3} = 2\frac{d\langle\hat{\pi}^2\rangle_0}{d\eta} - 4\omega\omega'\langle\hat{v}^2\rangle_0 - 2\omega^2\frac{d\langle\hat{v}^2\rangle_0}{d\eta}$$
$$- i\frac{dR(\{\hat{v}, \hat{\pi}\})}{d\eta} - i\frac{d^2R(\hat{v}^2)}{d\eta^2}, \tag{47}$$

where

$$\frac{d\langle\hat{\pi}^2\rangle_0}{d\eta} + iR(\hat{\pi}^2) = -i\langle[\hat{\pi}^2, \hat{H}]\rangle_0 = i\omega^2\langle[\hat{v}^2, \hat{H}]\rangle_0 \tag{48}$$

and

$$\langle[\hat{v}^2, \hat{H}]\rangle_0 = i\frac{d\langle\hat{v}^2\rangle_0}{d\eta} - R(\hat{v}^2). \tag{49}$$

Equation (47) finally becomes

$$0 = \frac{d^3\langle\hat{v}^2\rangle_0}{d\eta^3} + 4\omega^2\frac{d\langle\hat{v}^2\rangle_0}{d\eta} + 2(\omega^2)'\langle\hat{v}^2\rangle_0 + 2iR(\hat{\pi}^2)$$
$$+ 2i\omega^2R(\hat{v}^2) + i\frac{dR(\{\hat{v}, \hat{\pi}\})}{d\eta} + i\frac{d^2R(\hat{v}^2)}{d\eta^2}. \tag{50}$$

Let us note that Eq. (50) is exact [no simplifications have been done to obtain Eq. (50) starting from (39)]. Further Eq. (50) has been obtained without using any peculiar property of the vacuum state and is also valid for any state satisfying the modified Schroedinger equation (24).

A perturbative approach is needed in order to solve Eq. (50). To the first order in $\tilde{M}_P^{-2}$, one can then evaluate the quantum gravitational corrections on the unperturbed vacuum (37) and then identify $\rho \to \sqrt{2p}$. The differential master equation governing the evolution of the two-point function is finally

$$\frac{d^3p}{d\eta^3} + 4\omega^2\frac{dp}{d\eta} + 2\frac{d\omega^2}{d\eta}p + \Delta_p = 0 \tag{51}$$

with

$$\Delta_p = -\frac{1}{\tilde{M}_P^2}\left[\frac{d^3}{d\eta^3}\frac{h}{4a'^2} - \frac{d^2}{d\eta^2}\frac{p'(h+2)}{4pa'^2} - \frac{d}{d\eta}\frac{h^2+4p'^2}{8a'^2p^2} + \frac{\omega\omega'h}{a'^2}\right] \tag{52}$$

where

$$h \equiv p'^2 + 4\omega^2 p^2 - 1. \tag{53}$$

The above equation is valid to the first order in $\tilde{\mathrm{M}}_{\mathrm{P}}^{-2}$ and, in the $\tilde{\mathrm{M}}_{\mathrm{P}} \to \infty$ limit, it must reproduce the standard evolution of the two-point function, which is known to satisfy the second order differential equation

$$\frac{d^2 p}{d\eta^2} - \frac{1}{2p}\left(\frac{dp}{d\eta}\right)^2 + 2\omega^2 p - \frac{1}{2p} = 0 \tag{54}$$

as can be easily derived from (35), given the relation (38). Differentiation of Eq. (54) leads to the third order equation (51), without quantum gravitational effects.

The above master equation can be used for the evolution of the vacuum (and not of a generic quantum state). Let us observe that (51) is a third order differential equation for p and also contains unphysical solutions, which do not satisfy the unperturbed Eq. (54) in the $\tilde{\mathrm{M}}_{\mathrm{P}} \to \infty$ limit.

C. The vacuum prescription

In the short wavelength limit $-k\eta \gg 1$, the classical equation (28) admits plane wave solutions of the form $v_\pm = \frac{1}{\sqrt{2k}} \exp(\pm ik\eta)$ and an arbitrary linear combination provides a suitable initial state of the system. In particular, if one retains only positive frequency waves, correspondingly one has $p = \frac{1}{2k}$. The initial condition $p = \frac{1}{2k}$ corresponds to the so-called BD vacuum prescription. The BD vacuum state is the quantum state which coincides with the Hamiltonian vacuum, as initial condition. Let us note that, in the short wavelength limit, h is zero for $p = 1/2k$ and consequently, to the leading order, the quantum gravitational corrections calculated in our approach are $\Delta_p = 0$.

The general combination of plane wave solutions is

$$v = \frac{1}{\sqrt{2k}}(\alpha \exp(ik\eta) + \beta \exp(-ik\eta)), \tag{55}$$

corresponding to

$$p = \frac{1}{2k}[|\alpha|^2 + |\beta|^2 + 2\mathrm{Re}(\alpha\beta^* \exp(2ik\eta))]. \tag{56}$$

The integration constants α and β are complex numbers, constrained by the Wronskian condition (31) which leads to

$$|\alpha|^2 - |\beta|^2 = 1; \tag{57}$$

the BD vacuum simply corresponds to $|\beta| = 0$. One may rewrite the expression for p, given the condition (57), and find

$$p = \frac{1}{2k}\left[1 + 2|\beta|^2 + 2|\beta|\sqrt{1 + |\beta|^2}(\cos\delta \cos 2k\eta - \sin\delta \sin 2k\eta)\right], \tag{58}$$

where δ is the difference between the phases of α and β respectively. Let us note that 2 real parameters (δ and $|\beta|$) enter the final expression (58), playing the role of the 2 integration constants of the second order differential equation (54). Let us note that, only for $|\beta| = 0$, $h = 0$ and the quantum gravitational corrections are negligible in the short wavelength limit.

If one solves the third order differential equation (51), even on neglecting the quantum corrections, 3 integration constants are necessary for the general solution. However, only a subset of these solutions is physical, i.e. satisfy Eq. (54), and one then expects some relation holds among the three integration constants. On solving Eq. (51), in the short wavelength limit ($-k\eta \gg 1$) and $\tilde{\mathrm{M}}_{\mathrm{P}} \to \infty$, one finds

$$p \simeq \frac{1}{2k^2}[c_+ - c_- \cos(2k\eta) + c_0 \sin(2k\eta)]. \tag{59}$$

Then on comparing with (58) we have

$$\frac{c_+}{k} = 1 + 2|\beta|^2, \qquad \frac{c_-}{k} = -2|\beta|\sqrt{1 + |\beta|^2}\cos\delta,$$
$$\frac{c_0}{k} = -2|\beta|\sqrt{1 + |\beta|^2}\sin\delta \tag{60}$$

or equivalently

$$c_+^2 - c_-^2 - c_0^2 = k^2, \qquad c_+ > 0. \tag{61}$$

The BD vacuum corresponds to $c_- = c_0 = 0$.

IV. APPLICATIONS

In this section we apply our formalism to diverse inflationary backgrounds and calculate the quantum gravitational corrections to primordial spectra. In particular, we study a pure de Sitter evolution, power-law inflation and finally SR inflation. Our starting point is the equation

$$\frac{d^2 p}{d\eta^2} - \frac{1}{2p}\left(\frac{dp}{d\eta}\right)^2 + 2\omega^2 p - \frac{1}{2p} = -\frac{1}{p}\int_{-\infty}^{\eta} d\eta' p \Delta_p, \tag{62}$$

which is obtained by integrating (51) and imposing the BD initial conditions on p, i.e. $p(-\infty) = 1/(2k)$, $p'(-\infty) = p''(-\infty) = 0$.

A. De Sitter evolution

In order to illustrate the main effects of quantum gravity on the spectrum, starting from unperturbed exact expressions, the de Sitter case is first discussed. Such a case can be obtained from realistic inflationary models in the limit $\dot{H} \to 0$, at least for $\Delta_p = 0$.

When $H = \text{const}$, one has $\omega = \sqrt{k^2 - \frac{2}{\eta^2}}$ for both scalar and tensor perturbations (the equation for the scalar sector must be obtained by starting from a general background evolution and then taking the $\dot{H} \to 0$ limit).

The BD solution of Eq. (54) is

$$p = \frac{1 + k^2\eta^2}{2k^3\eta^2}, \tag{63}$$

leading to the following expression for Δ_p:

$$\Delta_p = \frac{4H^2}{\tilde{M}_P^2 k^4 \eta^3} = -\frac{4H^2}{k\tilde{M}_P^2} p' \tag{64}$$

to the first order in $\tilde{M}_P$. Then Eq. (62) can be rewritten as

$$\frac{d^2p}{d\eta^2} - \frac{1}{2p}\left(\frac{dp}{d\eta}\right)^2 + 2\omega^2 p - \frac{1}{2p} = \frac{2H^2}{k\tilde{M}_P^2 p}(p^2 - p_\infty^2) \tag{65}$$

with $p_\infty = 1/(2k)$. The latter equation can be recast in the form of the original, unperturbed equation (54), by defining

$$\tilde{p} = \frac{p}{\sqrt{1 - \frac{4H^2}{\tilde{M}_P^2 k} p_\infty^2}} \tag{66}$$

and

$$\tilde{\omega}^2 = \omega^2 - \frac{H^2}{\tilde{M}_P^2 k} \equiv \tilde{k}^2 - \frac{z''}{z} \tag{67}$$

with

$$\tilde{k} = k\sqrt{1 - \frac{H^2}{\tilde{M}_P^2 k^3}} \equiv N_k k. \tag{68}$$

The general solution of

$$\frac{d^2\tilde{p}}{d\eta^2} - \frac{1}{2\tilde{p}}\left(\frac{d\tilde{p}}{d\eta}\right)^2 + 2\tilde{\omega}^2\tilde{p} - \frac{1}{2\tilde{p}} = 0 \tag{69}$$

is known and is given by

$$\tilde{p} = \frac{1}{2\tilde{k}^4\eta^2}\Big\{\sqrt{\tilde{k}^2 + c_0^2 + c_-^2}(1 + \tilde{k}^2\eta^2) + \cos(2\tilde{k}\eta)[2c_0\tilde{k}\eta - c_-(\tilde{k}^2\eta^2 - 1)] + \sin(2\tilde{k}\eta)[c_0(\tilde{k}^2\eta^2 - 1) + 2c_-\tilde{k}\eta]\Big\}. \tag{70}$$

On setting the oscillatory contribution to zero ($c_- = c_0 = 0$), one finally finds the perturbed BD vacuum

$$p = \frac{1 + N_k^2 k^2\eta^2}{2N_k^2 k^3\eta^2}. \tag{71}$$

In the long wavelength limit, one finds the observable features of the primordial spectra

$$p \xrightarrow{-k\eta\to 0} \frac{1}{2k^3\eta^2(1 - \frac{H^2}{\tilde{M}_P^2 k^3})} \tag{72}$$

and, for $\frac{H^2}{\tilde{M}_P^2 k^3} \ll 1$, such spectra behave as

$$p \xrightarrow{-k\eta\to 0} \frac{1}{2k^3\eta^2}\left(1 + \frac{H^2}{\tilde{M}_P^2 k^3}\right) = p_0\left(1 + \frac{H^2}{\tilde{M}_P^2 k^3}\right), \tag{73}$$

i.e. quantum gravitational effects lead to a power enhancement with respect to the standard results in the spectrum for large scales.

In our previous paper [8] we obtained the solution (73) through a different approach. In such an approach the expression was obtained by simply taking the long wavelength limit of the exact solution of Eq. (51) with Δ_p evaluated using the zeroth order Bunch-Davies solution and on suitably choosing the initial conditions for the perturbed evolution. In that approach the quantum gravitational corrections appear as an additive contribution to the standard spectrum and in the long wavelength regime the modified spectrum has the form of expression (73). In this sense the result is quite robust; nonetheless it relies on the validity of the perturbative expansion employed to evaluate the quantum gravitational corrections to the first order in M_P^{-2}. We further note that, given the perturbative expansion used, the denominator of (72) is small. One may ask whether the long wavelength limit and the asymptotic M_P^{-2} expansion limit commute. However we cannot say anything since we do not know the exact solution which is necessary to check this.

Let us note that the length scale L, defined in Sec. II, is hidden in the expression for the quantum gravitational corrections. On returning to the original physical quantities one has $p \to p/L$, $a \to La$, $\eta \to \eta/L$ and $k \to Lk$. The scale $L \equiv \bar{k}^{-1}$ would then appear in the result, as an effect of the initial volume integration of the homogeneous dynamics.

B. Power law

Power-law inflation corresponds to the simplified case in which the Hubble parameter depends on time; yet still the equations of motions, for both the homogeneous part and the perturbations, can be solved exactly. In this case the evolution of the scale factor is given by

$$a(\eta) = a_0\left(\frac{\eta_0}{\eta}\right)^{\frac{q}{q-1}}, \tag{74}$$

where q is a constant parameter, which is related to the variation of H by $q \equiv (-\dot{H}/H^2)^{-1}$ and the de Sitter limit is recovered for $q \to \infty$. The dynamics of the scalar and the tensor perturbations are governed by the same equation, which is given by (62) with $\omega = \sqrt{k^2 - \frac{2}{\eta^2}\frac{1-\frac{\epsilon}{2}}{(1-\epsilon)^2}}$ and $\epsilon \equiv -\dot{H}/H^2$. The BD vacuum can be now expressed in term of the Hankel functions:

$$p = -\frac{\pi\eta}{4}[H_\nu^{(1)}(-k\eta)H_\nu^{(2)}(-k\eta)] \tag{75}$$

with $\nu = \frac{3}{2} + \frac{1}{q-1}$. The observable features of the primordial spectra can be calculated by taking the long wavelength limit of (75), finding

$$p \to -\frac{\eta}{4\pi}\Gamma(\nu)^2\left(-\frac{k\eta}{2}\right)^{-2\nu} = \left[2^{\frac{q+1}{q-1}}\frac{\Gamma(\nu)^2}{\pi}\right]\eta_0^{-\frac{2q}{q-1}}k^{-2\nu}\left(\frac{a}{a_0}\right)^2. \tag{76}$$

Alternatively one may solve (54) in the long wavelength regime. In this case one simply observes that $\omega^2 \to -a''/a$, $p \to C_0 a^2$ and the normalization, C_0, can be fixed by matching the long wavelength solution with the short wavelength prescription for the BD vacuum [$p \to 1/(2k)$] at the horizon crossing ($k = a_k H_k$), namely:

$$C_0\frac{k^2}{H_k^2} = \frac{1}{2k} \Rightarrow C_0 = \frac{1}{2ka_k^2}. \tag{77}$$

One then obtains

$$p \to \frac{1}{2k}\left(\frac{a}{a_k}\right)^2 = \frac{1}{2}\left[\frac{q}{(q-1)}\right]^{\frac{2q}{q-1}}\eta_0^{-\frac{2q}{q-1}}k^{-2\nu}\left(\frac{a}{a_0}\right)^2. \tag{78}$$

On comparing the results (76) and (78), we observe that the normalization constant C_0, obtained by the matching procedure, is very close to the exact normalization when q is large (and they coincide in the $q \to \infty$ limit, i.e. for the de Sitter case).

One can also adopt the matching procedure to solve the perturbed equation (51). We already observed that the quantum gravitational corrections are negligible to leading order, in the short wavelength limit. Conversely they can be evaluated perturbatively and then the long wavelength limit taken. In such a limit we find that

$$p\Delta_p \to A_0(p^2)' \tag{79}$$

with

$$A_0 = -\frac{2C_0k^2}{\tilde{\mathrm{M}}_\mathrm{P}^2}\frac{(q-1)(2q+1)}{q(q+1)} \tag{80}$$

where C_0 is the normalization of p.

On neglecting Δ_p in the interval $]-\infty, \eta_k]$, one is then led to the following perturbed equation, valid in the long wavelength regime

$$\frac{d^2p}{d\eta^2} - \frac{1}{2p}\left(\frac{dp}{d\eta}\right)^2 + 2\omega^2 p - \frac{1}{2p} = -\frac{1}{p}\int_{\eta_k}^{\eta} d\eta' p\Delta_p \tag{81}$$

where η_k is the conformal time at the horizon crossing $a_k H_k = k$. Let us note that (81) is now obtained by integrating (51) and, on imposing the conditions $p(\eta_k) = 1/(2k)$, $p'(\eta_k) = p''(\eta_k) = 0$.

The integral on the right-hand side of Eq. (81) can be easily performed, in the long wavelength regime given (79), and takes the form:

$$\frac{d^2p}{d\eta^2} - \frac{1}{2p}\left(\frac{dp}{d\eta}\right)^2 + 2\omega^2 p - \frac{1}{2p} + \frac{A_0}{2}p - \frac{A_0}{4k^2p} = 0. \tag{82}$$

On defining $\tilde{k} = k\sqrt{1 + \frac{A_0}{2k^2}}$ and $\tilde{p} = p/(\sqrt{1 + \frac{A_0}{2k^2}})$, this latter equation can be cast in the form of the unperturbed equation, having the following solution:

$$\tilde{p} = \tilde{C}_0 a^2. \tag{83}$$

The normalization factor $\tilde{C}_0$ can here be fixed, by matching the short and the long wavelength solutions at the horizon crossing, i.e. when $a_{\tilde{k}}H_{\tilde{k}} = \tilde{k}$. One then finds

$$\frac{1}{2k\left(\sqrt{1 + \frac{A_0}{2k^2}}\right)} = \tilde{C}_0 a_{\tilde{k}}^2 \tag{84}$$

and

$$\tilde{C}_0 = \frac{1}{2k\left(\sqrt{1 + \frac{A_0}{2k^2}}\right)}\left(\frac{2\nu - 1}{-2\eta_0}\right)^{2\nu-1}\frac{\tilde{k}^{-2\nu+1}}{a_0^2}. \tag{85}$$

The perturbed solution in then given by

$$p \to \frac{1}{2}\left(\frac{q}{q-1}\right)^{\frac{2q}{q-1}}(\eta_0)^{-\frac{2q}{q-1}}k^{-2\nu}\left(\frac{a}{a_0}\right)^2\left(1+\frac{A_0}{2k^2}\right)^{-\frac{q}{q-1}} \tag{86}$$

and the quantum gravitational corrections, which are encoded in the factor $(1+\frac{A_0}{2k^2})^{-\frac{q}{q-1}}$, are negligible for large k. Let us note that this behavior is simply dictated by the dependence of A_0 on k, that is, it is related to the dependence on k of the unperturbed solution (in the de Sitter limit one correctly reproduces the k^3 dependence).

We further note that, apart from the de Sitter case, the matching procedure leads to errors in the correct normalization of the long wavelength solution. However we expect that the peculiar dependence on k of the quantum correction to the long wavelength solution is unaffected by such an approximation. Indeed expression (86) correctly reproduces the quantum corrections for the de Sitter case where the calculation can be performed "exactly" (without any matching). We used the matching through the corrected wavelength since it seemed more appropriate to us and the results obtained coincide with those presented in one of our previous papers (see [8]) using a different approximation scheme.

C. Slow-roll inflation

The de Sitter and the power-law evolutions are fairly good approximations to the inflationary dynamics. Furthermore these models permit an almost exact treatment of the primordial fluctuations and are thus of pedagogical interest. A wider class of more realistic inflationary models is that associated with the slow-roll dynamics. In such a case the evolution of cosmological perturbations occurs during a generic inflationary phase having a slowly varying Hubble parameter and a scalar field. The diverse inflationary models are then treated within the slow-roll approximation and the features of the spectra of perturbations, generated during inflation, are accurately estimated in such a framework, with an accuracy comparable with the magnitude of the so-called SR parameters. It is then worth generalizing our procedure to such a case.

In the general relativity framework it is quite common to introduce the SR parameters

$$\epsilon_{\mathrm{SR}} \equiv -\frac{\dot{H}}{H^2} \quad \text{and} \quad \eta_{\mathrm{SR}} \equiv -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} \tag{87}$$

and calculate the spectra just in terms of these two. The SR approximation consists of neglecting their derivatives (that is treating them as constants) or, equivalently, to only keeping first order contributions in the SR variables.

To first order in the SR approximations, the scale factor evolution satisfies the equation

$$aH \simeq -\frac{1+\epsilon_{\mathrm{SR}}}{\eta} \tag{88}$$

and its solution is then given by

$$a = a_0\left(\frac{\eta_0}{\eta}\right)^{1+\epsilon_{\mathrm{SR}}}. \tag{89}$$

In terms of the above quantities one finds

$$\omega^2 = k^2 - \frac{z''}{z} = k^2 - \frac{2(1+3\epsilon_{\mathrm{SR}} - \frac{3}{2}\eta_{\mathrm{SR}})}{\eta^2} \tag{90}$$

for the scalar perturbation and

$$\omega^2 = k^2 - \frac{a''}{a} = k^2 - \frac{2(1+\frac{3}{2}\epsilon_{\mathrm{SR}})}{\eta^2} \tag{91}$$

for the tensor perturbations. In contrast with the de Sitter and power-law cases, the equations for the scalar and the tensor perturbations are now different. However, because of the forms of (90) and (91), it is possible to recover the equation/solution for the tensor perturbations starting from the equation/solution for the scalar perturbations and taking the limit $\eta_{\mathrm{SR}} \to \epsilon_{\mathrm{SR}}$. We shall then focus on the scalar case and finally extract the tensor case results in the above limit.

We proceed in a fashion analogous to the power-law case. In the short wavelength regime, the quantum gravitational corrections evaluated perturbatively are absent at the leading and next to leading order. We thus neglect their contribution in such a limit. Conversely, in the long wavelength regime, the quantum gravitational correction should be taken into account and can be evaluated perturbatively. Finally the matching at the horizon crossing is performed.

In the long wavelength regime, the quantum corrections may be rewritten as

$$\Delta_p = \frac{a^5H^7}{k^6\tilde{\mathrm{M}}_{\mathrm{P}}^2}\left(7(\epsilon_{\mathrm{SR}} - \eta_{\mathrm{SR}}) - 4\frac{k^2}{a^2H^2}\right) \equiv \Delta_1 + \Delta_2, \tag{92}$$

where the first term

$$\Delta_1 \equiv \frac{7a^5H^7}{k^6\tilde{\mathrm{M}}_{\mathrm{P}}^2}(\epsilon_{\mathrm{SR}} - \eta_{\mathrm{SR}}), \tag{93}$$

is peculiar for the scalar sector in the SR case and the second term

$$\Delta_2 \equiv -\frac{4a^3H^5}{k^4\tilde{\mathrm{M}}_{\mathrm{P}}^2} \tag{94}$$

is common for de Sitter and power-law cases. To the leading order, $p = C_0 a^2 \epsilon_{\mathrm{SR}}$ with $C_0 = H_k^2/(2k^3\epsilon_{\mathrm{SR}})$, $p'/p = 2aH$ and $p''/p = 6a^2H^2$. The perturbed second order equation for p is (81), where the integration on the right-hand side is taken from $\eta_k = -1/k$ to η.

On integrating by parts one then finds

$$\frac{1}{p}\int_{-1/k}^{\eta} d\eta' p\Delta_1 = \frac{A_0}{p}\int_{-1/k}^{\eta} d\eta' a^4 H^6 p' = \frac{A_0}{p}\left(\frac{a^4H^6 p}{3} - \frac{k^3 H_k^2}{6}\right) \tag{95}$$

with $A_0 = \frac{7(\epsilon_{SR}-\eta_{SR})}{2k^6\tilde{M}_P^2}$ and

$$\frac{1}{p}\int_{-1/k}^{\eta} d\eta' p\Delta_2 = -\frac{B_0}{p}\int_{-1/k}^{\eta} d\eta' a^2 H^4 p' = -\frac{B_0}{p}\left(\frac{a^2H^4 p}{2} - \frac{kH_k^2}{4}\right) \tag{96}$$

with $B_0 = \frac{2}{k^4\tilde{M}_P^2}$.

The equation for p then takes the following form:

$$\left[1 + \frac{7}{18}(\epsilon_{SR} - \eta_{SR})\frac{H^4}{H_k^2 k^3 \tilde{M}_P^2}\right]p'' - \frac{(p')^2}{2p} + 2\left(k^2 - \frac{H^4}{kH_k^2\tilde{M}_P^2} - \frac{z''}{z}\right)p = \frac{1}{2p}\left(1 - \frac{H_k^2}{k^3\tilde{M}_P^2}\right) \tag{97}$$

and can be rewritten as

$$\left(1 + \frac{\delta_k}{\tilde{M}_P^2}\right)\tilde{p}'' - \frac{(\tilde{p}')^2}{2\tilde{p}} + 2\left(\tilde{k}^2 - \frac{z''}{z}\right)\tilde{p} = \frac{1}{2\tilde{p}} \tag{98}$$

with

$$\delta_k \equiv \frac{7}{18}(\epsilon_{SR} - \eta_{SR})\frac{H_k^2}{k^3}, \tag{99}$$

$$\tilde{k} \equiv k\sqrt{1 - \frac{H_k^2}{k^3\tilde{M}_P^2}}, \tag{100}$$

$$\tilde{p} \equiv \left(1 - \frac{H_k^2}{k^3\tilde{M}_P^2}\right)^{-1/2} p \tag{101}$$

where, on replacing $H \to H_k$, we neglected, to the leading order in SR, the time dependence of H. The fact that $p'' = [3(p')^2]/(2p)$ implies an ambiguity in the final form adopted by Eq. (97), since this identity can be used to rewrite contributions proportional to the second derivative of p as terms proportional to the square of p'. This ambiguity will be resolved by demanding that the result coincide with the one obtained by a different approach in Ref. [8]. The equation for $\tilde{p}$ is very similar to (69), except for the contribution proportional to $\delta_k/\tilde{M}_P^2$. If $\delta_k/\tilde{M}_P^2 \ll 1$, which is consistent with our perturbative approach, one finds the following long wavelength solution for $\tilde{p}$

$$\tilde{p} \simeq \tilde{C}_0 z^{2(1-\frac{\delta_k}{\tilde{M}_P^2})} \tag{102}$$

and consequently one has

$$p = \tilde{C}_0\sqrt{1 - \frac{H_k^2}{k^3\tilde{M}_P^2}}\, z^{2(1-\frac{\delta_k}{\tilde{M}_P^2})}. \tag{103}$$

The integration constant $\tilde{C}_0$ is fixed by connecting the long wavelength solution to $p = 1/2k$, when each mode $\tilde{k}$ crosses the horizon ($\tilde{k} = a_{\tilde{k}}H_{\tilde{k}}$). Finally one has

$$p = \frac{1}{2k}\left[\frac{a^2H_k^2}{k^2(1 - \frac{H_k^2}{k^3\tilde{M}_P^2})}\right]^{1-\frac{7}{18}(\epsilon_{SR}-\eta_{SR})\frac{H_k^2}{k^3\tilde{M}_P^2}} \tag{104}$$

in the long wavelength regime and, given the smallness of the quantum gravitational corrections ($H_k/M_P \ll 1$), one finally finds the expression

$$p \simeq C_0 a^2 \epsilon_{SR}\left[1 + \frac{H^2}{k^3\tilde{M}_P^2}\left(1 - \frac{7}{18}(\epsilon_{SR} - \eta_{SR})\ln\frac{a^2H^2}{k^2}\right)\right] \tag{105}$$

valid for the scalar sector. In the tensor sector one easily obtains the corrections in the limit $\eta_{SR} \to \epsilon_{SR}$. For such a case

$$p = \frac{a^2H^2}{2k^3(1 - \frac{H^2}{k^3\tilde{M}_P^2})} \tag{106}$$

and

$$p \simeq C_0 a^2 \epsilon_{SR}\left(1 + \frac{H^2}{k^3\tilde{M}_P^2}\right). \tag{107}$$

V. QUANTUM GRAVITATIONAL CORRECTIONS

The effect of Δ_p on the evolution of the two-point function p is that of adding to the standard, unperturbed, BD solution p_{BD} a contribution of order $\tilde{M}_P^{-2}$. When realistic inflationary models are considered, these modified spectra are derived from (105) and (107) by replacing $k^3 \to (k/\bar{k})^3$, where $\bar{k}$ is an unspecified reference wave number. The appearance of $\bar{k} = L^{-1}$ in the quantum corrections can be traced back to the three volume integral in the original action for the homogeneous inflaton-gravity system plus perturbations [see the action (5)]. Such a volume, on a spatially flat homogeneous space-time, is formally infinite and consequently the value of $\bar{k}$ remains undetermined. Naively one may argue that $\bar{k}$ is related to an infrared problem (divergence) and indeed, in the literature, its value is taken to be the infrared cutoff for the perturbations, namely the largest observable scale in the CMB. Alternatively one may consider it to be the scale at which new effects or physics set in. We shall briefly return to this in the conclusions.

In the previous section we calculated the form of the quantum gravitational modifications to the primordial scalar spectrum, in the case of SR inflation

$$Q_k = 1 + \frac{H^2\bar{k}^3}{\tilde{\mathrm{M}}_{\mathrm{P}}^2 k^3}\left(1 - \frac{7}{18}(\epsilon_{\mathrm{SR}} - \eta_{\mathrm{SR}})\ln\frac{a^2H^2}{k^2}\right). \quad (108)$$

In such an expression, the wave number k necessarily refers to the scales, around the pivot scale k_*, which are probed by the CMB and exited from the horizon $N_* \sim 60$ e-folds before inflation ends. Its contribution to (108) is

$$\left(\frac{k}{aH}\right)^{-2(\epsilon_{\mathrm{SR}}-\eta_{\mathrm{SR}})} \simeq \left(\frac{k_*}{a_* \mathrm{e}^{N*} H_{k_*}}\right)^{-2(\epsilon_{\mathrm{SR}}-\eta_{\mathrm{SR}})} \simeq \mathrm{e}^{2N_*(\epsilon_{\mathrm{SR}}-\eta_{\mathrm{SR}})} \quad (109)$$

and may well lead to a contribution of $\mathcal{O}(1)$ for reasonable values of the SR parameters of the order of 1 percent. Let us note that the first equality, in (109), is strictly valid for the modes very close to the pivot scale $k \sim k_* = a_* H_{k_*}$. Away from the pivot scale, small deviations proportional to the SR parameters, $-2(\epsilon_{\mathrm{SR}} - \eta_{\mathrm{SR}})\ln(\frac{k}{k_*})$, are neglected. Depending on the SR parameters and on N_*, the quantum corrections Q_k may lead to a power loss or a power increase for large scales which can be generically parametrized in the following form:

$$p^{(L)} \simeq p_0^{(L)}\left[1 \pm q\left(\frac{k_*}{k}\right)^3\right], \quad (110)$$

where $p_0^{(L)}$ is p without quantum corrections and evaluated in the long wavelength regime. The quantity inside the square brackets is Q_k. An analogous parametrization holds for the tensor sector with a different q.

A. Extrapolation beyond NLO

The parametrization of the primordial spectra by (110) is still not suitable for comparison with observations. In the $k \ll k_*$ limit the quantum gravitational corrections are either negative or very large (infinite in the $k \to 0$ limit). Such an apparently pathological behavior is simply a consequence of the perturbative technique employed to evaluate the corrections. One may hope that resummation to all orders leads to a finite result. In any case we are not allowed to extend the validity of the perturbative corrections up to $\mathcal{O}(1)$.

Thus, instead of introducing a sharp cutoff on the NLO expressions for the modified spectra by multiplying q by an *ad hoc* step function which keeps the correction small but leads to a discontinuous spectrum, we interpolate our expression through a well-defined function, with a finite and reasonable behavior in the $k \to 0$ limit. Such a function, which must reproduce (110) when $q(k_*/k)^3 \ll 1$, may be regarded as a resummation of the perturbative series.

In order to restrict the number of parameters which will be fitted by the comparison with the data and still allow for different limits when $k \to 0$, we consider the following parametrization:

$$p^{(L)} \simeq p_0^{(L)} \frac{1 + \tilde{q}_1(\frac{k_*}{k})^3}{1 + \tilde{q}_2(\frac{k_*}{k})^3} \sim p_0^{(L)}\left[1 + (\tilde{q}_1 - \tilde{q}_2)\left(\frac{k_*}{k}\right)^3\right], \quad (111)$$

where one more parameter with respect to (110) has been added, in order to obtain a regular expression for k small. We have also examined diverse parametrizations and have chosen the one that led to the best possible fits. Let us note that the above modifications are substantially different from considering a running spectral index α_s, such as

$$p^{(L)} \simeq p_0^{(L)}\left(\frac{k_*}{k}\right)^{-\frac{\alpha_s}{2}\ln(\frac{k}{k_*})}. \quad (112)$$

Indeed, for the latter case, the standard power-law dependence is affected at both large and small scales and, in particular, a negative running would lead to a zero amplitude in the $k \to 0$ limit and a smaller amplitude with respect to a simple power law when $k \gg k_*$. On the other hand, the modified spectrum (111) reduces to the power-law case when $k \gg k_*$ and may lead to a nonzero amplitude when $k \to 0$, depending on the choice of the parameters $\tilde{q}_{1,2}$.

VI. DATA ANALYSIS

In this section we report the comparison between the theoretical predictions given by (111) and the Planck 2015 [17] data set. The analysis is performed using the Markov chain Monte Carlo code COSMOMC [18], which has been properly modified to take into account the estimated quantum gravitational effects.

Let us note that BD vacuum in the tensor sector gives a power increase for large scales in the tensor spectrum. Such an increase would be counterbalanced by a loss of power in the scalar sector, as far as temperature correlations are concerned. One may parametrize such a power increase in a suitable way, just as we did for the scalar sector, in order to eliminate the divergence for small k and fit the corresponding parameter with the data at our disposal. Since our main source of data comes from temperature correlations, which do not discriminate between scalar and tensor fluctuations, we neglect *a priori* quantum gravitational corrections in the tensor spectrum. Such a choice is a simplifying assumption done in order not to have to disentangle possible degenerate parameters. Let us note, however, that such a choice can be realized physically either by an appropriate vacuum choice, differing from a pure BD, or by a very long cutoff scale associated with tensor dynamics. Thus we limit our analysis to a subset of the more general case, for which the quantum gravitational corrections affect the tensor sector in a non-negligible way, thus minimizing the power loss in the scalar sector. The tensor spectrum is then given by the unperturbed power-law expression

$$p_t = A_t \left(\frac{k}{k_*}\right)^{n_t} \tag{113}$$

and we assume that the LO spectra are generated by the conventional SR mechanism and single field inflation. The consistency condition, relating scalar and tensor spectral indices and the tensor to scalar ratio, is valid when quantum gravitational corrections are neglected. Indeed throughout the analysis we assume that the consistency relation (already implemented in COSMOMC) between the spectral indices and the tensor to scalar ratio

$$n_t = -\frac{r}{8}\left(2 - n_s - \frac{r}{8}\right) \tag{114}$$

holds to the second order in the SR approximation and the amplitude of the spectrum of tensor perturbations is given by $A_t = rA_s$, to the leading order in ${\rm M_P}^{-1}$, i.e. on neglecting the quantum gravitational corrections. We then consider a primordial scalar spectrum $p^{(L)}$ parametrized by

$$p_s \simeq p_0^{(L)} \frac{1 + (1 - 2q_2)(\frac{k_*}{e^{q_1} k})^3}{1 + (\frac{k_*}{e^{q_1} k})^3} \tag{115}$$

where k_*/e^{q_1} is the scale at which the loss in power begins to be relevant and $1 - 2q_2$ simply fixes the limit of p_s when $k \to 0$. In the limit $q_1 \to \infty$ the quantum gravitational corrections are suppressed. Let us note that $q_2 = 0$, or $q_1 \to \infty$, correspond to the standard power-law case with no loss of power ($p_s = p_0^{(L)}$) and $q_2 = 0.5$ corresponds to zero power at $k = 0$. The expression (115) is a parametrization equivalent to (111), with $\tilde{q}_1 = \exp(-3q_1)(1 - 2q_2)$ and $\tilde{q}_2 = \exp(-3q_1)$, which we have found to be more convenient to be used in COSMOMC.

Our analysis is based on the Planck data sets released in 2015 and includes the Planck TT data with polarization at low l (PL), and the data of the BICEP2/*Keck Array*-Planck joint analysis (BK) [19]. In particular we use *plik_dx11dr2_HM_v18_TT*, *lowTEB* and *BKPlanck* publicly available Planck likelihoods. We find the best fit for our model with and without BK data and compare it with standard power-law predictions, and with those assuming a non-negligible running of the spectral index (112).

TABLE I. Range of parameters varied.

τ	$\ln(10^{10}A_s)$	n_s	r	α_s	q_1	q_2
[0.01,0.8]	[2.7,4.0]	[0.9,1.1]	[0,0.8]	[−0.1,0.1]	[0,21]	[0,0.5]

For simplicity we obtained the best fits for the parameters of the primordial spectra shown in Table I and the parameters are taken to vary with uniform priors in the intervals indicated in the same table. The priors for τ, A_s, n_s, r and α_s are those used by the Planck 2015 analysis. The remaining cosmological parameters are fixed to the Planck best fit and in particular we chose

$$100\theta_{MC} = 1.040, \qquad \Omega_b h^2 = 0.0222, \qquad \Omega_c h^2 = 0.119. \tag{116}$$

Let us note that the pivot scale k_* is 0.05 ${\rm Mpc}^{-1}$ and is the same for both the scalar and the tensor sectors. The additional parameters q_1 and q_2 are chosen to vary in the largest possible interval leading to a power loss for large scales (compared with the pivot scale), with the parametrization chosen. At present our theoretical predictions are not able to constrain the value of such parameters, or estimate possible allowed intervals where to let them vary (see [20] for an attempt to estimate priors from quantum gravity), thus the choice of broad enough priors seems reasonable.

In particular the prior for q_2 is chosen to let it vary between $q_2 = 0$, where the quantum gravitational corrections cancel out independently of q_1, and $q_2 = 1/2$. The values for q_2 with $q_2 > 1/2$, lead to an increase of power; those with $q_2 < 0$, lead to a physically unacceptable negative spectrum and are thus excluded from the analysis.

On expanding (115) to the first order in the quantum gravitational corrections and comparing the result with the theoretical predictions (108), one finds, after some algebra, the following relation among the parameters of our model

$$\exp(3q_1) = \frac{24 q_2}{\pi^2 r \cdot A_s \cdot Q(n_s, r, N_*)} \left(\frac{k_*}{\bar{k}}\right)^3 \tag{117}$$

with

$$Q(n_s, r, N_*) \equiv \frac{7}{18}\left(1 - n_s - \frac{r}{8}\right) N_* - 1, \tag{118}$$

where we have used the following standard SR relations for single field inflation:

$$\frac{H_*^2}{{\rm M_P}^2} \simeq \frac{\pi^2}{2} A_s \cdot r \tag{119}$$

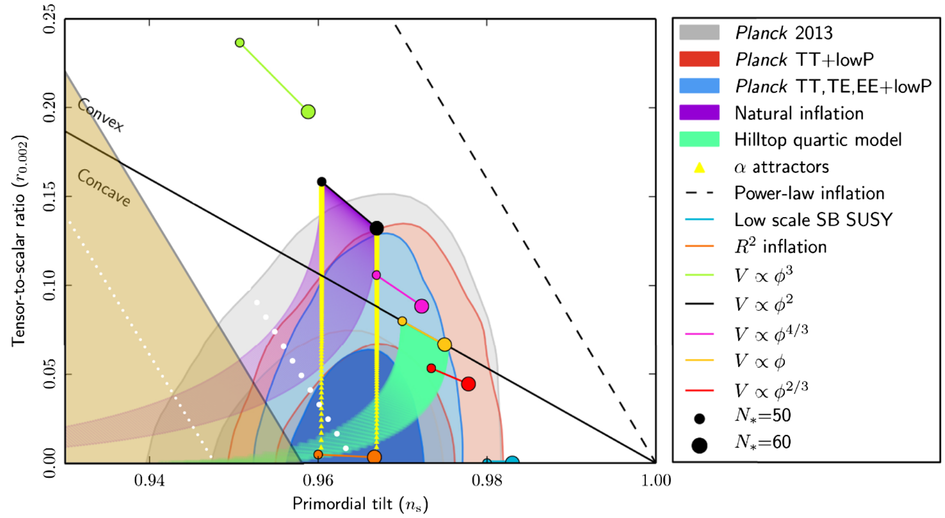

FIG. 1. The figure plots the region (yellow area) compatible with a loss of power in the (n_s, r) plane for $N_* = 60$. The dotted lines are the contours of the $N_* = 50$ (small dots) and $N_* = 70$ (large dots) areas. These contours are superimposed on the Planck 2015 analysis of various inflationary models. hilltop quartic models and natural inflation models lead to a loss in power; conversely chaotic inflation is not compatible with such a loss.

and

$$r = 16\epsilon_{\rm SR}, \qquad n_s = 1 + 2\eta_{\rm SR} - 4\epsilon_{\rm SR}. \qquad (120)$$

The parameter q_1 is related to the scale $\bar{k}$, i.e. that at which the quantum gravitational modifications of the spectrum become important, through the above relation (117). Let us first note that, with the priors considered for the quantities on the right-hand side of (117), such expression may vary from $-\infty$ to $+\infty$. The case of power loss, which we are investigating, is only reproduced by positive values of Q and, correspondingly, the right-hand side of (117) then varies in the interval $[0, +\infty]$ [let us note that the relation (117) is otherwise undefined]. Such a positivity requirement can be fulfilled only by particular inflationary models, as Fig. (1) shows, with larger values of N_* generically favored compared to smaller ones. For example consider the case of chaotic inflation, driven by a power-law potential $V \propto \phi^n$. For such a case

$$(n_s, r) = \left(1 - \frac{2(n+2)}{4N_* + n}, \frac{16n}{4N_* + n}\right) \overset{N_* \gg n}{\longrightarrow} \left(1 - \frac{n+2}{2N_*}, \frac{4n}{N_*}\right), \qquad (121)$$

$$Q \sim -\frac{11}{18} \qquad (122)$$

and (117) is undefined.

Conversely for the hilltop inflationary models, one has

$$(n_s, r) = \left(1 - \frac{2(n-1)}{N_*(n-2)}, \frac{1}{(N_*)^{2\frac{n-1}{n-2}}}\right), \qquad (123)$$

where n is defined by the shape of the potential

$$V_{\rm hilltop} = V_0\left[1 - \left(\frac{\phi}{\mu}\right)^n\right] \qquad (124)$$

and

$$Q \sim -\frac{11 - 2n}{18 - 9n}, \qquad (125)$$

leading to a loss in power for $2 < n < 11/2$.

Let us note that, with the form obtained for the quantum gravitational corrections, our model leads to severe constraints on the shape of the inflationary potential. As shown in Fig. 1, only a small subset of the inflationary models, satisfying the observed values of n_s and r, lead to a loss of power for large scales. The remaining models would give a power increase, which may be a distinguishing feature, unless $\bar{k}$ is too small to be observed in the CMB. More generally, on referring to the classification in [21], power loss is associated only with a subset of class I models, with $n_s \simeq 1 + 2\ln b/N_*$, $r \propto 1/N_*^{-2\ln b}$ and $0 < b < 0.277$ for $50 < N_* < 70$.

TABLE II. List of models.

Model number	Primordial spectra	Data sets	Parameters
1	Power law	PL	A_s, n_s, r
2	and tensors	PL + BK	
3	Running spectral index	PL	A_s, n_s, r, α_s
4	and tensors	PL + BK	
5	Quantum gravitational	PL	A_s, n_s, r, q_1, q_2
6	corrections and tensors	PL + BK	

TABLE III. Monte Carlo best fits.

Number	τ	$\ln(10^{10}A_s)$	n_s	r	α_s	q_1	q_2
1	7.7×10^{-2}	3.09	0.965	1.05×10^{-2}	...	...	...
2	8.3×10^{-2}	3.10	0.967	1.65×10^{-2}	...	...	...
3	7.8×10^{-2}	3.09	0.964	1.85×10^{-2}	-1.02×10^{-2}	...	...
4	8.9×10^{-2}	3.11	0.967	3.13×10^{-2}	-6.65×10^{-3}	...	...
5	8.0×10^{-2}	3.09	0.965	1.63×10^{-2}	...	3.48	1.3×10^{-1}
6	8.9×10^{-2}	3.12	0.966	4.7×10^{-2}	...	2.64	5.6×10^{-2}

Without specifying further assumptions on $k_*/\bar{k}$, which will be evaluated by inverting (117) once the parametrization of the primordial spectra is fixed by the data, the prior for q_1 is taken to be [0, 21]. The value $q_1 = 0$ corresponds to the assumption that the loss in power starts to be relevant below the pivot scale k_* (which is known to be the case given the outcome of previous independent analysis [10]). Conversely, the value $q_1 = 21$ corresponds to a scale $\sim 10^9$ smaller than the pivot scale. Such a choice is very conservative as it includes the entire CMB spectrum below the pivot scale.

The different combinations of primordial spectra and data sets considered are listed in Table II with an index specifying the model number. The best fits found, for the parameters we varied, are presented in Table III and the corresponding effective χ^2, defined as $-2\ln\mathcal{L}$ where $\mathcal{L}$ is the likelihood, are listed in Tables IV and V. The differences between the total χ^2 for the different cases are reported, using our model as reference. In particular the cases 1 and 3 are compared with 5 and the cases 2 and 4 are compared with 6.

A. Results

The MCMC results (see Tables IV and V, and Fig. 2) show that the quantum gravitational modification of the standard power-law form for the primordial scalar spectrum improves the fit to the data. Such improvements are much more significant with respect to the standard modifications of the primordial spectra obtained on considering a running spectral index. Let us note that the 2015 Planck data give constraints on the running, which are quite different from those coming from the 2013 data. In particular the fit to the 2015 data does not improve much if one considers a running spectral index in the scalar sector.

The comparison of the marginalized 1D likelihoods for the parameters q_1 and q_2 in Fig. (3) show that the two data sets lead to close predictions. In particular their marginalized maxima are

$$q_1 \simeq 3.4, \qquad 2q_2 \simeq 0.23, \tag{126}$$

when Planck data alone are considered and

$$q_1 \simeq 3.8, \qquad 2q_2 \simeq 0.20, \tag{127}$$

when BK data are added to the analysis. Correspondingly n_s and A_s also take very similar values for the best fit.

The value of q_2 indicates a ~20%–25% loss in power when k approaches zero. Let us note that the tensor to scalar ratio r is weakly constrained.

From Tables IV and V we observe that cases 3 and 4, with a running spectral index, are disfavored with respect to cases 1 and 2 respectively, since they have almost the same, effective, χ^2, but with one more independent d.o.f. to fit the

TABLE IV. Monte Carlo comparison (PL).

Number	$\chi^2_{\rm Tot}$	$\Delta\chi^2 \equiv \chi^2_\# - \chi^2_7$
1	11265.3	3.3
3	11265.1	3.1
5	11262.0	0

TABLE V. Monte Carlo comparison (PL + BK).

Number	$\chi^2_{\rm Tot}$	$\Delta\chi^2 \equiv \chi^2_\# - \chi^2_8$
2	11307.4	4.1
4	11307.3	4.0
6	11303.3	0

TABLE VI. Marginalized confidence intervals, case 5.

	68%	95%	99%
r	$[0.0, 6.2\times10^{-2}]$	$[0.0, 1.8\times10^{-1}]$	$[0.0, 3.2\times10^{-1}]$
q_1	$[1.7, 2.0\times10^{1}]$	$[2.5, 2.1\times10^{1}]$	$[2.1.2.1\times10^{1}]$
q_2	$[0.0, 2.8\times10^{-1}]$	$[0.0, 5.0\times10^{-1}]$	$[0.0, 5.0\times10^{-1}]$

TABLE VII. Marginalized confidence intervals, case 6.

	68%	95%	99%
r	$[0.0, 4.8\times10^{-2}]$	$[0.0, 8.4\times10^{-2}]$	$[0.0, 1.2\times10^{-1}]$
q_1	$[2.1, 1.6\times10^{1}]$	$[2.7, 2.1\times10^{1}]$	$[2.2, 2.1\times10^{1}]$
q_2	$[0.0, 5.0\times10^{-1}]$	$[0.0, 5.0\times10^{-1}]$	$[0.0, 5.0\times10^{-1}]$

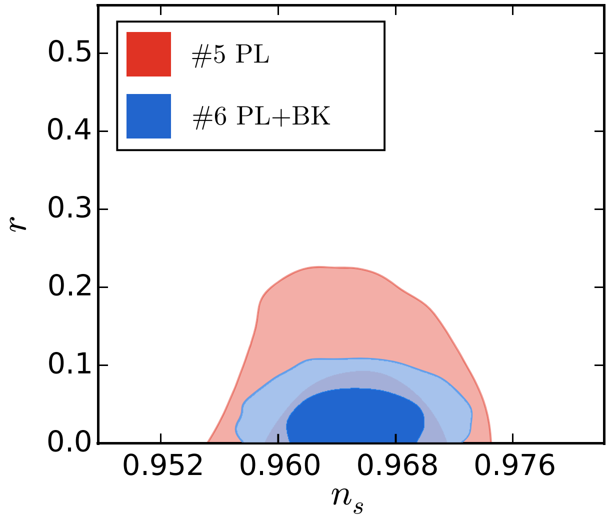

FIG. 2. The figure shows the 68% and 95% confidence level constraints on r and n_s.

data. Conversely the cases 5 and 6 ($\Delta\chi^2 > 2$) are favored with regard to the cases 1–4, as an improvement greater than 2 for the effective χ^2 is obtained, through the addition of 2 independent parameters.

In Fig. 3 we finally plot the marginalized likelihoods for r, q_1 and q_2. The corresponding marginalized 68%, 95% and 99% confidence intervals are listed in Tables VI and VII. The marginalized likelihoods for q_1 and q_2 show a 1σ deviation from standard power law for both cases 5 and 6. Let us note that, on comparing the results with those obtained from the Planck 2013 data, the constraints on q_1 and q_2 are now weaker [8].

Finally let us discuss the constraint on $\bar{k}$. On assuming, for example, Hilltop inflation (123), one can invert the relation (117) obtaining

$$\frac{\bar{k}}{k_*} \simeq \exp(-q_1)\left(\frac{24 q_2 N_*^{2\frac{n-1}{n-2}}}{\pi^2 A_s}\frac{9n-18}{11-2n}\right)^{1/3}. \quad (128)$$

Given that the amplitude A_s is quite constrained by observations and, on using $50 < N_* < 70$, $n = 4$, we obtain the corresponding values for $\bar{k}$, which are very large compared to the wave number associated with the largest observable scale in the CMB namely $k_{\min} \simeq 1.4 \times 10^{-4}\ \mathrm{Mpc}^{-1}$. These values are illustrated in Fig. 4 for cases 5 and 6 as functions of n [defined by (124)]. Let us note that the existence of such a (relatively) small fundamental length may have relevant consequences on astrophysical observations. Indeed, it is associated with distances which are comparable with the diameter of a large galaxy or a galaxy cluster. We further observe that a 3 order of magnitude variation of the value of $\bar{k}$ can be obtained on "retuning" the parameters used for its estimate. Such a variation is illustrated in the Fig. 4 where $\ln_{10}\bar{k}$ is plotted for the case of hilltop inflation on allowing n in (125) and N_* to vary. Let us note that the estimate for $\bar{k}$, although illustrated for a specific inflationary model, is quite general and can also be found for other diverse power loss compatible models.

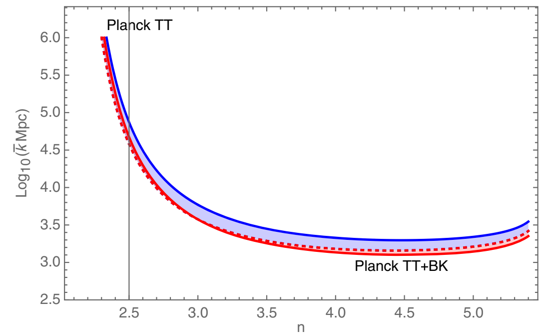

FIG. 4. Constraints on $\bar{k}$ for hilltop inflation as a function of n without (blue region) and with (red region) BK data. The region spans different $N_* \in [50, 70]$.

VII. CONCLUSIONS

As we mentioned in the Introduction the matter-gravity system is amenable to a Born-Oppenheimer treatment, wherein gravitation is associated with the heavy (slow) degrees of freedom and matter with the light (fast) degrees

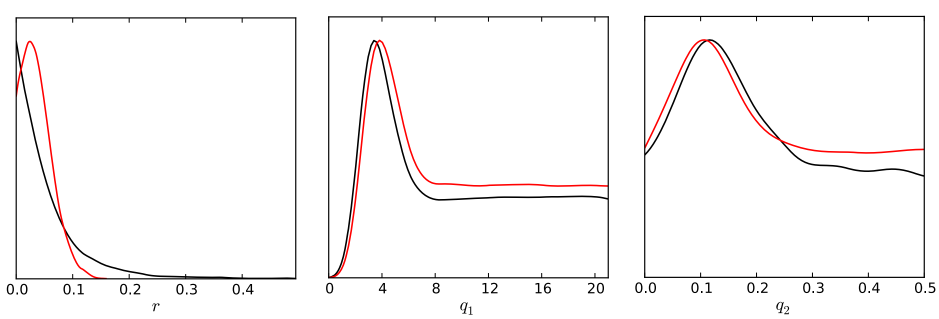

FIG. 3. Marginalized 1D likelihoods for r, q_1 and q_2 without (black line) and with (red line) BK data.

of freedom. Once the system is canonically quantized and the associated wave function suitably decomposed, one obtains that, on neglecting terms due to fluctuations (nonadiabatic effects), in the semiclassical limit gravitation is driven by the mean matter Hamiltonian and matter follows gravitation adiabatically, while evolving according to the usual Schwinger-Tomonaga (or Schrödinger) equation. Our scope in this paper has been to study perturbatively the effect of the nonadiabatic contributions, for different inflationary backgrounds. In particular we wished to see such effects on the observable features of the scalar/tensor fluctuations generated during inflation. In order to do this we obtained a master equation for the two-point function for such fluctuations, which includes the lowest order quantum gravitational corrections in an asymptotic expansion in inverse powers of the Planck mass. These corrections manifest themselves on the largest scales, since the associated perturbations are more affected by quantum gravitational effects, as they exit the horizon at the early stages of inflation and are exposed to high energy and curvature effects for a longer period of time. Interestingly, the very short wavelength part of the spectrum remains unaffected and one may consistently assume the BD vacuum as an initial condition for the evolution of the quantum fluctuations. Computationally this feature is relevant as it allows one to find the long wavelength part of the spectrum of the fluctuation through a matching procedure (similar to the standard case without quantum gravitational corrections).

In particular, one finds, for a de Sitter evolution, a power enhancement with respect to the standard results for the spectrum at large scales, with corrections behaving as k^{-3}. Such a k^{-3} was also found with similar approaches [22] and may appear to be a peculiarity of such quantum gravity models. However, the case of power-law inflation is different: while power enhancement is also true for power-law inflation, of interest for this case is that one finds that the k dependence of the quantum gravitational corrections differs from k^{-3} and is, perhaps not surprisingly, directly related to the k dependence of the unperturbed spectra.

Finally it is the slow roll case that is more realistic and of greatest interest. The quantum gravitational corrections for the SR case have peculiar features and are very different from the de Sitter case. In particular, for the case of the scalar fluctuations, their form is not simply a deformation of the de Sitter result proportional to the SR parameters. New contributions arise due to SR and their effect is comparable with the de Sitter-like contributions for very large wavelengths. The new contributions are proportional to $\epsilon_{\rm SR} - \eta_{\rm SR}$ and are zero for the de Sitter and power-law cases. They can lead to a power-loss term for low k in the spectrum of the scalar curvature perturbations at the end of inflation, providing the difference $\epsilon_{\rm SR} - \eta_{\rm SR} > 0$. The evolution of the primordial gravitational waves has also been addressed. The quantum gravitational corrections also affect the dynamics of tensor perturbations and determine a deviation from the standard results in the low multipole region, which always leads to a power enhancement. In performing the analysis, for simplicity, we restricted ourselves to the particular case of negligible quantum gravitational contributions to the spectrum of primordial gravitational waves. Further, since our corrections are perturbative,in order to keep them so for all values of k, we have suitably extrapolated our predictions for the scalar sector beyond the leading order, describing this in terms of two parameters, and examined them down to $k \to 0$. Other parametrizations have also been considered; however, the one we presented is the simplest and leads to the best results.

It is found that, given the form obtained for the quantum gravitational corrections, our model imposes severe constraints on the shape of the inflationary potential, as a loss in power at large scales is compatible with observations, whereas a power enhancement must be zero or extremely small to fit the data. Only a small subset of the inflationary models, satisfying the observed values for n_s and r, lead to a loss of power at large scales. The remaining models give a power increase which may be a distinguishable feature, unless $\bar{k}$ is too small to be observed in the CMB.

Finally the analysis performed was based on Planck data sets released in 2015 and include the Planck TT data with polarization at low l (PL) and the data of the BICEP2/*Keck Array*-Planck joint analysis (BK) [19]. In our preceding paper [8] our model predictions were tested through Planck 2013 and BICEP2 earlier data and the results were different. The MCMC results (see Tables IV, V) show that the quantum gravitational modification of the standard power-law form for the primordial scalar spectrum improves the fit to the data. Such improvements are much more significant with respect to the standard modifications of the primordial spectra, obtained by considering a running spectral index. Let us note that the 2015 Planck data give constraints on the running, which are quite different from those coming from 2013 data. In particular the fit to the 2015 data does not improve much if one considers a running spectral index in the scalar sector. On including the BK data in our analysis, we find that the results take very similar values for the best fit. Furthermore, comparison with the data predicts, for our model, a loss in power of about 20%–25% with respect to the standard power law as k approaches zero. and fixes the scale $\bar{k}$, which necessarily appears in the theoretical model. One finds values for $\bar{k}$ which are very large, compared to the wave number associated with the largest observable scale in the CMB (namely $k_{\rm min} \simeq 1.4 \times 10^{-4}$ Mpc). Let us note that the existence of such a small fundamental length may have relevant consequences on astrophysical observation. Indeed it is associated with distances which are comparable with the diameter of a large galaxy or a galaxy cluster. We further observe that a 3 order of magnitude variation of the value of $\bar{k}$ can be obtained on "retuning" the parameters used for its estimate. Further we observe that the value of $\bar{k}$, although illustrated for a specific inflationary model, is quite general and is found for diverse power loss compatible models. This is rather surprising and of course, assuming our proposed mechanism is correct, indicates the possible presence of new physics at such scales. Actually such a result is not new. Indications for this have been seen both from a study of the stability of clusters of galaxies or is associated with the running of Newton's constant [23].

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[1] A. A. Starobinsky, Lect. Notes Phys. **246**, 107 (1986); A. D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Chur, Switzerland, 1990).

[2] E. D. Stewart and D. H. Lyth, Phys. Lett. B **302**, 171 (1993).

[3] M. Born and J. R. Oppenheimer, Ann. Phys. (Berlin) **389**, 457 (1927); C. A. Mead and D. G. Truhlar, J. Chem. Phys. **70**, 2284 (1979); C. A. Mead, Chem. Phys. **49**, 23 (1980); **49**, 33 (1980).

[4] R. Brout and G. Venturi, Phys. Rev. D **39**, 2436 (1989); G. Venturi, Classical Quantum Gravity **7**, 1075 (1990); G. L. Alberghi, R. Casadio, and A. Tronconi, Phys. Rev. D **74**, 103501 (2006); G. L. Alberghi, C. Appignani, R. Casadio, F. Sbisa, and A. Tronconi, Phys. Rev. D **77**, 044002 (2008); A. Tronconi, G. P. Vacca, and G. Venturi, Phys. Rev. D **67**, 063517 (2003).

[5] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).

[6] J. Martin and R. H. Brandenberger, Phys. Rev. D **63**, 123501 (2001); A. A. Starobinsky, Pis'ma Zh. Eksp. Teor. Fiz. **73**, 415 (2001) [JETP Lett. **73**, 371 (2001)].

[7] U. H. Danielsson, Phys. Rev. D **66**, 023511 (2002); G. L. Alberghi, R. Casadio, and A. Tronconi, Phys. Lett. B **579**, 1 (2004); D. Martín de Blas and J. Olmedo, J. Cosmol. Astropart. Phys. 06 (2016) 029.

[8] A. Y. Kamenshchik, A. Tronconi, and G. Venturi, Phys. Lett. B **726**, 518 (2013); **734**, 72 (2014); J. Cosmol. Astropart. Phys. 04 (2015) 046.

[9] P. A. R. Ade *et al.* (Planck Collaboration), Astron. Astrophys. **571**, A22 (2014).

[10] C. R. Contaldi, M. Peloso, L. Kofman, and A. D. Linde, J. Cosmol. Astropart. Phys. 07 (2003) 002; M. Cicoli, S. Downes, B. Dutta, F. G. Pedro, and A. Westphal, J. Cosmol. Astropart. Phys. 12 (2014) 030; D. K. Hazra, A. Shafieloo, G. F. Smoot, and A. A. Starobinsky, J. Cosmol. Astropart. Phys. 06 (2014) 061; F. G. Pedro and A. Westphal, J. High Energy Phys. 04 (2014) 034; E. Dudas, N. Kitazawa, S. P. Patil, and A. Sagnotti, J. Cosmol. Astropart. Phys. 05 (2012) 012; M. Cicoli, S. Downes, and B. Dutta, J. Cosmol. Astropart. Phys. 12 (2013) 007; C. R. Contaldi, M. Peloso, and L. Sorbo, J. Cosmol. Astropart. Phys. 07 (2014) 014; A. Gruppuso, P. Natoli, F. Paci, F. Finelli, D. Molinari, A. De Rosa, and N. Mandolesi, J. Cosmol. Astropart. Phys. 07 (2013) 047; A. Gruppuso, Mon. Not. R. Astron. Soc. **437**, 2076 (2014); Y.-S. Piao, B. Feng, and X.-m. Zhang, Phys. Rev. D **69**, 103520 (2004); Z.-G. Liu, Z.-K. Guo, and Y.-S. Piao, Phys. Rev. D **88**, 063539 (2013); Z. G. Liu, Z. K. Guo, and Y. S. Piao, Eur. Phys. J. C **74**, 3006 (2014).

[11] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[12] V. F. Mukhanov, Sov. Phys. JETP **68**, 1297 (1988); J. M. Maldacena, J. High Energy Phys. 05 (2003) 013; V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992); V. F. Mukhanov, Phys. Lett. B **218**, 17 (1989); L. Castelló Gomar, M. Martín-Benito, and G. A. Mena Marugán, J. Cosmol. Astropart. Phys. 06 (2015) 045; L. Castelló Gomar, M. Fernández-Méndez, G. A. Mena Marugán, and J. Olmedo, Phys. Rev. D **90**, 064015 (2014).

[13] F. Finelli, G. P. Vacca, and G. Venturi, Phys. Rev. D **58**, 103514 (1998).

[14] C. Bertoni, F. Finelli, and G. Venturi, Classical Quantum Gravity **13**, 2375 (1996).

[15] T. S. Bunch and P. C. W. Davies, Proc. R. Soc. A **360**, 117 (1978).

[16] V. P. Ermakov, Univ. Izv. Kiev **9**, 1 (1880); E. Pinney, Proc. Am. Math. Soc. **1**, 681 (1950); H. R. Lewis and W. B. Riesenfeld, J. Math. Phys. (N.Y.) **10**, 1458 (1969); R. Goodall and P. G. Leach, J. Nonlinear Math. Phys. **12**, 15 (2005).

[17] P. A. R. Ade *et al.* (Planck Collaboration), Astron. Astrophys. **594**, A20 (2016); **594**, A13 (2016).

[18] A. Lewis and S. Bridle, Phys. Rev. D **66**, 103511 (2002).

[19] P. A. R. Ade *et al.* (BICEP2 and Planck Collaborations), Phys. Rev. Lett. **114**, 101301 (2015).

[20] T. Hertog, J. Cosmol. Astropart. Phys. 02 (2014) 043.

[21] D. Roest, J. Cosmol. Astropart. Phys. 01 (2014) 007.

[22] G. Calcagni, Ann. Phys. (Berlin) **525**, 323 (2013); C. Kiefer and M. Krämer, Phys. Rev. Lett. **108**, 021301 (2012); D. Bini, G. Esposito, C. Kiefer, M. Krämer, and F. Pessina, Phys. Rev. D **87**, 104008 (2013); G. L. Alberghi, R. Casadio, and A. Tronconi, Phys. Rev. D **74**, 103501 (2006); C. Kiefer and M. Krämer, Phys. Rev. Lett. **108**, 021301 (2012); D. Brizuela, C. Kiefer, and M. Krämer, Phys. Rev. D **93**, 104035 (2016).

[23] D. Reeb, Subnucl. Ser. **46**, 651 (2011); M. Reuter and H. Weyer, Phys. Rev. D **70**, 124028 (2004); S. Capozziello and M. De Laurentis, Ann. Phys. (Berlin) **524**, 545 (2012).