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Bouncing universe in the presence of an extended Chaplygin gas

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In this paper, we investigate the possibility of setting a model of a nonsingular universe in the context of the extended Chaplygin gas model with the equation of state $p = A\rho - \frac{B}{\rho^\alpha}$ through the framework of four-dimensional Friedmann-Robertson-Walker background. We find the following solutions of the singularity-free cosmological model: a cyclic universe with the minimal and maximal values of the scale factor that remains the same in every cycle, for an open universe with $k = -1$ and a negative cosmological constant; a nonsingular oscillating universe as a single bouncing solution for the cases of $k = 0$ and $k = 1$ curvature; and an oscillating universe with the minimal and maximal values of the scale factor that periodically rises up and down in the presence of a self-interacting scalar field model for all cases of the curved universe (with $k = -1$, $k = 0$ and $k = 1$). We also study whether a nonsingular bounce requires violation of the null energy condition.

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I. INTRODUCTION

The idea of an oscillating universe was initially introduced in the 1930s by Richard Tolman [1] as an alternative to standard big bang cosmology [2–4] to avoid the big bang singularity and replace it with a cyclical evolution. One of the primary motivations for such a study was to circumvent the need for initial conditions in cosmology. This was, however, shown to be extremely difficult to achieve within the context of general relativity (GR) without encountering singularities [5–9]. As shown by Hawking and Penrose [10,11] using their singularity theorem, GR predicts a spacetime singularity if a certain condition is satisfied. Once a singularity is formed, general relativity is no longer valid and it should be replaced by a more fundamental gravity theory [4,12].

Even in the framework of an inflationary scenario [13], which resolves several problems of standard big bang cosmology in the early Universe [14,15], the initial singularity cannot be avoided [16]. Since the inflation is realized by the dynamics of the scalar matter fields coupled to Einstein gravity [17], a new gravitational theory may be required to describe the beginning of the Universe [6]. Many researchers have attempted to resolve this singularity problem through the generalized/modified general relativity theory [18].

Superstring theory—which is one of the most promising candidates for a unified theory of fundamental interactions—may solve this problem; however, it is not yet complete and is not able to describe any realistic strong gravitational phenomena. Loop quantum gravity theory may resolve the problem of the big bang singularity via loop quantum cosmology (LQC) [5], which at present is the main background-independent and nonperturbative candidate for a quantum theory of gravity (for example see Refs. [19,20]).

Oscillating universes have been explored in several contexts in an attempt to solve some problems of the standard cosmological model (SCM). There have been many discussions on this topic, and a number of models have been proposed in the literature, including nonsingular models of universes in teleparallel theories [21], cyclic models in a braneworld scenario [22–24], a closed oscillating universe [25,26], a cyclic braneworld in LQC [27–29], and cyclic cosmologies with spinor matter [30] (see Refs. [31–39] for recent developments). Even though the bouncing models can solve many of the shortcomings of the SCM—since in principle the problems come from a "shortage of time" for things to happen early after the big bang [17,18,40]—obtaining a bouncing solution is very dependent on the choice of scenario and background. However, each model can be useful for extracting the characteristics of more general behavior [17,41]. Here we do not discuss all of the bouncing scenarios; for a more complete survey of older models see Refs. [18,42] for a recent review.

This paper is organized as follows. In Sec. II, we briefly introduce the bouncing framework by the action of the extended Chaplygin gas model. In Sec. III, we study the structure of the dynamical system via phase plane analysis to obtain a spectrum of cosmological perturbations. In Sec. IV, we discuss an oscillating universe in the presence of a scalar field and some realizations of cosmological bounces. In this situation, physics which goes beyond general relativity and/or standard matter theory (i.e., models of matter which obey the null energy condition) is required [17]. This is argued in general in Sec. V, where we discuss the possibility of conditions to obtain a bounce and their direct connection with the null energy condition (NEC) for all curved universes. Also, the signatures of this bouncing cosmology scenario are illustrated in the field plots of the pertinent

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systems, so that they show in some detail the bouncing trajectories and also oscillating universes near the critical points. We conclude in Sec. VI with a discussion of some key results for a bouncing universe in the presence of the extended Chaplygin gas model.

II. THE MODEL

In this section we consider the Friedmann-Robertson-Walker (FRW) cosmological model in the presence of an extended Chaplygin gas and cosmological constant, by assuming the equation of state

$$p_{\rm ch} = A\rho_{\rm ch} - \frac{B}{\rho_{\rm ch}^{\alpha}}, \quad \text{where } A, B \geq 0 \text{ and } 0 \leq \alpha \leq 1 \quad (1)$$

from the Friedmann equations

$$3\left(H^2 + \frac{k}{a^2}\right) = \rho_{\rm ch} + \Lambda, \quad (2)$$

$$\left(2\dot{H} + 3H^2 + \frac{k}{a^2}\right) = -p_{\rm ch} + \Lambda \quad (3)$$

and the conservation equation

$$\dot{\rho_{\rm ch}} + 3H(\rho_{\rm ch} + p_{\rm ch}) = 0 \quad (4)$$

By inserting Eq. (1) into Eq. (4), one can obtain

$$\rho_{\rm ch} = \left[\frac{B}{1+A} + \frac{C}{a^{3(1+A)(1+\alpha)}}\right]^{\frac{1}{1+\alpha}}, \quad (5)$$

where C is an arbitrary integration constant, and

$$\omega_{\rm ch} = \frac{P_{\rm ch}}{\rho_{\rm ch}} = A - \frac{B}{\rho_{\rm ch}^{\alpha+1}}. \quad (6)$$

III. PERTURBATION AND STABILITY ANALYSIS

The Jacobian stability of a dynamical system can be regarded as the robustness of the system to small perturbations of the whole trajectory. This is a very convenient way of regarding the resistance of limit cycles to small perturbation of trajectories. It is especially important in cosmology where there is the problem of initial conditions, as it gives us the possibility of studying all of the evolution paths admissible for all initial conditions [43–45]. Furthermore, phase planes are useful in visualizing the behavior of the system, particularly in oscillatory systems where the phase paths can "spiral in" towards zero, "spiral out" towards infinity, or reach neutrally stable situations called centers.

In this section, therefore, we study the structure of the dynamical system via phase plane analysis, by introducing the following new variables:

$$\chi = H, \qquad \zeta = a, \qquad \eta^2 = \rho_{\rm ch}. \quad (7)$$

From Eqs. (2) and (3), the evolution equations of these variables become

$$\dot{\chi} = \frac{k}{\zeta^2} - \frac{(1+A)\eta^2}{2} + \frac{B}{2\eta^{2\alpha}}, \quad (8)$$

$$\dot{\zeta} = \zeta\chi, \quad (9)$$

$$2\eta\dot{\eta} = -3\chi\left(\frac{(1+A)\eta^2}{2} - \frac{B}{2\eta^{2\alpha}}\right), \quad (10)$$

where a dot denotes a derivative with respect to the cosmic time. The Friedmann equation (2) in terms of the new variables would be

$$\eta^2 = 3\chi^2 + \frac{3k}{\zeta^2} - \Lambda. \quad (11)$$

Therefore, Eqs. (8)–(10) reduce to the following relations:

$$\dot{\chi} = \frac{k}{\zeta^2}\frac{(-1-3A)}{2} - 3\frac{(1+A)\chi^2}{2} + \frac{\Lambda(A+1)}{2} + \frac{B}{(3\chi^2 + \frac{3k}{\zeta^2} - \Lambda)^{\alpha}}, \quad (12)$$

$$\dot{\zeta} = \zeta\chi. \quad (13)$$

By solving them, one can obtain the critical points (χ_c, ζ_c):

$$\chi_c = 0, \quad (14)$$

and ζ_c is the root of

$$(3k\zeta_c^{-2} - \Lambda)^{\alpha}(\zeta_c^2\Lambda(A+1) + k(1+3A)) - 2B\zeta_c^2 = 0. \quad (15)$$

It can be seen that the number of critical points in the model depends on the value of α. It is more convenient to investigate the properties of the dynamical system (12)–(13) than Eqs. (8)–(10). Now, we can obtain the critical points (or fixed points) and study their stability. The critical points are always exact constant solutions in the context of the autonomous dynamical systems, and are often the extreme points of the orbits and therefore describe the asymptotic behavior of the systems. In the following, we find the fixed points by simultaneously solving $\chi' = 0$ and $\zeta' = 0$.

By means of the Jacobian, one can linearly approximate the nonlinear systems near the hyperbolic fixed points, such that a linear stability analysis holds (see the Appendix).

In the modified Chaplygin gas model the Jacobian is (note that $\chi_c = 0$)

$$\text{Jacobian} = \begin{pmatrix} -3(1+A)\chi_c - \frac{3\beta\chi_c}{(3\chi_c^2 + 3\frac{k}{\zeta_c^2} - \Lambda)^2}, & \frac{k(1+3A)}{\zeta_c^3} + \frac{3\beta k}{(3\chi_c^2 + 3\frac{k}{\zeta_c^2} - \Lambda)^2 \zeta_c^3} \\ \zeta_c & \chi_c \end{pmatrix}.$$

Hence, there are two eigenvalues for any critical point:

$$\lambda_1 = \frac{1}{\zeta_c}\sqrt{k\left((1+3A) + \frac{3B}{(-\Lambda + \frac{3k}{\zeta_c^2})^2}\right)},$$

$$\lambda_2 = -\frac{1}{\zeta_c}\sqrt{k\left((1+3A) + \frac{3B}{(-\Lambda + \frac{3k}{\zeta_c^2})^2}\right)}. \quad (16)$$

For $k = -1$ and positive values of A and B, they are complex with zero real part, and the critical points are centers,

$$\lambda_1 = \frac{i}{\zeta_c}\sqrt{\left((1+3A) + \frac{3B}{(-\Lambda + \frac{3k}{\zeta_c^2})^2}\right)},$$

$$\lambda_2 = \frac{-i}{\zeta_c}\sqrt{\left((1+3A) + \frac{3B}{(-\Lambda + \frac{3k}{\zeta_c^2})^2}\right)}.$$

In the case of centers (nonhyperbolic critical points), curves are closed trajectories around the center in the phase space and eigenvalues are purely imaginary and conjugated. The real parts of the eigenvalues are zero for oscillating solutions, as the corresponding critical points are centers and marginally stable. In our model, depending on α, k, and Λ, the number of critical points and their properties may change. In the following, we focus on models that represent the oscillating evolutions, in the two special cases of $\alpha = 1$ and $\alpha = 1/2$.

A. Case 1: $k = -1$, $\alpha = 1/2$

In this case, we have two critical points in the phase space: both are centers and can be written as

$$\chi_c^+ = 0, \quad \zeta_c^+ = \frac{1}{3}\left\{\frac{-84\Lambda^2 + 6D^{\frac{1}{3}}}{4\Lambda^3 + B^2} + \frac{24\Lambda(\Lambda^3 - 12B^2)}{(4\Lambda^3 + B^2)D^{\frac{1}{3}}}\right\}^{\frac{1}{2}},$$

$$\chi_c^- = 0, \quad \zeta_c^- = -\frac{1}{3}\left\{\frac{-84\Lambda^2 + 6D^{\frac{1}{3}}}{4\Lambda^3 + B^2} + \frac{24\Lambda(\Lambda^3 - 12B^2)}{(4\Lambda^3 + B^2)D^{\frac{1}{3}}}\right\}^{\frac{1}{2}},$$

where

$$D = -8\Lambda^6 + 360\Lambda^3 B^2 - 81B^4 + (12\Lambda^3\sqrt{3}B + 3\sqrt{3}B^3)\sqrt{243B^2 - 8\Lambda^3}.$$

In Fig. 1, the region III is not accepted because the scale factor becomes negative. In Fig. 2, we have magnified region I of Fig. 1 which shows the oscillating scale factor in the configuration space. In Fig. 3, the Hubble parameter H and energy density of the Chaplygin gas $\rho_{\rm ch}$ are shown for this region. The forbidden region (a region is forbidden because it implies a negative energy) which is when $a^2 < \frac{3}{3H^2-\Lambda}$ is performed—shown in Figs. 1 and/or 2 implies that the system does not have any real critical point.

From Figs. 1 and/or 2, we observe the trajectories that have a closed orbit in the phase space and give an oscillating solution. One can see that none of the trajectories in the phase space wind around the center critical points. More specifically, the red trajectory hits the forbidden region and does not give an oscillating solution.

B. Case 2: $k = -1$, $\alpha = 1$

In this case we have four critical points in the phase space as follows:

$$\zeta_{1c} = \left(\frac{2\Lambda + 3A\Lambda - \sqrt{\Lambda^2 + 3B + 9AB}}{\Lambda^2 + A\Lambda^2 - B}\right)^{\frac{1}{2}},$$

$$\zeta_{3c} = -\left(\frac{2\Lambda + 3A\Lambda - \sqrt{\Lambda^2 + 3B + 9AB}}{\Lambda^2 + A\Lambda^2 - B}\right)^{\frac{1}{2}},$$

$$\zeta_{2c} = \left(\frac{-2\Lambda - 3A\Lambda - \sqrt{\Lambda^2 + 3B + 9AB}}{\Lambda^2 + A\Lambda^2 - B}\right)^{\frac{1}{2}},$$

$$\zeta_{4c} = -\left(\frac{-2\Lambda - 3A\Lambda - \sqrt{\Lambda^2 + 3B + 9AB}}{\Lambda^2 + A\Lambda^2 - B}\right)^{\frac{1}{2}},$$

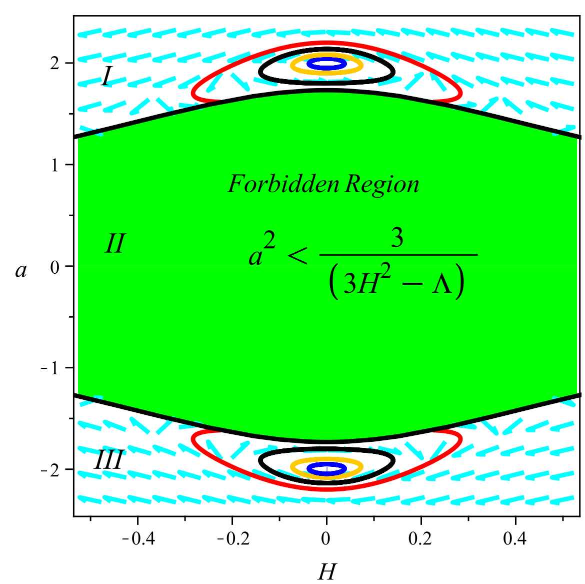

FIG. 1. Phase plane of the system around the critical points (χ_c^+, ζ_c^+) and (χ_c^-, ζ_c^-) for $k = -1$, $\alpha = 1/2$, $A = 1$, $B = 0.5$, and $\Lambda = -1$.

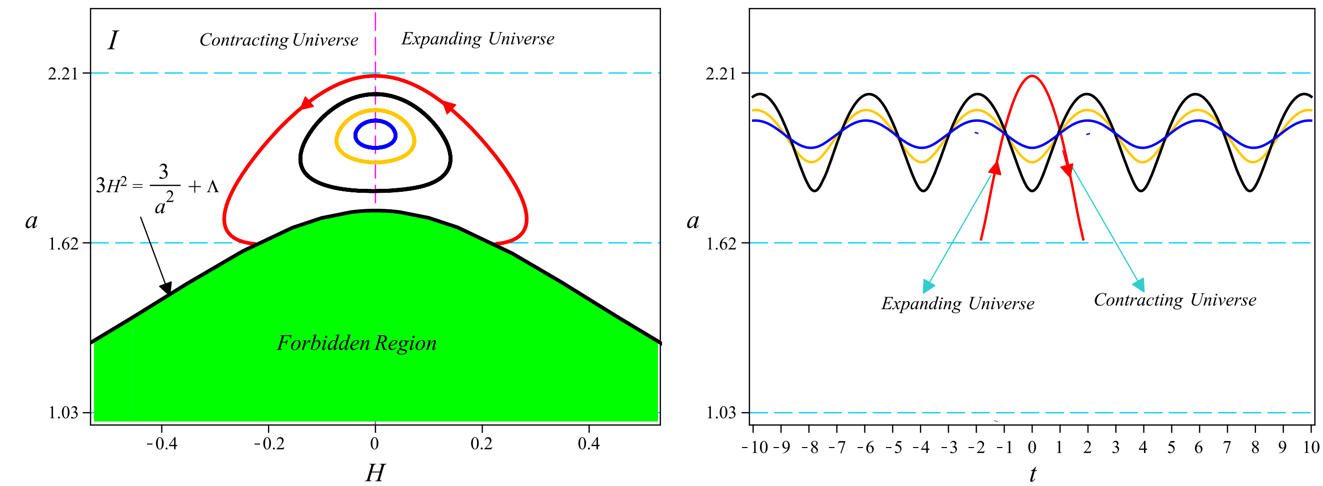

FIG. 2. (Left) Dynamical behavior of the system around the critical point (χ_c^+, ζ_c^+). (Right) Time evolution of the scale factor a corresponding to the trajectories in phase space for the case of $k = -1$, $\alpha = 1/2$, $A = 1$, $B = 0.5$, and $\Lambda = -1$.

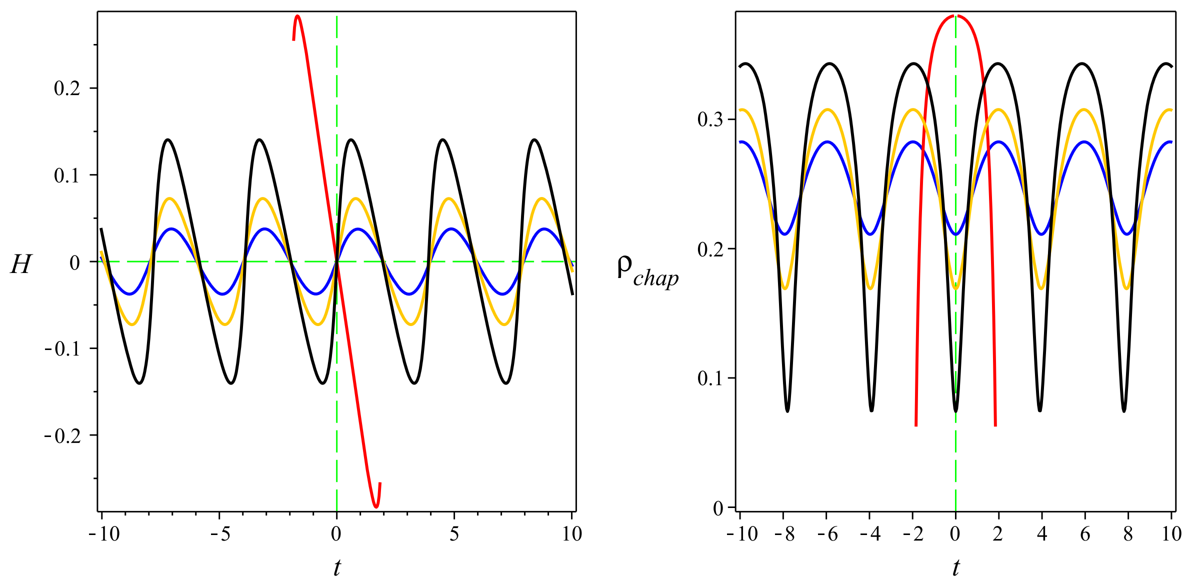

FIG. 3. Time evolution of the Hubble parameter H and energy density of the Chaplygin gas ρ_{chap}, corresponding to the trajectories of the phase space in region I of Fig. 1.

which, again, are all centers. In Fig. 4, regions III and IV are not accepted since the scale factor becomes negative. Region I of Fig. 4 has been magnified in Fig. 5, and we have also shown the oscillating scale factor in configuration space. The Hubble parameter H and energy density of the Chaplygin gas ρ_{chap} are shown for this region in Fig. 6. Figures 7 and 8 illustrate the phase space and scale factor a, Hubble parameter H, and energy density of the Chaplygin gas ρ for region II, as $\rho_{\text{chap}} < 0$ may not be favored for this region.

C. Case 3: $k = 0$, $k = 1$

In both the $k = 0$ and $k = 1$ cases the eigenvalues are real,

$$\lambda_1 = \frac{k}{\zeta_c}\sqrt{\left((1+3A) + \frac{3B}{(-\Lambda + \frac{3}{\zeta_c^2})^2}\right)},$$
$$\lambda_2 = -\frac{k}{\zeta_c}\sqrt{\left((1+3A) + \frac{3B}{(-\Lambda + \frac{3}{\zeta_c^2})^2}\right)}, \tag{17}$$

and thus the universe does not oscillate. However, a single bounce without oscillation can occur under the appropriate conditions. Considering that the minimal conditions require a bounce with $\dot{a}_b = 0$, $\ddot{a}_b \geq 0$ (where the subindex b denotes the quantities that are evaluated at the bounce), it can also follow equivalently at the bounce, as $\chi_b = 0$ and $\frac{d\chi_b}{dt} > 0$ in terms of new variables. Applying this condition to the right-hand side of Eq. (12) yields

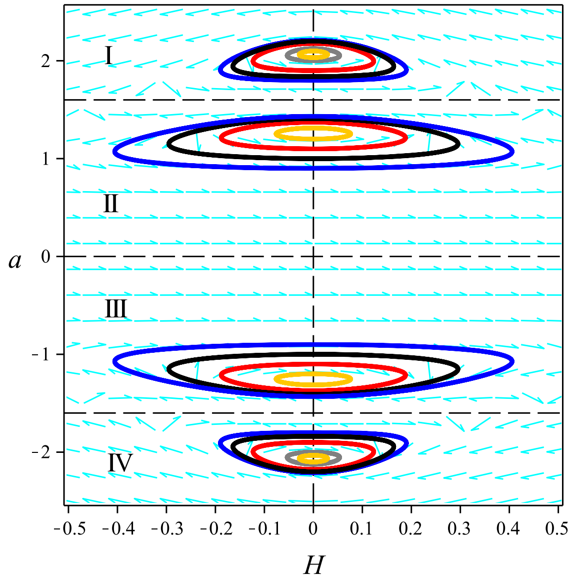

FIG. 4. Dynamical behavior of the system around the critical points (χ_1, ζ_1), (χ_2, ζ_2), (χ_3, ζ_3), and (χ_4, ζ_4) for $k = -1$, $\alpha = 1$, $A = 0.8$, $B = 0.3$, and $\Lambda = -1$.

$$B > \frac{1}{a_b^2}\left[(k - \Lambda a_b^2 + 3Ak - A\Lambda a_b^2)\left(\frac{3k}{a_b^2} - \Lambda\right)^{\alpha}\right]. \quad (18)$$

Accordingly, by an appropriate choice of operation parameters based on the condition (18), it is possible to obtain a bounce solution. Also, by assuming $8\pi G = 1$, from Eq. (2) we can obtain the energy density at the bounce as

$$\rho_b = \left(\frac{3k}{a_b^2} - \Lambda\right). \quad (19)$$

In order to avoid a negative energy density at bounce, the condition $(\rho_b = (\frac{3k}{a_b^2} - \Lambda) > 0)$ must be satisfied. This implies that the bounce in a flat universe requires a negative cosmological constant. Considering the bounce condition (18), we have plotted the phase plane diagram for the $k = 0$ and $k = 1$ cases in Fig. 9. It is interesting to note that it gives us the opportunity to study all of the evolution paths admissible for all initial conditions, which is an appealing feature of phase plane analysis. Since each line in the phase plane corresponds to a certain initial condition (here, $H = 0$ and $a = a_b$), it is possible to obtain different bounce solutions by taking different initial conditions.

IV. AN OSCILLATING UNIVERSE IN THE PRESENCE OF A SCALAR FIELD

It is possible to generate bouncing models in a wide choice of scenarios, essentially by any of the mechanisms presented in Ref. [18]. Obviously, the outcome is strongly dependent on the choice, but specific models can sometimes be useful for extracting characteristics of a more general behavior. In this sense, scalar, vector, and tensor perturbations have been studied in many exact backgrounds displaying a bounce [46–50]. The role of scalar fields in cosmology has been examined in Ref. [51]. Nonsingular solutions for a scalar field in the presence of a potential were also studied in Ref. [52]. Here, we consider a model of the universe filled with an extended Chaplygin gas in the presence of a scalar field ϕ and a self-interacting potential $V(\phi)$ with the effective Lagrangian $L_\phi = \frac{1}{2}\dot{\phi}^2 - V$. In this case, the Friedmann equations are

$$3H^2 + \frac{3k}{a^2} = \rho_{\rm ch} + \frac{1}{2}\dot{\phi}^2 + V, \quad (20)$$

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -p_{\rm ch} - \frac{1}{2}\dot{\phi}^2 + V. \quad (21)$$

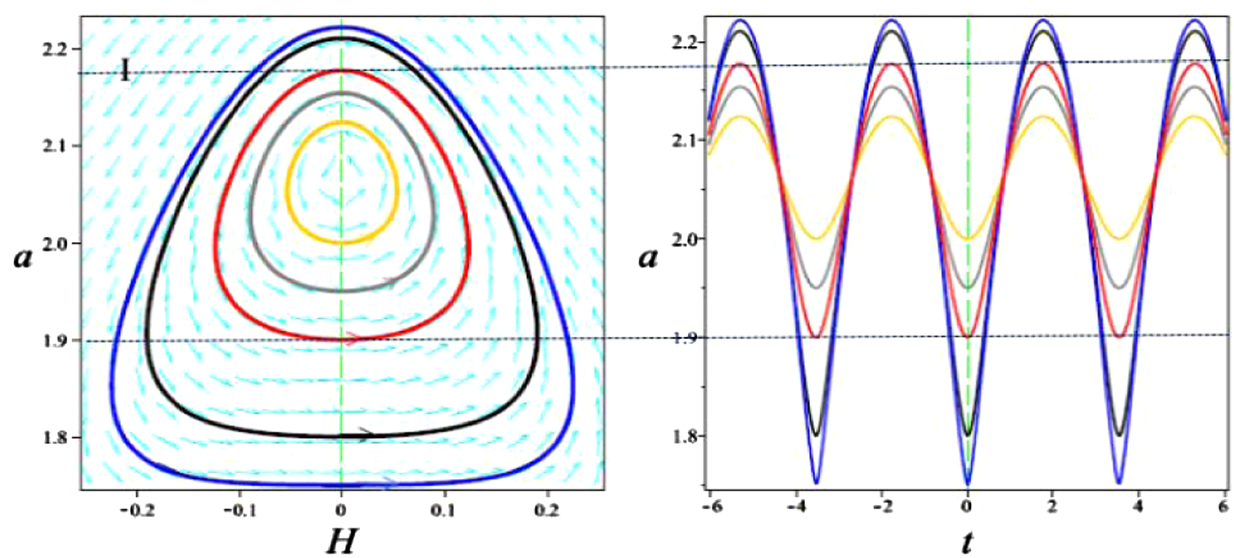

FIG. 5. Dynamical behavior of the system around the critical point (χ_1, ζ_1) for $k = -1$, $\alpha = 1$, and $\Lambda = -1$.

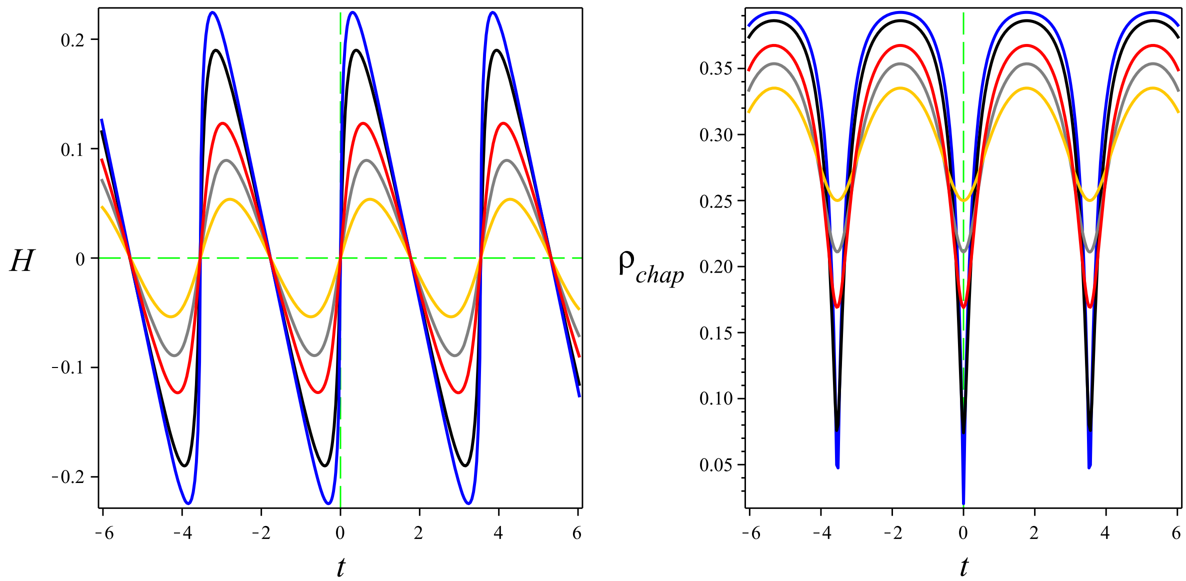

FIG. 6. Time evolution of the Hubble parameter H and energy density of the Chaplygin gas $\rho_{\rm chap}$, corresponding to the trajectories of the phase space in Fig. 5 for $k=-1$, $\alpha=1$, and $\Lambda=-1$.

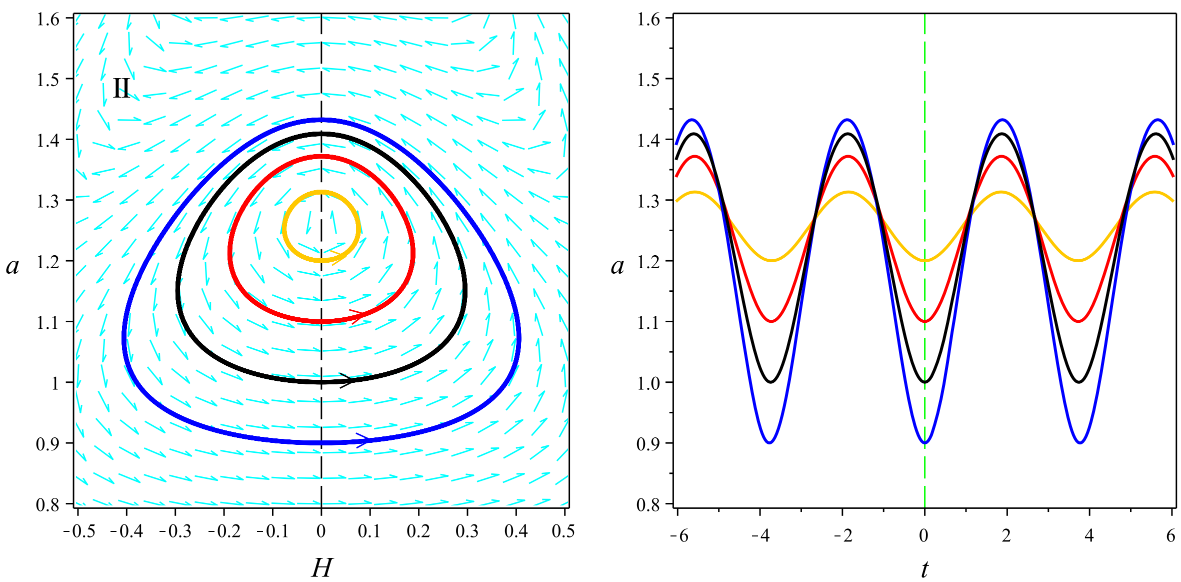

FIG. 7. Dynamical behavior of the system around the critical point (χ_2, ζ_2) for $k=-1$, $\alpha=1$, and $\Lambda=-1$.

The conservation equation is

$$\dot{\rho_{\rm ch}} + 3H(\rho_{\rm ch} + p_{\rm ch}) = 0. \tag{22}$$

The field equation for scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{23}$$

For simplicity, we use the following new variables:

$$x = H, \qquad y = \rho_{\rm ch}, \qquad \dot{\phi} = g(t), \qquad z = V. \tag{24}$$

Now, from the equations of motion (20)–(23), we obtain

$$\frac{dx}{dt} = -\frac{f(y)}{2} - \frac{1}{2}g(t)^2 + \frac{k}{a^2}, \tag{25}$$

$$\frac{dy}{dt} = -3xf(y), \tag{26}$$

$$\frac{da}{dt} = ax, \tag{27}$$

$$\frac{dz}{dt} = \frac{\dot{g}}{-3xg(t) - g(t)}, \tag{28}$$

where $f(y) = (\rho_{\rm ch} + p_{\rm ch})$ is given by $f(y) = (1+A)y - \frac{B}{y^{\alpha}}$.

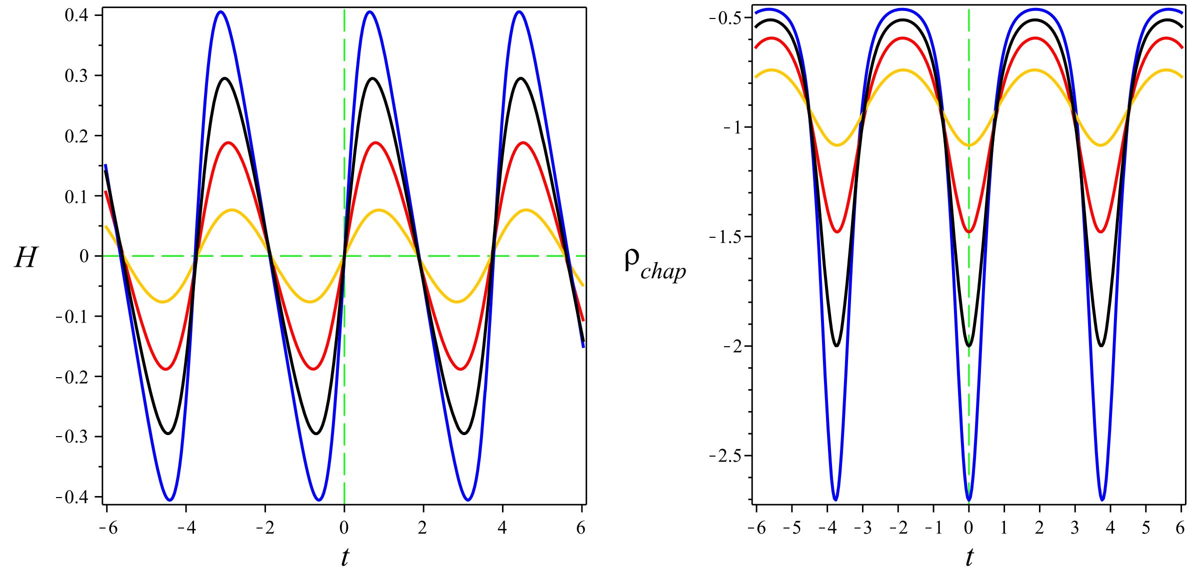

FIG. 8. Time evolution of the Hubble parameter H and energy density of the Chaplygin gas ρ_{chap}, corresponding to the trajectories of the phase space in Fig. 7 for $k = -1$, $\alpha = 1$, and $\Lambda = -1$.

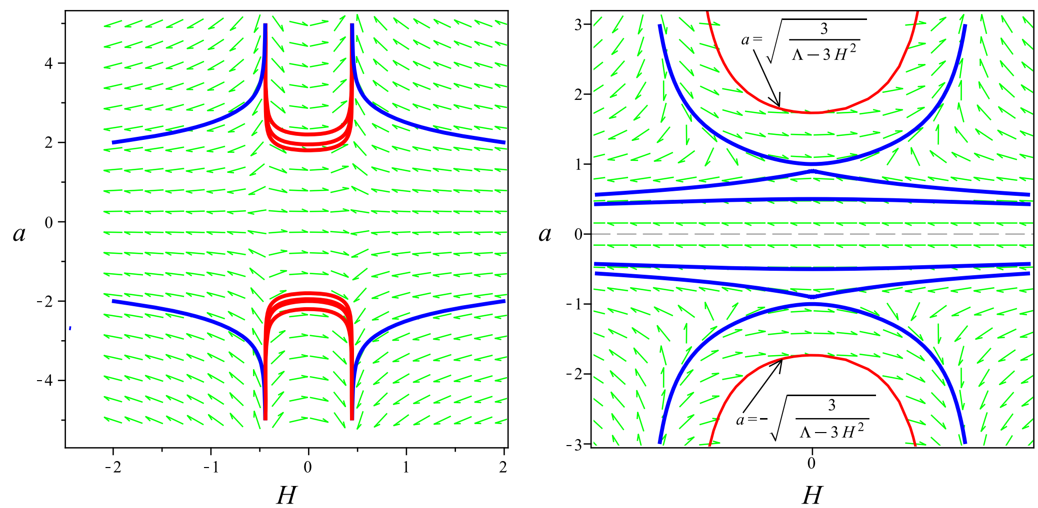

FIG. 9. Phase plane analysis for the (left) $k = 0$ and (right) $k = 1$ cases. For $k = 0$, we have chosen the parameters $A = 1$, $\alpha = \frac{1}{2}$, $B = 4$, and $\Lambda = -1$; however, for $k = 1$ we have choosen $A = .1$, $\alpha = \frac{1}{2}$, $B = .8$, and $\Lambda = 1$. The plots show that there is a single bouncing trajectory which cannot undergo repeated expansions and contractions.

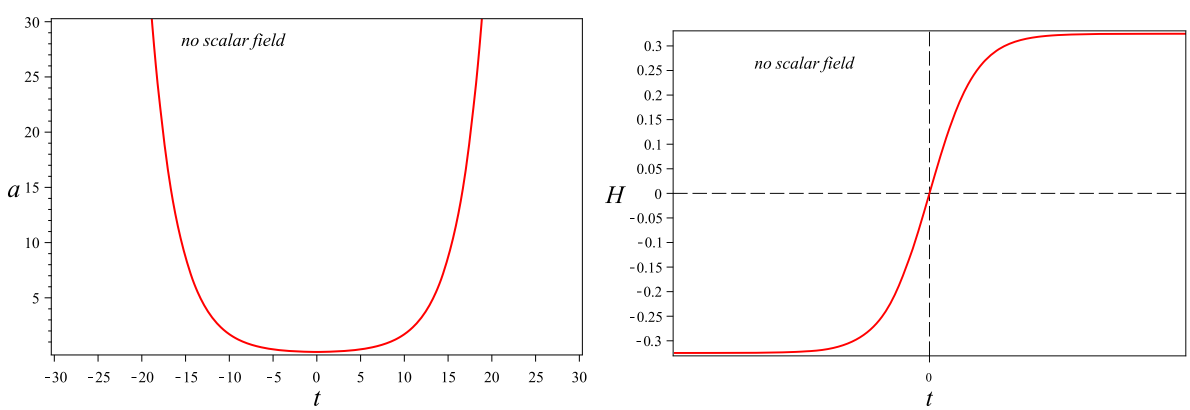

FIG. 10. Dynamical behavior of the scale factor and Hubble parameter for $k = 0$, $g(t) = 0$, $A = 0$, $B = 0.8$, and $\alpha = 0.8$.

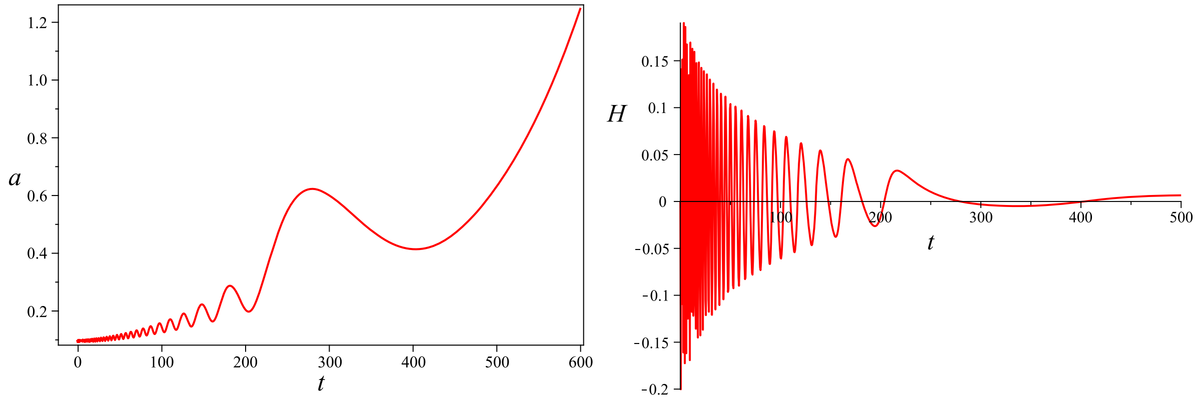

FIG. 11. Dynamical behavior of the scale factor and Hubble parameter for $k = 0$, $g(t) = e^{-\lambda t}$, $\lambda = .03$, $A = 0$, $B = 0.8$, $\alpha = 0.8$.

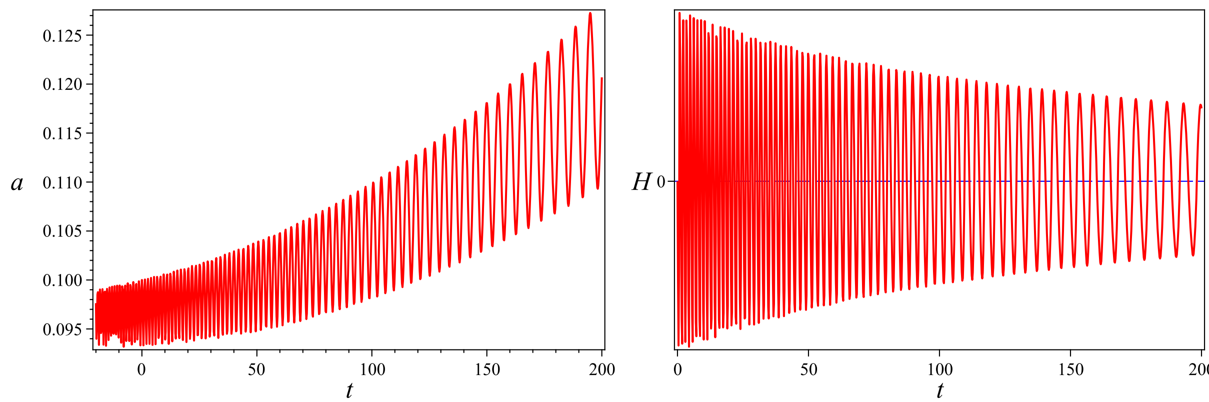

FIG. 12. Dynamical behavior of the scale factor and Hubble parameter for $k = 0$, $g(t) = e^{-\lambda t}$, $\lambda = .015$, $A = 0$, $B = 0.8$, and $\alpha = 0.8$.

For $g(t) = 0$, the scale factor has a single bounce: the universe does not undergo repeated cycles (see Fig. 10). Note that this case represents the universe with a cosmological constant in the absence of scalar field.

Assuming $g(t) = e^{-\lambda t}$ (where λ is a constant which can be called the damping coefficient), Eqs. (25)–(27) give an oscillating universe with minimal and maximal values of the scale factor increasing cycle by cycle. By recalling that

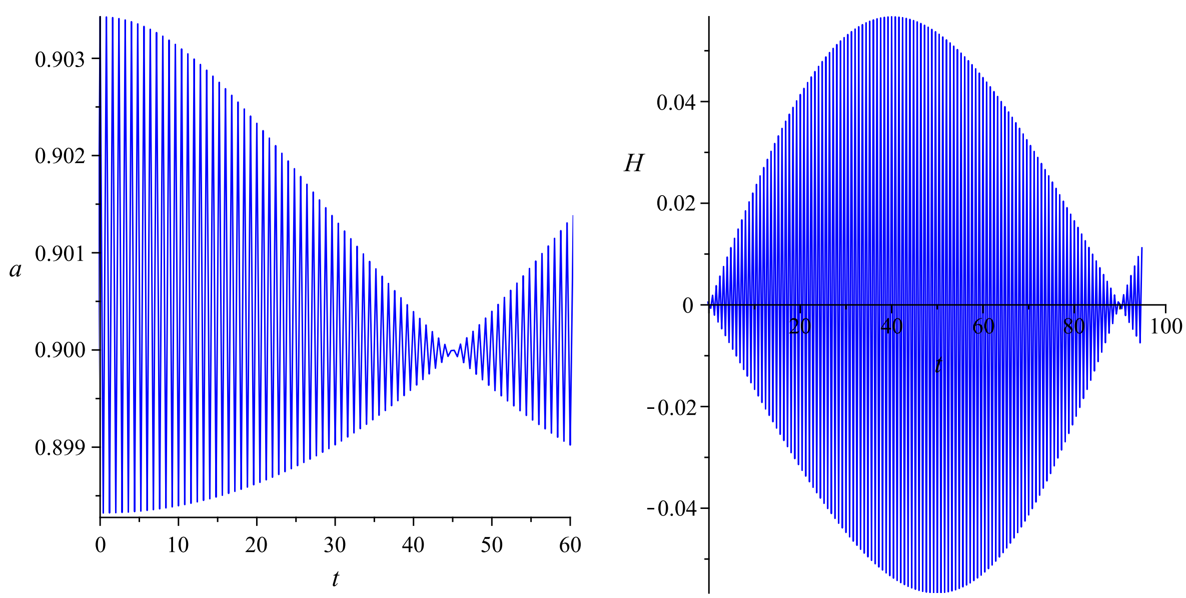

FIG. 13. Dynamical behavior of the scale factor and Hubble parameter for $k = 0$, $g(t) = e^{-\lambda t}$, $\lambda \to 0$, $A = 1$, $B = 0.8$, and $\alpha = 0.8$.

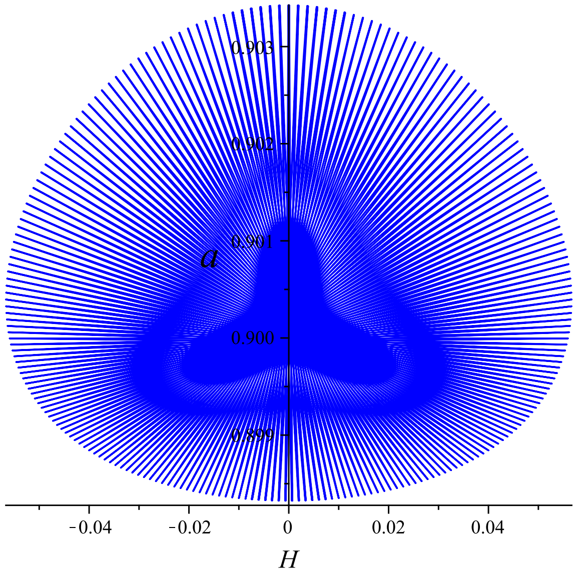

FIG. 14. Phase plane of $(a-H)$ for $g(t)=e^{-\lambda t}$, $k=0$, $\lambda\to 0$, $A=1$, $B=0.8$, and $\alpha=0.8$.

the minimal conditions require a bounce with ($x_b=0$, $(\frac{dx}{dt}|_{t_b}=\frac{\ddot{a}_b}{a_b}-H_b^2)>0$) and applying it in Eq. (25), we can obtain a bounce condition in terms of the parameters and initial conditions of the model as

$$B>\left[\frac{-2k}{a_b^2}+g_b^2+(A+1)y_b\right](y_b)^{\alpha}, \tag{29}$$

where $g_b=e^{-\lambda t_b}$. Figures 11 and 12 show the oscillating behavior of the scale factor for different initial conditions and different constant parameters. It is also interesting that the Hubble parameter H oscillates and its amplitude decays exponentially with time and then holds a steady-state value at late times (as in a damped harmonic oscillator). It is worth mentioning that a larger value for λ leads to a faster decay of oscillations. In the limit $\lambda\to\infty$, the system does not oscillate at all. Also, for $\lambda\to 0$ the system will continue oscillating forever (see Figs. 13 and 14). The cyclic behavior of the universe can also be obtained for the cases of $k=1$ and $k=-1$; see Figs. 15 and 16, where we illustrate the dynamical behavior of the scale factor and phase plane of $(a-H)$.

It is interesting to note that $g(t)=e^{-\lambda t}$ with a positive value of λ gives an oscillating universe with minimal and maximal values of the scale factor increasing cycle by cycle, while $g(t)=e^{-\lambda t}$ with a negative value of λ [identical to $g(t)=e^{\lambda t}$ with a positive value of λ] gives an oscillating universe with minimal and maximal values of the scale factor decreasing cycle by cycle (as observed in Fig. 17).

V. ENERGY CONDITIONS

Up to here, the possibility of conditions that lead to bouncing solutions surrounding the critical points was discussed regardless of whether we had considered all of the energy conditions or restrictions on the matter energy-momentum tensor $T_{\mu\nu}$, which play an important role in general relativity. In this section we are going to demonstrate a direct connection between a bounce and the NEC in the presence of an extended Chaplygin gas model.

In this discussion, some quantities of interest are

$$\rho_{\rm ch}+p_{\rm ch}=\frac{1}{4\pi G}\left[-\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right], \tag{30}$$

$$\rho_{\rm ch}-p_{\rm ch}=\frac{1}{4\pi G}\left[\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}+2\frac{k}{a^2}-\Lambda\right], \tag{31}$$

$$\rho_{\rm ch}+3p_{\rm ch}=-\frac{3}{4\pi G}\left[\frac{\ddot{a}}{a}-\frac{\Lambda}{3}\right]. \tag{32}$$

By the strict inequalities discussed above, there will be open temporal regions surrounding the bounce for which

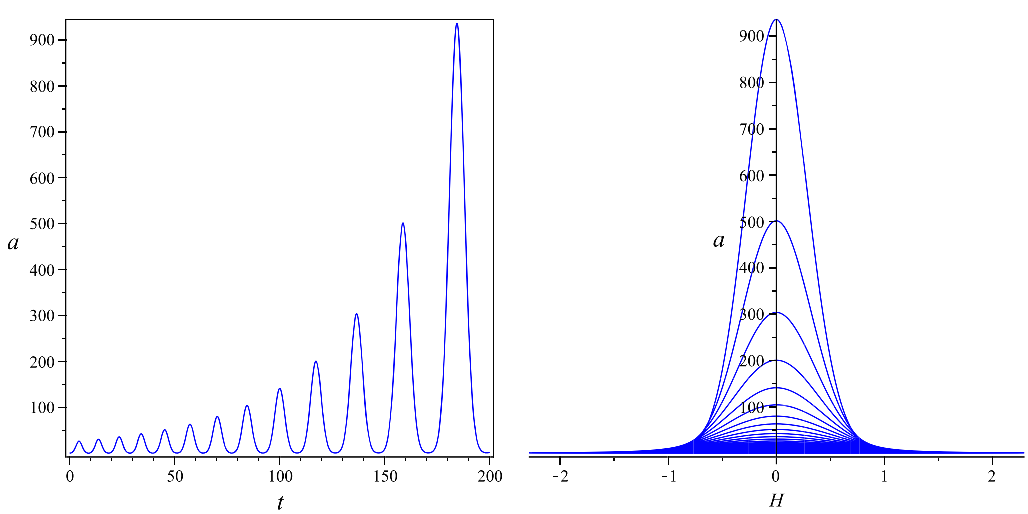

FIG. 15. Dynamical behavior of the scale factor and phase plane of $(a-H)$ for $k=1$, $g(t)=e^{-\lambda t}$, $\lambda.02$, $A=.1$, $B=0.1$, and $\alpha=0.5$.

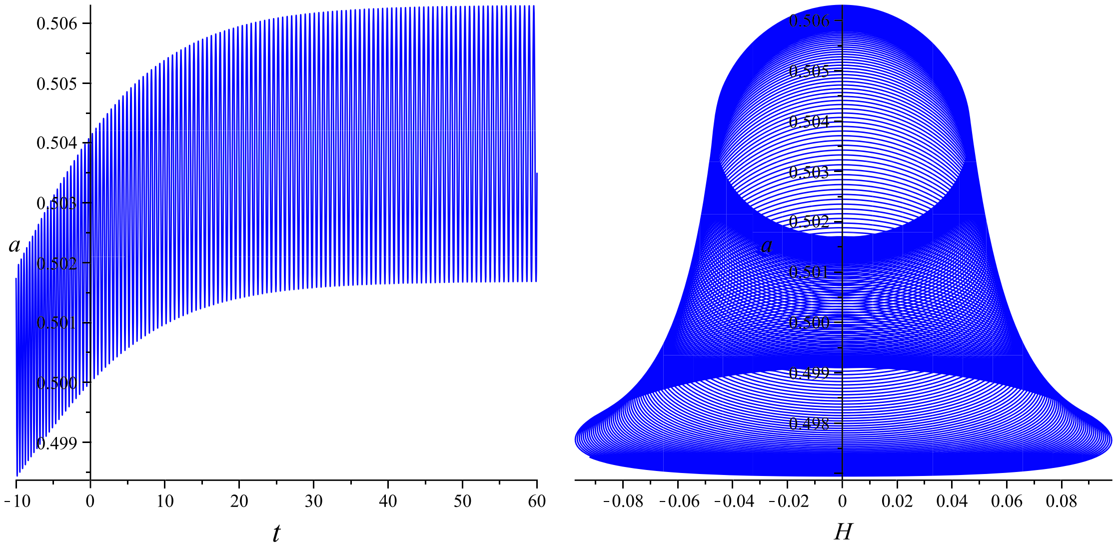

FIG. 16. Dynamical behavior of the scale factor and phase plane of $(a - H)$ for $k = -1$, $g(t) = e^{-\lambda t}$, $\lambda = .1$, $A = .1$, $B = 6$, and $\alpha = 0.5$.

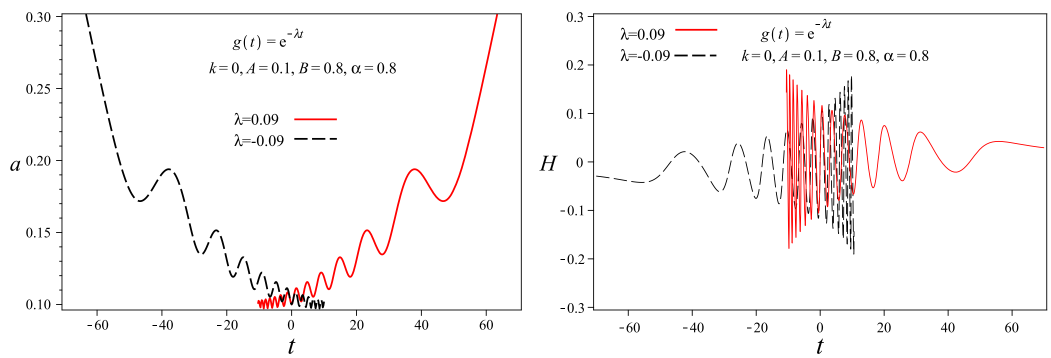

FIG. 17. Dynamical behavior of the scale factor and Hubble parameter for positive and negative values of λ.

$$\rho_b + p_b < \frac{1}{4\pi G}\left[\frac{k}{a^2}\right], \tag{33}$$

$$\rho_b - p_b > \frac{1}{4\pi G}\left[\frac{2k}{a^2} - \Lambda\right], \tag{34}$$

$$\rho_b + 3p_b < \frac{\Lambda}{4\pi G}. \tag{35}$$

The standard pointwise energy conditions are the NEC, weak energy condition (WEC), strong energy condition (SEC), and dominant energy condition (DEC). Their specializations to a FRW universe have previously been discussed in Refs. [53,54] and their basic definitions are given in Ref. [55]:

$$\text{NEC} \Leftrightarrow (\rho_{\rm ch} + p_{\rm ch} \geq 0), \tag{36}$$

$$\text{WEC} \Leftrightarrow (\rho_{\rm ch} \geq 0) \text{ and } (\rho_{\rm ch} + p_{\rm ch} \geq 0), \tag{37}$$

$$\text{SEC} \Leftrightarrow (\rho_{\rm ch} + 3p_{\rm ch} \geq 0) \text{ and } (\rho_{\rm ch} + p_{\rm ch} \geq 0). \tag{38}$$

$$\text{DEC} \Leftrightarrow (\rho_{\rm ch} \geq 0) \text{ and } (\rho_{\rm ch} \pm p_{\rm ch} \geq 0). \tag{39}$$

Note that if the NEC is violated, then all of the other pointwise energy conditions are violated as well [19].

A. NEC in a spatially flat ($k = 0$) and hyperbolic ($k = -1$) universe

With the above discussion points and using Eq. (33), it follows that the presence of the bounce in an open and flat universe implies the violation of the NEC. If the NEC is satisfied, a very general property of an expanding universe is that it always evolves from a state with a high energy density towards a state with a lower one. Here, we are interested in studying the relations imposed by the Einstein equations between extrema of the scale factor, the energy density, and the energy conditions in the extended Chaplygin gas fluid. We start from the conservation equation

$$\dot{\rho}_b = -3H_b(\rho_b + p_b). \tag{40}$$

Since $H_b = 0$ and hence $\dot{\rho}_b = 0$, the energy density reaches its extremum at the bounce point. To complete our understanding of the behavior of the energy density at the bounce, we need to find the second derivative of the energy density at the bounce. From Eq. (40), we find that

$$\ddot{\rho}_b = -3\dot{H}_b(\rho_b + p_b) - 3H_b(\dot{\rho}_b + \dot{p}_b) = -3\dot{H}_b(\rho_b + p_b). \tag{41}$$

Using Eqs. (41) and (33), the discussion can be classified as

$$\ddot{a}_b > 0 \Rightarrow \begin{cases} \dot{H}_b > 0, \\ \\ (\rho_b + p_b) < 0, \end{cases} \Rightarrow \text{NEC violated and } \ddot{\rho}_b > 0 \Rightarrow \rho_b = \rho_{\min} = -\tfrac{3k}{a_b^2} - \Lambda.$$

The first and second derivatives of the Chaplygin gas pressure at the bounce are, respectively,

$$\dot{p}_b = \dot{\rho}_b\left(A + \frac{\alpha B}{\rho_b^{\alpha+1}}\right), \tag{42}$$

$$\ddot{p}_b = \ddot{\rho}_b\left(A + \frac{\alpha B}{\rho_b^{\alpha+1}}\right). \tag{43}$$

We can say that in the extended Chaplygin gas model, the minimum ρ_b leads to the minimum $(\rho_b + p_b)$. (It is obvious that, for a positive value of ρ_b, the quantities $\ddot{p}_b$ and $\ddot{\rho}_b$ have the same sign.) The equation of state (1) implies the violation of the NEC in an extended Chaplygin gas only if

$$A - \frac{B}{\rho_{\mathrm{ch}}^{\alpha+1}} < -1. \tag{44}$$

It follows from Eqs. (44) and (19) that the presence of a bounce in an open and flat universe requires a negative cosmological constant in the specific range of $\Lambda < -(\frac{B}{A+1})^{\frac{1}{1+\alpha}}$. It is worth mentioning that although recent observations point toward a positive cosmological constant, it is still possible that in the very early universe the cosmological constant was negative. Several important theoretical results and predictions in quantum cosmology have been obtained with a negative cosmological constant.

In Ref. [56], the oscillating behavior of the scale factor for different curvatures with $\Lambda = -0.1$ was shown. Also, the authors of Ref. [57] explained that a bounce in loop quantum gravity requires a negative potential or negative cosmological constant. Also, a unique way of realizing inflation with a negative cosmological constant in a cyclic universe was presented in Ref. [58].

B. NEC in a hyperspherical ($k = +1$) universe

As discussed in the previous section, the presence of a bounce in both flat and open universes (with $k = 0$ or $k = -1$) automatically implies the violation of the NEC. However, in a closed universe ($k = 1$), by making the bounce sufficiently gentle $\ddot{a}_b \leq \frac{1}{a_b}$, the NEC will be satisfied. Rewriting Eq. (30) at the bounce as

$$\rho_b + p_b = \frac{-1}{4\pi G}\left[\frac{\ddot{a}_b}{a_b} - \frac{1}{a_b^2}\right], \tag{45}$$

it follows that the NEC would be violated if $\ddot{a}_b \geq \frac{1}{a_b}$. Thus, depending on the magnitude of $\ddot{a}_b$, the different possibilities can be classified as

$$\begin{cases} \ddot{a}_b > \frac{1}{a_b} \Rightarrow \begin{cases} \dot{H}_b > 0, \\ \\ (\rho_b + p_b) > 0, \end{cases} \Rightarrow \text{NEC violated and } \ddot{\rho}_b > 0 \Rightarrow \rho_b = \rho_{\min} = \frac{3}{a_b^2} - \Lambda, \\ \ddot{a}_b < \frac{1}{a_b} \Rightarrow \begin{cases} \dot{H}_b > 0, \\ \\ (\rho_b + p_b) > 0, \end{cases} \Rightarrow \text{NEC satisfied and } \ddot{\rho}_b < 0 \Rightarrow \rho_b = \rho_{\max} = \frac{3}{a_b^2} - \Lambda. \end{cases}$$

The argument shows that the presence of a nonsingular universe does not necessarily imply the violation of the NEC and it is possible to obtain a bounce solution without violating the NEC (see Fig. 18). It also indicates that, if the NEC is violated, the energy density reaches its minimum at bounce points as ($\rho_b = \rho_{\min} = \frac{3}{a_b^2} - \Lambda$). By taking this and Eq. (44), the NEC would be violated in the presence of an extended Chaplygin gas if

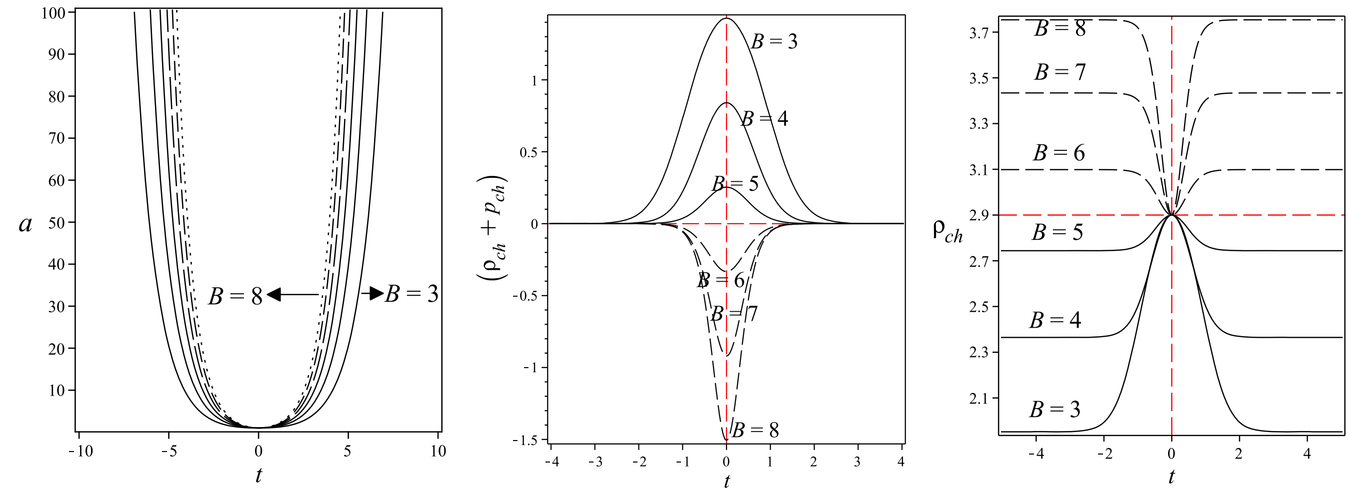

FIG. 18. Dynamical behavior of the scale factor a, the sum of the energy density and pressure ($\rho_{\rm ch}+p_{\rm ch}$), and energy density $\rho_{\rm ch}$ during the bounce phase for $k=1$, $A=.1$, $\alpha=\frac{1}{2}$, $\Lambda=.1$, and $a_b=1$ for different values of B. The plots show that although for all values of B from 3–8 a bounce can occur, for $B=6$, 7, and 8 the NEC is violated.

$$B>(A+1)\left[\frac{3}{a_b^2}-\Lambda\right]^{\alpha+1}. \tag{46}$$

In a closed universe ($k=1$), however, the following biconditional is established:

$$B>(A+1)\left[\frac{3}{a_b^2}-\Lambda\right]^{\alpha+1}\Leftrightarrow \ddot{a_b}>\frac{1}{a_b}. \tag{47}$$

For the parameters $A=0.1$, $\alpha=\frac{1}{2}$, $\Lambda=0.1$, and $a_b=1$ in the case of $k=1$, for instance, the NEC would be violated in an extended Chaplygin gas if $B>5.656854250$. Figure 18 depicts the behavior of the scale factor, the quantity $\rho_{\rm ch}+p_{\rm ch}$, and the energy density $\rho_{\rm ch}$ for different values of B and the same value of a_b. One can see that in the plots where $B<5.656854250$ is satisfied, the NEC is violated.

It is interesting to note that the scale factor for a $k=1$ universe will approach infinity in the future. From Eq. (5), therefore, the Chaplygin gas energy density will approach the constant value $\rho_{\rm ch}\to(\frac{B}{1+A})^{\frac{1}{1+\alpha}}$ in the future. From Eq. (2), consequently, the Hubble parameter approaches a constant value as

$$H\to\pm\sqrt{\frac{8\pi G}{3}\left(\frac{B}{1+A}\right)^{\frac{1}{1+\alpha}}+\frac{\Lambda}{3}}, \tag{48}$$

in which the minus and plus signs represent contracting and expanding universes, respectively. Also, Eq. (1) shows that when $t\to\infty$, $p_{\rm ch}\to-(\frac{B}{1+A})^{\frac{1}{1+\alpha}}$, $\omega_{\rm ch}\to-1$, and $(\rho_{\rm ch}+p_{\rm ch})\to0$ are established, so that the conservation equation (40) tells us $\frac{d\rho}{dt}\to0$. Therefore, the energy density at two situations (bounce point and infinity) approximates its extremum ($\frac{d\rho}{dt}=0$) limit. Depending on the violation or satisfaction of the NEC, one can determine the maximum or minimum of the energy density at these points, such that

$$\begin{cases}\text{NEC violated}\Rightarrow\rho_b=\rho_{\min}=\frac{3}{a_b^2}-\Lambda, & \rho_\infty=\rho_{\max}=\left(\frac{B}{1+A}\right)^{\frac{1}{1+\alpha}},\\ \text{NEC satisfied}\Rightarrow\rho_b=\rho_{\max}=\frac{3}{a_b^2}-\Lambda, & \rho_\infty=\rho_{\min}=\left(\frac{B}{1+A}\right)^{\frac{1}{1+\alpha}},\end{cases}$$

where ρ_∞ denotes the energy density evaluated at infinity.

C. NEC in the presence of a scalar field (for $k=-1$, $k=0$, and $k=+1$)

Violations of some of the energy conditions are produced by some scalar field theories. A universe filled with radiation and pressureless matter coupled to a classical conformal massless scalar field was studied in Ref. [59]. Another nonsingular universe based on a scalar field was presented in Ref. [60]. Here, we want to study the NEC in the extended Chaplygin gas model in the presence of a scalar field. The field equations for this case have been obtained in Sec. IV, so that the total energy density and pressure are written as

$$\rho_T+p_T=(\rho_{\rm ch}+p_{\rm ch})+(\rho_\phi+p_\phi), \tag{49}$$

where

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \quad (50)$$

From Eqs. (20) and (21), we can obtain the sum of the total energy density and pressure at the bounce point as

$$(\rho_T + p_T)|_{t_b} = \frac{-1}{4\pi G}\left[-\frac{\ddot{a}_b}{a_b} + \frac{k}{a_b^2}\right]. \quad (51)$$

This equation immediately implies the violation of the NEC for the $k = 0$ and $k = -1$ cases. These conditions are exactly the same as those we obtained for an extended Chaplygin gas in the absence of a self-interacting scalar field. However, the violation of the NEC implies the condition $\ddot{a}_b > \frac{1}{a}$ for the $k = 1$ case, as we can find specific ranges of the parameters and initial conditions of the model which satisfy this condition. In this respect, by substituting $(\rho_{\rm ch} + p_{\rm ch}) = (1 + A)y - \frac{B}{y^\alpha}$ and $(\rho_\phi + p_\phi) = \dot{\phi}^2$ into Eq. (49), we can obtain

$$(\rho_T + p_T)|_{t_b} = (A + 1)y_b - \frac{B}{y_b^\alpha} + g_b^2. \quad (52)$$

Therefore, the equation indicates that the violation of the NEC needs the right-hand side of Eq. (52) to be negative, which yields

$$B > [(A + 1)y_b + g_b^2](y_b)^\alpha. \quad (53)$$

VI. CONCLUSION

In this paper we have studied the possibility of obtaining singularity-free cosmological solutions in the context of an extended Chaplygin gas, with curvature and in some cases in the presence of a self-interacting scalar field. We found nonsingular solutions that can be periodic or bouncing by employing dynamical system techniques. Using the phase plane analysis, the full classification of the solutions was expressed based on the different curvatures. The phase plane diagram together with the phase plane trajectories were plotted in order to understand some general features of the system. One of the advantages of using this method in the oscillating models is that, using the properties of eigenvalues (with no need to find the exact solution of the differential equations), it is possible to determine whether the universe is cyclic or not. Also, the oscillating behavior can be observed in the phase space landscape as closed or spiral trajectories. For instance, it was shown that the eigenvalues are purely imaginary and hence the trajectories in the $(a - H)$ phase space are circles in an extended Chaplygin gas for an open universe $(k = -1)$ with a negative cosmological constant. This type of picture is sometimes called a center and it represents a cyclic universe where the minimal and maximal values of the scale factor remain the same in every cycle.

It was also demonstrated that the eigenvalue of the corresponding critical points for the cases of $k = 0$ and $k = 1$ are not complex. Thus, the phase trajectories cannot spiral in towards the critical points and/or spiral out towards infinity and there is no oscillating behavior for a and H, while a single bounce (a bouncing evolution without regular repetition) can occur under some appropriate conditions.

A combination of field equations gives the equation $\dot{H} = -4\pi G(\rho_T + p_T) + \frac{k}{a^2}$ which determines how the Hubble parameter changes with time. This equation implies that the presence of a bounce $(\dot{H}_b > 0)$ leads to the violation of the NEC $[(\rho_T + p_T) < 0]$ in the $k = 0$ and $k = -1$ cases. However, it does not necessarily imply the violation of the NEC in the $k = 1$ case, where there is the possibility of having a bounce while the NEC is satisfied. It can be achieved by taking proper values for the parameters and initial conditions of the model.

We distinguished three main types of evolution of the universe in the extended Chaplygin gas model:

(i) A cyclic universe where the minimal and maximal values of the scale factor remain the same in every cycle, for an open universe with $k = -1$ and a negative cosmological constant.

(ii) A nonsingular oscillating universe as a single bouncing solution for $k = 0$ and $k = 1$, where for $k = 0$ a negative cosmological constant is required and the NEC is violated. However, both positive and negative cosmological constants are allowed for the $k = 1$ case, while the NEC is violated if $(\ddot{a}_b > \frac{1}{a_b})$ or equivalently when the condition $B > (A + 1)[\frac{3}{a_b^2} - \Lambda]^{\alpha+1}$ is satisfied.

(iii) An oscillating universe where the minimal and maximal values of the scale factor periodically increase and decrease in the presence of a self-interacting scalar field model for all curvatures $(k = -1$, $k = 0$, and $k = 1)$. The NEC is automatically violated for $k = -1$ and $k = 0$, and under the condition $B > [(A + 1)y_b + g_b^2](y_b)^\alpha$ for $k = 1$.

Thus, it can be concluded that although a bouncing solution can occur for different curvatures, it can occur without violating the null energy condition only for $k = 1$, as is usual in the context of general relativity.

As discussed in Sec. V, the relations imposed by the Einstein equations between extrema of the scale factor, the energy density, and the energy conditions in an extended Chaplygin gas fluid imply that if the NEC is violated, the energy density reaches its minimum value at the bounce point and never reaches high densities where quantum gravity is important. An appealing feature of this study is that, we have obtained two different types of singularity-free cosmological solutions in the context of the extended Chaplygin gas with negative curvature. In one solution the NEC is violated and the energy density reaches its minimum value at the bounce, whereas in another the NEC is satisfied and the energy density reaches its maximum value at the bounce.

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APPENDIX: JACOBIAN STABILITY ANALYSIS FOR TWO DIMENSIONAL DYNAMICAL SYSTEMS

Consider the autonomous system

$$\begin{cases} \frac{d\chi}{dt} = f(\chi, \zeta), \\ \frac{d\zeta}{dt} = g(\chi, \zeta), \end{cases}$$

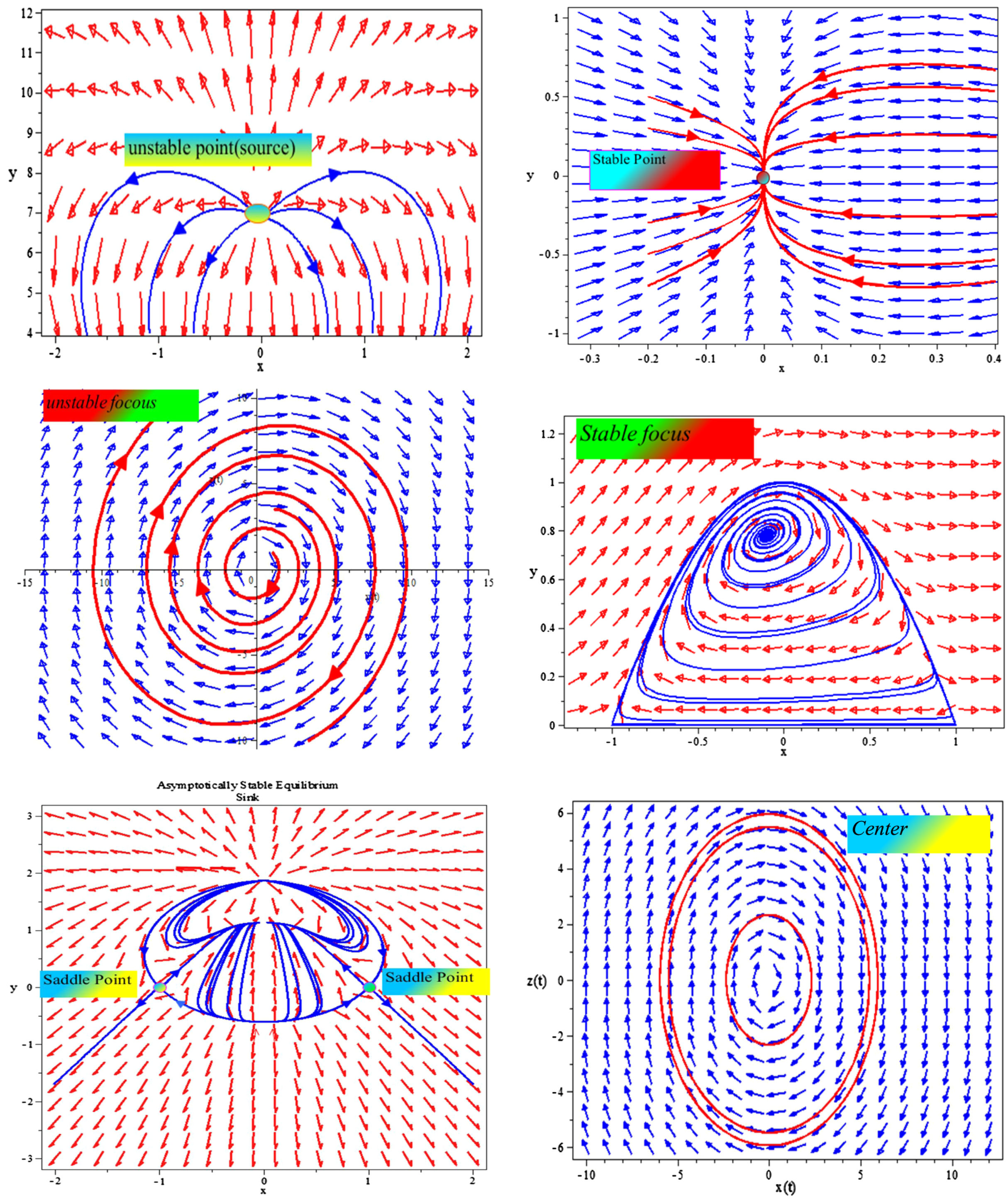

FIG. 19. Classification of phase plane portraits for two dimensional systems around the fixed points.

Then the nonlinear system may be approximated by the system

$$\begin{cases} f(\chi,\zeta) \approx f(\chi_c,\zeta_c) + \frac{\partial f}{\partial \chi}\Big|_{(\chi_c,\zeta_c)}(\chi-\chi_c) + \frac{\partial f}{\partial \zeta}\Big|_{(\chi_c,\zeta_c)}(\zeta-\zeta_c), \\ g(\chi,\zeta) \approx g(\chi_c,\zeta_c) + \frac{\partial g}{\partial \chi}\Big|_{(\chi_c,\zeta_c)}(\chi-\chi_c) + \frac{\partial g}{\partial \zeta}\Big|_{(\chi_c,\zeta_c)}(\zeta-\zeta_c). \end{cases}$$

But since (χ_c,ζ_c) is an equilibrium point, we have $f(\chi_c,\zeta_c) = g(\chi_c,\zeta_c) = 0$. Hence we have

$$\begin{cases} \frac{d\chi}{dt} = \frac{\partial f}{\partial \chi}\Big|_{(\chi_c,\zeta_c)}(\chi-\chi_c) + \frac{\partial f}{\partial \zeta}\Big|_{(\chi_c,\zeta_c)}(\zeta-\zeta_c), \\ \frac{d\zeta}{dt} = \frac{\partial g}{\partial \chi}\Big|_{(\chi_c,\zeta_c)}(\chi-\chi_c) + \frac{\partial g}{\partial \zeta}\Big|_{(\chi_c,\zeta_c)}(\zeta-\zeta_c). \end{cases}$$

This is a linear system. Its coefficient matrix (Jacobian) is

$$\text{Jacobian} = \begin{pmatrix} \frac{\partial f}{\partial \chi}\Big|_{(\chi_c,\zeta_c)} & \frac{\partial f}{\partial \zeta}\Big|_{(\chi_c,\zeta_c)} \\ \frac{\partial g}{\partial \chi}\Big|_{(\chi_c,\zeta_c)} & \frac{\partial g}{\partial \zeta}\Big|_{(\chi_c,\zeta_c)} \end{pmatrix}.$$

We wish to find the eigenvalues of the Jacobian matrix. The following classification of the fixed point p is standard.

(1) λ_1, λ_2 *are real and distinct*

1.1. $\lambda_1 \cdot \lambda_2 > 0$ (*the eigenvalues have the same sign*): p is called a *node* or type I singularity; that is, every orbit tends to the origin in a definite direction as $t \to \infty$.

1.1.1. λ_1, $\lambda_2 > 0$: p is an *unstable node*.

1.1.2. λ_1, $\lambda_2 < 0$: p is a *stable node*.

1.2. $\lambda_1 \cdot \lambda_2 < 0$ (*the eigenvalues have different signs*): p is an *unstable fixed point*, or a *saddle* point singularity.

(2) λ_1, λ_2 *are complex*, i.e., $\lambda_{1,2} = \alpha \pm i\beta$, $\beta \neq 0$

2.1. $\alpha \neq 0$: p is a *spiral*, or a *focus*; that is, the solutions approach the origin as $t \to \infty$, but not from a definite direction.

2.1.1. $\alpha < 0$: p is a *stable focus*.

2.1.2. $\alpha > 0$: p is an *unstable focus*.

2.2 $\alpha = 0$: p is a *center*, which means it is not stable in the usual sense, and we have to look at higher-order derivatives. The phase portraits for different types of the fixed points have been shown in Fig. 19.

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