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Nonleptonic kaon decays at large N_c

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We study the scaling with the number of colors, N_c, of the weak amplitudes mediating kaon mixing and decay. We evaluate the amplitudes of the two relevant current-current operators on the lattice for $N_c = 3$–7. We conclude that the subleading $1/N_c$ corrections in $\hat{B}_K$ are small, but those in the $K \to \pi\pi$ amplitudes are large and fully anticorrelated in the $I = 0, 2$ isospin channels. We briefly comment on the implications for the $\Delta I = 1/2$ rule.

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I. INTRODUCTION

The prediction of flavor violating processes involving kaons remains elusive. In particular, there is still no satisfactory explanation of the striking $\Delta I = 1/2$ rule, nor a reliable prediction of ϵ'/ϵ. In spite of the spectacular progress in lattice QCD calculations in the past decade, few attempts have been made at these difficult observables, and the systematic uncertainties in the existing results [1] remain too large. On the other hand, a rather precise determination of the $K - \bar{K}$ mixing amplitude (given by $\hat{B}_K$) has emerged [2,3].

The large N_c limit of QCD [4] has been invoked in many phenomenological approaches to this problem (some relevant references are [5–9]). This seems counterintuitive since the strict large N_c limit of the $\Delta I = 1/2$ rule fails completely. The predictions therefore rely on significant subleading N_c effects, which are however very difficult to predict accurately. As a result, these approaches typically involve further approximations beyond the strict large-N_c expansion.

In [10], the results of the most ambitious lattice computation of $K \to \pi\pi$ to date were presented, and a significant $\Delta I = 1/2$ dominance was observed. It was noted that the $\Delta I = 1/2$ rule seems to be originating in an approximate cancellation of the two diagrams (color connected and disconnected) contributing to the $\Delta I = 3/2$ amplitude. Unfortunately it is not possible to isolate these two contributions physically, so it is not clear what to extract from this finding. In the large N_c expansion however this is possible since the leading scaling in N_c of the contributions is different. The cancellation can therefore be phrased in terms of the sign and size of the $1/N_c$ corrections in the isospin amplitudes. In fact, it was in the context of phenomenological approaches using the large N_c expansion where the opposite sign of these contributions was first pointed out [6]. There is however a strong correlation between the $\Delta I = 3/2$ amplitude and $\hat{B}_K$, and therefore this suggest that the same cancellation in the former should be affecting the latter, suggesting a value of $\hat{B}_K$ significantly smaller than the $N_c \to \infty$ value. The role of the $1/N_c$ expansion in the interpretation of the results in [10] is also discussed in the latest update of RBC/UKQCD's results for the $K \to \pi\pi \Delta I = 3/2$ decay amplitude [11]. A study of the—related—issue of deviations from the naïve factorization approximation to $K \to \pi\pi$ amplitudes can be found in [12].

The goal of this paper is to study from first principles the large N_c behavior of certain $\Delta S = 1$ and $\Delta S = 2$ amplitudes. More concretely we consider K-π and K-$\bar{K}$ transitions mediated by the four-fermion current-current operators on the lattice varying the number of colors $N_c = 3$–7. As it is well-known, these amplitudes fix $\hat{B}_K$ [up to $SU(3)$ flavor breaking effects by quark masses] and, up to chiral corrections, also the $\Delta I = 3/2$ contribution to the nonleptonic kaon decay, $K \to \pi\pi$ [13]. Furthermore, in the GIM limit of degenerate charm and up quarks, the $\Delta I = 1/2$ contribution to the nonleptonic decays can also be determined from the current-current operator matrix elements, only [14,15]. In fact this is the limit where the cancellation of [10] can be more clearly isolated. For this reason, we will consider only the $SU(4)$-flavor limit $m_c = m_u = m_d = m_s$. We miss in this way the effects of a heavy charm, which were originally argued to be the origin of the $\Delta I = 1/2$ rule [16]. This fact, however, has not been confirmed by nonperturbative studies [1,17].

The paper is organized as follows. In Sec. II we introduce our method and set our notation. We present the main results in Sec. III and conclude in IV.

II. FORMALISM

The operator product expansion allows us to represent the weak Hamiltonian that mediates CP-conserving $\Delta S = 1$ transitions by an effective Hamiltonian in terms of four-fermion operators. At the electroweak scale, $\mu \simeq M_W$, we can neglect all quark masses and the weak Hamiltonian takes the simple form

$$H_{\rm w}^{\Delta S=1} = \int d^4x \frac{g_{\rm w}^2}{4M_W^2} V_{us}^* V_{ud} \sum_{\sigma=\pm} k^\sigma(\mu) \bar{Q}^\sigma(x,\mu), \quad (1)$$

where $g_{\rm w}^2 = 4\sqrt{2}G_{\rm F}M_W^2$. Only two four-quark operators of dimension six can appear with the correct symmetry properties under the flavor symmetry group $\mathrm{SU}(4)_{\rm L}\times\mathrm{SU}(4)_{\rm R}$, namely

$$\bar{Q}^{\pm}(x,\mu) = Z_Q^{\pm}(\mu)(J_\mu^{su}(x)J_\mu^{ud}(x) \pm J_\mu^{sd}(x)J_\mu^{uu}(x) - [u\leftrightarrow c]), \tag{2}$$

where J_μ is the left-handed current, $J_\mu^{\alpha\beta} = (\bar{\psi}_\alpha\gamma_\mu P_-\psi_\beta)$, $P_\pm = \frac{1}{2}(\mathbf{1}\pm\gamma_5)$, and parentheses around quark bilinears indicate that they are traced over spin and color. Eventually, $Z_Q^{\pm}(\mu)$ is the renormalization constant of the bare operator $Q^{\pm}(x)$ computed in some regularization scheme as, for example, the lattice. There are other bilinear operators of lower dimensionality that could mix with those above: however, they vanish in the GIM limit [14].

The operators $\bar{Q}^{\sigma}(\mu)$ are renormalized at a scale μ in some renormalization scheme, being their μ-dependence exactly canceled by that of the Wilson coefficients $k^{\sigma}(\mu)$. It is common practice to define renormalization group invariant (RGI) operators, which are defined by canceling their perturbative μ-dependence, as derived from the Callan-Symanzik equations,

$$\hat{Q}^{\sigma} \equiv \hat{c}^{\sigma}(\mu)\bar{Q}^{\sigma}(\mu), \tag{3}$$

with

$$\hat{c}^{\sigma}(\mu) \equiv \left(\frac{N_c}{3}\frac{g^2(\mu)}{4\pi}\right)^{-\frac{\gamma_0^{\sigma}}{2b_0}} \exp\left\{-\int_0^{g(\mu)} \mathrm{d}g\left[\frac{\gamma^{\sigma}(g)}{\beta(g)} - \frac{\gamma_0^{\sigma}}{b_0 g}\right]\right\}, \tag{4}$$

where $g(\mu)$ is the running coupling and $\beta(g) = -g^3\sum_n b_n g^{2n}$, $\gamma^{\sigma}(g) = -g^2\sum_n \gamma_n^{\sigma} g^{2n}$ are the β-function and the anomalous dimension, respectively. The one- and two-loop coefficients of the β-function, and the one-loop coefficient of the anomalous dimensions, are renormalization scheme-independent. Their values for the theory with N_f flavors are [18,19]

$$b_0 = \frac{1}{(4\pi)^2}\left[\frac{11}{3}N_c - \frac{2}{3}N_f\right], \tag{5}$$

$$b_1 = \frac{1}{(4\pi)^4}\left[\frac{34}{3}N_c^2 - \left(\frac{13}{3}N_c - \frac{1}{N_c}\right)N_f\right], \tag{6}$$

and for the operators $Q^{\pm}$ [20]

$$\gamma_0^{\pm} = \frac{1}{(4\pi)^2}\left[\pm 6 - \frac{6}{N_c}\right]. \tag{7}$$

The normalization of $\hat{c}^{\sigma}(\mu)$ coincides with the most popular one for $N_c = 3$, whilst using the 't Hooft coupling $\lambda = N_c g^2(\mu)$ in the first factor instead of the usual coupling, so that the large N_c limit is well-defined.

Defining similarly an RGI Wilson coefficient

$$\hat{k}^{\sigma} \equiv \frac{k^{\sigma}(\mu)}{\hat{c}^{\sigma}(\mu)}, \tag{8}$$

we can rewrite the Hamiltonian in terms of RGI quantities, which no longer depend on the scale, so we can write

$$\hat{k}^{\sigma}\hat{Q}^{\sigma} = \left[\frac{k^{\sigma}(M_W)}{\hat{c}^{\sigma}(M_W)}\right][\hat{c}^{\sigma}(\mu)\bar{Q}^{\sigma}(\mu)] = k^{\sigma}(M_W)U^{\sigma}(\mu, M_W)\bar{Q}^{\sigma}(\mu), \tag{9}$$

where μ is a convenient renormalization scale for the nonperturbative computation of matrix elements of $Q^{\pm}$, which will be later set to the inverse lattice scale a^{-1}. The factor $U^{\sigma}(\mu, M_W) = \hat{c}^{\sigma}(\mu)/\hat{c}^{\sigma}(M_W)$, therefore, measures the running of the renormalized operator between the scales μ and M_W. Ideally one would like to evaluate this factor nonperturbatively, as has been done for $N_c = 3$ [21], but this is beyond the scope of this paper. We will instead use the perturbative results at two loops in the RI scheme [22] to evaluate the $\hat{c}^{\sigma}(\mu)$ factors. This implies relying on perturbation theory at scales above $\mu = a^{-1} \sim 2$ GeV.

Our goal is to compute the $K\to\pi$ amplitudes mediated by $H_{\rm w}^{\Delta S=1}$. The hadronic contribution is encoded in the ratios of three- and two-point functions

$$\hat{R}^{\pm} \equiv \frac{\langle\pi|\hat{Q}^{\pm}|K\rangle}{f_K f_\pi m_K m_\pi} = \hat{c}^{\pm}(\mu)Z_R^{\pm}(\mu)R^{\pm}, \tag{10}$$

where $Z_R^{\pm}(\mu)$ are the renormalization factors for the ratios and $R^{\pm}$ is the ratio of matrix elements of bare operators. In the $SU(3)$ limit $m_s = m_d = m_u$, from R^+ we can determine $\hat{B}_K$ as

$$\hat{B}_K = \frac{3}{4}\hat{R}^{+}. \tag{11}$$

Concerning $K\to\pi\pi$ decays, the two very different isospin amplitudes

$$iA_I e^{i\delta_I} \equiv \langle(\pi\pi)_I|H_W|K_0\rangle, \qquad I = 0, 2 \tag{12}$$

can be related in chiral perturbation theory, and in the GIM limit, to the $K\to\pi$ amplitudes $A^{\pm} \equiv \hat{k}^{\pm}\hat{R}^{\pm}$ [14]

$$\frac{A_0}{A_2} = \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \frac{3}{2}\frac{A^-}{A^+}\right). \tag{13}$$

The $\Delta I = 1/2$ rule, i.e. the large enhancement of the ratio $|A_0/A_2| \sim 22$, is therefore related in this limit to the ratio of the amplitudes A^-/A^+.

At this point, it is necessary to comment on the chiral corrections. The relation between the $K-\bar{K}$ and $K\to(\pi\pi)|_{I=2}$ amplitudes is well-known to break down away from the chiral limit for the physical case $m_s \gg m_{u,d}$, since the chiral logarithmic corrections are much larger for the former amplitude [13]. On the other hand, this is not the case in the $SU(3)$ limit $m_s = m_u = m_d$, where the chiral logs are the same for both amplitudes both in the full as in the quenched case [23]. The following relation holds up to one loop in ChPT in the leading-log approximation:

$$\left.\frac{\langle\pi^+\pi^0|H_W|K\rangle}{m_K^2-m_\pi^2}\right|_{m_s=m_d} = \frac{iF}{\sqrt{2}}A^+G_F V_{ud}V_{us}^*, \qquad (14)$$

where F is the decay constant in the chiral limit and A^+ contains one loop corrections. This shows that, in this approximation, the $1/N_c$ corrections in the physical amplitude are fixed [24] by those in A^+. At the same order in ChPT, we can relate the amplitudes for both choices of quark masses

$$\langle\pi^+\pi^0|H_W|K^+\rangle_{m_\pi\to 0} = m_K^2\left.\frac{\langle\pi^+\pi^0|H_W|K^+\rangle}{m_K^2-m_\pi^2}\right|_{m_s=m_d} \times\left(1+\frac{9}{4}\frac{m_K^2}{(4\pi F)^2}\log\frac{m_K^2}{(4\pi F)^2}\right). \qquad (15)$$

The chiral log term gives an additional *negative* $1/N_c$ contribution to the amplitude at the physical point with respect to that in the degenerate case. Another important point to note is that, in the GIM limit, the chiral logs have been shown to be fully anticorrelated in $A^\pm$ [29], and therefore an extrapolation to the chiral limit using chiral perturbation theory will not change the anticorrelation found at larger masses. Unfortunately the computation of chiral logs in $K\to(\pi\pi)_{I=0}$ in the GIM limit is not available, although it is likely that the same anticorrelation holds also there.

III. RESULTS

We compute the ratios $\hat{R}^\pm$ on the lattice from the ratio of correlation functions

$$R^\pm = \lim_{\substack{z_0-x_0\to\infty \\ y_0-z_0\to\infty}} \frac{\sum_{\mathbf{x},\mathbf{y}}\langle P^{du}(y)Q^\pm(z)P^{us}(x)\rangle}{\sum_{\mathbf{x},\mathbf{y}}\langle P^{du}(y)A_0^{ud}(z)\rangle\langle A_0^{su}(z)P^{us}(x)\rangle}, \qquad (16)$$

where $P^{ab}(x) = \bar{\psi}^a(x)\gamma_5\psi^b(x)$, and $A_0^{ab}(x) = Z_A\bar{\psi}^a(x)\times \gamma_0\gamma_5\psi^b(x)$. The renormalized ratios $\hat{R}^\pm$ have been computed in $SU(N_c)$ for $N_c = 3$–7 and in the quenched approximation. Note that the latter does not modify the leading large N_c result, but it can modify the first subleading $1/N_c$ corrections. We have implemented the required correlation functions in the source code first developed in [30] and further optimized in [31]. The number of colors and the lattice size are given in the first two columns of Table I. The spatial volume, $L/a = 16$, is kept fixed in all simulations. On the other hand, $T/a = 48$ for $N_c = 3$, 4, 5 and $T/a = 32$ for $N_c = 6$, 7. Following [32] the bare coupling, $\beta = 2N_c/g_0^2$, is tuned with N_c in such a way that the string tension remains constant $a\sqrt{\sigma}\simeq 0.2093$; this results in $a\simeq 0.093$ fm with $\sigma = 1$ GeV/fm. The bare 't Hooft coupling λ is found to be well described by the following scaling:

$$\lambda = N_c g_0^2 = 2.775(3) + \frac{1.90(3)}{N_c^2}. \qquad (17)$$

TABLE I. Lattice simulation results. Lattice sizes are $(L/a)^3\times(T/a)$, with $L/a = 16$ throughout. The twisted bare mass is fixed to $a\mu = 0.02$. The lattice spacing is fixed by the string tension through $a\sqrt{\sigma}\simeq 0.2093$ [32]. $m_{\rm PCAC}$ is the current mass obtained from the axial Takahashi-Ward identity in twisted quark field variables. $m_{\rm PS}$ is the kaon and pion mass in our $m_u = m_d = m_s$ limit. $R^\pm$ are our results for the bare ratios given in Eq. (16).

N_c	T/a	β	$am_{\rm PCAC}$	$am_{\rm PS}$	$R^+_{\rm bare}$	$R^-_{\rm bare}$
3	48	6.0175	−0.002(14)	0.2718(61)	0.774(21)	1.218(31)
4	48	11.028	−0.0015(11)	0.2637(39)	0.783(15)	1.198(19)
5	48	17.535	0.0028(9)	0.2655(31)	0.839(8)	1.145(12)
6	32	25.452	0.0013(7)	0.2676(28)	0.871(6)	1.125(7)
7	32	34.8343	−0.0034(6)	0.2819(19)	0.880(5)	1.122(5)

The coupling β as a function of N_c is given in the third column of Table I. In order to preserve the multiplicative renormalization of $Q^\pm$, while avoiding the high computational cost of a simulation with exactly chiral lattice fermions, we use a Wilson twisted-mass fermion regularization [33]. (For the gauge sector we employ the standard plaquette action.) This allows us to devise a formulation of valence quarks that not only preserves good renormalization properties, but also prevents the appearance of linear cutoff effects in a [34]. The full-twist condition amounts to having a vanishing current quark mass $m_{\rm PCAC}$ from the axial Takahashi-Ward identity in so-called twisted quark field variables. The value of $am_{\rm PCAC}$ in our simulations is given in the fourth column of Table I, where we can see that the full-twist condition $am_{\rm PCAC} = 0$, expected from an accurate tuning of the Wilson critical mass (which we again take from [32]), is satisfied to a varying degree of accuracy; the deviations present are however irrelevant within the precision of our results. The bare quark mass is chosen to provide a pseudoscalar mass not far from the physical kaon mass in all cases (see the fifth column of Table I). Eventually, our results for the bare ratios $R^\pm$ defined in Eq. (16), computed in the $SU(3)$ limit, are shown in the last two columns of the table.

In Table II we show the various renormalization constants and renormalization group (RG) running factors needed to compute the renormalized amplitudes $\hat{B}_K$ and $A^\pm$ as a function of the number of colors. First of all, in order to

TABLE II. Perturbative renormalization constants and RG running factors. $Z^\sigma(a^{-1})$ at one-loop have been extracted from [35], whereas U^σ and k^σ are computed using the two-loop $\overline{\mathrm{MS}}$ coupling [with $\Lambda_{\overline{\mathrm{MS}}}$ taken from Eq. (18) from Ref. [36]].

N_c	$\hat{k}^+$	$k^+(M_W)$	$U^+(a^{-1},M_W)$	$\hat{c}^+(a^{-1})$	$Z^+(a^{-1})$
3	0.642	1.030	0.875	1.404	0.983
4	0.658	1.025	0.895	1.394	0.988
5	0.679	1.021	0.910	1.368	0.991
6	0.700	1.018	0.921	1.340	0.994
7	0.719	1.016	0.930	1.315	0.996
N_c	$\hat{k}^-$	$k^-(M_W)$	$U^-(a^{-1},M_W)$	$\hat{c}^-(a^{-1})$	$Z^-(a^{-1})$
3	2.398	0.940	1.319	0.517	1.059
4	1.998	0.958	1.210	0.580	1.043
5	1.780	0.968	1.156	0.620	1.035
6	1.643	0.974	1.124	0.666	1.030
7	1.550	0.978	1.103	0.696	1.026

get the renormalized ratios $\hat{R}^\pm$ from the bare ones computed on the lattice, we have used the known one-loop lattice renormalization constants in the RI scheme of Ref. [35]. Note that, due to the breaking of chiral symmetry in the adopted regularization, the axial current requires a finite, N_c-dependent, renormalization constant Z_A, that has to be included in the factors $Z_R^\pm$ in Eq. (10). This has also been taken from Ref. [35]. The values of $Z^\pm(a^{-1})$ are given in the rightmost column of Table II. The values of the normalization coefficients $\hat{c}^\pm(a^{-1})$ and of the running of the renormalized operators from the scale of lattice computations, $\mu = a^{-1}$, to the scale of the effective theory, M_W, computed using perturbative results at two-loops in the RI scheme [22], are given in the fifth and fourth columns of Table II, respectively. In the evaluation of the $\hat{c}^\sigma(\mu)$ factors we have used the large N_c scaling of the Λ parameter found in Ref. [36],

$$\frac{\Lambda_{\overline{\mathrm{MS}}}}{\sqrt{\sigma}} = 0.503(2)(40) + \frac{0.33(3)(3)}{N_c^2}. \tag{18}$$

Eventually, the Wilson coefficients $k^\pm(M_W)$, also computed following Ref. [22], are given in the third column of Table II, while their RGI counterparts $\hat{k}^\pm$, defined in Eq. (8), are given in the second column.

Our results for $\hat{B}_K$ as a function of $1/N_c$ are shown in Fig. 1 together with a linear fit to the data, represented by a solid black line. The grey band shows the 1σ error on the fit. We compare our results with our own evaluation of the predictions of the phenomenological analysis in Ref. [5], represented by a light red band for $N_f = 3$ and by a blue band for $N_f = 0$. For $N_f = 3$ we use in the latter the same values for hadronic masses and decay constants as in [5], and obtained the decay constant for $N_c \neq 3$ by rescaling $F_K = F_K(N_c = 3)\sqrt{N_c/3}$. For $N_f = 0$ we use as input for the hadronic quantities, including their N_c dependence, the interpolating formulas provided in [32], matched to our

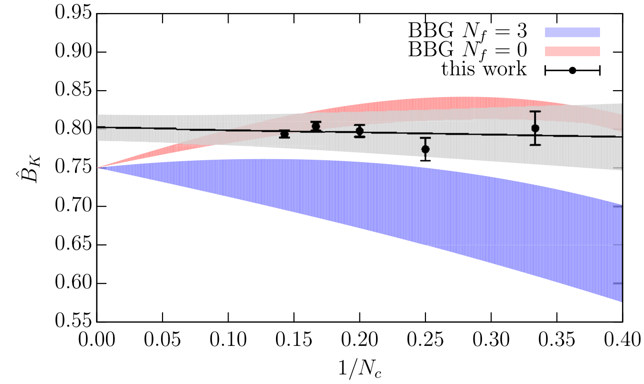

FIG. 1. $\hat{B}_K$ versus $1/N_c$. The grey band (solid line) is a linear fit to our five data points. The red and blue bands use the model prediction of [5].

measured values of M_K. In both cases the band covers the difference between setting the matching scale M in Eq. (62) of [5] at 0.6 GeV and at 1 GeV; for $N_f = 0$ it also comprises the uncertainty due to our value of M_K not being constant within errors as a function of N_c. Notice that both theoretical predictions give $\hat{B}_K = 3/4$ in the $N_c \to \infty$ limit. From Fig. 1 we can see that the subleading $1/N_c$ corrections in $\hat{B}_K$ are small (which goes in the direction of the predictions in [5], but not those in [7], that correspond to the chiral limit). The parameter of the linear fit to the data are shown in the first two lines of Table III for a different choice of the data points included in the fit, together with the corresponding p-values. The third line of the same table shows our result for a quadratic fit to the data. We can see that, in this case, the large N_c limit obtained is consistent with the theoretical expectation, albeit with large errors. Note that a significant $O(a^2)$ uncertainty for R^+ can be expected, cf. the $\mathcal{O}(10\%)$ effect for $N_c = 3$, $N_f = 2$ shown by the data of [37] at a lattice spacing comparable to ours.

The smallness of $1/N_c$ corrections in $\hat{B}_K$ is related to the RGI normalization of this quantity, $\hat{c}^+(a^{-1})$: the significant N_c-dependence of R^+ (see Table I) is canceled to a large extent by the RGI Wilson coefficient $\hat{k}^+$ (see Table II). In contrast, the total $K \to \pi$ amplitudes show very significant

TABLE III. Fit parameters of A^σ assuming a linear (l) or quadratic (q) dependence, and various fit ranges. The order at which each coefficient enters in the polynomial ansatz in powers of $1/N_c$ is indicated, alongside with the p-value for each fit.

obs	fit	1	$1/N_c$	$1/N_c^2$	p-value
$\hat{B}_K$	l, $N_c \geq 3$	0.802(17)	−0.03(10)	⋯	0.24
	l, $N_c \geq 4$	0.808(27)	−0.07(16)	⋯	0.14
	q, $N_c \geq 3$	0.788(79)	0.12(78)	−0.3(1.8)	0.12
A^+	l, $N_c \geq 3$	0.956(20)	−0.89(11)	⋯	0.10
	l, $N_c \geq 4$	0.981(18)	−1.05(11)	⋯	0.39
A^-	l, $N_c \geq 3$	0.984(28)	1.77(17)	⋯	0.21
	l, $N_c \geq 4$	0.996(39)	1.69(24)	⋯	0.14

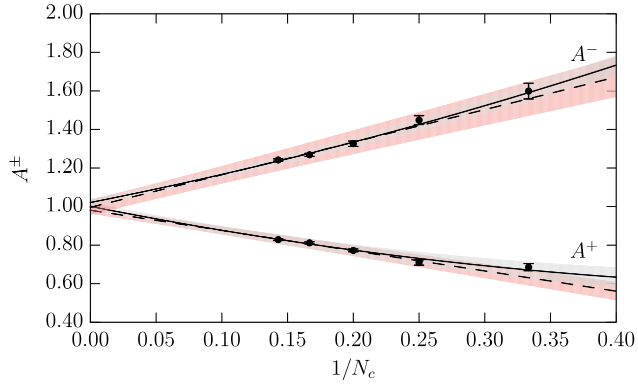

FIG. 2. $A^\pm$ versus $1/N_c$. The grey bands (solid lines) are obtained from the results of the fits to $1/2(A^- \pm A^+)$ in Eqs. (19); the red bands (dashed lines) are linear fits including $N_c = 4$–7 from Table III.

subleading $1/N_c$ corrections, as shown in Fig. 2. In Fig. 2, we present our data for $A^\pm$ obtained from the ratios $R^\pm$ of Eq. (16) and the results of a linear (dashed lines) and quadratic (solid lines) fit to the data. The parameters of the linear fit for A^+ and A^- are shown in the fourth and fifth (sixth and seventh) lines of Table III, respectively. We can see from Fig. 2 that the corrections at $N_c = 3$ are naturally ~30% and that they are strongly anticorrelated in $A^\pm$. For the quadratic fit, and in order to clarify further this correlation, we have considered the combinations $\frac{1}{2}(A^- \pm A^+)$; the results are shown in Fig. 3. The curves correspond to the following best fits:

$$\frac{A^- + A^+}{2} = 1.01(3) + \frac{1.08(11)}{N_c^2} \quad (p\text{-value} = 0.81),$$
$$\frac{A^- - A^+}{2} = 0.01(2) + \frac{1.35(11)}{N_c} \quad (p\text{-value} = 0.12). \qquad (19)$$

The subleading $1/N_c$ effects seem to cancel in the first combination, while they are the only visible corrections in the second one. The parameters of the quadratic fit in Fig. 2 are obtained from the results of Eq. (19).

We have not included any systematic error in these results. There are two obvious sources: finite lattice spacing and the quenched approximation. Although it is impossible to quantify those errors, we do not expect them to be larger that those observed at $N_c = 3$, where they have been studied. We have already commented above on the expected size of $O(a^2)$ discretization effects, based on the results of [37]. Concerning the quenching error, it is well-known that $\hat{B}_K$ is remarkably insensitive to the number of dynamical quark flavors, cf. [2] and benchmark quenched studies [38]; we thus expect a small effect in A^+. The pioneering large-N_c study of dynamical QCD in [39] shows that an extension of our work to take into account unquenching effects is feasible.

IV. CONCLUSIONS

We have presented the first computation on the lattice of the $1/N_c$ corrections to the $\Delta S = 1$ amplitudes $K - \pi$ in the GIM and SU(3) limit $m_c = m_u = m_s = m_d$. The size and sign of $1/N_c$ corrections are relevant to give a solid physical basis to the observation made in [10] that suggests that the $\Delta I = 1/2$ rule might originate in a near cancellation of two contributions to the $K \to (\pi\pi)_{I=2}$ amplitude, that add up in the $I = 0$ channel. The observed cancellation can be traced to large and anticorrelated $1/N_c$ corrections in the two isospin amplitudes. We have quantified the subleading $1/N_c$ dependence of the simpler $K - \pi$ amplitudes, $A^\pm$, that are closely related to the $K - \pi\pi$ ones in the degenerate light quark limit, $m_s = m_d$. Our results show that the subleading $1/N_c$ corrections in A^σ are large and consistent with being equal and opposite in sign for A^+ and A^-, supporting the observation in [10]. However, the size of these corrections is natural, i.e. $\mathcal{O}(1)/N_c$ and not large enough to explain the $\Delta I = 1/2$ rule, although we have argued that larger $1/N_c$ corrections could be present at the physical point, $m_s \gg m_d$, suggested by a large chiral log. We have also studied the subleading N_c corrections to $\hat{B}_K$ and found that they are significantly smaller than those in the closely related amplitude A^+, because of the different normalization. This shows that a value of $\hat{B}_K$ close to the

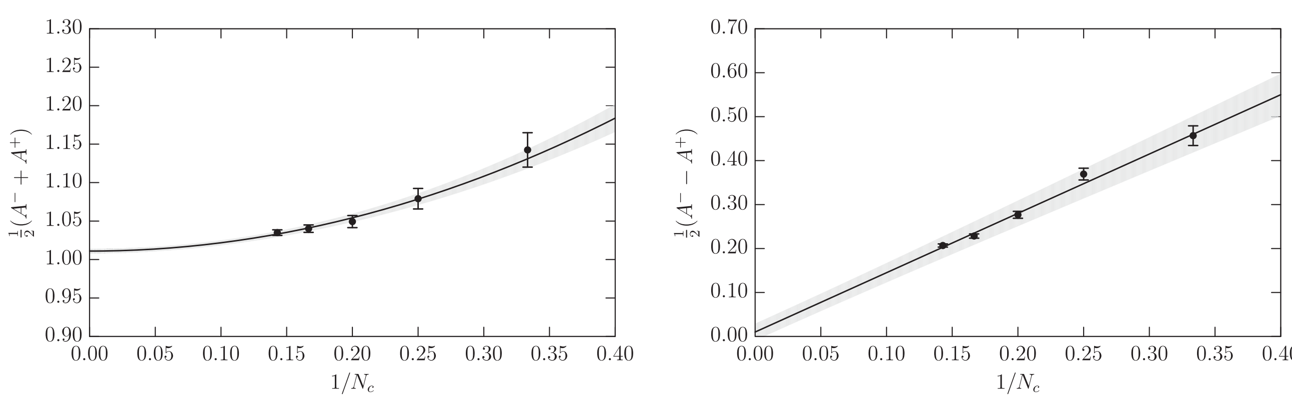

FIG. 3. $\frac{A^- \pm A^+}{2}$ versus $1/N_c$. The bands (solid lines) are quadratic and linear fits in $1/N_c$, respectively.

$N_c \to \infty$ value is consistent with large $1/N_c$ corrections in the $\Delta S = 1$ amplitudes.

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