SU(3) flavor symmetry breaking in large N_c excited hyperons

Fl. Stancu[*]

University of Liège, Institute of Physics B5, Sart Tilman, B-4000 Liège 1, Belgium

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The $1/N_c$ expansion method for studying the mass spectrum of excited baryons is shortly reviewed together with applications to mixed symmetric states. The $[\mathbf{70}, \ell^+]$ multiplet, belonging to the $N = 2$ band, is reanalyzed, with emphasis on hyperons and the SU(3) symmetry breaking operators entering the mass formula to first order. An important result is that the hierarchy of masses as a function of strangeness is correctly reproduced for all multiplets. Predictions for unknown excited hyperons to $SU(6) \times O(3)$ multiplets are made.

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I. INTRODUCTION

Understanding the baryon resonances and their group theory classification is an essential and current topic in hadronic physics. It is well known that the number of observed resonances is smaller than the number of excited baryons predicted by the quark model. The number of "missing" resonances is much larger in the strange sector. The question is whether or not the missing hyperons with strangeness $\mathcal{S} = -1, -2, -3$ are due to the lack of experimental data or due to models based on SU(3) symmetry breaking. Experimentally, the hyperons are difficult to produce. In particular, for $\mathcal{S} = -2$ hyperons, kaon-nucleon or Σ-hyperon induced reactions are required, and the planned kaon beams at Thomas Jefferson National Acceleration Facility (JLAB) and the Japan Proton Accelerator Research Complex (J-PARK) are expected to improve the situation [1].

Here, we discuss a theoretical approach attempting to make an SU(3) classification of excited baryons in the framework of the $1/N_c$ expansion method, where N_c is the number of colors. This method, proposed by 't Hooft [2] and applied to baryons by Witten [3], is a powerful tool to study baryon spectroscopy. The underlying symmetry is $SU(2N_f)$ which results from the discovery that, for N_f flavors, the ground state baryons display an exact contracted $SU(2N_f)$ spin-flavor symmetry in the large N_c limit of QCD [4,5]. The Skyrme model, the strong coupling theory [6] and the static quark model share a common symmetry with QCD baryons in the large N_c limit [7].

The $1/N_c$ expansion method has been applied with great success to the ground state baryons, described by the symmetric representation **56** of SU(6) [5,8–12]. At $N_c \to \infty$, the ground state baryons are degenerate. At large, but finite N_c, the mass splitting starts at order $1/N_c$ as first observed in Ref. [7]. For a review regarding the ground state, see, for example, Ref. [13].

The extension of the $1/N_c$ expansion method to excited states requires the symmetry group $SU(2N_f) \times O(3)$ [14], in order to introduce orbital excitations. The practice shows that the experimentally observed resonances can approximately be classified as $SU(2N_f) \times O(3)$ multiplets, grouped into excitation bands, $N = 1, 2, 3, \ldots$, each band containing a number of $SU(6) \times O(3)$ multiplets, as in quark models. In addition, lattice QCD studies have shown that the number of each spin and flavor states in the lowest energy bands is in agreement with the expectations based on a weakly broken $SU(6) \times O(3)$ symmetry [15], used in quark models and in the treatment of excited states in large N_c QCD. Presently, the lattice QCD report errors bars on the baryon masses larger than the next order corrections in the mass formula of the $1/N_c$ expansion [16].

Some symmetric multiplets of $SU(6) \times O(3)$, in particular $[\mathbf{56}, 2^+]$ and $[\mathbf{56}, 4^+]$, containing two and four units of orbital excitations, were analyzed by analogy to the ground state in Refs. [17] and [18] respectively. In this case, the splitting starts at order $1/N_c$ as well.

For mixed symmetric states, the situation is more intricate. Two approaches have been proposed so far. The first one is based on the Hartree approximation and describes the N_c quark system as a ground state symmetric core of $N_c - 1$ quarks and an excited quark [19]. This implies the split of $SU(2N_f)$ generators into two parts, one acting on the core and the other on the excited quark. Naturally, the number of generators entering the mass formula becomes larger, and hence the applicability of the method beyond the $N = 1$ band becomes more problematic [20].

The second procedure, where the Pauli principle is implemented to all N_c identical quarks, has been proposed in Refs. [21,22]. There is no physical reason to separate the excited quark from the rest of the system. The method can straightforwardly be applied to all excitation bands N. It requires the knowledge of the matrix elements of all the $SU(2N_f)$ generators acting on mixed symmetric states described by the partition $[N_c - 1, 1]$. In both cases, the

[*]fstancu@ulg.ac.be

mass splitting starts at order N_c^0. The latest achievements for the ground state and the current status of large N_c excited baryons can be found in Ref. [23].

The present work considers as an example the mixed symmetric $[\mathbf{70}, \ell^+]$ multiplet in the spirit of the procedure of Refs. [21,22]. This multiplet has already been analyzed in Ref. [24] by using the 2014 version of the Review of Particle Properties (PDG2014) [25]. We use the same formalism as in Ref. [24] but propose a new assignment to the $\Lambda(2110)5/2^{+***}$ resonance. Here, we suggest that it belongs to the quartet ${}^4\Lambda[\mathbf{70}, 2^+]\frac{5}{2}^+$ instead of the ${}^4\Lambda[\mathbf{70}, 2^+]\frac{5}{2}^+$ doublet. In addition, for its experimental mass, we use the average value of the 2016 Review of Particle Properties (PDG2016) [26] instead of the mass found by Zhang *et al.* [27]. This cures the previous anomaly that in some sectors the hyperon Λ appears with a smaller mass than the nucleon partner [24]. As a benefit, predictions for a few unknown hyperons are made.

In Sec. II, we recall the mass formula of the $1/N_c$ expansion, and in Sec. III, we shortly review the applications of the method to $N = 1$, 2, 3 and 4 excitation bands. The matrix elements of the SU(3) flavor symmetry breaking operators B_i for the mixed symmetric $[\mathbf{70}, \ell^+]$ multiplet of the $N = 2$ band are presented in Sec. IV. The spectrum of $[\mathbf{70}, \ell^+]$ is reanalyzed in Sec. V. The last section is devoted to conclusions.

II. MASS OPERATOR

The general form of the mass operator, where the SU(3) symmetry is broken, has the following form [12]:

$$M = \sum_i c_i O_i + \sum_i d_i B_i. \tag{1}$$

The rotational invariant operators O_i are defined as the scalar products

$$O_i = \frac{1}{N_c^{n-1}} O_\ell^{(k)} \cdot O_{SF}^{(k)}, \tag{2}$$

where $O_\ell^{(k)}$ is a k-rank tensor in SO(3) and $O_{SF}^{(k)}$ is a k-rank tensor in SU(2) spin, but invariant in SU(N_f). For the ground state, one has $k = 0$. The excited states also require $k = 1$ and $k = 2$ terms. The $k = 1$ tensor components are the generators L^i of SO(3). In a spherical basis, the components of the $k = 2$ tensor operator of SO(3) (i, $j = -1$, 0, 1) read (see the Appendix)

$$L^{(2)ij} = \frac{1}{2}\{L^i, L^j\} - \frac{1}{3}(-)^i \delta_{i,-j} \vec{L} \cdot \vec{L}. \tag{3}$$

The operators $O_{SF}^{(k)}$ are constructed from the SU(N_f) generators, S^i, T^a and G^{ia} obeying the su($2N_f$) algebra

$$[S^i, S^j] = i\varepsilon^{ijk} S^k, \quad [T^a, T^b] = i f^{abc} T^c, \quad [S^i, T^a] = 0,$$
$$[S^i, G^{ja}] = i\varepsilon^{ijk} G^{ka}, \quad [T^a, G^{jb}] = i f^{abc} G^{jc},$$
$$[G^{ia}, G^{jb}] = \frac{i}{4}\delta^{ij} f^{abc} T^c + \frac{i}{2}\varepsilon^{ijk}\left(\frac{1}{N_f}\delta^{ab} S^k + d^{abc} G^{kc}\right), \tag{4}$$

In the symmetric core + excited quark procedure [19], each SU($2N_f$) generator is split into two parts,

$$S^i = S_c^i + s^i, \quad T^a = T_c^a + t^a, \quad G^{ia} = G_c^{ia} + g^{ia}, \tag{5}$$

where the operators carrying a lower index c act on a symmetric ground state core and s^i, t^a and g^{ia} act on the excited quark. The procedure has the algebraical advantage that it reduces the problem of the knowledge of the matrix elements of the SU($2N_f$) generators acting on a system described by a mixed symmetric representation of SU($2N_f$) to the knowledge of the matrix elements of S_c^i, T_c^a and G_c^{ia}, acting on symmetric states of partition $[N_c - 1]$, which are simpler to find than the matrix elements of the SU($2N_f$) generators for $[N_c - 1, 1]$ mixed symmetric states. Then, the operator reduction rules for the ground state [10] may be used for the core operators. However, the number of terms to be included in operators describing observables remains usually very large as compared to experimental data, so that the method cannot easily be applied to mixed symmetric highly excited baryons. It should be remembered that the spin-orbit operator O_2 of symmetric multiplets is defined in terms of angular momentum L^i components acting on the whole system as in Ref. [17] and is order $\mathcal{O}(1/N^c)$,

$$O_2 = \frac{1}{N_c} L \cdot S, \tag{6}$$

while for mixed symmetric multiplets, it is defined as a single-particle operator [19],

$$O_2 = \ell \cdot s = \sum_{i=1}^{N_c} \ell(i) \cdot s(i), \tag{7}$$

the matrix elements of which are order $\mathcal{O}(N_c^0)$. The reason to mention O_2 is that, although its contribution to the mass is generally small, like in quark models, here it plays an important role in proving the compatibility between the meson-nucleon scattering picture and the quark model-type picture, legitimating in this way the extension of the $1/N^c$ expansion to excited states of mixed symmetry [28].

An extra complication for $N_f = 3$ (u, d, s quarks) is that the effects of the SU(3) flavor symmetry breaking are comparable to $1/N_c$ corrections. The second term in the mass formula (1) is designed to introduce the symmetry breaking. The operators B_i break the SU(3) flavor symmetry and are defined to have zero expectation values for

TABLE I. List of dominant operators and their coefficients (MeV) c_i and d_i from the mass formula (1) obtained in a numerical fit for the $[\mathbf{70}, \ell^+]$ multiplet. The spin-orbit operator O_2 is defined by Eq. (7) for $[\mathbf{70}, \ell^+]$.

Operator	Coefficient (MeV)
$O_1 = N_c \mathbb{1}$	630 ± 11
$O_2 = \ell \cdot s$	62 ± 26
$O_3 = \frac{1}{N_c} S^i S^i$	95 ± 31
$O_4 = \frac{1}{N_c} [T^a T^a - \frac{1}{12} N_c (N_c + 6)]$	108 ± 43
$O_6 = \frac{1}{N_c} L^{(2)ij} G^{ia} G^{ja}$	137 ± 57
$B_1 = n_s$	40 ± 33
$B_2 = \frac{1}{N_c} (L^i G^{i8} - \frac{1}{2\sqrt{3}} L^i S^i)$	-37 ± 122
$B_3 = \frac{1}{N_c} (S^i G^{i8} - \frac{1}{2\sqrt{3}} S^i S^i)$	60 ± 162
$\chi^2_{\rm dof}$	1.80

nonstrange baryons. The SU(3) flavor symmetry breaking is implemented at order $\mathcal{O}(\epsilon)$ where $\epsilon \sim 0.3$ is a measure of the SU(3) flavor symmetry breaking by the strange quark mass [12]. Thus, ϵ and $1/N_c$ at $N_c = 3$ are of similar size, and both corrections have to be included. Corrections of order ϵ/N_c are neglected.

In the context of our approach, where the baryon is treated as a system of N_c quarks irrespective of its spin-flavor symmetry, the SU(3) breaking operators are defined as

$$B_1 = n_s, \tag{8}$$

where n_s is the number of strange quarks and

$$B_2 = \frac{1}{N_c}\left(L^i G^{i8} - \frac{1}{2\sqrt{3}} L \cdot S\right), \tag{9}$$

$$B_3 = \frac{1}{N_c}\left(S^i G^{i8} - \frac{1}{2\sqrt{3}} S \cdot S\right), \tag{10}$$

where the angular momentum operator L^i, the spin operator S^i and the component 8 of the spin-flavor operator G^{i8} act on the entire system of N_c quarks.

Then, in Eq. (1), the coefficients c_i encode the quark dynamics, and d_i measure the SU(3) breaking. They are determined from a numerical fit to data. An example, containing the commonly used O_i and B_i operators together with the coefficients c_i and d_i, can be found in Table I.

III. STATUS OF EXCITED HYPERONS IN THE $1/N_c$ EXPANSION

Here, we briefly recall some important achievements in the study of baryons spectra for the $N = 1$, 2, 3 and 4 bands.

A. $N = 1$ band

The $N = 1$ band has been the most studied so far. It is the best known experimentally, and it contains only one SU(6) × O(3) multiplet, the $[\mathbf{70}, \mathbf{1}^-]$. The first application of the $1/N_c$ expansion was a phenomenological analysis of strong decays of resonances with one unit of orbital excitation [29]. There were no operators to distinguish the strange quark from u and d, but the decay of some hyperons was considered via an explicit SU(3)-flavor breaking.

In the symmetric core + excited quark procedure, the $N_f = 3$ case has been thoroughly studied by Goity *et al.* [30] where 11 SU(3) exact flavor symmetry and 4 first order SU(3)-flavor symmetry breaking operators were included. Two of them, proportional to the generators t^8 and T^8_c, thus giving a measure of the strangeness, bring significant contributions, and the other two bring small contributions. The fit was made to 19 empirical quantities (17 masses and 2 mixing angles) associated to three- and four-star resonances. Predictions were made for unknown hyperons having strangeness $\mathcal{S} = -1$, -2 and -3. The masses of $\Lambda(1405)$ and $\Lambda(1520)$ were well reproduced, but this was due to the simplicity of the wave function in the symmetric core + excited quark procedure where the part corresponding to $S_c = 1$ is missing [23]. In addition, one should note the absence of the pure flavor operator $t \cdot T_c$, coupling the core flavor operator T_c to the excited quark flavor operator t.

A much smaller number of operators was needed for the $[\mathbf{70}, \mathbf{1}^-]$ multiplet in the approach of Refs. [21,22]. There were seven exact SU(3)-flavor symmetries, one SU(3)-flavor symmetry breaking representing the total strangeness and one isospin breaking operator. This approach, based on an exact wave function, accommodates a slightly heavier $\Lambda(1405)$ at 1421 ± 14 MeV. However, both procedures predict too large a mass (of 1790 MeV in Ref. [31]) for the three-star puzzling $\Xi(1690)$ resonance, a situation similar to quark models [32]. The Skyrme model gives a lower mass and possibly a more natural interpretation of $\Xi(1690)$ [33].

We note that in both approaches the Λ-N splitting is similar, around 150 MeV for octets. In decuplets, the Σ-Δ splitting is about 130 MeV in Ref. [30] and about 170 MeV in Ref. [31] where a different choice of B_i operators has been made, as implied by arguments given in the Introduction.

B. $N = 2$ band

The $N = 2$ band has the following multiplets: $[\mathbf{56}', 0^+]$, $[\mathbf{56}, 2^+]$, $[\mathbf{70}, 0^+]$, $[\mathbf{70}, 2^+]$ and $[\mathbf{20}, 1^+]$. The observed resonances are usually assigned to the symmetric $[\mathbf{56}]$ or the mixed symmetric $[\mathbf{70}]$ SU(6) multiplets. The antisymmetric SU(6) × O(3) multiplet $[\mathbf{20}, 1^+]$ has been ignored so far, on the basis that it does not have a real counterpart.

The multiplet $[\mathbf{56}', 0^+]$ describes states with a radial excitation, in particular the Roper resonance. It was the first to be studied in the large N_c limit [34], by using a simplified mass formula of the Gürsey-Radicati type. The analysis was free of any assumption regarding the wave function except its symmetry in SU(6). Strong decay widths were calculated as well.

The analysis of the $[\mathbf{56}, 2^+]$ baryon masses has first been performed in Ref. [17]. It has been reconsidered in Ref. [18] with nearly identical results, and the analysis has been extended to the higher multiplet $[\mathbf{56}, 4^+]$ of the $N = 4$ band in the same paper.

The $[\mathbf{70}, 0^+]$ and $[\mathbf{70}, 2^+]$ baryon masses were first analyzed in Ref. [35] for $N_f = 2$ and extended in Ref. [20] to $N_f = 3$, both studies being performed within the symmetric core + excited quark procedure [19]. The $[\mathbf{70}, \ell^+]$ ($\ell = 0, 2$) multiplets were revisited [36] within the approach of Ref. [21] where the Pauli principle was fully taken into account.

In Refs. [35] and [36], Regge-type trajectories have been drawn for the most dominant coefficient in the mass formula, c_1 and c_1^2 respectively, and somewhat conflicting results have been obtained. The trajectories were derived as a function of the band number $N = 0$, 1, 2, 3 and 4. While in Ref. [35] a single trajectory has been obtained (note that large N_c results for the $N = 3$ band were not available yet), in Ref. [36], two distinct, nearly parallel, Regge trajectories have been obtained, the lower one for symmetric [**56**]-plets and the higher one for mixed symmetric [**70**]-plets.

In Ref. [24], a combined analysis of the $[\mathbf{56}, 2^+]$ and $[\mathbf{70}, \ell^+]$ multiplets of the $N = 2$ band has been made. An important aspect was that the same set of linearly independent operators in the mass formula has been used which was not the case before. Distinct Regge trajectories resulted again. The data were from PDG2014 which sometimes gives more precise values for the resonance masses with smaller error bars than before.

C. $N = 3$ and 4 bands

The $N = 3$ band contains eight $\mathrm{SU}(6) \times \mathrm{O}(3)$ multiplets [37]. Those belonging to the mixed symmetric $[\mathbf{70}, \ell^-]$ multiplets ($\ell = 1$, 2, 3) were studied in Ref. [38]. They were all nonstrange baryons. It is premature to perform an extended $1/N_c$ analysis to the $N = 3$ band, due to lack of experimental data.

The $N = 4$ band has 17 $\mathrm{SU}(6) \times \mathrm{O}(3)$ multiplets [39] from which only the lowest, the $[\mathbf{56}, 4^+]$ multiplet, has been analyzed in the $1/N_c$ expansion method [18]. Being described by a symmetric representation of SU(6), it is technically simple, as mentioned in the Introduction. Despite the lack of data for highly excited hyperons, tentative predictions have been made in Ref. [18] by including only B_1 and a single experimentally known hyperon, the $\Lambda(2350)9/2^{+***}$.

IV. MATRIX ELEMENTS OF B_i OPERATORS FOR $[\mathbf{70}, \ell^+]$

Here, we are concerned with the $[\mathbf{70}, \ell^+]$ multiplet. The matrix elements of O_i for $[\mathbf{70}, \ell^+]$, as a function of N_c, were derived in Ref. [36]. Note that in the case of mixed symmetric states the matrix elements of O_6 are $\mathcal{O}(N_c^0)$, in contrast to the symmetric case where they are $\mathcal{O}(N_c^{-1})$, and nonvanishing only for octets, while for the symmetric case they are nonvanishing for decuplets. Thus, at large N_c, the splitting starts at order $\mathcal{O}(N_c^0)$ for mixed symmetric states due both to O_2 and O_6.

The SU(3) flavor breaking operators B_i were chosen to have identical definitions for mixed symmetric multiplets [24] to those for symmetric multiplets [17]. The expectation value of B_1 is

$$B_1 = n_s, \tag{11}$$

where n_s is the number of strange quarks in a baryon. The diagonal matrix elements of B_2 and B_3 for $[\mathbf{70}, \ell^+]$ at arbitrary N_c were first calculated in Ref. [24] where they were exhibited in Table IV. For practical purposes, we do not reproduce that table. At $N_c = 3$, we have summarized those results by two simple analytic formulas. The diagonal matrix elements of B_2 take the following form,

$$B_2 = -n_s \frac{\langle L \cdot S \rangle}{6\sqrt{3}}, \tag{12}$$

where $\langle L \cdot S \rangle$ is the expectation value of the spin-orbit operator acting on the whole system. Thus, the contribution of B_2 is positive or negative depending on the sign of $\langle L \cdot S \rangle$. The diagonal matrix elements of B_3 take the simple analytic form

$$B_3 = -n_s \frac{S(S+1)}{6\sqrt{3}}, \tag{13}$$

where S is the total spin. The contribution of B_3 is always negative, otherwise vanishing for nonstrange baryons. These formulas can be applied to ^28_J, ^48_J, ^210_J and $^21_{1/2}$ baryons of the $[\mathbf{70}, \ell^+]$ multiplet.

Interestingly, for the decuplet members of the symmetric $[\mathbf{56}, 2^+]$ multiplet, the expressions of B_2 and B_3 at $N_c = 3$ given by Eqs. (12) and (13) of Ref. [24] are the same as those of Eqs. (12) and (13) shown above.

V. SPECTRUM OF $[\mathbf{70}, \ell^+]$

Presently, we use the PDG2016 [26] to reanalyze the mixed symmetric multiplet $[\mathbf{70}, \ell^+]$ with $\ell = 0$ or 2. The values of the fitted coefficients c_i and d_i are exhibited in Table I together with the value of $\chi^2_{\mathrm{dof}} = 1.80$. The results can only roughly be compared to those presented in Table I, Fit 2, of Ref. [36], because B_2 and B_3 were missing there. Note that the factor 15 of O_6 has been removed here, which

TABLE II. Partial contribution and the total mass (MeV) predicted by the $1/N_c$ expansion with matrix elements of O_i from Ref. [24] and of B_i given in the text. The column Ref. [24] gives the total mass of Ref. [24]. The last two columns give the empirically known masses and status from the 2016 Review of Particles Properties [26] unless specified by (A) from Ref. [40], (L) from Ref. [42], (G1) from Ref. [43], (B) from Ref. [44], (AB) from Ref. [41] and (G2) from Ref. [45].

	Partial contribution (MeV)											
	c_1O_1	c_2O_2	c_3O_3	c_4O_4	c_6O_6	d_1B_1	d_2B_2	d_3B_3	Total (MeV)	Ref. [24]	Experiment (MeV)	Name, status
$^4N[\mathbf{70},2^+]\frac{7}{2}^+$	1889	62	118	27	−23	0	0	0	2073 ± 38	2080 ± 39	2060 ± 65(A)	$N(1990)7/2^{+**}$
$^4\Lambda[\mathbf{70},2^+]\frac{7}{2}^+$						40	11	−22	2102 ± 19	2105 ± 19	2100 ± 30(L)	$\Lambda(2020)7/2^{+*}$
$^4\Xi[\mathbf{70},2^+]\frac{7}{2}^+$						79	22	−43	2131 ± 8	2130 ± 8	2130 ± 8	$\Xi(2120)?^{?*}$
$^4N[\mathbf{70},2^+]\frac{5}{2}^+$	1889	−10	118	27	57	0	0	0	2081 ± 33	2042 ± 41	2000 ± 50	$N(2000)5/2^{+**}$
$^4\Lambda[\mathbf{70},2^+]\frac{5}{2}^+$						40	−2	−22	2097 ± 18	2009 ± 40	2110 ± 20	$\Lambda(2110)5/2^{+***}$
$^4\Xi[\mathbf{70},2^+]\frac{5}{2}$						79	−4	−43	2113 ± 41			
$^4N[\mathbf{70},2^+]\frac{3}{2}^+$	1889	−62	118	27	0	0	0	0	1972 ± 29	1955 ± 32		
$^4\Lambda[\mathbf{70},2^+]\frac{3}{2}^+$					0	40	−11	−22	1979 ± 42			
$^4\Xi[\mathbf{70},2^+]\frac{3}{2}$						79	−22	−43	1986 ± 99			
$^4N[\mathbf{70},2^+]\frac{1}{2}^+$	1889	−93	118	27	−80	0	0	0	1861 ± 33	1878 ± 34	1870 ± 35(A)	$N(1880)1/2^{+**}$
$^4\Lambda[\mathbf{70},2^+]\frac{1}{2}^+$						40	−16	−22	1863 ± 79			
$^4\Xi[\mathbf{70},2^+]\frac{1}{2}^+$						79	−32	−43	1865 ± 153			
$^2N[\mathbf{70},2^+]\frac{5}{2}^+$	1889	21	23	27	0	0	0	0	1960 ± 29	1959 ± 29	$1860 \pm^{120}_{60}$ (A)	$N(1860)5/2^{+**}$
$^2\Sigma[\mathbf{70},2^+]\frac{5}{2}^+$					0	40	4	−4	2000 ± 18	2031 ± 11	2051 ± 25(G1)	$\Sigma(2070)5/2^{+*}$
$^2\Xi[\mathbf{70},2^+]\frac{5}{2}$						79	7	−8	2038 ± 45			
$^2N[\mathbf{70},2^+]\frac{3}{2}^+$	1889	−31	23	27	0	0	0	0	1908 ± 21	1902 ± 22	1900 ± 30(A)	$N(1900)3/2^{+***}$
$^2\Sigma[\mathbf{70},2^+]\frac{3}{2}^+$					0	40	−6	−4	1938 ± 16	1933 ± 11	1941 ± 18	$\Sigma(1940)?^{?*}$
$^2\Xi[\mathbf{70},2^+]\frac{3}{2}^+$					0	79	−11	−8	1968 ± 7	1964 ± 70	1967 ± 7(B)	$\Xi(1950)?^{?***}$
$^4N[\mathbf{70},0^+]\frac{3}{2}^+$	1889	0	118	27	0	0	0	0	2034 ± 18	2024 ± 20	2040 ± 28(AB)	$N(2040)3/2^{+*}$
$^4\Sigma[\mathbf{70},0^+]\frac{3}{2}^+$						40	0	−22	2052 ± 22	2000 ± 23	2100 ± 69	$\Sigma(2080)3/2^{+**}$
$^4\Xi[\mathbf{70},0^+]\frac{3}{2}^+$						79	0	−43	2070 ± 46			
$^2\Delta[\mathbf{70},2^+]\frac{5}{2}^+$	1889	−21	24	134	0	0	0	0	2026 ± 48	2086 ± 37	1962 ± 139	$\Delta(2000)5/2^{+**}$
$^2\Sigma^*[\mathbf{70},2^+]\frac{5}{2}^+$					0	40	3	−4	2065 ± 52			
$^2\Xi^*[\mathbf{70},2^+]\frac{5}{2}^+$					0	79	7	−8	2104 ± 73			
$^2\Delta[\mathbf{70},0^+]\frac{1}{2}^+$	1889	0	24	134	0	0	0	0	2047 ± 49			
$^2\Sigma^*[\mathbf{70},0^+]\frac{1}{2}^+$		0				40	0	−4	2083 ± 46	2119 ± 25	1902 ± 96	$\Sigma(1880)1/2^{+**}$
$^2\Sigma^*[\mathbf{70},0^+]\frac{1}{2}^+$		0				79	0	−8	2118 ± 53			
$^2\Lambda'[\mathbf{70},2^+]\frac{5}{2}^+$	1889	62	24	−81	0	40	3	−4	1933 ± 47			
$^2\Lambda'[\mathbf{70},0^+]\frac{1}{2}^+$	1889	0	24	−81	0	40	0	−4	1868 ± 43	1865 ± 19	1853 ± 20(G2)	$\Lambda(1810)1/2^{+***}$

explains the larger value of c_6 now. In fact, the product c_6O_6 matters in the mass. The value of c_2 is similar to that of Ref. [36]. The $1/N_c$ corrections are dominated by O_3 in octets and by O_4 in decuplets. The SU(3) flavor breaking is dominated by B_1 for all hyperons.

The PDG2016 as well as PDG2014 incorporate the new multichannel partial wave analysis of the Bonn-Gatchina group [40]. Accordingly, the resonance $P_{13}(1900)$ has been upgraded from two to three stars with a Breit-Wigner mass of 1905 ± 30 MeV. The resonance $N(2000)5/2^+$ has been split into two two-star resonances, namely $N(1860)5/2^+$ and $N(2000)5/2^+$, with masses indicated in Table II. There is a new one-star resonance $N(2040)3/2^+$ observed in the decay $J/\psi \to p\bar{p}\pi^0$ [41]. There is also a new two-star resonance $N(1880)1/2^+$ observed by the Bonn-Gatchina group with a mass of 1870 ± 35 MeV [40].

In a previous work [36], only 11 resonances have been included in the numerical fit. Here, as well as in Ref. [24], 16 resonances have been included, with a status of three, two or one star. These extra resonances are the hyperons $\Xi(2120)?^{?*}$, $\Sigma(2070)5/2^{+*}$, $\Sigma(1940)?^{?*}$, $\Xi(1950)?^{?***}$ and $\Sigma(2080)3/2^{+**}$. For the three-star resonances, we use the Breit-Wigner mass of PDG2016 except for $\Xi(1950)?^{?***}$ where we take the value found in Ref. [44] which reduces the $\chi^2_{\rm dof}$ value from 1.96 to 1.80. For the spectrum, such a choice would not make much difference.

For the resonances omitted from the summary table of PDG2016, the masses and the error bars considered in the fit correspond to averages over those data taken into account in the particle listings, except for a few which favor specific experimental values cited in the headings of Table II.

The $N(1710)1/2^{+***}$ and $\Sigma(1770)1/2^{+*}$ resonances have been ignored in this fit. The theoretical argument is that their masses are too low, leading to unnatural sizes for the coefficients c_i or d_i [46]. Experimentally, the controversial $N(1710)1/2^{+***}$ resonance has not been seen in the latest George Washington University (GWU) analysis of Arndt *et al.* [47]. We have also ignored $\Delta(1750)1/2^{+*}$, inasmuch as neither Arndt *et al.* [47] nor Anisovich *et al.* [40] find evidence for it.

The partial contributions and the calculated total masses obtained from the fit are presented in Table II. One can see that the fit is generally good except for $\Sigma(1880)1/2^{+**}$ where the calculated mass somewhat too high. The operator B_2 has a vanishing expectation value, and the contribution of B_3, although negative, is negligible. The mass of the $N(1860)5/2^{+**}$ seems large, too, but it is within the large error bars of Ref. [40].

The good fit for the $N(1880)1/2^{+**}$ resonance was due to the negative contributions of -93 and -80 MeV of the spin-orbit operator O_2 and of O_6 operators respectively. However, its strange partners are almost degenerate because the positive contribution of B_1 is accidentally cancelled out by the negative contribution of $B_2 + B_3$.

The assignment of $\Sigma(1940)?^{?*}$ and $\Xi(1950)?^{?***}$ to the $^2[\mathbf{70}, 2^+]3/2^+$ multiplet seems reasonable. Thus, these resonances may have $J^P = 3/2^+$, hopefully to be confirmed experimentally in future analyses.

Some predictions have also been made for experimentally unknown strange partners in octets and decuplets. Note that Λ and Σ are degenerate in our approach because the expectation values of B_2 and B_3 are identical at $N_c = 3$, although they are different at arbitrary N_c. This is not the case for the $[\mathbf{56}, 2^+]$ multiplet. Also, the total contribution of B_i is generally of about 30 MeV which is much less than for the $[\mathbf{56}, 2^+]$ multiplet. We did not present predictions for the Ω's in the $[\mathbf{70}, \ell^+]$ multiplet because we thought them irrelevant at this stage of theory and experiment.

A useful remark is that the contributions of B_2 and B_3 mutually cancel out for hyperons belonging to decuplets with $\ell \neq 0$. In that case, B_1 is enough in the mass formula, like in Ref. [36]. The contributions of B_2 and B_3 are generally small. This is due to the smallness of the coefficients d_2 and d_3 of Table I, having sizes of a similar order of magnitude to the corresponding ones from Ref. [30] obtained for the $N = 1$ band in the excited quark + ground state core method.

Presently, the smaller negative contribution of B_3 [see Eq. (13)] makes the hyperons masses larger than those derived in Ref. [24] and helps in restoring the correct hierarchy as a function of strangeness.

It is important to make a comparison between the present results and those of Ref. [24] where a different assignment and mass have been chosen for $\Lambda(2110)5/2^{+***}$. For this purpose, we have included in Table II the column called Ref. [24] which gives the total masses obtained in our previous work. One can notice that presently the fit to the resonances $N(2000)5/2^{+**}$ and $\Sigma(2070)5/2^{+*}$ slightly deteriorates, which may be a reason for the increase of $\chi^2_{\rm dof}$ from 1.48 to 1.80. Note that all these resonances have $J = 5/2^+$.

Our suggestions for assignments of resonances in the $[\mathbf{70}, \ell^+]$ multiplet can be compared to those made in Ref. [48] as educated guesses. The assignment of $\Sigma(1880)1/2^{+**}$ as a $[\mathbf{70}, 0^+]1/2^+$ decuplet resonance is confirmed as well as the assignment of $\Lambda(1810)1/2^{+***}$ as a flavor singlet. We agree with Ref. [48] regarding $\Lambda(2110)5/2^{+***}$ as a partner of $N(2000)5/2^{+**}$ in a spin quartet. We disagree with Ref. [48] that $N(1900)3/2^{+***}$ is a member of a spin quartet. We propose it as a partner of $\Sigma(1940)?^{?*}$ and $\Xi(1950)?^{?***}$ in a spin doublet.

However, one has to keep in mind that at the same J spin doublets and quartets can mix, for example, for $N[70, 2^+]$ at $J = 3/2$ or $5/2$. The mixing would be due to the off-diagonal matrix elements of the spin-orbit operator O_2 and the tensor operator O_6.

The problem of assignment is not trivial. Within the $1/N_c$ expansion method, Ref. [17] suggested that $\Sigma(2080)3/2^{+**}$ and $\Sigma(2070)5/2^{+*}$ could be members of two distinct decuplets in the $[\mathbf{56}, 2^+]$ multiplet, while here and in Ref. [48], they seem to be good candidates for mixed symmetric states.

VI. CONCLUSIONS

The inclusion of three SU(3) symmetry breaking operators, B_1, B_2 and B_3, in the mass formula of the $[\mathbf{70}, \ell^+]$ multiplet helps to brings more insight into the $\mathrm{SU}(6) \times \mathrm{O}(3)$ classification of highly excited baryons when accompanied by realistic assignments. Presently, it seems that the evolution of the Λ-N or Σ-N splitting with excitation energy in baryon multiplets described by the $1/N_c$ expansion remains an open problem.

Alternative suggestions for assignments of the known baryons should be studied, and more data for excited hyperons are highly desirable. The continuing study of the presently available data and the production of new hyperons are needed for understanding the structure of baryons and disentangle between various models. At the Workshop on Physics with the Neutral Kaon beam at JLAB [1], it was pointed out that a K_L beam at JLAB would open new opportunities for studying excited hyperons which may help in understanding the multiplet structure of excited baryons. Similar hopes are at J-PARK.

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APPENDIX: THE SECOND RANK TENSOR OF SO(3)

In this Appendix, we derive the expression (3) of the second rank tensor $L^{(2)ij}$ of SO(3) in a spherical basis. Let us denote the spherical components of the SO(3) generators by L^i. Then, the product L^iL^j can be written as

$$L^iL^j = \sum_{k=0}^{2}\sum_{\mu=-k}^{k} C^{1\ 1\ k}_{i\ j\ \mu} T^k_\mu, \tag{A1}$$

in terms of a Clebsch-Gordan coefficient and the irreducible k-rank tensor T^k_μ. In the anticommutator $\{L^i, L^j\}$, only the tensors $k = 0$ and 2 survive for symmetry reasons. Then, one can write

$$\frac{1}{2}\{L^i, L^j\} = \sum_\mu C^{1\ 1\ 2}_{i\ j\ \mu} T^2_\mu + C^{1\ 1\ 0}_{i\ j\ 0} T^0_0. \tag{A2}$$

The second term contains the Clebsch-Gordan coefficient

$$C^{110}_{ij0} = (-)^{1-i}\frac{1}{\sqrt{3}}\delta_{i,-j}. \tag{A3}$$

The standard definition of T^0_0 is (see, for example, Eq. (4.7) of Ref. [49])

$$T^0_0 = -\frac{1}{\sqrt{3}}\vec{L}\cdot\vec{L}. \tag{A4}$$

Then, shifting the second term of Eq. (A2) from right to left, we obtain the second rank tensor $L^{(2)ij}$ of SO(3) as

$$L^{(2)ij} = \sum_\mu C^{1\ 1\ 2}_{i\ j\ \mu} T^2_\mu, \tag{A5}$$

or alternatively

$$L^{(2)ij} = \frac{1}{2}\{L^i, L^j\} - \frac{1}{3}(-)^i\delta_{i,-j}\vec{L}\cdot\vec{L}. \tag{A6}$$

Equation (A5) can be used to calculate the matrix elements of $L^{(2)ij}$ defined as an irreducible second rank tensor. Using the Wigner-Eckart theorem and a spherical harmonic basis, one has

$$\langle \ell' m' | L^{(2)ij} | \ell m\rangle = \sum_{\mu,m} C^{112}_{ij\mu} C^{\ell 2 \ell'}_{m\mu m'} \langle \ell' || T^2 || \ell\rangle. \tag{A7}$$

The reduced matrix element $\langle \ell' || T^2 || \ell\rangle$ can be easily calculated. The result leads to

$$\langle \ell' m' | L^{(2)ij} | \ell m\rangle = \delta_{\ell'\ell}\sqrt{\frac{\ell(\ell+1)(2\ell-1)(2\ell+3)}{6}} \times \sum_{\mu,m} C^{1\ 1\ 2}_{i\ j\ \mu} C^{\ell\ 2\ \ell'}_{m\ \mu\ m'}, \tag{A8}$$

which has been used in deriving the matrix elements of O_6 and is consistent with Eq. (A5) of Ref. [19].

Equation (A6) indicates that the definition of $L^{(2)ij}$ from Ref. [35] contains a typographic error in the second term on the right-hand side; namely the phase $(-)^i$ is missing. Previous and present results are not affected by this inadvertence.

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