Angular momentum distribution in the transverse plane

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We discuss several possibilities to relate the t dependence of generalized parton distributions to the distribution of angular momentum in the transverse plane. Using a simple spectator model, we demonstrate that none of them correctly describes the orbital angular momentum distribution that for a longitudinally polarized nucleon is obtained directly from light-front wave functions.

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I. INTRODUCTION

Since the famous European Muon Collaboration experiment [1] there has been great interest in understanding the contributions from orbital angular momentum and from gluon spin to the nucleon spin. Meanwhile, generalized parton distributions (GPDs) have been introduced as a novel tool to describe the internal structure of hadrons. For more than a decade, there has been a strong interest in GPDs as many observables can be linked to them. Specifically, GPDs have been used extensively since they were first identified with the total angular momentum of the quarks and gluons within a nucleon. The total angular momentum carried by the quarks is calculated using the second moment of GPDs as [2]

$$J^z = \frac{1}{2}\int dxx[H(x,0,0) + E(x,0,0)]. \tag{1}$$

However, this famous Ji relation or sum rule yields the z component of the total angular momentum of the quarks in a nucleon that is polarized in the $+z$ direction only when GPDs are extrapolated to $t = 0$ in the general expression

$$J(t) \equiv \frac{1}{2}\int dxx[H(x,\xi,t) + E(x,\xi,t)], \tag{2}$$

where $t = \Delta^2$, x is the light-cone momentum fraction carried by the quark (averaged between the initial and final states), $\xi \equiv \frac{p^+ - p'^+}{p^+ + p'^+} = -\frac{\Delta^+}{p^+ + p'^+}$ is the longitudinal momentum transfer, p and p' are initial and final state momenta of the nucleon, respectively, and $\Delta \equiv p' - p$ is the momentum transfer.

GPDs have also been used to visualize nucleons in three dimensions after doing the suitable Fourier transform (FT) of these GPDs [3–5]. These images are in a space where one dimension describes the light-cone momentum fraction (x) and the other two dimensions describe the transverse position ($\vec{b}_\perp$) of the parton (relative to the transverse center of momentum). This distribution of partons in the transverse plane has a probabilistic interpretation in the same sense and with the same limitations as the usual parton distributions. For an unpolarized nucleon, the two-dimensional FT of GPD $H(x,0,t)$ in the transverse plane reads [3]

$$q(x,\vec{b}_\perp) = \int \frac{d^2\vec{\Delta}_\perp}{(2\pi)^2} e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp} H(x,0,-\vec{\Delta}_\perp^2), \tag{3}$$

where the impact parameter $\vec{b}_\perp$, which is the Fourier conjugate to $\vec{\Delta}_\perp$, is defined in the two-dimensional transverse plane perpendicular to the light-cone direction. Here, $|\vec{b}_\perp| \equiv b_\perp$ is introduced as the displacement of the active quark (q) from the transverse center of momentum of the entire nucleon. The transverse center of momentum $\vec{R}_\perp$ is defined as the weighted average of the transverse positions of all partons, where the weight factor is the respective momentum fraction [6]

$$\vec{R}_\perp = \sum_{i\in q,g} x_i \vec{r}_{\perp i} = x\vec{r}_\perp + (1-x)\vec{R}_{\perp s}, \tag{4}$$

where x and $\vec{r}_\perp \equiv \vec{r}_{\perp 1}$ are the momentum fraction and transverse position of the active quark and $1-x$ and $\vec{R}_{\perp s}$ are those of the spectator(s). Therefore, for a quark, one can write

$$\vec{b}_\perp = \vec{r}_\perp - \vec{R}_\perp = (1-x)(\vec{r}_\perp - \vec{R}_{\perp s}). \tag{5}$$

Further details of $q(x,\vec{b}_\perp)$ can be found in Refs. [3–5,7].

In the context of $t \neq 0$, there have been many discussions on the connection between the current theoretical GPD framework and deeply virtual Compton scattering (DVCS) experiments. One can find the details in Ref. [8]. Therefore, it is useful to study the t dependence of GPDs with the total angular momentum in the coordinate space to check if the partonic interpretation still holds in that space. Furthermore, the work presented in Ref. [9] (see also Ref. [10]) suggests that the three-dimensional FT of Eq. (2) can be used to

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calculate the distribution of angular momentum in coordinate space. The "chiral quark soliton model" that was used in Ref. [10] had an infinite target mass; therefore, there was no issue of relativistic corrections. The relativistic corrections would potentially be an issue upon taking the three-dimensional FT of the generalized form factors [available in Eq. (2)] for finite nucleon mass. The interpretation of the three-dimensional FT of form factors (FFs) as a distribution in three-dimensional space becomes ambiguous and suffers from relativistic corrections at length scales equal to or smaller than the Compton wavelength of the target [3,11]. However, a two-dimensional FT of FFs does not suffer from such relativistic corrections. As a corollary, one finds that the distribution of charge in the transverse plane is given by the two-dimensional FT of the Dirac form factor [11]. Because of these results, performing a two-dimensional FT of FFs to study the angular momentum distribution in the transverse plane has been suggested (for example, see Ref. [12]). Furthermore, there are potential issues due to total derivative terms linking canonical and symmetric energy momentum tensors [13]. We are therefore motivated by these investigations and suggestions, as mentioned above, to investigate the t dependence of GPDs with the total angular momentum distributions of the quark in the transverse plane for a longitudinally polarized nucleon. Before proceeding, we should note that while L_z does not commute with $\vec{b}_\perp$, it does commute with $|\vec{b}_\perp|$, and it is thus meaningful to discuss $L_z(|\vec{b}_\perp|)$.

In the following sections, we prescribe four different techniques to study the angular momentum distributions in the context of the scalar diquark model (SDQM). First, we define a two-dimensional FT of Eq. (2) as $\tilde{J}(\vec{b}_\perp)$. This technique will be referred to as the naive technique in this paper. In the second technique, using $\tilde{J}(\vec{b}_\perp)$, we derive the two-dimensional FT of the result that was originally suggested for the purpose of the three-dimensional FT in Refs. [9,10]. This technique will be referred to as the Polyakov-Goeke (PG) technique. In the third technique, we present an independent derivation of the distribution of angular momentum in the transverse plane in the infinite momentum frame (IMF). Finally, we compare these three different distributions with the one calculated directly from light-front wave functions (LFWFs) using the widely recognized Jaffe-Manohar (JM) definition for orbital angular momentum (OAM). In momentum space, this definition preserves the partonic interpretation.

II. DISTRIBUTION OF ANGULAR MOMENTUM IN THE TRANSVERSE PLANE

A. Naive technique

We take the two-dimensional FT of Eq. (2) to calculate the distribution of total angular momentum (TAM) in the transverse plane in the Drell-Yan frame as

$$\tilde{J}(\vec{b}_\perp) = \int \frac{d^2\vec{\Delta}_\perp}{(2\pi)^2} e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp} J(t = -\vec{\Delta}_\perp^2) \tag{6}$$

where the TAM distribution depends only on the distance $b_\perp \equiv |\vec{b}_\perp|$ from the origin, which is the transverse center of the entire nucleon. It is evident that

$$\int d^2\vec{b}_\perp \tilde{J}(\vec{b}_\perp) = J(t=0) \equiv J^z, \tag{7}$$

but that does not automatically imply that $\tilde{J}(\vec{b}_\perp)$ represents the distribution of TAM in the transverse plane.

B. Polyakov-Goeke technique: Two-dimensional reduction of Polyakov-Goeke prescription

The relation between the symmetric energy-momentum tensor (EMT) and the total angular momentum distribution, which is defined and available in Ref. [9], reads [2,14,15]

$$J(t) + \frac{2t}{3}\frac{d}{dt}J(t) = \int d^3\vec{b} e^{i\vec{b}\cdot\vec{\Delta}} \epsilon^{ijk} s_i b_j T_{0k}(\vec{b},\vec{s}), \tag{8}$$

where $T^{\mu\nu}$ is the EMT, $T_{0k}(\vec{b},\vec{s})$ represents the momentum distribution of the quarks within the nucleon (and thus the integrand on the right-hand side is interpreted as the angular momentum density), $\vec{b}$ is three-dimensional coordinate space, and $\vec{s}$ is the nucleon spin. In our case, the nucleon is polarized in the $+z$ direction.

References [9,10] suggest that the three-dimensional FT of Eq. (8) yields the distribution of angular momentum in three-dimensional coordinate space. As mentioned before, there is an issue of relativistic corrections for the three-dimensional FT. However, there are no such corrections to the two-dimensional FT of FFs in the infinite momentum frame. We thus define the two-dimensional FT in the coordinate space ($\vec{b}_\perp$) as

$$\rho_J^{\mathrm{PG}}(\vec{b}_\perp) \equiv \int db_z \epsilon^{ijk} s_i b_j T_{0k}(\vec{b},\vec{s}). \tag{9}$$

With the help of Eq. (6), the two-dimensional FT of Eq. (8) yields

$$\rho_J^{\mathrm{PG}}(b_\perp) = \frac{1}{3}\tilde{J}(b_\perp) - \frac{1}{3} b_\perp \frac{d}{db_\perp}\tilde{J}(b_\perp). \tag{10}$$

In the following, the total angular momentum distribution defined by Eq. (10) will be referred to as the PG technique because it was derived as the two-dimensional reduction of the result available in Ref. [9].

C. Infinite momentum frame technique

As an alternative, one can derive the distribution of TAM directly in the IMF. For this purpose, we define the EMT

$T^{\mu\nu}$ in terms of several form factors. Note that it is the same $T^{\mu\nu}$ that was used to derive Eq. (8). The form factors of the symmetric EMT (valid for quarks and gluons) read [2,10]

$$\langle p'|T^{\mu\nu}|p\rangle = \bar{u}(p')\bigg[M_2(t)\frac{P^\mu P^\nu}{M_N} + J(t)\frac{i(P^\mu\sigma^{\nu\rho} + P^\nu\sigma^{\mu\rho})}{2M_N}\Delta_\rho + d_1(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{5M_N} \pm \bar{c}(t)g^{\mu\nu}\bigg]u(p), \tag{11}$$

where $u(p)$ is the nucleon spinor, M_N is the mass of the nucleon, and $2P = p + p'$.

The z component of the total angular momentum of the quarks in terms of angular momentum density $M^{\alpha\mu\nu}$ is defined as [16]

$$J^z = \epsilon^{zxy}\int d^2\vec{b}_\perp M^{+xy}, \qquad M^{+xy} = T^{+y}b_x - T^{+x}b_y. \tag{12}$$

To define TAM distribution in the transverse plane, one needs to localize the nucleon in the transverse direction. For this purpose, states $|p^+, \vec{R}_\perp, s_z\rangle$, which are the eigenstates of the transverse center of momentum, are introduced in terms of the light-cone helicity eigenstates as

$$|p^+, \vec{R}_\perp = \vec{0}_\perp, s_z\rangle = \frac{\mathcal{N}}{(2\pi)^2}\int d^2\vec{p}_\perp |p^+, \vec{p}_\perp, s_z\rangle, \tag{13}$$

where $\mathcal{N}$ is a normalization constant satisfying $|\mathcal{N}|^2\int\frac{d^2\vec{P}_\perp}{(2\pi)^2} = 1$ [3–7].

For these transversely localized states the TAM distribution in impact parameter space (in the IMF) can be defined as

$$\rho_J^{\mathrm{IMF}}(\vec{b}_\perp) \equiv \frac{1}{p^+}\langle p^+, \vec{R}_\perp = \vec{0}_\perp, s_z|[T^{+y}(\vec{b}_\perp)b_x - T^{+x}(\vec{b}_\perp)b_y]|p^+, \vec{R}_\perp = \vec{0}_\perp, s_z\rangle. \tag{14}$$

In order to evaluate the matrix elements we note the phase factor

$$\langle p^+, \vec{p}'_\perp, s_z|T^{+j}(\vec{b}_\perp)|p^+, \vec{p}_\perp, s_z\rangle = \langle p^+, p'_\perp, s_z|T^{+j}(0_\perp)|p^+, p_\perp, s_z\rangle e^{-i\vec{b}_\perp\cdot\vec{\Delta}_\perp}, \qquad j = x, y, \tag{15}$$

where the matrix elements of the EMT are evaluated using Eq. (11). We note that only the term involving $p^+\sigma^{xy}\Delta^y$ in the second term of Eq. (11) survives, taking the matrix elements between helicity eigenstates that have the same light-cone helicity $[\langle p^+, p'_\perp, s_z|\sigma^{+j}|p^+, p_\perp, s_z\rangle = 0]$ and symmetric integration around the z axis.

Equation (14) after simplification thus yields

$$\rho_J^{\mathrm{IMF}}(b_\perp) = \mp\frac{1}{2}\bigg[b_\perp\frac{d}{db_\perp}\tilde{J}(b_\perp)\bigg], \tag{16}$$

where the TAM distribution depends only on the distance $b_\perp \equiv |\vec{b}_\perp|$.

Here, it is noted that Eq. (8), which was derived in Ref. [9], has been used in Ref. [10] to study the angular momentum density in the infinite target mass frame. Equation (8) was later modified to its relativistic version in Ref. [13]. One can derive Eq. (16) directly from the relativistic version of Eq. (8) available in Ref. [13].

III. MODEL CALCULATIONS AND RESULTS

We use the SDQM to calculate the proposed OAM densities to test if any of the densities agree with the distribution of OAM in the transverse plane obtained using light-front wave functions which has a partonic interpretation. The SDQM is not a good approximation for quantum chromodynamics (QCD). However, it is perfect to illustrate a point of principle: None of the above proposed distributions agrees with the partonic calculation to be derived below. What is particularly useful about the SDQM in this context is that maintaining Lorentz invariance (which is important here) is straightforward and there are no added complications due to the absence of gauge fields. Of course QCD is a gauge theory, but if a certain interpretation fails already in a nongauge theory—as we will demonstrate—it is very unlikely to hold in a gauge theory.

The relevant GPDs contained in Eq. (2) are calculated using the following LFWFs for the SDQM [17–19]:

$$\begin{aligned}
\psi^\uparrow_{+\frac{1}{2}}(x, \vec{k}_\perp) &= \left(M + \frac{m}{x}\right)\phi(x, \vec{k}_\perp^2),\\
\psi^\uparrow_{-\frac{1}{2}}(x, \vec{k}_\perp) &= -\frac{k^1 + ik^2}{x}\phi(x, \vec{k}_\perp^2),\\
\psi^\downarrow_{+\frac{1}{2}}(x, \vec{k}_\perp) &= \frac{k^1 - ik^2}{x}\phi(x, \vec{k}_\perp^2),\\
\psi^\downarrow_{-\frac{1}{2}}(x, \vec{k}_\perp) &= \left(M + \frac{m}{x}\right)\phi(x, \vec{k}_\perp^2),
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
\phi(\vec{k}_\perp^2) \equiv \phi(x, \vec{k}_\perp^2) &= \frac{g/\sqrt{1-x}}{M^2 - \frac{\vec{k}_\perp^2 + m^2}{x} - \frac{\vec{k}_\perp^2 + \lambda^2}{1-x}}\\
&= \frac{-gx\sqrt{(1-x)}}{\vec{k}_\perp^2 + u(\lambda^2)}\\
\text{and}\quad u(\lambda^2) &= m^2(1-x) - M^2x(1-x) + x\lambda^2.
\end{aligned} \tag{18}$$

Here, g is the Yukawa coupling and M, m, and λ are the masses of the “nucleon,” “quark,” and “diquark,” respectively. Furthermore, $\vec{k}_\perp \equiv \vec{k}_{\perp q} - \vec{k}_{\perp \mathrm{scalar}}$ represents the relative transverse momentum between the quark and the diquark (scalar). The upper wave function index $\uparrow$ refers to the helicity of the "nucleon" and the lower index to that of the "quark."

GPDs using overlap integrals of LFWFs in the Drell-Yan frame read [18]

$$H(x,0,-\vec{\Delta}_\perp^2) = \int \frac{d^2\vec{k}_\perp}{16\pi^3}[\psi^{\uparrow}_{+\frac{1}{2}}(x,\vec{k}'_\perp)\psi^{\uparrow}_{+\frac{1}{2}}(x,\vec{k}_\perp) + \psi^{\uparrow}_{-\frac{1}{2}}(x,\vec{k}'_\perp)\psi^{\uparrow}_{-\frac{1}{2}}(x,\vec{k}_\perp)] \tag{19}$$

$$= \frac{g^2}{16\pi^3}(1-x)\int d^2\vec{k}_\perp\left[\frac{(m+xM)^2}{[(\vec{k}'^2_\perp + u(\lambda^2))(\vec{k}^2_\perp + u(\lambda^2)]} + \frac{k'^1 - ik'^2}{(\vec{k}'^2_\perp + u(\lambda^2))}\cdot\frac{k^1+ik^2}{(\vec{k}^2_\perp + u(\lambda^2))}\right] \tag{20}$$

and

$$E(x,0,-\vec{\Delta}_\perp^2) = \frac{-2M}{\Delta^1 - i\Delta^2}\int\frac{d^2\vec{k}_\perp}{16\pi^3} \times [\psi^{\uparrow}_{+\frac{1}{2}}(x,\vec{k}'_\perp)\psi^{\downarrow}_{+\frac{1}{2}}(x,\vec{k}_\perp) + \psi^{\uparrow}_{-\frac{1}{2}}(x,\vec{k}'_\perp)\psi^{\downarrow}_{-\frac{1}{2}}(x,\vec{k}_\perp)] \tag{21}$$

$$= \frac{-2Mg^2}{16\pi^3(\Delta^1 - i\Delta^2)}\int d^2\vec{k}_\perp \times \frac{(Mx+m)(1-x)}{[(\vec{k}'^2_\perp + u(\lambda^2))(\vec{k}^2_\perp + u(\lambda^2)]}[(k^1 - ik^2) - (k'^1 - ik'^2)], \tag{22}$$

where $\vec{k}'_\perp = \vec{k}_\perp + (1-x)\vec{\Delta}_\perp$ is the relative transverse momentum of the quark in the final state of a nucleon and $u(\lambda^2)$ is defined in Eq. (18). Since some of the above $\vec{k}_\perp$ integrals diverge, a manifestly Lorentz-invariant Pauli-Villars regularization is applied to regularize the divergent pieces of the integrals. For this purpose, $\lambda^2 \to \Lambda^2 (= 10\lambda^2)$ is employed throughout this paper.

Using Eq. (1) and the GPDs available in Eqs. (20) and (22), one can calculate the quark OAM for a nucleon polarized in the $+z$ direction as

$$L^z = \frac{1}{2}\int_0^1 dx[xH(x,0,0) + xE(x,0,0) - \Delta q(x)],$$

where

$$\Delta q(x) = \int\frac{d^2\vec{k}_\perp}{16\pi^3}[|\psi^{\uparrow}_{+\frac{1}{2}}(x,\vec{k}_\perp)|^2 - |\psi^{\uparrow}_{-\frac{1}{2}}(x,\vec{k}_\perp)|^2], \tag{23}$$

and the quark spin angular momentum reads [17,20]

$$S = \frac{1}{2}\int_0^1 \Delta q(x)dx = \frac{g^2}{32\pi^2}\int_0^1(1-x) \times \left[(Mx+m)^2\left(\frac{1}{u(\lambda^2)} - \frac{1}{u(\Lambda^2)}\right) - \ln\left(\frac{u(\Lambda^2)}{u(\lambda^2)}\right)\right]dx. \tag{24}$$

Now, from Eqs. (2) and (6), one can define GPDs and TAM in impact parameter space $b_\perp$ as [21]

$$\tilde{J}(\vec{b}_\perp) = \frac{1}{2}\int_0^1 x[\mathcal{H}(x,\vec{b}_\perp) + \mathcal{E}(x,\vec{b}_\perp)]dx, \tag{25}$$

where

$$\mathcal{H}(x,\vec{b}_\perp) = \int\frac{d^2\vec{\Delta}_\perp}{(2\pi)^2}e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp}H(x,0,-\vec{\Delta}_\perp^2) = [\psi^{\uparrow *}_{+\frac{1}{2}}(x,\vec{r}_\perp)\psi^{\uparrow}_{+\frac{1}{2}}(x,\vec{r}_\perp) + \psi^{\uparrow *}_{-\frac{1}{2}}(x,\vec{r}_\perp)\psi^{\uparrow}_{-\frac{1}{2}}(x,\vec{r}_\perp)] \times \frac{1}{(1-x)^2} \tag{26}$$

and

$$\mathcal{E}(x,\vec{b}_\perp) = \int\frac{d^2\vec{\Delta}_\perp}{(2\pi)^2}e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp}E(x,0,-\vec{\Delta}_\perp^2).$$

$$\frac{(-i\frac{\partial}{\partial b^x} - \frac{\partial}{\partial b^y})}{2M}\mathcal{E}(x,\vec{b}_\perp) = [\psi^{\uparrow *}_{+\frac{1}{2}}(x,\vec{r}_\perp)\psi^{\downarrow}_{+\frac{1}{2}}(x,\vec{r}_\perp) + \psi^{\uparrow *}_{-\frac{1}{2}}(x,\vec{r}_\perp)\psi^{\downarrow}_{-\frac{1}{2}}(x,\vec{r}_\perp)]\frac{1}{(1-x)^2}, \tag{27}$$

where $\vec{b}_\perp$ and $\vec{r}_\perp$ are related by $\vec{b}_\perp = (1-x)\vec{r}_\perp$ [21]. This relation was recently discussed in Ref. [22] using LFWFs constructed from the soft-wall AdS/QCD prediction.

Here, in order to describe distributions in impact parameter space, we introduce LFWFs in the impact parameter space $b_\perp$ as

$$\psi^{\uparrow\downarrow}_{\pm\frac{1}{2}}(x,\vec{b}_\perp) \equiv \frac{1}{2\pi(1-x)}\int d^2\vec{k}_\perp e^{-i\frac{\vec{k}_\perp\cdot\vec{b}_\perp}{1-x}}\psi^{\uparrow\downarrow}_{\pm\frac{1}{2}}(x,\vec{k}_\perp),$$
$$\psi^{\uparrow\downarrow}_{\pm\frac{1}{2}}(x,\vec{r}_\perp) \equiv \frac{1}{2\pi}\int d^2\vec{k}_\perp e^{-i\vec{k}_\perp\cdot\vec{r}_\perp}\psi^{\uparrow\downarrow}_{\pm\frac{1}{2}}(x,\vec{k}_\perp). \tag{28}$$

The factor $\frac{1}{1-x}$ in the exponent accounts for the fact that the variable $\vec{k}_\perp$ is Fourier conjugate to $\vec{r}_\perp = \vec{r}_{\perp 1} - \vec{r}_{\perp 2}$, the displacement between the active quark and the spectator (scalar). The prefactor $\frac{1}{(1-x)}$ also ensures the proper normalization of the wave functions,

$$\int |\psi^{\uparrow\downarrow}_{\pm\frac{1}{2}}(x,\vec{b}_\perp)|^2 d^2\vec{b}_\perp = \int |\psi^{\uparrow\downarrow}_{\pm\frac{1}{2}}(x,\vec{r}_\perp)|^2 d^2\vec{r}_\perp = \int |\psi^{\uparrow\downarrow}_{\pm\frac{1}{2}}(x,\vec{k}_\perp)|^2 d^2\vec{k}_\perp. \quad (29)$$

Inserting Eqs. (26), (27), (28), and (17) into Eq. (25) yields

$$\tilde{J}(b_\perp) = \frac{g^2}{32\pi^3}\int_0^1 \frac{x}{1-x} \times [(Mx+m)^2[\mathbf{K}_0(Z)]^2 + u(\lambda^2)[\mathbf{K}_1(Z)]^2]dx + \frac{2Mg^2}{32\pi^3}\int_0^1 x(Mx+m)[\mathbf{K}_0(Z)]^2 dx, \quad (30)$$

where $Z = |\frac{b_\perp}{1-x}|\sqrt{u(\lambda^2)}$, $u(\lambda^2)$ is defined in Eq. (18), and $\mathbf{K}_n(Z)$ is the modified Bessel function of the second kind.

Using Eqs. (2), (6), (20), (22), and (25), it is straightforward to verify the following relation numerically for a consistency check of our expressions.

$$\int d^2b_\perp \tilde{J}(b_\perp) = \frac{1}{2}\int dxx[H(x,0,0) + E(x,0,0)]. \quad (31)$$

Using the LFWFs available in Eq. (28), one can evaluate the spin angular momentum distribution in the transverse plane $(b_\perp)$ for a nucleon polarized in the $+z$ direction as

$$S(b_\perp) = \frac{1}{2}\int_0^1 dx[|\psi^{\uparrow}_{+\frac{1}{2}}(x,\vec{b}_\perp)|^2 - |\psi^{\uparrow}_{-\frac{1}{2}}(x,\vec{b}_\perp)|^2], \quad (32)$$

where

$$|\psi^{\uparrow}_{+\frac{1}{2}}(x,\vec{b}_\perp)|^2 = \frac{g^2}{16\pi^3}\frac{(Mx+m)^2}{(1-x)}[\mathbf{K}_0(Z)]^2,$$
$$|\psi^{\uparrow}_{-\frac{1}{2}}(x,\vec{b}_\perp)|^2 = \frac{g^2}{16\pi^3}\frac{u(\lambda^2)}{(1-x)}[\mathbf{K}_1(Z)]^2. \quad (33)$$

If one assumes that $\tilde{J}(b_\perp)$, $\rho_J^{\mathrm{PG}}(b_\perp)$, and $\rho_J^{\mathrm{IMF}}(b_\perp)$ can be interpreted as TAM densities, then the differences

$$L^{\mathrm{naive}}(b_\perp) \equiv \tilde{J}(b_\perp) - S(b_\perp), \quad (34)$$

$$L^{\mathrm{PG}}(b_\perp) \equiv \rho_J^{\mathrm{PG}}(b_\perp) - S(b_\perp), \quad \text{and} \quad (35)$$

$$L^{\mathrm{IMF}}(b_\perp) \equiv \rho_J^{\mathrm{IMF}}(b_\perp) - S(b_\perp) \quad (36)$$

would have to represent the respective OAM densities.

In the following section, we will investigate if that is indeed the case by comparing these distributions with the one calculated directly from LFWFs in impact parameter space using the JM definition.

A. Impact parameter space distribution directly from light-front wave functions

With the LFWFs available in Eq. (17), one can compute the orbital angular momentum $\mathcal{L}^z$ of the "quark" for a "nucleon" polarized in the $+z$ direction directly as [16,23–25]

$$\mathcal{L}^z = \int_0^1 dx \int \frac{d^2\vec{k}_\perp}{16\pi^3}(1-x)|\psi^{\uparrow}_{-\frac{1}{2}}(x,\vec{k}_\perp)|^2 = \frac{g^2}{16\pi^2}\int_0^1 (1-x)^2 \ln\left[\frac{u(\Lambda^2)}{u(\lambda^2)}\right]dx. \quad (37)$$

Lorentz-invariant Pauli-Villars regularization is manifestly applied to evaluate the above integral. It is straightforward to show

$$\mathcal{L}^z = L^z \quad (38)$$

as was expected since L^z in the SDQM does not contain a vector potential; therefore, no gauge-related issues arise [23,26]. Likewise, one can define the OAM density directly from the LFWFs available in Eq. (28) as

$$\mathcal{L}(b_\perp) = \int_0^1 dx(1-x)|\psi^{\uparrow}_{-\frac{1}{2}}(x,\vec{b}_\perp)|^2 = \frac{g^2}{16\pi^3}\int_0^1 dxu(\lambda^2)[\mathbf{K}_1(Z)]^2. \quad (39)$$

Here, $\mathcal{L}(b_\perp)$ represents the orbital angular momentum density for the active quark as a function of the distance from the transverse center of momentum in a "nucleon" that is polarized in the $+z$ direction.

The TAM distribution of the active quark in the transverse plane is shown in Fig. 1. For the different techniques, obviously, the area under the graphs is the only feature that all four distributions have in common, i.e.,

$$\int_0^\infty db_\perp b_\perp \tilde{J}(b_\perp) = \int_0^\infty db_\perp b_\perp \rho_J^{\mathrm{PG}}(b_\perp) = \int_0^\infty db_\perp b_\perp \rho_J^{\mathrm{IMF}}(b_\perp) = \int_0^\infty db_\perp b_\perp(\mathcal{L}(b_\perp) + S(b_\perp)). \quad (40)$$

Similarly, the OAM distributions of the active quark in the transverse plane are shown in Fig. 2; also in this case, the area under the graphs for all four distributions is the only common feature, i.e.,

$$\int_0^\infty db_\perp b_\perp L^{\mathrm{naive}}(b_\perp) = \int_0^\infty db_\perp b_\perp L^{\mathrm{PG}}(b_\perp) = \int_0^\infty db_\perp b_\perp L^{\mathrm{IMF}}(b_\perp) = \int_0^\infty db_\perp b_\perp \mathcal{L}(b_\perp) = \frac{\mathcal{L}^z}{2\pi}. \quad (41)$$

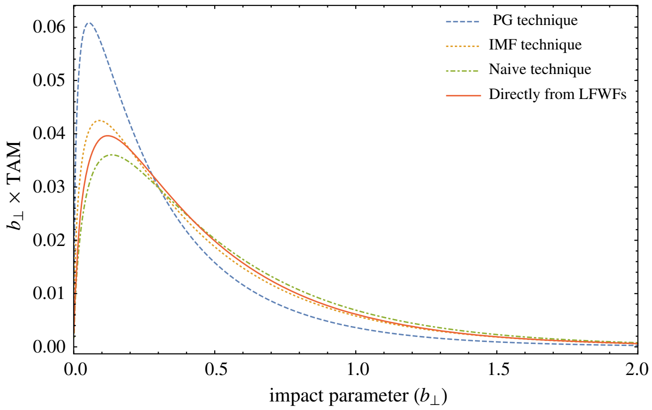

FIG. 1. TAM distribution of the quark in the scalar diquark model for a nucleon polarized in the $+z$ direction. Solid line: $\mathcal{L}(b_\perp) + S(b_\perp)$, directly from LFWFs [Eq. (39)+Eq. (32)]; dashed line: $\rho_J^{\mathrm{PG}}(b_\perp)$, PG technique [Eq. (10)]; dotted line: $\rho_J^{\mathrm{IMF}}(b_\perp)$, IMF technique [Eq. (16)]; dash-dotted line: $\tilde{J}(b_\perp)$, naive technique [Eq. (16)]. The plots are in units of $\frac{g^2}{16\pi}$.

Equations (40) and (41) ensure that the TAM/OAM does not change regardless of whether one uses a different frame or technique to perform the calculations.

The $b_\perp$ distributions of OAM presented in Fig. 2 for the three different techniques (naive, PG, and IMF) do not agree with the one associated with the definition introduced by Jaffe-Manohar. Therefore, these results clearly demonstrate that $L^{\mathrm{naive}}(b_\perp)$, $L^{\mathrm{PG}}(b_\perp)$, and $L^{\mathrm{IMF}}(b_\perp)$ do not represent the orbital angular momentum distribution for a longitudinally polarized nucleon since $\mathcal{L}(b_\perp)$ already has that interpretation in momentum space. Furthermore, using the naive technique in the SDQM, we also conclude that the FT of $J(t)$ does not represent the distribution of angular momentum in the transverse plane regardless of whether the FT is two dimensional or three dimensional. All three techniques discussed above are associated with the FT of $J(t)$. While $J(t)$ itself indeed is identified with the second moment of GPDs in the limiting case $t \to 0$, our investigation exhibits three different possibilities of relating the t dependence of GPDs to the angular momentum distributions in the transverse plane. None of them turns out to yield the distribution one would expect from the Jaffe-Manohar definition for a longitudinally polarized nucleon.

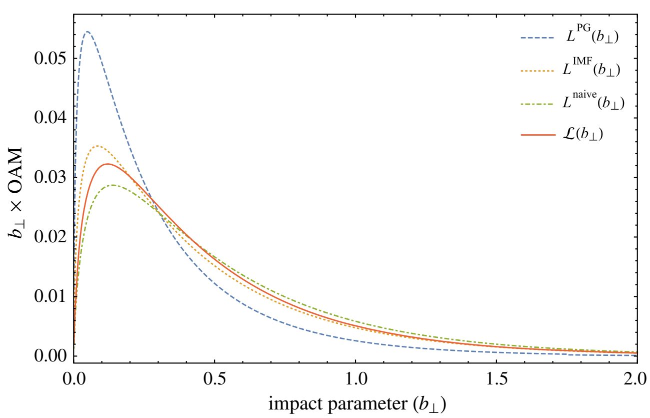

FIG. 2. OAM distribution of the quark in the scalar diquark model for a nucleon polarized in the $+z$ direction. Solid line: $\mathcal{L}(b_\perp)$, directly from LFWFs [Eq. (35)]; dashed line: $L^{\mathrm{PG}}(b_\perp)$, PG technique [Eq. (36)]; dotted line: $L^{\mathrm{IMF}}(b_\perp)$, IMF technique [Eq. (36)]; dash-dotted line: $L^{\mathrm{naive}}(b_\perp)$, naive technique [Eq. (34)]. The plots are in units of $\frac{g^2}{16\pi}$.

IV. DISCUSSION

A. Naive technique

It was demonstrated using the SDQM that, although $J(t)$ yields the z component of the total angular momentum of the quarks for a nucleon polarized in the $+z$ direction in the limit $t \to 0$, the two-dimensional FT of its t dependence does not yield the distribution of angular momentum in the transverse plane.

This result is best understood by recalling that Lorentz or rotational invariance was heavily used in Ref. [2] for deriving Eq. (1) as it restricts the allowed tensor structure. In Ref. [27], Ji's angular momentum sum rule was rederived by considering the transverse deformation of parton distributions in a transversely polarized nucleon, and in several steps of the derivation, rotational invariance was used for rotations that mix "longitudinal" and "transverse" directions. When one considers distributions in the transverse plane, rotational invariance is no longer fully applicable. This is analogous to the observation that the unintegrated Ji relation, i.e., $J(x) \equiv \frac{x}{2}[H(x,0,0) + E(x,0,0)]$, is not the x distribution of $J^z(x)$ for a longitudinally polarized nucleon [23].

B. Polyakov-Goeke technique and IMF technique

In hadron spin structure studies, the total angular momentum of a quark is decomposed into spin and orbital parts, and the spin distribution of the quark in the transverse plane can be obtained using a two-dimensional FT of axial form factors. To study angular momentum distribution in the transverse plane, one may be tempted to interpret the two distributions (densities), proposed in Eqs. (10) and (16), as a sum of spin and orbital angular momentum distributions (densities). In particular, the observation that $\rho_J^{\mathrm{IMF}}(b_\perp)$ differs from the light-front wave-function–based result may thus appear surprising. However, both proposed densities have in common that they are based on the symmetric energy momentum tensor $T^{\mu\nu}$. On the other hand, the symmetric energy momentum tensor $T^{\mu\nu}$ can be expressed in terms of canonical energy momentum tensor $\mathcal{T}^{\mu\nu}$ and spin current $S^{\mu}_{\nu\lambda}$ [28,29]. The total angular momentum density, which preserves the interpretation as a sum of spin and orbital angular momentum densities, is based on the canonical energy momentum tensor. Therefore, it is important to illustrate the role of the total divergence term available in $T^{\mu\nu}$.

The symmetric energy momentum tensor for the massless Dirac particle can be expressed as

$$T^{\mu\nu} = \frac{i}{2}\overline{q}(\gamma^{\mu}\overset{\leftrightarrow}{D}^{\nu} + \gamma^{\mu}\overset{\leftrightarrow}{D}^{\nu})q. \tag{42}$$

Using the equations of motion (valid in matrix elements), one finds

$$\overline{q}\gamma^{x}\overset{\leftrightarrow}{D}^{+}q = \overline{q}\gamma^{+}\overset{\leftrightarrow}{D}^{x}q - \partial^{y}(\overline{q}\gamma^{+}\gamma^{x}\gamma^{y}q) + \frac{1}{4}\partial_{-}[q_{-}^{\dagger}\gamma^{0}\gamma^{x}\gamma^{+}\gamma^{-}q_{+} - q_{+}^{\dagger}\gamma^{0}\gamma^{x}\gamma^{-}\gamma^{+}q_{-}]. \tag{43}$$

Inserting this into the total angular momentum density, one finds

$$xT^{+y} - yT^{+x} = x\overline{q}\gamma^{+}i\overset{\leftrightarrow}{D}^{y}q - y\overline{q}\gamma^{+}i\overset{\leftrightarrow}{D}^{x}q + \overline{q}\gamma^{+}i\gamma^{x}\gamma^{y}q + i\partial_{x}(x\overline{q}\gamma^{+}\gamma^{x}\gamma^{y}q) + i\partial_{y}(y\overline{q}\gamma^{+}\gamma^{x}\gamma^{y}q) + \frac{i}{4}\partial_{-}[x\overline{q}\gamma^{y}\gamma^{+}\gamma^{-}q - y\overline{q}\gamma^{x}\gamma^{+}\gamma^{-}q]. \tag{44}$$

Therefore, there are two terms that together have the physical interpretation as an orbital angular momentum density: a term that represents the spin density, as well as a total derivative term. While the presence of these total derivative terms has no consequences for the integrated quantities, they cause a profound dilemma when attempting to study angular momentum densities. Though $xT^{+y} - yT^{+x}$ seems to be a perfect candidate for the total angular momentum density, one has to be careful not to interpret that density as a simple sum of orbital angular momentum density and spin density. This statement may sound paradoxical but it is due to the presence of terms that are total derivatives and thus do not contribute to the overall angular momentum. Nevertheless, these contributions play an important role in studying the angular momentum density in the transverse plane. Note that, although the issue of total derivative terms was raised and discussed in Ref. [13], it was illustrated explicitly in this paper using model calculations.

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[1] J. Ashman *et al.* (European Muon Collaboration), Phys. Lett. B **206**, 364 (1988).
[2] X. D. Ji, Phys. Rev. Lett. **78**, 610 (1997).
[3] M. Burkardt, Phys. Rev. D **62**, 071503 (2000); **66**, 119903 (E) (2002).
[4] M. Burkardt, Int. J. Mod. Phys. A **18**, 173 (2003).
[5] M. Diehl, Eur. Phys. J. C **25**, 223 (2002); **31**, 277(E) (2003).
[6] D. E. Soper, Phys. Rev. D **15**, 1141 (1977).
[7] M. Diehl, Phys. Rep. **388**, 41 (2003).
[8] B. L. G. Bakker and C. R. Ji, Phys. Rev. D **83**, 091502 (2011).
[9] M. V. Polyakov, Phys. Lett. B **555**, 57 (2003).
[10] K. Goeke, J. Grabis, J. Ossmann, M. V. Polyakov, P. Schweitzer, A. Silva, and D. Urbano, Phys. Rev. D **75**, 094021 (2007).
[11] G. A. Miller, Phys. Rev. Lett. **99**, 112001 (2007).
[12] V. D. Burkert, Proc. Int. Sch. Phys. Fermi **180**, 303 (2012).
[13] E. Leader and C. Lorcé, Phys. Rep. **541**, 163 (2014).
[14] H. Pagels, Phys. Rev. **144**, 1250 (1966).
[15] X. D. Ji, Phys. Rev. D **55**, 7114 (1997).
[16] R. L. Jaffe and A. Manohar, Nucl. Phys. **B337**, 509 (1990).
[17] S. J. Brodsky and S. D. Drell, Phys. Rev. D **22**, 2236 (1980).
[18] S. J. Brodsky, M. Diehl, and D. S. Hwang, Nucl. Phys. **B596**, 99 (2001).
[19] S. J. Brodsky, H. C. Pauli, and S. S. Pinsky, Phys. Rep. **301**, 299 (1998).
[20] S. J. Brodsky, D. S. Hwang, B. Q. Ma, and I. Schmidt, Nucl. Phys. **B593**, 311 (2001).
[21] M. Burkardt and D. S. Hwang, Phys. Rev. D **69**, 074032 (2004).
[22] C. Mondal, N. Kumar, H. Dahiya, and D. Chakrabarti, Phys. Rev. D **94**, 074028 (2016).
[23] M. Burkardt and H. BC, Phys. Rev. D **79**, 071501 (2009).
[24] S. Bashinsky and R. L. Jaffe, Nucl. Phys. **B536**, 303 (1998).
[25] A. Harindranath and R. Kundu, Phys. Rev. D **59**, 116013 (1999).
[26] X. Ji, A. Schfer, F. Yuan, J. H. Zhang, and Y. Zhao, Phys. Rev. D **93**, 054013 (2016).
[27] M. Burkardt, Phys. Rev. D **72**, 094020 (2005).
[28] F. J. Belinfante, Physica (Utrecht) **6**, 887 (1939).
[29] L. Rosenfeld, Mem. Cl. Sci., Acad. R. Belg., Collect. 8 **18**, 1 (1940); L. Rosenfeld, On the energy-momentum tensor, *Selected Papers of Léon Rosenfeld*, edited by R. S. Cohen and J. J. Stachel (Reidel, Dordrecht, 1979), pp. 711–735 (English Translation).