

# Large- $p_T$ production of $D$ mesons at the LHCb in the parton Reggeization approach

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The production of  $D$  mesons in proton-proton collisions at the LHCb detector is studied. We consider the single production of  $D^0/\bar{D}^0$ ,  $D^\pm$ ,  $D^{*\pm}$ , and  $D_s^\pm$  mesons and correlation spectra in the production of  $D\bar{D}$  and  $DD$  pairs at the  $\sqrt{S} = 7$  TeV and  $\sqrt{S} = 13$  TeV. In case of the single  $D$ -meson production we calculate differential cross sections over transverse momentum  $p_T$  while in the pair  $D\bar{D}$ ,  $DD$ -meson production the cross sections are calculated over the azimuthal angle difference  $\Delta\phi$ , rapidity difference  $\Delta y$ , invariant mass of the pair  $M$  and over the  $p_T$  of the one meson from a pair. The cross sections are obtained at the leading order of the parton Reggeization approach using Kimber-Martin-Ryskin unintegrated parton distribution functions in a proton. To describe the  $D$ -meson production we use universal scale-dependent  $c$ -quark and gluon fragmentation functions fitted to  $e^+e^-$  annihilation data from CERN LEP1. Our predictions find a good agreement with the LHCb Collaboration data within uncertainties and without free parameters.

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## I. INTRODUCTION

The open charm production in collisions of high energy hadrons is an appropriate class of processes to test the perturbative quantum chromodynamics (QCD) [1]. A large mass of charm quark  $m_c$  as a lower bound of hard energy scale  $\mu \geq m_c$  leads to the small value of the strong coupling constant  $\alpha_S(\mu)$  due to the condition  $m_c \gg \Lambda_{\text{QCD}}$ , where  $\Lambda_{\text{QCD}}$  is the asymptotic scale parameter of QCD. Production of charm quark with large transverse momentum ( $p_T \gg m_c$ ) is a typical multiscale hard process for which the fixed-order QCD calculations should be corrected by the fragmentation contribution to resum large logarithmic terms  $\sim \alpha_S \log p_T/m_c$ , where  $\alpha_S = g_S^2/4\pi$  is the strong coupling constant.

Nowadays, the theoretical study of inclusive  $D$ -meson hadroproduction were performed at the leading order (LO) for double production ( $D\bar{D}$ ,  $DD$ ) and the next-to-leading order (NLO) for single production in the collinear parton model (CPM) of QCD within the two approaches: the general-mass variable-flavor-number (GM-VFN) scheme [2] and the so-called fixed order scheme improved with next-to-leading logarithms (FONLL scheme) [3]. In the first one, realized in the Refs. [4–6], the large transverse momenta collinear logarithms are resummed through the evolution of the fragmentation functions (FFs), satisfying

the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [7]. The  $D$ -meson FFs were extracted both at the LO and NLO in the fixed factorization scheme from the fit of  $e^+e^-$  data taken by the OPAL Collaboration at CERN LEP1 [8]. Opposite, in the FONLL approach, the NLO  $D$ -meson production cross sections are calculated with a nonperturbative  $c$ -quark FF, that is not a subject to DGLAP evolution. The FONLL scheme was implemented in the Refs. [9,10], and its main ingredients are the following: the NLO fixed order calculation with resummation of large transverse momentum logarithms at the next-to-leading level for heavy quark production. The important difference of these two fragmentation approaches is in following: in the case of scale-depended GM-VFN scheme  $D$  mesons are produced both from  $c$ -quarks and from gluons, in case of scale-independent FONLL scheme  $D$  mesons are produced only from  $c$ -quarks.

At the high-energy Regge limit  $\sqrt{S} \gg p_T \gg m_c$ , where  $\sqrt{S}$  is the invariant collision energy, one has new dynamic regime of particle production in the multi-Regge kinematics (MRK) or in the quasi-multi-Regge kinematics (QMRK), when one particle or a group of particles are produced in the rapidity region, being strongly separated in rapidity from other particles. Radiative corrections in this regime are dominated by the production of additional hard jets. The only one way in CPM to treat these processes is to calculate higher-order corrections, which is a challenging task for some processes even at the NLO level, such as for relevant to our study process of double charm production in the gluon fusion,  $gg \rightarrow cc\bar{c}\bar{c}$ .

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Alternatively, to solve this problem we should change a factorization scheme from the collinear approximation of the PM to the high-energy or  $k_T$ -factorization [11–13] and take into account a large part of the higher-order corrections which are incorporated in the transverse-momentum-dependent parton distribution functions (PDFs) of off shell initial partons. Recently, the studies of single and double  $D$ -meson hadroproduction at the LHC were performed in the  $k_T$ -factorization approach with the scale-independent Peterson  $c$ -quark FF in Refs. [14,15] and with the scale-dependent  $c$ -quark and gluon FFs from [6] in Refs. [16,17]. The latter are based on the parton Reggeization approach (PRA), which is a combination of  $k_T$ -factorization framework with the fully gauge-invariant amplitudes with Reggeized partons in the initial state.

In our previous study of the single  $D$ -meson production at the Tevatron and the LHC [16] in the central rapidity region of produced mesons ( $|y| < 1$  and  $|y| < 0.5$ , respectively) in the PRA we obtained quite a good agreement between theoretical calculations and the experimental data. In this work we continue the investigation and extend it into the region of large rapidity for single and double  $D$ -meson production. Not so long ago the LHCb Collaboration has provided the forward  $D$ -meson production data [18]. The differential cross sections on  $D$ -meson transverse momentum  $d\sigma/dp_T$  were measured for  $D^0/\bar{D}^0$ ,  $D^\pm$ ,  $D^{*\pm}$ , and  $D_s^\pm$  mesons in proton-proton collisions at the CERN LHC at a center-of-mass energy of  $\sqrt{S} = 7$  TeV and with rapidities in the range of  $2.0 < y < 4.5$ . Nowadays, the recent data come from the LHC Run II and the newest measurements of  $D$ -meson production were presented by the LHCb Collaboration at  $\sqrt{S} = 13$  TeV [19]. They are presented as  $p_T$  distributions of  $D$  mesons at the same rapidity region and in a wider  $p_T$  range than in the LHC Run I. The data are presented in three kinds: absolute double differential cross sections  $d^2\sigma/(dp_T dy)$ , ratios of  $D$ -meson production cross sections between different center-of-mass energies  $R_{13/7}$ , and ratios of the cross sections for different mesons.

Not only the single but also a pair production of  $D$  mesons is under consideration. The first observations of double charm production were performed at the  $\sqrt{S} = 7$  TeV by the LHCb Collaboration [20]. The published results include the cross sections differential in azimuthal angle difference, transverse momentum, rapidity difference and invariant mass of various combinations of  $D$ -meson pairs. The spectra were measured at the region of large rapidities  $2.0 < y < 4.0$  at the collision energy  $\sqrt{S} = 7$  TeV. We continue our recent study of double  $D$ -meson production in PRA [17] testing the new contributions from Reggeized quark—Reggeized antiquark annihilation processes and studying different combinations of  $D$  mesons in the measured pair production spectra.

Recently, PRA was successfully applied for the analysis of inclusive production of single jet [21], pair of jets [22], prompt-photon [23,24], photon plus jet [25], Drell-Yan

lepton pairs [26], bottom-flavored jets [27,28], charmonium and bottomonium production [29–34] at the Tevatron and LHC. This study has demonstrated the advantages of the high-energy factorization scheme based on PRA in the descriptions of data comparing to the CPM calculations.

This paper is organized as follows. In Sec. II we present the relevant Reggeized amplitudes as they are obtained from the Lipatov's effective high-energy action in QCD. In Sec. III the formalism of our calculation in the PRA and the fragmentation model are discussed. In Secs. IV and V our results for single and double  $D$ -meson production, respectively, are presented in comparison with the experimental data and discussed. In Sec. VI we summarize our conclusions.

## II. REGGEIZED AMPLITUDES

The Reggeization of amplitudes at the high energy is a natural property of gauge-invariant quantum field theories, such as quantum electrodynamics [35] and QCD [36,37]. At large  $\sqrt{S}$  the dominant contribution to cross sections of QCD processes give MRK or QMRK parton scattering processes with the gluon or quark  $t$ -channel exchange. Due to the Reggeization of quarks and gluons, an important role is dedicated to the vertices of Reggeon-particle interactions. Nowadays, they can be straightforwardly derived from the non-Abelian gauge-invariant effective action for the interactions of the Reggeized partons with the usual QCD partons, which was firstly introduced in Ref. [38] for Reggeized gluons only, and then extended by inclusion of Reggeized quark fields in the Ref. [39]. The full set of the induced and effective vertices together with Feynman rules can be found in Refs. [39,40].

In our recent paper [16] we have shown that gluon and  $c$ -quark fragmentation give the leading contribution to the  $D$ -meson production in the PRA. The same result was also obtained in the Ref. [4] in NLO of the CPM, as well as the light quark fragmentation turns out to be negligible. According to this statement in case of the forward production of charmed mesons in the framework of the PRA we will also consider the gluon and  $c$ -quark fragmentation. The lowest order in  $\alpha_s$  parton subprocesses of PRA which give a contribution to a single  $D$ -meson production via gluon fragmentation are the following:

$$\mathcal{R} + \mathcal{R} \rightarrow g, \quad (1)$$

and

$$\mathcal{Q} + \bar{\mathcal{Q}} \rightarrow g. \quad (2)$$

The corresponding lowest order parton subprocesses which contribute to a single  $D$ -meson production via  $c$ -quark fragmentation are charm quark-antiquark pair production in the gluon-gluon fusion

$$\mathcal{R} + \mathcal{R} \rightarrow c + \bar{c}, \quad (3)$$

and in the quark-antiquark annihilation

$$\mathcal{Q} + \bar{\mathcal{Q}} \rightarrow c + \bar{c}, \quad (4)$$

where  $\mathcal{R}$  are the Reggeized gluons and  $\mathcal{Q}$  denotes Reggeized  $u$ ,  $d$  and  $s$  quarks.

The double  $D\bar{D}$ -meson production at the lowest order of PRA is described by the parton subprocesses (3) and (4) of  $c\bar{c}$ -pair production as well as by the parton subprocesses of  $gg$ -pair production

$$\mathcal{R} + \mathcal{R} \rightarrow g + g, \quad (5)$$

$$\mathcal{Q} + \bar{\mathcal{Q}} \rightarrow g + g. \quad (6)$$

In the case of  $DD$ -meson production we should consider contributions from six parton subprocesses. At first, these are (5) and (6). There are also  $2 \rightarrow 3$  and  $2 \rightarrow 4$  subprocesses

$$\mathcal{R} + \mathcal{R} \rightarrow c + \bar{c} + g, \quad (7)$$

$$\mathcal{Q} + \bar{\mathcal{Q}} \rightarrow c + \bar{c} + g, \quad (8)$$

$$\mathcal{R} + \mathcal{R} \rightarrow c + \bar{c} + c + \bar{c}, \quad (9)$$

$$\mathcal{Q} + \bar{\mathcal{Q}} \rightarrow c + \bar{c} + c + \bar{c}. \quad (10)$$

Formally, the matrix elements of the  $2 \rightarrow 2$ ,  $2 \rightarrow 3$  and  $2 \rightarrow 4$  subprocesses are of different order in the strong coupling constant  $\alpha_s$ , but their contributions can be of the same order because the fragmentation probabilities for gluon and  $c$ -quark fragmentation to a  $D$  meson are related as  $P_{g \rightarrow D} \sim \alpha_s P_{c \rightarrow D}$ . Taking into account our previous study [16,17] and the results of Ref. [14], we conclude that gluon fragmentation dominates over the  $c$ -quark fragmentation in the  $DD$ -pair production at the LHC energy, so we will consider the contributions of (5) and (6) subprocesses only.

Let us define four-vectors  $(n^+)^\mu = P_2^\mu/E_1$  and  $(n^-)^\mu = P_1^\mu/E_2$ , where  $P_{1,2}$  are the four-momenta of the colliding protons, and  $E_{1,2}$  are their energies. We have  $(n^\pm)^2 = 0$ ,  $n^+ \cdot n^- = 2$ , and  $S = (P_1 + P_2)^2 = 4E_1E_2$ . For any four-momentum  $k^\mu$ , we define  $k^\pm = k \cdot n^\pm$ . It is easy to see that the four-momenta of the Reggeized gluons can be represented as

$$\begin{aligned} q_1^\mu &= \frac{q_1^+}{2} (n^-)^\mu + q_{1T}^\mu, \\ q_2^\mu &= \frac{q_2^-}{2} (n^+)^\mu + q_{2T}^\mu, \end{aligned} \quad (11)$$

where  $q_T = (0, \mathbf{q}_T, 0)$ . Then the amplitude of gluon production in a fusion of two Reggeized gluons can be presented as a convolution of the Fadin-Kuraev-Lipatov effective PRR vertex  $C_{\mathcal{R}\mathcal{R}}^{g,\mu}(q_1, q_2)$  and polarization four-vector of final gluon  $\varepsilon_\mu(k)$

$$\mathcal{M}(\mathcal{R} + \mathcal{R} \rightarrow g) = C_{\mathcal{R}\mathcal{R}}^{g,\mu}(q_1, q_2) \varepsilon_\mu(k), \quad (12)$$

where

$$\begin{aligned} C_{\mathcal{R}\mathcal{R}}^{g,\mu}(q_1, q_2) &= -\sqrt{4\pi\alpha_s} f^{abc} \frac{q_1^+ q_2^-}{2\sqrt{t_1 t_2}} \\ &\times \left[ (q_1 - q_2)^\mu + \frac{(n^+)^\mu}{q_1^+} (q_2^2 + q_1^+ q_2^-) \right. \\ &\left. - \frac{(n^-)^\mu}{q_2^-} (q_1^2 + q_1^+ q_2^-) \right], \end{aligned} \quad (13)$$

$a$  and  $b$  are the color indices of the Reggeized gluons with incoming four-momenta  $q_1$  and  $q_2$ , and  $f^{abc}$  with  $a = 1, \dots, N_c^2 - 1$  is the antisymmetric structure constant of color gauge group  $SU_c(3)$ . Similarly, we can write down the amplitudes for subprocess (2) with massless spinors of Reggeized quark  $\mathcal{Q}$  and antiquark  $\bar{\mathcal{Q}}$ ,  $U(x_1 P_1)$  and  $V(x_2 P_2)$  respectively

$$\mathcal{M}(\mathcal{Q} + \bar{\mathcal{Q}} \rightarrow g) = \varepsilon_\mu(k) \bar{V}(x_2 P_2) C_{\mathcal{Q}\bar{\mathcal{Q}}}^{g,\mu}(q_1, q_2) U(x_1 P_1). \quad (14)$$

The Fadin-Sherman effective vertex [41] is given by

$$\begin{aligned} C_{\mathcal{Q}\bar{\mathcal{Q}}}^{g,\mu}(q_1, q_2) &= -i\sqrt{4\pi\alpha_s} T^a \\ &\times \left[ \gamma^\mu - \hat{q}_1 \frac{(n^-)^\mu}{q_1^- + q_2^-} - \hat{q}_2 \frac{(n^+)^\mu}{q_1^+ + q_2^+} \right], \end{aligned} \quad (15)$$

where  $T^a$  are the generators of the color gauge group  $SU_c(3)$ ,  $k = q_1 + q_2$  and

$$\gamma^{(-)\mu}(q_1, q_2) = \gamma^\mu + \hat{q}_1 \frac{(n^-)^\mu}{q_2^-}. \quad (16)$$

Then the squared amplitudes have a simple form and read

$$|\overline{\mathcal{M}(\mathcal{R} + \mathcal{R} \rightarrow g)}|^2 = \frac{3}{2} \pi \alpha_s \mathbf{k}_T^2, \quad (17)$$

$$|\overline{\mathcal{M}(\mathcal{Q} + \bar{\mathcal{Q}} \rightarrow g)}|^2 = \frac{16}{3} \pi \alpha_s (t_1 + t_2), \quad (18)$$

where  $t_1 = -q_1^2 = |\mathbf{q}_{1T}|^2$ ,  $t_2 = -q_2^2 = |\mathbf{q}_{2T}|^2$ .

The amplitudes for the subprocesses (3), (4), (5), and (6), can be written as a convolution of corresponding effective vertices with final gluon polarization four-vectors  $\varepsilon_\mu(k_i)$  in case of gluon production or with spinors  $U(x_1 P_1)$  and  $V(x_2 P_2)$  in case of initial-state Reggeized quark  $\mathcal{Q}$  and antiquark  $\bar{\mathcal{Q}}$

$$\mathcal{M}(\mathcal{R} + \mathcal{R} \rightarrow c + \bar{c}) = \varepsilon_\mu(k_1) \varepsilon_\nu(k_2) C_{\mathcal{R}\mathcal{R}}^{c\bar{c}}(q_1, q_2, k_1, k_2), \quad (19)$$

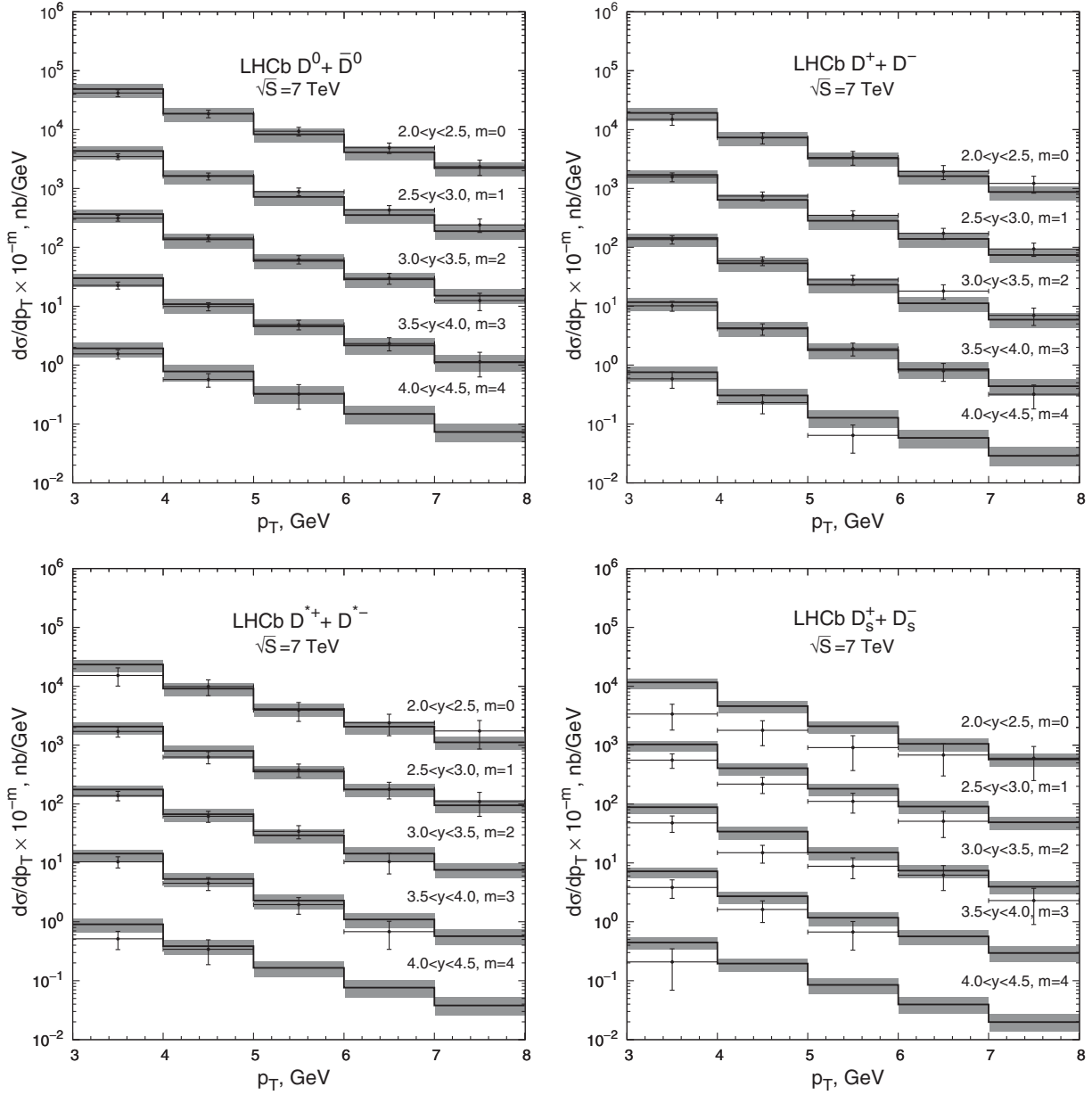


FIG. 1. Transverse momentum distributions of  $D^0/\bar{D}^0$ ,  $D^\pm$ ,  $D^{*\pm}$ , and  $D_s^\pm$  mesons in different rapidity regions at the energy  $\sqrt{S} = 7$  TeV. The experimental data from the LHCb Collaboration [18].

$$\begin{aligned} \mathcal{M}(Q + \bar{Q} \rightarrow c + \bar{c}) \\ = \bar{V}(x_2 P_2) C_{Q\bar{Q}}^{c\bar{c}}(q_1, q_2, k_1, k_2) U(x_1 P_1), \end{aligned} \quad (20)$$

$$\mathcal{M}(\mathcal{R} + \mathcal{R} \rightarrow g + g) = \varepsilon_\mu(k_1) \varepsilon_\nu(k_2) C_{\mathcal{R}\mathcal{R}}^{gg,\mu\nu}(q_1, q_2, k_1, k_2), \quad (21)$$

$$\begin{aligned} \mathcal{M}(Q + \bar{Q} \rightarrow g + g) \\ = \varepsilon_\mu(k_1) \varepsilon_\nu(k_2) \bar{V}(x_2 P_2) C_{Q\bar{Q}}^{gg,\mu\nu}(q_1, q_2, k_1, k_2) U(x_1 P_1), \end{aligned} \quad (22)$$

where  $C_{\mathcal{R}\mathcal{R}, Q\bar{Q}}$  are the effective vertices of the transition of two Reggeized partons into the gluon pair or charm

quark-antiquark pair. The general form of the squared amplitudes is

$$|\overline{\mathcal{M}}|^2 = \pi^2 \alpha_s^2 A \sum_{n=0}^4 w_n S^n, \quad (23)$$

where  $A$  and  $w_n$  are functions which depend on variables  $s, t, u, a_1, a_2, b_1, b_2, S$ . The corresponding effective vertices and matrix elements squared are too large to be presented here and can be found in an explicit form in Ref. [22] in the case of the massless final charm quarks (antiquarks). We perform our analysis in the region of  $\sqrt{S}, p_T \gg m_c$ , that allows us to

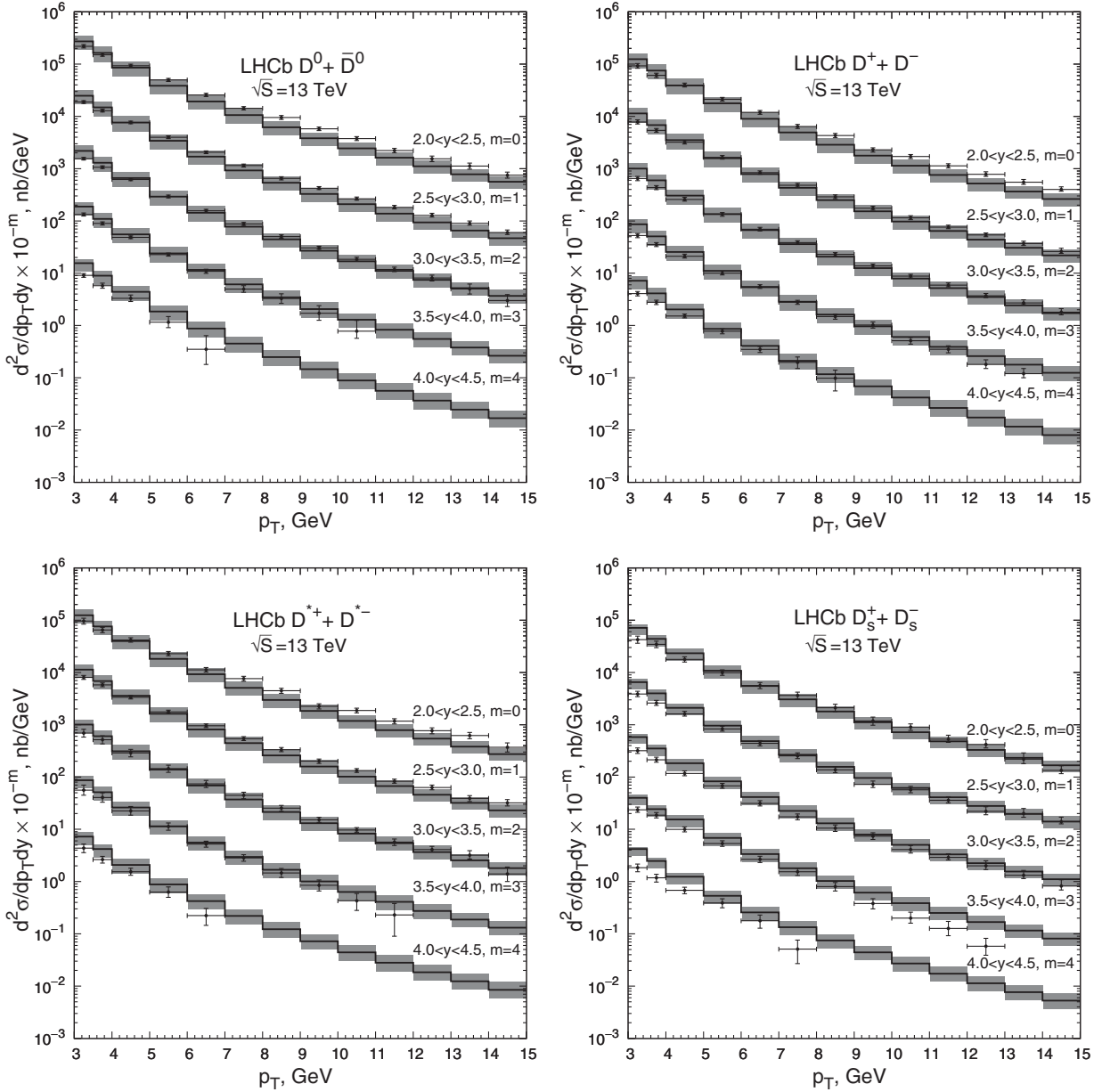


FIG. 2. Transverse momentum distributions of  $D^0/\bar{D}^0$ ,  $D^\pm$ ,  $D^{*\pm}$ , and  $D_s^\pm$  mesons in different rapidity regions at the energy  $\sqrt{S} = 13$  TeV. The experimental data from the LHCb Collaboration [19].

use ZM VFNS scheme, where the masses of the charm quarks in the hard-scattering amplitude are neglected.

For all the squared amplitudes of  $2 \rightarrow 2$  subprocesses we checked that in the collinear limit, i.e.  $\mathbf{q}_{(1,2)T} = 0$ , the Reggeized squared amplitudes transform to the squared amplitudes of the corresponding parton subprocesses in CPM. It is evident that the squared amplitudes for  $2 \rightarrow 1$  subprocesses are vanishing in the collinear limit.

### III. HADRONIC CROSS SECTIONS

In the conventional CPM the factorization theorem allows us to present the cross sections of processes which take place

in high-energy hadron-hadron collisions and have a one hard scale  $\mu^2$  as a convolution of scale-dependent parton (quark or gluon) *collinear* distributions  $f(x, \mu^2)$  and squared amplitude of the hard parton scattering. These distributions represent the density of partons carrying the longitudinal momentum fraction  $x$  of the proton, integrated over the parton transverse momentum  $k_T$  up to  $k_T = \mu$ . The evolution of this density from some scale  $\mu_0$ , fixing a nonperturbative regime, to the typical scale  $\mu$  is described by DGLAP [7] evolution equations where the large logarithms of type  $\log(\mu^2/\Lambda_{\text{QCD}}^2)$  (collinear logarithms) are summed. The typical scale  $\mu$  of the hard-scattering processes is usually of order of the transverse momentum of the produced particle,  $p_T$ .

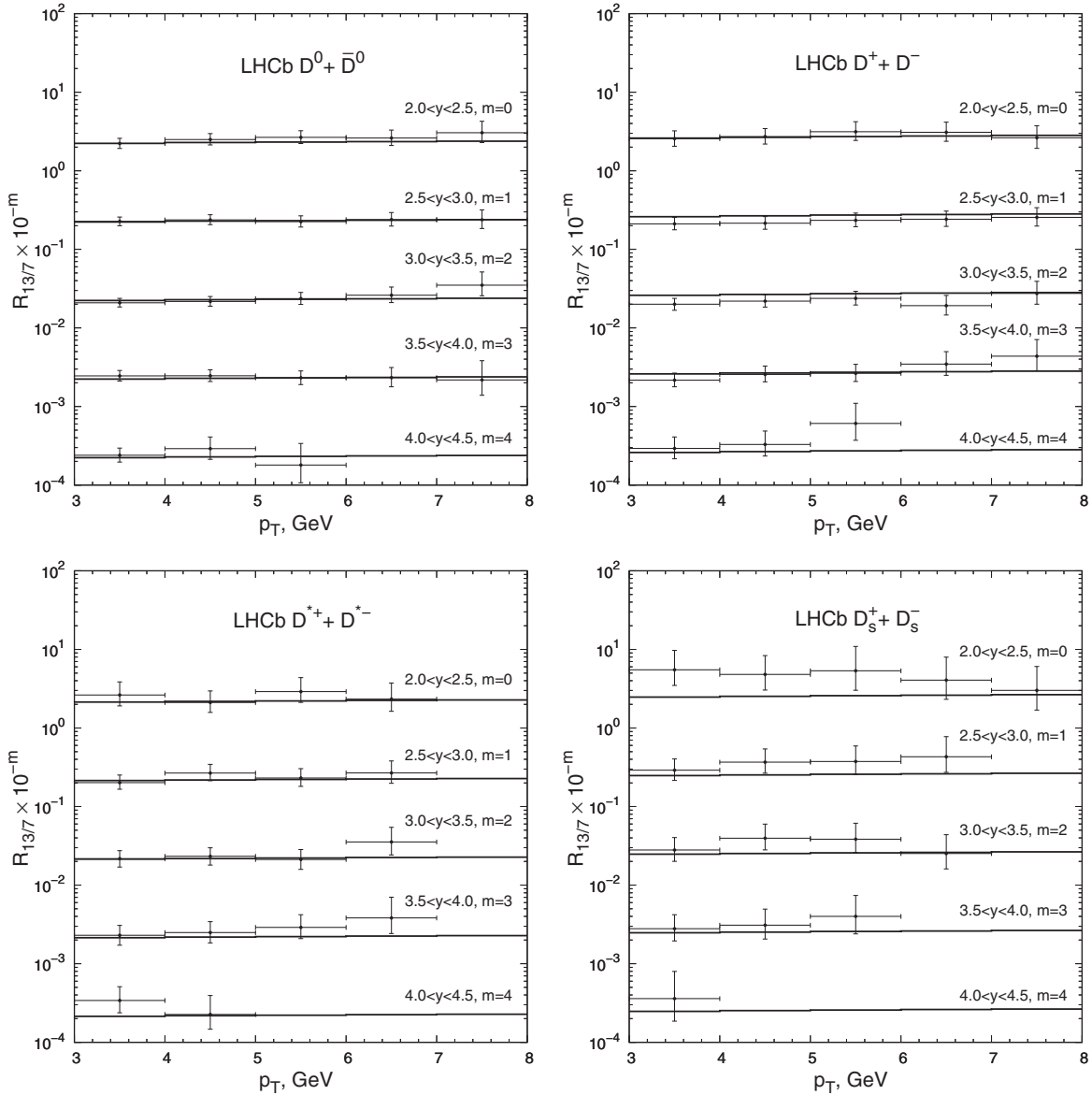


FIG. 3. The ratios of prompt  $D^0/\bar{D}^0$ ,  $D^\pm$ ,  $D^{*\pm}$ , and  $D_s^\pm$  cross section at  $\sqrt{S} = 13$  TeV to the same at  $\sqrt{S} = 7$  TeV for different regions of rapidity. The experimental data from the LHCb Collaboration [19].

The PRA gives the description of QCD parton scattering amplitudes in the region of large  $S$  and fixed momentum transfer  $t$ ,  $S \gg |t|$  (Regge region), with various color states in the  $t$ -channel. In the Regge region we should keep the transverse momenta of the initial partons and their virtualities. It becomes possible introducing the unintegrated over transverse momenta parton distribution functions (unPDFs)  $\Phi(x, t, \mu^2)$ , which depend on parton transverse momentum  $\mathbf{q}_T$  while its virtuality is  $t = -|\mathbf{q}_T|^2$ . The unPDFs are defined to be related with collinear ones through the equation

$$xf(x, \mu^2) = \int^{\mu^2} dt \Phi(x, t, \mu^2). \quad (24)$$

The factorization formula in the PRA reads

$$d\sigma = \sum_{i,j} \int \frac{dx_1}{x_1} \int \frac{d^2\mathbf{q}_{1T}}{\pi} \Phi_i(x_1, t_1, \mu_F^2) \times \int \frac{dx_2}{x_2} \int \frac{d^2\mathbf{q}_{2T}}{\pi} \Phi_j(x_2, t_2, \mu_F^2) d\hat{\sigma}_{ij}(q_1, q_2, \mu_F, \mu_R), \quad (25)$$

where  $t_1 = -|\mathbf{q}_{1T}|^2$ ,  $t_2 = -|\mathbf{q}_{2T}|^2$ ,  $\mu_F$  and  $\mu_R$  are the factorization and renormalization scales respectively,  $\Phi_i(x, t, \mu_F^2)$  is the unPDF of an  $i$ -parton in the initial state and  $d\hat{\sigma}_{ij}$  is the cross section of the hard off shell partonic subprocess. The cross section (25) is normalized to be in accordance with the CPM at the collinear limit, when  $\mathbf{q}_{(1,2)T} = 0$ .

According to the factorization formula (25) the partonic cross section of the  $2 \rightarrow 1$  subprocess [as an example we

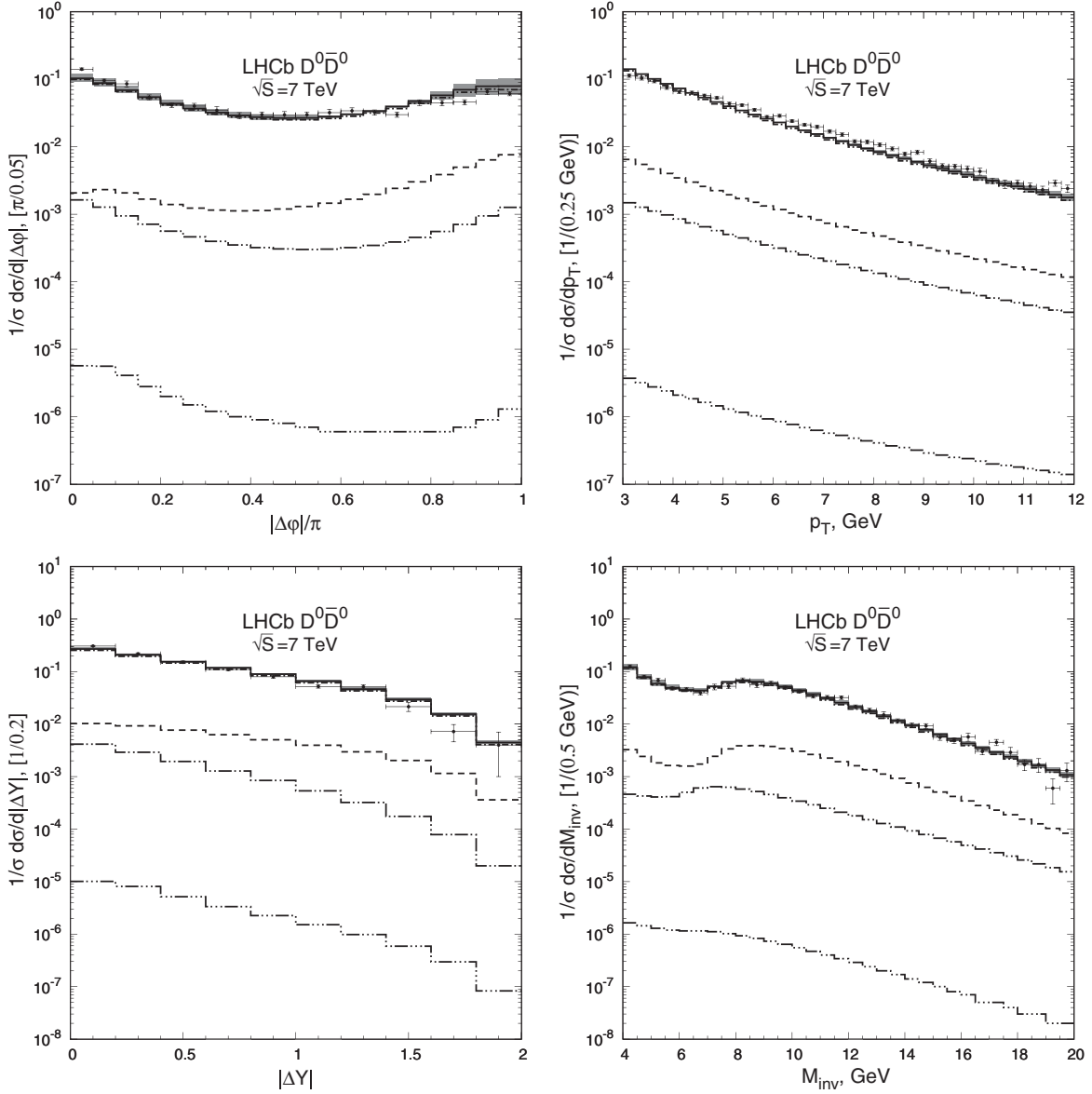


FIG. 4. The spectra of  $D^0\bar{D}^0$  pairs differential in azimuthal angle difference (left, top), transverse momentum (right, top), rapidity distance (left, bottom) and invariant mass of the pair (right, bottom) at the  $2 < y < 4$  and  $\sqrt{S} = 7$  TeV. The LHCb data at LHC are from the Ref. [20]. Dashed line represents the contribution of gluon fragmentation in gluon-gluon fusion, dash-dotted line—the  $c$ -quark fragmentation contribution in gluon-gluon fusion, double-dot-dashed line is the  $c$ -quark fragmentation contribution in quark-antiquark annihilation (sum of  $u$ -,  $d$ - and  $s$ -quark contributions), and triple-dot-dashed—the same for gluon fragmentation, solid line is their sum.

take (1)] is being a convolution of gluon production squared amplitude (17) with unPDFs, and it can be simplified to the following formula:

$$\begin{aligned} \frac{d\sigma}{dydk_T}(p+p \rightarrow g+X) \\ = \frac{1}{k_T^3} \int d\phi_1 \int dt_1 \Phi(x_1, t_1, \mu^2) \\ \times \Phi(x_2, t_2, \mu^2) |\mathcal{M}(\mathcal{R} + \mathcal{R} \rightarrow g)|^2, \end{aligned} \quad (26)$$

where  $\phi_1$  is the azimuthal angle between  $\mathbf{k}_T$  and  $\mathbf{q}_{1T}$ .

The hadronic cross section of any  $2 \rightarrow 2$  subprocess, here we show the case of heavy-quark pair production through the process (3), can be written as follows:

$$\begin{aligned} \frac{d\sigma}{dy_1 dy_2 dk_{1T} dk_{2T}}(p+p \rightarrow c(k_1) + \bar{c}(k_2) + X) \\ = \frac{k_{1T} k_{2T}}{16\pi^3} \int d\phi_1 \int d\Delta\phi \int dt_1 \\ \times \Phi(x_1, t_1, \mu^2) \Phi(x_2, t_2, \mu^2) \frac{|\mathcal{M}(\mathcal{R} + \mathcal{R} \rightarrow c + \bar{c})|^2}{(x_1 x_2 S)^2}, \end{aligned} \quad (27)$$

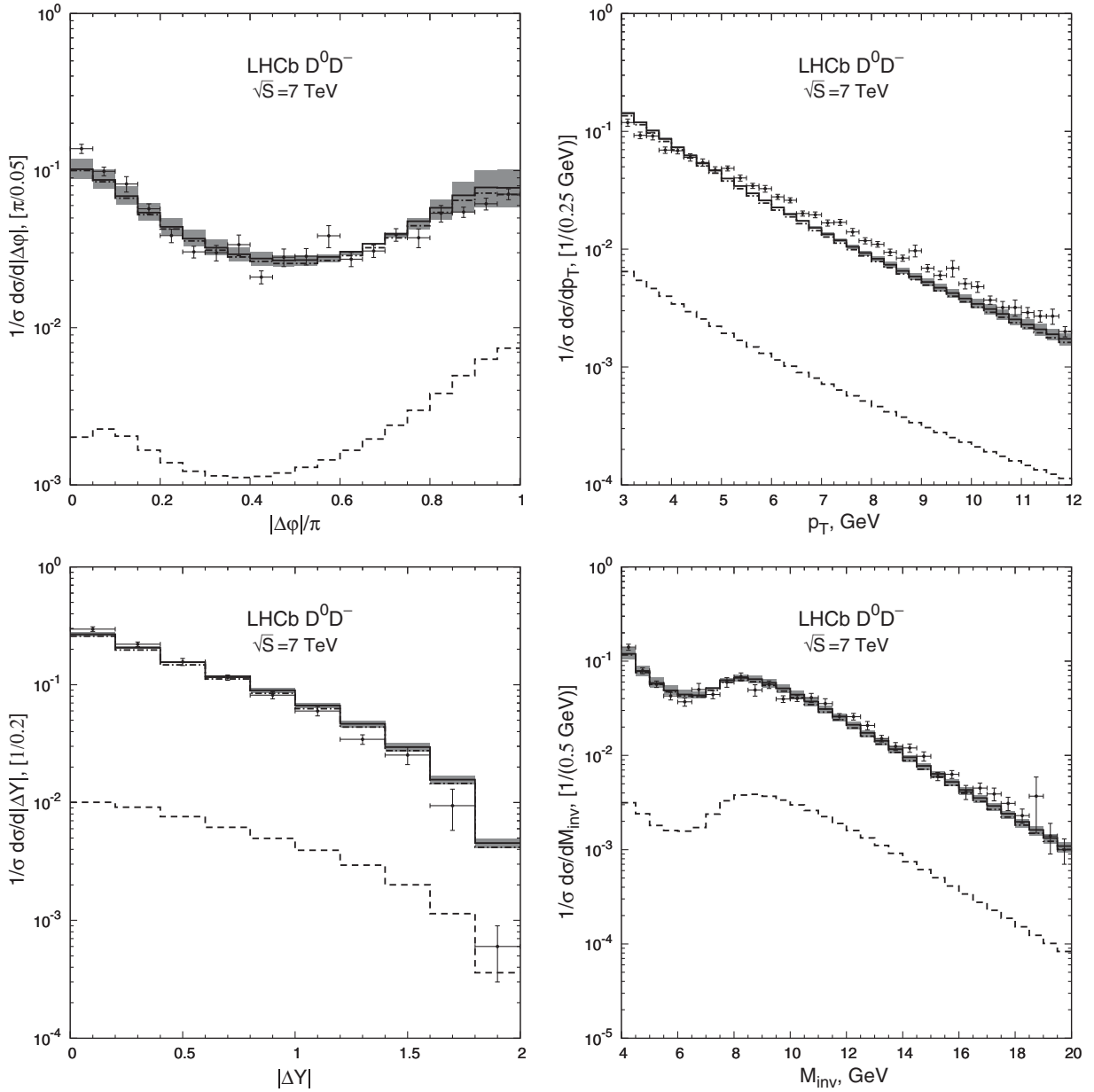


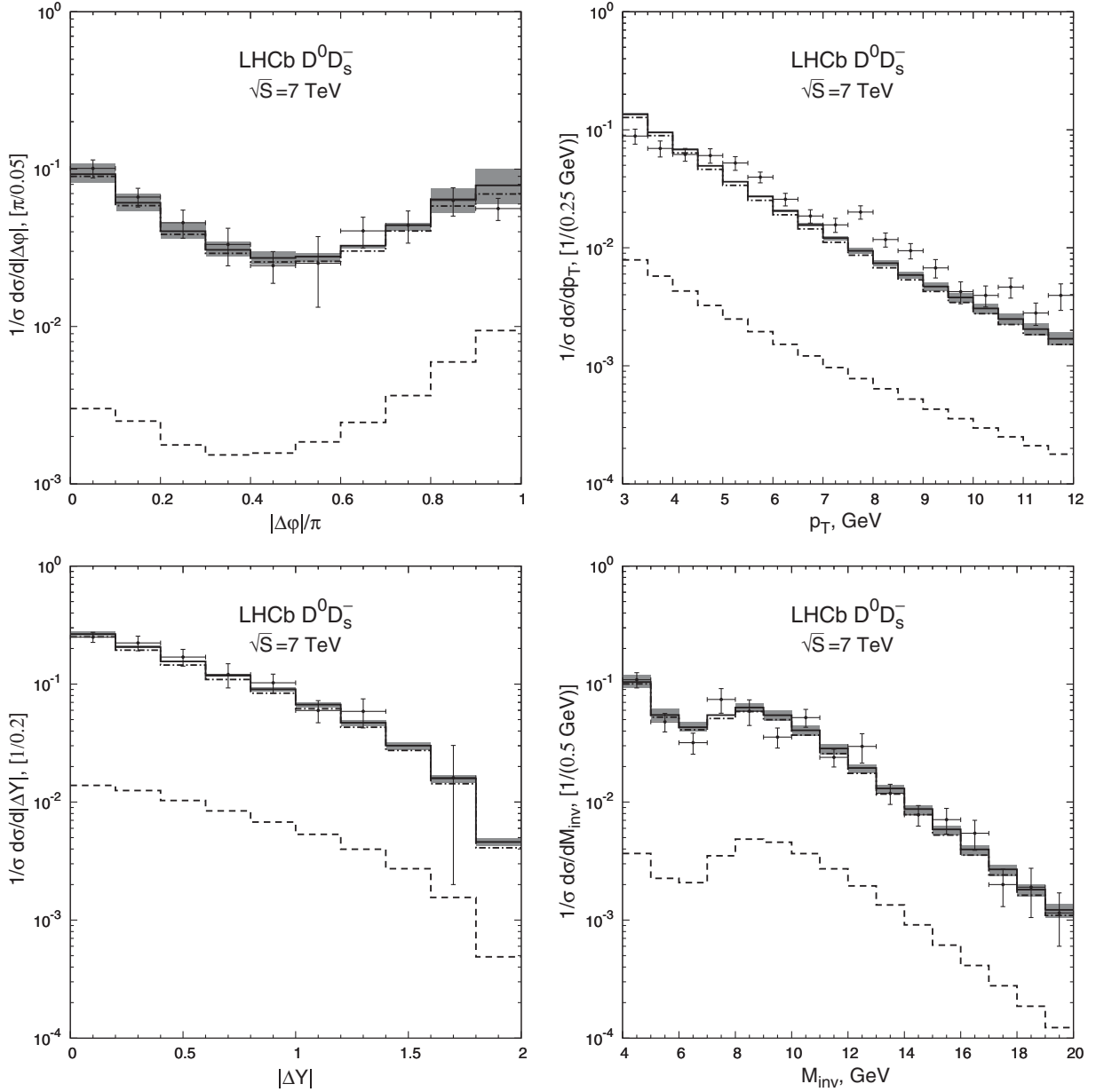
FIG. 5. The spectra of  $D^0 D^-$  pairs differential in azimuthal angle difference (left, top), transverse momentum (right, top), rapidity distance (left, bottom) and invariant mass of the pair (right, bottom) at the  $2 < y < 4$  and  $\sqrt{S} = 7$  TeV. The LHCb data at LHC are from the Ref. [20]. Dashed line represents the contribution of gluon fragmentation in gluon-gluon fusion, dash-dotted line—the  $c$ -quark fragmentation contribution in gluon-gluon fusion, solid line is their sum.

where  $x_1 = q_1^+/P_1^+$ ,  $x_2 = q_2^-/P_2^-$ ,  $\Delta\phi$  is the azimuthal angle between  $\mathbf{k}_{1T}$  and  $\mathbf{k}_{2T}$ , the rapidity of the final parton with four-momentum  $k$  can be presented as  $y = \frac{1}{2} \ln(\frac{k^+}{k^-})$ . Again, we have checked a fact that in the limit of  $t_{1,2} \rightarrow 0$ , we reproduce the conventional factorization formula of the collinear parton model from (26) and (27).

The unintegrated parton distribution function  $\Phi(x, t, \mu^2)$  is an important ingredient in our scheme, and we obtain that using the mixed DGLAP-BFKL approach of Kimber,

Martin and Ryskin (KMR) [42] developed to extract unPDFs from conventional integrated ones. The KMR approach is based on the assumption that at small  $x$  the  $k_T$ -ordered DGLAP chain of emissions is followed by at least a few steps of BFKL evolution, with subsequent emissions ordered in rapidity and with broken  $k_T$ -ordering. This assumption allows us to justify the  $k_T$ -factorization ansatz for the cross section, together with particular procedure to calculate the hard-scattering part of it, based on the Reggeization of partons in the initial state. In our



FIG. 6. The spectra of  $D^0 D_s^-$  pairs. The notations as in the Fig. 5.

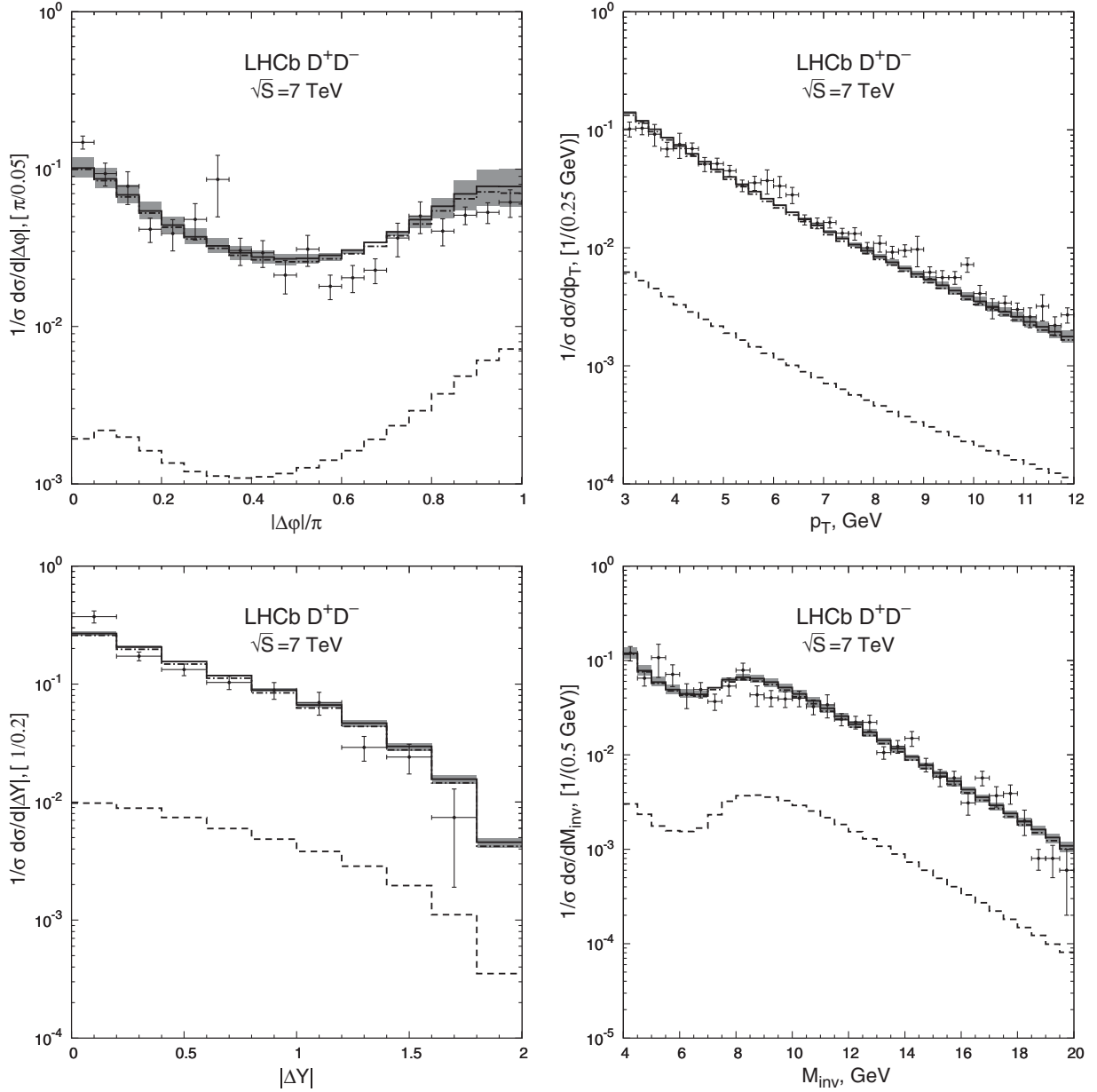
numerical analysis, as input for KMR procedure, we use the LO set of the Martin-Roberts-Stirling-Thorne (MRST) [43] proton PDFs as our default.

This procedure to obtain unPDFs requires less computational efforts than the precise solution of two-scale evolution equations such as, for instance, Ciafaloni-Catani-Fiorani-Marchesini equation [44], but we found it to be suitable and adequate to the physics of processes under study. In our previous study [16] devoted to the similar processes of  $D$ -meson production we proved that they give the best description of  $p_T$ -spectra measured at the central region of rapidity at the LHC.

To describe the hadronization stage we use the fragmentation model [45]. So, the transition from the produced

gluon or  $c$ -quark to the  $D$  meson is described by corresponding fragmentation function  $D_i(z, \mu^2)$ . According the factorization theorem in QCD, in the fragmentation model the basic formula for the single  $D$ -meson production cross section reads

$$\begin{aligned} & \frac{d\sigma(p + p \rightarrow D + X)}{dp_T dy} \\ &= \sum_i \int_0^1 \frac{dz}{z} D_{i \rightarrow D}(z, \mu^2) \\ & \times \frac{d\sigma(p + p \rightarrow i(k_i = p_D/z) + X)}{dk_{iT} dy_i}, \end{aligned} \quad (28)$$

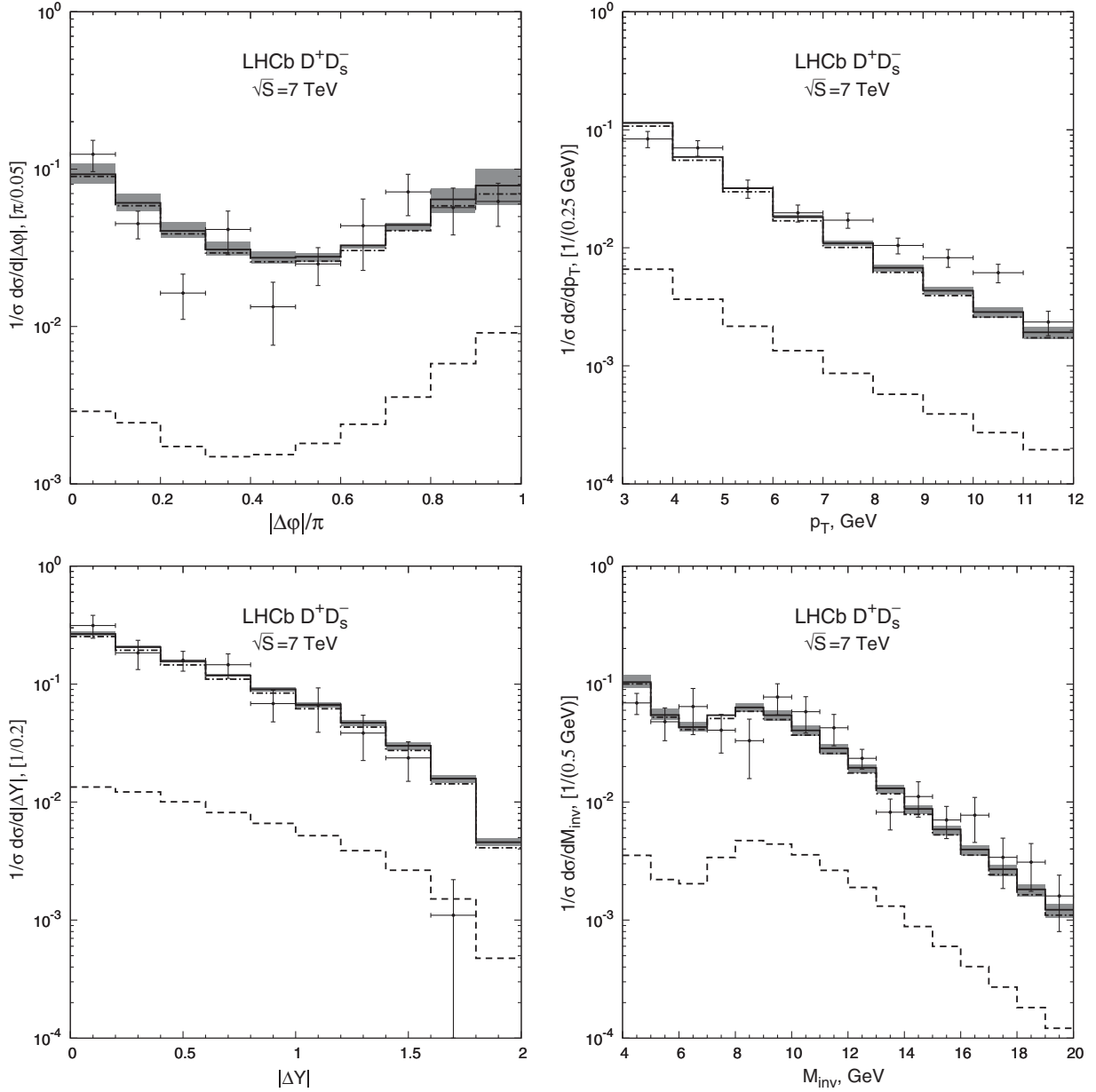
FIG. 7. The spectra of  $D^+D^-$  pairs. The notations as in the Fig. 5.

where  $D_{i \rightarrow D}(z, \mu^2)$  is the fragmentation function for the parton  $i$  splitting into  $D$  meson at the scale  $\mu^2$ ,  $z$  is the longitudinal momentum fraction of a fragmenting particle carried by the  $D$  meson. In the zero-mass approximation the fragmentation parameter  $z$  can be defined as follows:  $p_D^\mu = z k_i^\mu$ ,  $p_D$  and  $k_i$  are the  $D$ -meson and  $i$ -parton four-momenta, and  $y_D = y_i$ .

In the case of pair  $D$ -meson production an additional integral over the momentum fraction of second parton appears. The fragmentation formula now has the following form:

$$\begin{aligned} & \frac{d\sigma(p + p \rightarrow D + \bar{D} + X)}{dp_{DT} dy_D dp_{\bar{D}T} dy_{\bar{D}}} \\ &= \sum_{ij} \int_0^1 \frac{dz_1}{z_1} \int_0^1 \frac{dz_2}{z_2} D_{i \rightarrow D}(z_1, \mu^2) D_{j \rightarrow \bar{D}}(z_2, \mu^2) \\ & \times \frac{d\sigma(p + p \rightarrow i(k_i = p_D/z_1) + j(k_j = p_{\bar{D}}/z_2) + X)}{dk_{iT} dy_i dk_{jT} dy_j}, \end{aligned} \quad (29)$$

where  $D_{i \rightarrow D}(z_1, \mu^2)$  and  $D_{j \rightarrow \bar{D}}(z_2, \mu^2)$  are the fragmentation functions for the transitions of parton  $i = g, c, \bar{c}$  into

FIG. 8. The spectra of  $D^+D_s^-$  pairs. The notations as in the Fig. 5.

$D$  meson with momentum fraction  $z_1$  and of parton  $j = g, c, \bar{c}$  into  $\bar{D}$  meson with momentum fraction  $z_2$ , respectively, at the scale  $\mu^2$ . In our calculations we use the LO FFs from Ref. [4], where the fits of nonperturbative  $D^0, D^+, D^{*+}$ , and  $D_s^+$  FF's to OPAL data from LEP1 [8] were performed. These FFs satisfy two desirable properties: at first, their scaling violations are ruled by DGLAP evolution equations; at second, they are universal.

As the contribution of gluon fragmentation at  $\mu > \mu_0$  is initiated by the perturbative transition of gluons to  $c\bar{c}$ -pairs encountered by DGLAP evolution equations, the part of  $c$ -quarks produced in the subprocess (3) with their subsequent transition to  $D$  mesons are already taken into

account considering  $D$ -meson production via gluon fragmentation in the subprocess (1). Such a way, to avoid double counting, we must subtract this contribution, that can be effectively done by the lower cut of the amplitude in formula (27) as  $\hat{s} > 4m_c^2$ , i.e. at the threshold of the production of  $c\bar{c}$ -pair.

#### IV. SINGLE $D$ -MESON PRODUCTION

The forward rapidity region in  $pp$  collisions at the LHC became available due to the specially designed LHCb detector where the measurements of differential cross sections of  $D^0/\bar{D}^0, D^\pm, D^{*\pm}$  mesons with  $2.0 < y < 4.5$  at the  $\sqrt{S} = 7$  TeV were performed [18,20]. The observed

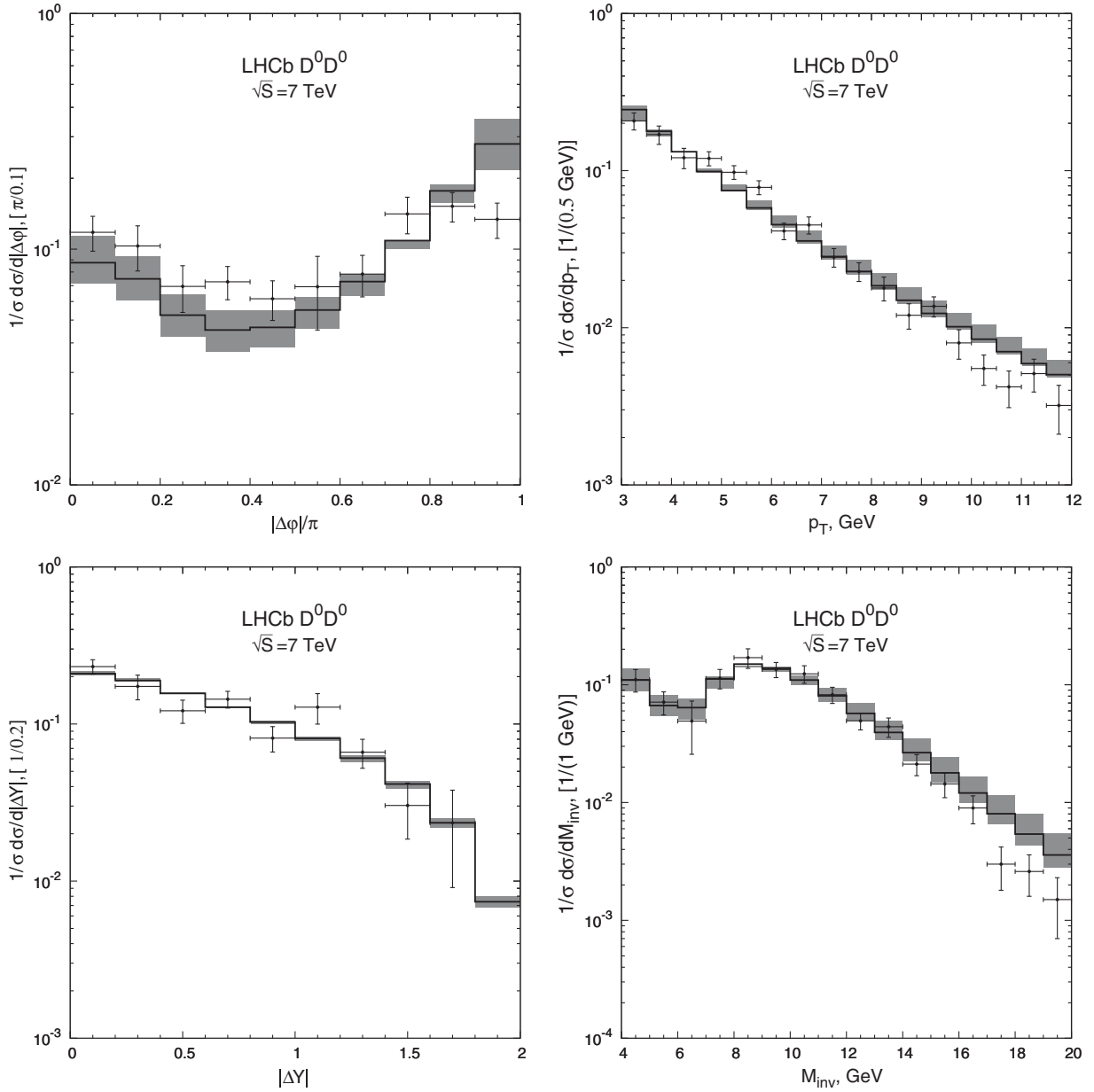


FIG. 9. The spectra of  $D^0 D^0$  pairs differential in azimuthal angle difference (left, top), transverse momentum (right, top), rapidity distance (left, bottom) and invariant mass of the pair (right, bottom) at the  $2 < y < 4$  and  $\sqrt{S} = 7$  TeV. The LHCb data at LHC are from the Ref. [20]. Solid line represents the leading contribution of gluon fragmentation in gluon-gluon fusion.

data divided into five rapidity regions are presented together with our results obtained in the LO of PRA in the Fig. 1. The sum of all contributions from subprocesses (1)–(3) is shown as solid line. We estimated a theoretical uncertainty arising from uncertainty of definition of factorization and renormalization scales ( $\mu = \mu_R = \mu_F$ ) by varying them between  $1/2p_T$  and  $2p_T$  around their central value of  $p_T$ , the transverse momentum of fragmenting parton. The resulting uncertainty is depicted in the figures by shaded bands. We find a good agreement between our predictions and experimental data in the whole  $p_T$  interval

of  $D$ -meson transverse momenta within experimental and theoretical uncertainties only with one exclusion for  $D_s$  mesons, when our results overestimate data by a factor of 2. It may be connected with the bad quality of gluon or  $c$ -quark FF extracted from the  $e^+e^-$  data, see [4] for details.

In the calculations of Ref. [14], which were done in the similar approach with KMR unPDFs [42] and using scale-dependent FFs KKKS08 [4], the underestimation of experimental data from LHCb Collaboration for  $D^0$ -meson transverse momentum spectra by a factor of 2 or more was found. But, in the Ref. [14] the only  $c$ -quark

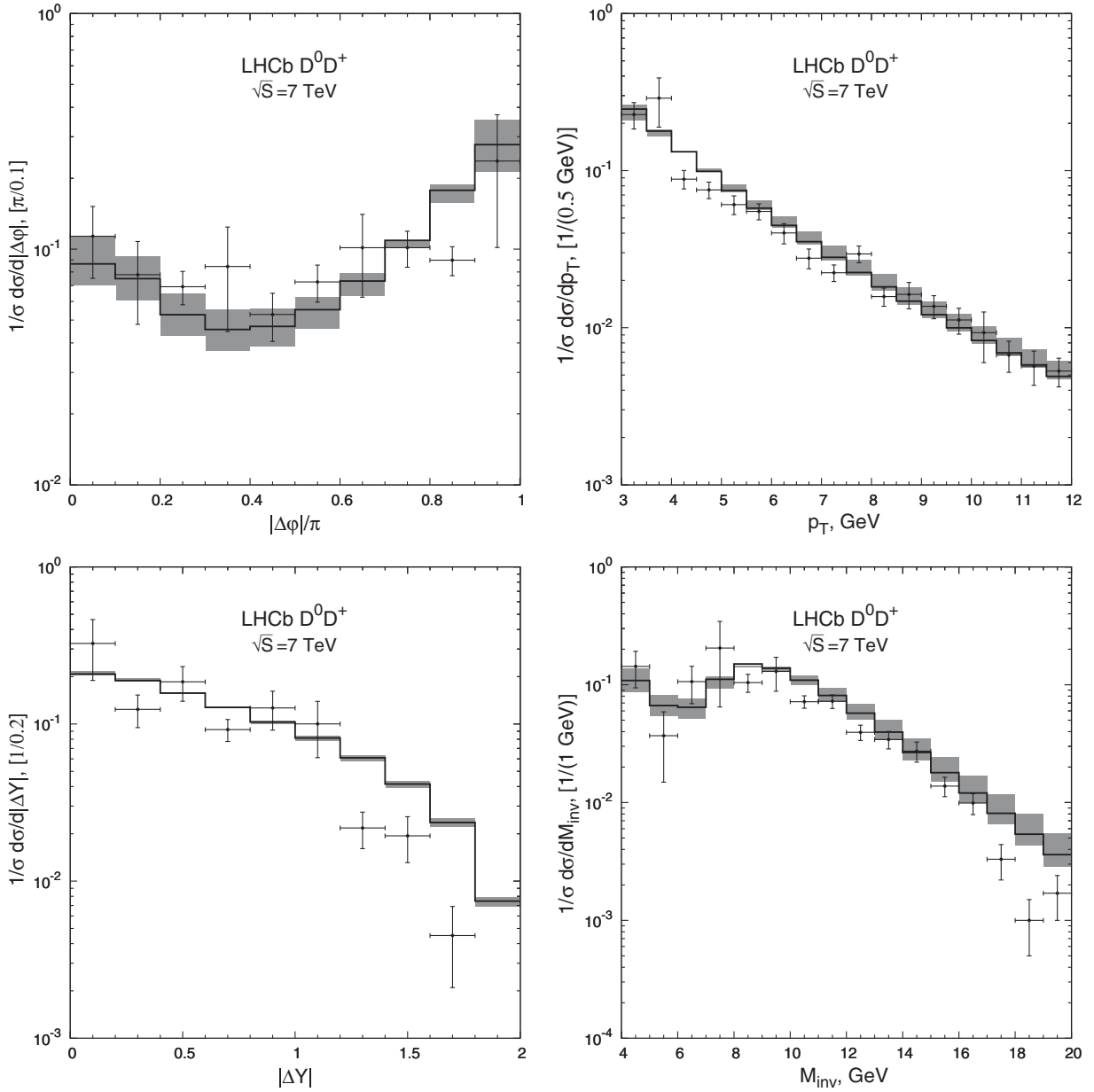


FIG. 10. The spectra of  $D^0 D^+$  pairs. The notations as in the Fig. 9.

fragmentation into  $D$  mesons was applied. We correct this results taking into account gluon to  $D$ -meson fragmentation and obtain good agreement with the data.

We also provide theoretical calculations for  $D$  production within the LHCb acceptance at the energy  $\sqrt{S} = 13$  TeV. We compare them with the most recent data from the LHCb [19]. The results are presented as double differential distributions  $d^2\sigma/(dp_T dy)$  for  $D^0/\bar{D}^0$ ,  $D^\pm$ ,  $D^{*\pm}$  mesons in the same rapidity region as at the  $\sqrt{S} = 7$  TeV in the Fig. 2. Our predictions for  $D_s$  mesons became better at the  $\sqrt{S} = 13$  TeV than at the smaller energy, and they are very close to data. Again we estimate a theoretical uncertainty varying the factorization and

renormalization scales up and down by the factor of 2 around the central value and depict it by the shaded bands at the plot. In addition to the differential cross sections the LHCb Collaboration presented ratios between differential cross sections at 13 and 7 TeV in the transverse momentum region  $3 < p_T < 8$  GeV.

$$R_{13/7}(p_T, y) = \frac{d^2\sigma_{13}}{dp_T dy} / \frac{d^2\sigma_7}{dp_T dy}. \quad (30)$$

We present our calculations of the ratio in the Fig. 3. We obtain a good agreement of our predictions with the experimental data for the  $D^0$ ,  $D^+$ , and  $D^*$  mesons at the

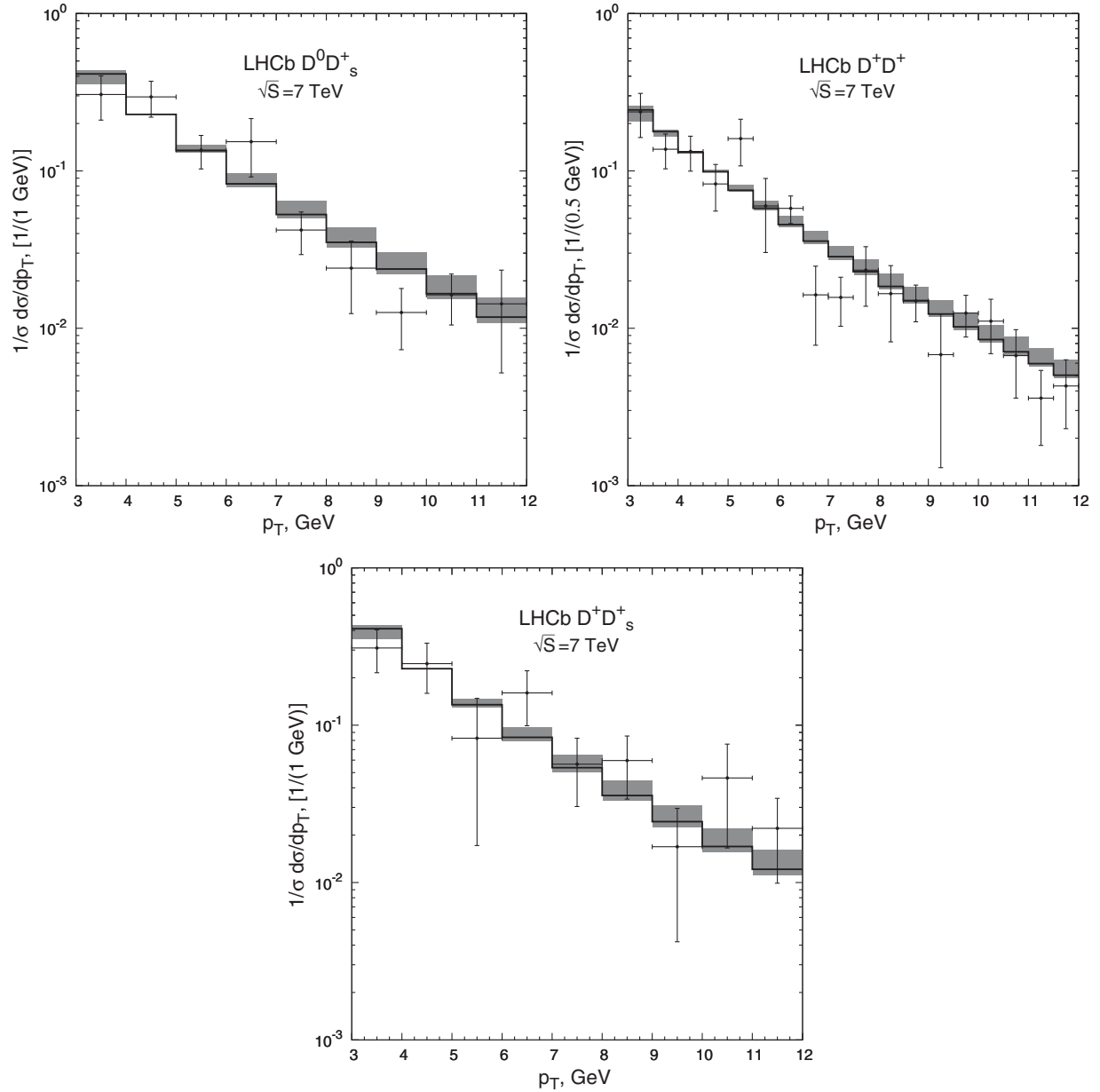


FIG. 11. The transverse momentum distributions of  $D^0 D_s^+$  (left, top),  $D^+ D^+$  (right, top) and  $D^+ D_s^+$  (bottom) pairs at the  $2 < y < 4$  and  $\sqrt{S} = 7$  TeV. The LHCb data at LHC are from the Ref. [20]. Solid line represents the leading contribution of gluon fragmentation in gluon-gluon fusion.

whole ranges of their transverse momenta and rapidities. In case of  $D_s$  mesons, the agreement is not so good because of the overestimation of data by theoretical predictions at the  $\sqrt{S} = 7$  TeV.

## V. PAIR PRODUCTION OF $D$ MESONS

A comparative study of correlation observables in  $D\bar{D}$ - and  $DD$ -pair production is an important test of  $D$ -meson production mechanism. As it was found earlier in Ref. [14] in case of particle plus antiparticle production ( $D\bar{D}$ ) the

data from LHCb Collaboration [20] can be described well in the  $k_T$ -factorization approach with inclusion of subprocess (3) only and using the scale-independent  $c$ -quark Peterson FF. Working in the same manner, to describe  $DD$  pair production via  $c$ -quark fragmentation into  $D$ -meson, we should include contribution from  $2 \rightarrow 4$  subprocess (9) as it was done in Ref. [15]. It was obtained that predictions based on the model of  $D$ -meson production via  $c$ -quark fragmentation can not describe data for correlation observables in  $DD$ -pair production absolutely and inclusion of double parton scattering (DPS) production mechanism is

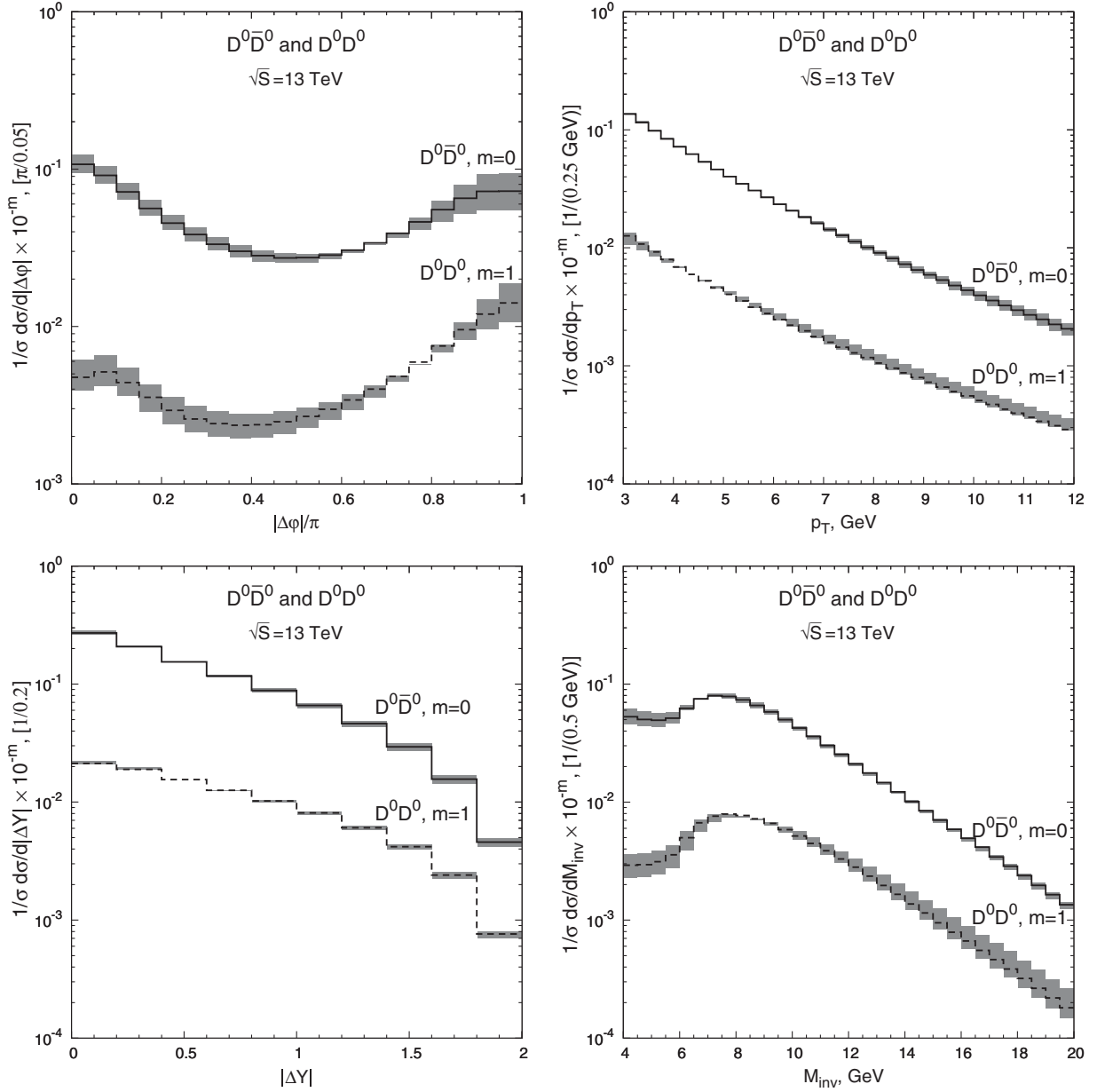


FIG. 12. The predicted spectra of  $D^0\bar{D}^0$  (solid line,  $m = 0$ ) and  $D^0D^0$  (dashed line,  $m = 1$ ) pairs differential in azimuthal angle (left, top), transverse momentum (right, top), rapidity distance (left, bottom) and invariant mass of the pair (right, bottom) at the  $2 < y < 4$  and  $\sqrt{S} = 13$  TeV.

needed. The inclusion of a gluon fragmentation channel, which is the dominant one in the  $D$ -meson production at the LHC energy [16], allows us to describe data for  $DD$ -pair production well without hypothesis about DPS production mechanism [17].

Now we start with discussion of  $D\bar{D}$ -pair production at the LHCb. In Fig. 4, we compare our predictions with the LHCb data at the  $\sqrt{S} = 7$  TeV for  $D^0\bar{D}^0$ -pair production normalized spectra over the azimuthal angle difference  $\Delta\phi$ ,  $D$ -meson transverse momenta  $p_T$ , rapidity distance  $\Delta Y$  and invariant mass of  $D\bar{D}$  pair  $M$ . As it is estimated, the

dominating contribution is gluon-gluon fusion into  $c\bar{c}$  pair with  $c$ -quark fragmentation into the  $D^0$  meson and with  $\bar{c}$ -quark fragmentation into the  $\bar{D}^0$  meson (dash-dotted line). The contribution from gluon-gluon fusion into two gluons when the first gluon fragments into  $D^0$  meson and second gluon fragments into  $\bar{D}^0$  meson (dashed line) is lying below the leading contribution by the order of magnitude. Two next by value contributions from quark-antiquark annihilation into  $c\bar{c}$  pair (double-dot-dashed line) or into two gluons (triple-dot-dashed line) are even smaller, and we will ignore these ones below. In the Figs. 5–8, we

demonstrate that our predictions for correlation spectra of  $D^0D^-$ ,  $D_0D_s^-$ ,  $D^+D^-$  and  $D^+D_s^-$  are in a good agreement with the experimental data.

The production of  $DD$  pairs is mostly generated by the gluon fragmentation into the  $D$  meson in the subprocess of gluon-gluon fusion. The contribution of two-gluon production in the quark-antiquark annihilation is suppressed by 2 orders of magnitude in the same way as the contributions from the channel of  $c\bar{c}c\bar{c}$  double-pair production. In the Figs. 9–10, we compare our predictions based on two-gluon fragmentation mechanism with the experimental data for correlation spectra of  $D^0D^0$  and  $D^0D^+$ , correspondingly. In the Fig. 11, the transverse momentum spectra for  $D^0D_s^+$ ,  $D^+D^+$  and  $D^+D_s^+$  pairs are presented. We found a good agreement between predictions obtained in the LO PRA calculations and experimental data from LHCb Collaboration for all  $DD$ -pair correlation observables, and there is no place for contribution of DPS production mechanism, as it is discussed in literature. In the Fig. 12, the predicted spectra of  $D^0\bar{D}^0$  and  $D^0D^0$  pairs at the  $\sqrt{S} = 13$  TeV are shown. The LHCb Collaboration future measurements of these two-particle correlation spectra will be additional test of the PRA and the gluon to  $D$ -meson fragmentation model.

## VI. CONCLUSIONS

We introduce a comprehensive study of single and pair fragmentation production of  $D$  mesons in proton-proton collisions at the energies  $\sqrt{S} = 7$  and  $\sqrt{S} = 13$  TeV in the forward rapidity region at the LHC in the framework of parton Reggeization approach. We use the gauge invariant amplitudes of hard parton subprocesses in the LO level of theory with Reggeized gluons and Reggeized quarks in the initial state in a self-consistent way together with unintegrated parton distribution functions proposed by Kimber, Martin and Ryskin. To describe the nonperturbative transition of produced gluons and  $c$ -quarks into the  $D$  mesons we use the universal fragmentation functions obtained from the fit of  $e^+e^-$  annihilation data from CERN LEP1. We found our results for  $D$ -meson and  $D\bar{D}(DD)$ -pair production to be in the good coincidence with experimental data from the LHCb Collaboration. The predictions for the  $D^0\bar{D}^0(D^0D^0)$ -pair production correlation spectra in the large rapidity region for the energy  $\sqrt{S} = 13$  TeV are also presented. We have found that in case of single  $D$ -meson production at the LHC energies, both mechanisms, gluon and  $c$ -quark fragmentation, are important with some preference for the gluon fragmentation production. In the  $D\bar{D}$ -pair production we found  $c$ -quark production

mechanism to be the main one, while in the  $DD$ -pair production the gluon fragmentation is a totally dominant source. Additionally, we arrive at the important conclusion that  $DD$ -pair production can not be used for differentiation between single and double parton scattering approaches of high-energy QCD.

The good agreement between predictions of LO calculations in the PRA and experimental data for  $D$  meson production at the LHCb follows to the question on the role of NLO correction for such processes in the PRA. It is clear that for the self-consistency of PRA last one should be small in the relevant kinematical region. As because the PRA is the scheme of  $k_T$ -factorization for multiscale hard processes, which is based on the Lipatov's gauge invariant effective field theory for high energy processes in QCD, this question has not only phenomenological but also a general importance for development of the QCD at high energies. The NLO calculation will include the real part of corrections from emission of additional gluons or quarks as well as the virtual correction from loop diagrams. The scheme of extraction of double counting between LO and real NLO contributions are suggested in our recent article [46], where we studied prompt diphoton production in the PRA. The new type of rapidity divergences, associated with the  $\log(1/x)$ -corrections, appears in the loop corrections in the NLO PRA. The covariant procedure of regularization of such rapidity divergences, preserving the gauge invariance of effective action, is described in publications [47–50]. So, the NLO calculations in the PRA can be made and their comparison with presented here LO results will be an important test of the high-energy approach of the QCD.

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