Asymptotic 3-loop heavy flavor corrections to the charged current structure functions $F_L^{W^+-W^-}(x,Q^2)$ and $F_2^{W^+-W^-}(x,Q^2)$

A. Behring,[1] J. Blümlein,[1] G. Falcioni,[1] A. De Freitas,[1] A. von Manteuffel,[2,3] and C. Schneider[4]

[1]*Deutsches Elektronen-Synchrotron, DESY, Platanenallee 6, D-15738 Zeuthen, Germany*
[2]*Department of Physics and Astronomy, Michigan State University, East Lansing, Michigan 48824, USA*
[3]*PRISMA Cluster of Excellence, Johannes Gutenberg University, 55099 Mainz, Germany*
[4]*Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Altenbergerstraße 69, A-4040 Linz, Austria*

(Received 20 September 2016; published 8 December 2016)

We derive the massive Wilson coefficients for the heavy flavor contributions to the nonsinglet charged current deep-inelastic scattering structure functions $F_L^{W^+}(x,Q^2) - F_L^{W^-}(x,Q^2)$ and $F_2^{W^+}(x,Q^2) - F_2^{W^-}(x,Q^2)$ in the asymptotic region $Q^2 \gg m^2$ to 3-loop order in quantum chromodynamics at general values of the Mellin variable N and the momentum fraction x. Besides the heavy quark pair production, also the single heavy flavor excitation $s \to c$ contributes. Numerical results are presented for the charm quark contributions, and consequences on the unpolarized Bjorken sum rule and Adler sum rule are discussed.

DOI: 10.1103/PhysRevD.94.114006

I. INTRODUCTION

The flavor nonsinglet charged current structure functions $F_{1,2}^{W^+-W^-}(x,Q^2)$ can be measured in deep-inelastic neutrino (antineutrino)-nucleon scattering and in high energy charged lepton-nucleon scattering in ep or μp collisions. They are associated with the well-known unpolarized Bjorken sum rule [1] and Adler sum rule [2] by their first moment, the former of which can be used for QCD tests measuring the strong coupling constant $a_s = \alpha_s/(4\pi) = g_s^2/(4\pi)^2$. These structure functions also allow for an associated determination of the valence quark distributions of the nucleon. The massless contributions to these combinations of structure functions have been derived recently to 3-loop order [3]. In the present paper, we compute the asymptotic heavy flavor corrections to these flavor nonsinglet structure functions in the region $Q^2 \gg m^2$ to the same order, with m the heavy quark mass and Q^2 the virtuality of the process, and present numerical results in the case of charm quark contributions.[1]

We would like to point out that, contrary to some folklore, the calculation of the massive Wilson coefficients in the asymptotic region $Q^2 \gg m^2$ is *not* a massless calculation, in which mass effects are just obtained by external scale setting identifying, e.g., the factorization scale μ^2 with a heavy quark mass parameter m^2 in the logarithms. The calculation in the region $Q^2 \gg m^2$ uses massive propagators for massive lines, and the respective mass terms have a strong impact on the resulting Feynman parameterization and the types of integral arising. This holds not only for the intermediary terms but also for the results, where iterated integrals appear (cf., e.g., [6–8]), that are much more involved than the usual harmonic polylogarithms appearing in massless Wilson coefficients at 3-loop order. The only terms we will disregard in the following are power corrections $\sim m^2/Q^2$, which, however, have been dealt with completely in the nonsinglet case in Ref. [9] to 2-loop order. In the numerical illustrations, we will account for these contributions.

The massless and massive QCD corrections at first order in the coupling constant have been computed in Refs. [10–13][2] and in Refs. [4,15–18] to $O(a_s^2)$[3] The massive $O(a_s^2)$ corrections were calculated in the asymptotic representation [19], which is valid at high scales Q^2. To obtain an estimate of the range of validity, one may perform an $O(a_s)$ comparison with the complete result for the process of single heavy quark excitation [12,13]. Likewise, a comparison is possible for the $O(a_s^2)$ corrections, which were given in complete form in Ref. [9] for the Wilson coefficient with the gauge boson coupling to the massless fermions and assuming an approximation for the Cabibbo-suppressed flavor excitation term $s' \to c$, where the additional charm quark in the final state has been dealt with as being massless.

[1]It would also be desirable to compute the heavy flavor corrections to the combinations $F_{1,2}^{W^++W^-}(x,Q^2)$. As it is well known from Ref. [4], the 3-loop corrections to this combination receive contributions from the massive operator matrix elements $A_{Qg}^{(3)}$ and $A_{gg,Q}^{(3)}$, the calculation of which still will need a longer time to be completed. Both the $-$ and $+$ combinations form observables and can be measured individually. These combinations have to be formed to get the correct current-crossing relations (see, e.g., [5]) and even have therefore a preference compared to the individual structure functions.

[2]The massive 1-loop corrections given in [14] were corrected in [13]; see also [12].

[3]Some results given in [15] have been corrected in Ref. [4].

 © 2016 American Physical Society

The charged current scattering cross sections are given by [20,21]

$$\frac{d\sigma^{\nu(\bar{\nu})}}{dxdy} = \frac{G_F^2 s}{4\pi}\frac{M_W^4}{(M_W^2+Q^2)^2}\{(1+(1-y)^2)F_2^{W^\pm}(x,Q^2) - y^2 F_L^{W^\pm}(x,Q^2) \pm (1-(1-y)^2)xF_3^{W^\pm}(x,Q^2)\}, \tag{1.1}$$

$$\frac{d\sigma^{l(\bar{l})}}{dxdy} = \frac{G_F^2 s}{4\pi}\frac{M_W^4}{(M_W^2+Q^2)^2}\{(1+(1-y)^2)F_2^{W^\mp}(x,Q^2) - y^2 F_L^{W^\mp}(x,Q^2) \pm (1-(1-y)^2)xF_3^{W^\mp}(x,Q^2)\}, \tag{1.2}$$

where $x = Q^2/ys$ and $y = q.P/l.P$ denote the Bjorken variables, l and P are the incoming lepton and nucleon 4-momenta, respectively, and $s = (l+P)^2$. G_F is the Fermi constant and M_W the mass of the W boson. $F_i^{W^\pm}(x,Q^2)$ are the structure functions, where the $+(-)$ signs refer to incoming neutrinos (antineutrinos) and charged antileptons (leptons), respectively. We will consider the combination of structure functions

$$F_{1,2}^{W^+-W^-}(x,Q^2) = F_{1,2}^{W^+}(x,Q^2) - F_{1,2}^{W^-}(x,Q^2) \tag{1.3}$$

in the following. The longitudinal structure function is obtained by

$$F_L(x,Q^2) = F_2(x,Q^2) - 2xF_1(x,Q^2). \tag{1.4}$$

The combinations (1.3) can be measured projecting onto the kinematic factor $Y_+ = 1+(1-y)^2$ in the case of F_2 for the differential cross sections at x, $Q^2 = \text{const}$ and by varying s in addition, in the case of F_L.

The main formalism to obtain the massive Wilson coefficients in the asymptotic range $Q^2 \gg m^2$, i.e., $L_{q,L,2}^{W^+-W^-,\mathrm{NS}}$ and $H_{q,L,2}^{W^+-W^-,\mathrm{NS}}$, has been outlined in Refs. [4,19,22]. They are composed of the massive nonsinglet operator matrix elements (OMEs) [23] and the massless Wilson coefficients [3] up to 3-loop order. The following representation of the structure functions is obtained:

$$F_{L,2}^{W^+-W^-}(x,Q^2) = 2x\{[|V_{du}|^2(d-\bar{d}) + |V_{su}|^2(s-\bar{s}) - V_u(u-\bar{u})]\otimes[C_{q,L,2}^{W^+-W^-,\mathrm{NS}} + L_{q,L,2}^{W^+-W^-,\mathrm{NS}}] + [|V_{dc}|^2(d-\bar{d}) + |V_{sc}|^2(s-\bar{s})]\otimes H_{q,L,2}^{W^+-W^-,\mathrm{NS}}\}, \tag{1.5}$$

with one massless Wilson coefficient $C_{q,L,2}^{W^+-W^-,\mathrm{NS}}$ and two massive Wilson coefficients $L_{q,L,2}^{W^+-W^-,\mathrm{NS}}$ and $H_{q,L,2}^{W^+-W^-,\mathrm{NS}}$; see Secs. II and III. The coefficients V_{ij} are the Cabibbo-Kobayashi-Maskawa [24,25] matrix elements, where $V_u = |V_{du}|^2 + |V_{su}|^2$, and the present numerical values are [26]

$$|V_{du}| = 0.97425, \qquad |V_{su}| = 0.2253,$$
$$|V_{dc}| = 0.225, \qquad |V_{sc}| = 0.986, \tag{1.6}$$

with

$$u - \bar{u} = u_v, \tag{1.7}$$

$$d - \bar{d} = d_v, \tag{1.8}$$

$$s - \bar{s} \approx 0. \tag{1.9}$$

In the following, we will consider only the charm quark corrections with $m \equiv m_c$ the charm quark mass in the on-shell scheme. The transformation to the $\overline{\mathrm{MS}}$ scheme has been given in Ref. [23]. We note that the 3-loop asymptotic charm quark corrections to the combination of structure functions $xF_3^{W^++W^-}(x,Q^2)$ have been computed in Ref. [27] and related corrections to the twist-2 contributions of the polarized structure functions $g_{1,2}(x,Q^2)$ in Ref. [28].

A series of asymptotic 3-loop heavy flavor Wilson coefficients have also been calculated for neutral current scattering along with the transition matrix elements in the variable flavor number scheme; see Refs. [29] for recent surveys.

II. THE STRUCTURE FUNCTION $F_L(x,Q^2)$

The massive Wilson coefficients depend on the logarithms

$$L_M = \ln\left(\frac{m^2}{\mu^2}\right), \qquad L_Q = \ln\left(\frac{Q^2}{\mu^2}\right). \tag{2.1}$$

Here μ denotes the factorization scale. For the Wilson coefficients in Mellin N space, we consider the following series in the strong coupling constant:

$$L_{q,2(L)}^{W^+-W^-,\mathrm{NS}} = \delta_{2,0} + \sum_{k=1}^{\infty} a_s^k L_{q,2(L)}^{W^+-W^-,\mathrm{NS},(k)}, \tag{2.2}$$

$$H_{q,2(L)}^{W^+-W^-,\mathrm{NS}} = \delta_{2,0} + \sum_{k=1}^{\infty} a_s^k H_{q,2(L)}^{W^+-W^-,\mathrm{NS},(k)}, \tag{2.3}$$

$$C_{q,2(L)}^{W^+-W^-,\mathrm{NS}} = \delta_{2,0} + \sum_{k=1}^{\infty} a_s^k C_{q,2(L)}^{W^+-W^-,\mathrm{NS},(k)}. \tag{2.4}$$

In the following, we drop the arguments of the nested harmonic sums [30] and harmonic polylogarithms [31] by defining $S_{\vec{a}}(N) \equiv S_{\vec{a}}$ and $H_{\vec{b}}(x) \equiv H_{\vec{b}}$.

The 3-loop contributions to the Wilson coefficient $L_{q,L}^{W^+-W^-,\mathrm{NS},(3)}$ in Mellin N space are given by

$$
\begin{aligned}
L_{q,L}^{W^+-W^-,\mathrm{NS},(3)} ={}& C_F T_F^2 (2N_F+1)\left[L_Q^2 \frac{64}{9(N+1)} - L_Q\left(\frac{64(19N^2+7N-6)}{27N(N+1)^2} + \frac{128 S_1}{9(N+1)}\right)\right] \\
&+ C_A C_F T_F \Bigg[-L_Q^2 \frac{352}{9(N+1)} + L_Q\Bigg(\frac{1088}{9(N+1)} S_1 + \frac{128}{3(N+1)} S_3 + \frac{128}{3(N+1)} S_{-3} \\
&+ \left(-\frac{256(N^2+N+1)}{3(N-1)(N+1)(N+2)} + \frac{256}{3(N+1)} S_1\right) S_{-2} - \frac{256}{3(N+1)} S_{-2,1} \\
&+ \frac{16 P_4}{27(N-1)N^2(N+1)^3(N+2)} - \frac{128}{N+1}\zeta_3\Bigg)\Bigg] \\
&+ C_F^2 T_F \Bigg[L_Q^2\left(\frac{8(3N^2+3N+2)}{N(N+1)^2} - \frac{32}{N+1} S_1\right) \\
&+ L_M^2 \left(\frac{8(3N^2+3N+2)}{3N(N+1)^2} - \frac{32}{3(N+1)} S_1\right) \\
&+ L_M \left(\frac{8P_1}{9N^2(N+1)^3} - \frac{320}{9(N+1)} S_1 + \frac{64}{3(N+1)} S_2\right) \\
&+ L_Q \Bigg(\frac{128}{3(N+1)} S_1^2 - \frac{32 P_2}{9(N-1)N^2(N+1)^3(N+2)} - \frac{16(N+10)(5N+3)}{9N(N+1)^2} S_1 \\
&- \frac{256}{3(N+1)} S_3 + \left(\frac{512(N^2+N+1)}{3(N-1)(N+1)(N+2)} - \frac{512}{3(N+1)} S_1\right) S_{-2} \\
&- \frac{128}{3(N+1)} S_2 - \frac{256}{3(N+1)} S_{-3} + \frac{512}{3(N+1)} S_{-2,1} + \frac{256}{N+1}\zeta_3\Bigg) \\
&+ \frac{2P_3}{27N^3(N+1)^4} - \frac{896}{27(N+1)} S_1 + \frac{160}{9(N+1)} S_2 - \frac{32}{3(N+1)} S_3\Bigg] + \hat{c}_{q,L}^{(3)},
\end{aligned}
\tag{2.5}
$$

where $\hat{c}_{q,L}^{(3)} = c_{q,L}^{(3)}(N_F+1) - c_{q,L}^{(3)}(N_F)$ is the 3-loop massless contribution; cf. [3]. The color factors in the case of QCD are $C_A = N_c = 3$, $C_F = (N_c^2-1)/(2N_c)$, $T_F = 1/2$, and $N_c = 3$, and N_F denotes the number of massless flavors. Except for $\hat{c}_{q,L}^{(3)}(N_F)$, the Wilson coefficient is expressed by harmonic sums up to weight $\mathsf{w} = 3$. The polynomials P_i above read

$$P_1 = 3N^4 + 6N^3 + 47N^2 + 20N - 12, \tag{2.6}$$

$$P_2 = 36N^6 + 81N^5 - 125N^4 - 319N^3 - 211N^2 - 14N - 24, \tag{2.7}$$

$$P_3 = 219N^6 + 657N^5 + 1193N^4 + 763N^3 - 40N^2 - 48N + 72, \tag{2.8}$$

$$P_4 = 469N^6 + 1143N^5 - 515N^4 - 2055N^3 - 746N^2 + 120N - 144. \tag{2.9}$$

By performing a Mellin inversion, the corresponding representation in x space is obtained, which reads

$$
\begin{aligned}
L_{q,L}^{W^+-W^-,\mathrm{NS},(3)} = {} & C_A C_F T_F \Bigg[-L_Q^2 \frac{352x}{9} + L_Q\Bigg(\frac{16}{27}(781x-312) + \left(\frac{128(2x^3+x^2-1)H_0}{3x} + \frac{64}{3}xH_0^2\right)H_{-1} \\
& + \frac{832}{9}xH_0 - \frac{128}{3}xH_{-1}^2H_0 + \left(\frac{1088x}{9} + \frac{64}{3}xH_0^2\right)H_1 - \frac{128}{3}x^2H_0^2 \\
& + \left(-\frac{128(2x^3+x^2-1)}{3x} + \frac{256}{3}xH_{-1} + \frac{128}{3}xH_0\right)H_{0,-1} - \frac{128}{3}xH_0H_{0,1} - \frac{256}{3}xH_{0,-1,-1} - 128xH_{0,0,-1} \\
& + \frac{128}{3}xH_{0,0,1} + \left(\frac{256x^2}{3} - \frac{128}{3}xH_{-1} - \frac{128}{3}xH_1\right)\zeta_2\Bigg)\Bigg] \\
& + C_F T_F^2 N_F\left[L_Q^2\frac{128x}{9} - L_Q\left(\frac{128}{27}(25x-6) + \frac{512}{9}xH_0 + \frac{256}{9}xH_1\right)\right] \\
& + C_F T_F^2\left[L_Q^2\frac{64x}{9} - L_Q\left(\frac{64}{27}(25x-6) + \frac{256}{9}xH_0 + \frac{128}{9}xH_1\right)\right] \\
& + C_F^2 T_F\Bigg[L_Q^2\Big(8(x+2) - 16xH_0 - 32xH_1\Big) + L_M^2\left(\frac{8(x+2)}{3} - \frac{16}{3}xH_0 - \frac{32}{3}xH_1\right) \\
& + L_M\left(\frac{32}{9}(x+3)H_0 - \frac{8}{9}(53x-56) - \frac{16}{3}xH_0^2 + \frac{64}{3}xH_{0,1} + \left(-\frac{320x}{9} - \frac{64}{3}xH_0\right)H_1\right) \\
& + L_Q\Bigg(-\frac{32}{9}(83x-47) + \left(\frac{256}{3}xH_1 + \frac{256}{3}xH_{-1} - \frac{32}{3}x(16x+11)\right)\zeta_2 \\
& - \left(\frac{256(2x^3+x^2-1)H_0}{3x} + \frac{128}{3}xH_0^2\right)H_{-1} + \frac{256}{3}xH_{-1}^2H_0 + \frac{16}{9}(89x-24)H_0 + \frac{16}{3}x(16x+7)H_0^2 \\
& + \left(\frac{80}{9}(5x-6) + \frac{256}{3}xH_0 - \frac{128}{3}xH_0^2\right)H_1 + \frac{128}{3}xH_1^2 + \Bigg(\frac{256(2x^3+x^2-1)}{3x} - \frac{512}{3}xH_{-1} \\
& - \frac{256}{3}xH_0\Bigg)H_{0,-1} + \left(32x + \frac{256}{3}xH_0\right)H_{0,1} + \frac{512}{3}xH_{0,-1,-1} + 256xH_{0,0,-1} - \frac{256}{3}xH_{0,0,1}\Bigg) \\
& - \frac{2}{27}(653x-872) + \frac{16}{27}(11x+42)H_0 + \frac{8}{9}(x+3)H_0^2 - \frac{8}{9}xH_0^3 + \left(-\frac{896x}{27} - \frac{160}{9}xH_0 - \frac{16}{3}xH_0^2\right)H_1 \\
& + \left(\frac{160x}{9} + \frac{32}{3}xH_0\right)H_{0,1} - \frac{32}{3}xH_{0,0,1}\Bigg] + \hat{c}_{q,L}^{(3)}. \qquad (2.10)
\end{aligned}
$$

Here ζ_k, $k \in \mathbb{N}$, $k \geq 2$, are the values of the Riemann ζ function at integer argument. Except for $\hat{c}_{q,L}^{(3)}(N_F)$, the Wilson coefficient is expressed by weighted harmonic polylogarithms of up to weight $\mathsf{w} = 3$.

The contribution of the massive Wilson coefficient $H_{q,L}$ is found by combining the massless Wilson coefficient $C_{q,L}$ and $L_{q,L}$:

$$
H_{q,L}(x,Q^2) = C_{q,L}(N_F,x,Q^2) + L_{q,L}(x,Q^2). \qquad (2.11)
$$

Equation (1.4) provides the relation to the Wilson coefficients of the structure function $F_1(x,Q^2)$.

III. THE STRUCTURE FUNCTION $F_2(x,Q^2)$

The asymptotic massive 3-loop Wilson coefficient $L_{q,2}^{W^+-W^-,\mathrm{NS},(3)}$ in Mellin N space reads

$$
\begin{aligned}
L_{q,2}^{W^+-W^-,\mathrm{NS},(3)} = {} & C_F T_F^2 \Bigg\{ -\frac{2008(3N^2+3N+2)}{243N(N+1)} + L_M^3\left(\frac{8(3N^2+3N+2)}{9N(N+1)} - \frac{32}{9}S_1\right) \\
& + L_M^2\left(\frac{8P_1}{27N^2(N+1)^2} - \frac{320}{27}S_1 + \frac{64}{9}S_2\right) + L_M\left(\frac{2P_3}{81N^3(N+1)^3} - \frac{896}{81}S_1 + \frac{160}{27}S_2 - \frac{32}{9}S_3\right) \\
& + \left(\frac{8032}{243} - \frac{128\zeta_3}{3}\right)S_1 + \frac{32(3N^2+3N+2)\zeta_3}{3N(N+1)} + \left(N_F + \frac{1}{2}\right)\Bigg[\frac{4P_{41}}{729N^4(N+1)^4} \\
& + L_Q^3\left(\frac{32(3N^2+3N+2)}{27N(N+1)} - \frac{128}{27}S_1\right) + L_M^3\left(\frac{16(3N^2+3N+2)}{27N(N+1)} - \frac{64}{27}S_1\right) \\
& + L_Q^2\left(\frac{64}{9}S_1^2 - \frac{64}{3}S_2 - \frac{16P_{11}}{27N^2(N+1)^2} + \frac{32(29N^2+29N-6)S_1}{27N(N+1)}\right) + L_M\Bigg(\frac{4P_{33}}{81N^3(N+1)^3} - \frac{2176}{81}S_1 \\
& - \frac{320}{27}S_2 + \frac{64}{9}S_3\Bigg) + L_Q\Bigg(\frac{16P_{34}}{81N^3(N+1)^3} + \left(-\frac{32P_{15}}{81N^2(N+1)^2} + \frac{128}{9}S_2\right)S_1 - \frac{128}{27}S_1^3 \\
& - \frac{32(29N^2+29N-6)S_1^2}{27N(N+1)} + \frac{32(35N^2+35N-2)S_2}{9N(N+1)} - \frac{1792}{27}S_3 + \frac{256}{9}S_{2,1}\Bigg) \\
& + \left(\frac{512\zeta_3}{27} - \frac{24064}{729}\right)S_1 + \frac{128}{81}S_2 + \frac{640}{81}S_3 - \frac{128}{27}S_4 - \frac{128(3N^2+3N+2)\zeta_3}{27N(N+1)}\Bigg]\Bigg\} \\
& + C_F^2 T_F \Bigg\{ \frac{P_{47}}{162N^5(N+1)^5} - \frac{S_{-2,1}}{N^2(N+1)}\frac{128}{81}(112N^3+112N^2-39N+18) \\
& + L_Q^3\left(\frac{1}{N^2(N+1)^2}\frac{2}{3}(3N^2+3N+2)^2 - \frac{16(3N^2+3N+2)S_1}{3N(N+1)} + \frac{32}{3}S_1^2\right) \\
& + L_Q^2\Bigg(-\frac{2P_{31}}{9N^3(N+1)^3} + \left(\frac{2P_{16}}{9N^2(N+1)^2} + \frac{176}{3}S_2\right)S_1 - 16S_1^3 + \frac{64}{3}S_{-3} + \frac{64}{3}S_3 \\
& - \frac{4(107N^2+107N-54)S_1^2}{9N(N+1)} - \frac{44(3N^2+3N+2)S_2}{3N(N+1)} - \frac{128}{3}S_{-2,1} + \left(\frac{128}{3}S_1 - \frac{64}{3N(N+1)}\right)S_{-2}\Bigg) \\
& + L_M^2\Bigg(L_Q\left(\frac{1}{N^2(N+1)^2}\frac{2}{3}(3N^2+3N+2)^2 - \frac{16(3N^2+3N+2)S_1}{3N(N+1)} + \frac{32}{3}S_1^2\right) - \frac{2(N-1)P_{25}}{3N^3(N+1)^3} \\
& + \left(\frac{2P_9}{3N^2(N+1)^2} + \frac{80}{3}S_2\right)S_1 - \frac{4(N-1)(N+2)S_1^2}{N(N+1)} - \frac{16}{3}S_1^3 - \frac{20(3N^2+3N+2)S_2}{3N(N+1)} + \frac{64}{3}S_3 \\
& + \left(\frac{128}{3}S_1 - \frac{64}{3N(N+1)}\right)S_{-2} + \frac{64}{3}S_{-3} - \frac{128}{3}S_{-2,1}\Bigg) + L_M\Bigg(L_Q\Bigg(\left(-\frac{8P_{10}}{9N^2(N+1)^2} - \frac{64}{3}S_2\right)S_1 \\
& + \frac{1}{N^3(N+1)^3}\frac{2}{9}(3N^2+3N+2)P_1 + \frac{320}{9}S_1^2 + \frac{16(3N^2+3N+2)S_2}{3N(N+1)}\Bigg) \\
& + \frac{P_{40}}{9N^4(N+1)^4} - \left(\frac{256}{3}S_3 + \frac{256}{3}S_{-2,1} - 64\zeta_3 - \frac{2P_{32}}{9N^3(N+1)^3} - \frac{16(59N^2+59N-6)S_2}{9N(N+1)}\right)S_1 \\
& + \left(\frac{32}{3}S_2 - \frac{4P_7}{3N^2(N+1)^2}\right)S_1^2 - \frac{160}{9}S_1^3 - \frac{4P_{13}S_2}{9N^2(N+1)^2} - 32S_2^2 + \frac{32(29N^2+29N+12)S_3}{9N(N+1)} - \frac{256}{3}S_4 \\
& + \left(-\frac{64(16N^2+10N-3)}{9N^2(N+1)^2} + \frac{1280}{9}S_1 - \frac{128}{3}S_2\right)S_{-2} - \frac{128}{3}S_{-4} + \frac{128}{3}S_{3,1}
\end{aligned}
$$

$$+\left(\frac{64(10N^2+10N+3)}{9N(N+1)}-\frac{128}{3}S_1\right)S_{-3}-\frac{128(10N^2+10N-3)S_{-2,1}}{9N(N+1)}-\frac{128}{3}S_{-2,2}+\frac{512}{3}S_{-2,1,1}$$
$$-\frac{16(3N^2+3N+2)\zeta_3}{N(N+1)}\Bigg)+L_Q\Bigg[\frac{4P_{44}}{27N^4(N+1)^4(N+2)}+\left(-\frac{4P_{37}}{27N^3(N+1)^3}+\frac{640}{9}S_3+\frac{64}{3}S_{2,1}\right.$$
$$\left.-\frac{32(67N^2+67N-21)S_2}{9N(N+1)}+\frac{512}{3}S_{-2,1}+64\zeta_3\right)S_1+\left(\frac{2P_{19}}{27N^2(N+1)^2}-\frac{224}{3}S_2\right)S_1^2+\frac{32(4N-1)(4N+5)S_1^3}{9N(N+1)}+\frac{80}{9}S_1^4$$
$$+\frac{2P_{18}S_2}{9N^2(N+1)^2}+48S_2^2-\frac{32(53N^2+77N+4)S_3}{9N(N+1)}+\frac{352}{3}S_4+64S_{-2}^2+\frac{448}{3}S_{-4}$$
$$+\left(-\frac{64P_{23}}{9N^2(N+1)^2(N+2)}-\frac{128(10N^2+22N-9)S_1}{9N(N+1)}-\frac{256}{3}S_1^2+\frac{256}{3}S_2\right)S_{-2}$$
$$+\left(\frac{256}{3}S_1-\frac{64(10N^2+22N+3)}{9N(N+1)}\right)S_{-3}+64S_{3,1}+\frac{16(9N^2+9N-2)S_{2,1}}{3N(N+1)}+\frac{128(10N^2+22N-9)S_{-2,1}}{9N(N+1)}$$
$$-\frac{256}{3}S_{-3,1}-64S_{2,1,1}-\frac{512}{3}S_{-2,1,1}-\frac{16(9N^2-7N+6)\zeta_3}{N(N+1)}\Bigg]-\frac{1}{N(N+1)}\frac{48}{5}(3N^2+3N+2)\zeta_2^2$$
$$+\left(\frac{P_{42}}{162N^4(N+1)^4}+\frac{8P_{20}S_2}{81N^2(N+1)^2}-\frac{64}{9}S_2^2-\frac{8(347N^2+347N+54)S_3}{27N(N+1)}+\frac{704}{9}S_4+\frac{128S_{2,1}}{9N(N+1)}\right.$$
$$\left.-\frac{320}{9}S_{3,1}-\frac{256(10N^2+10N-3)S_{-2,1}}{27N(N+1)}-\frac{256}{9}S_{-2,2}+\frac{64}{3}S_{2,1,1}+\frac{1024}{9}S_{-2,1,1}+\frac{192\zeta_2^2}{5}-\frac{1208\zeta_3}{9}\right)S_1$$
$$+\left(\frac{P_{29}}{9N^3(N+1)^3}+\frac{16(5N^2+5N-4)S_2}{9N(N+1)}+16S_3-\frac{128}{9}S_{2,1}-\frac{256}{9}S_{-2,1}\right)S_1^2+\left(-\frac{16P_8}{27N^2(N+1)^2}+\frac{128}{27}S_2\right)S_1^3$$
$$+\frac{512}{9}S_5+\left(\frac{P_{28}}{81N^3(N+1)^3}+\frac{400}{27}S_3+\frac{256}{3}S_{-2,1}-\frac{64\zeta_3}{3}\right)S_2+\frac{8P_{21}S_3}{81N^2(N+1)^2}-\frac{32(23N^2+23N-3)S_2^2}{27N(N+1)}$$
$$-\frac{176(17N^2+17N+6)S_4}{27N(N+1)}+\left(-\frac{64P_{14}}{81N^3(N+1)^3}+\frac{128P_{12}S_1}{81N^2(N+1)^2}-\frac{128S_1^2}{9N(N+1)}+\frac{256}{27}S_1^3-\frac{1280}{27}S_2\right.$$
$$\left.+\frac{512}{27}S_3-\frac{512}{9}S_{2,1}\right)S_{-2}+\left(\frac{1}{N(N+1)^2}\frac{64}{81}(112N^3+224N^2+169N+39)+\frac{128}{9}S_1^2+\frac{128}{9}S_2\right.$$
$$\left.-\frac{128(10N^2+10N+3)S_1}{27N(N+1)}\right)S_{-3}+\left(-\frac{128(10N^2+10N+3)}{27N(N+1)}+\frac{256}{9}S_1\right)S_{-4}+\frac{256}{9}S_{-5}+\frac{16P_5S_{2,1}}{9N^2(N+1)^2}$$
$$+\frac{256}{9}S_{2,3}-\frac{512}{9}S_{2,-3}-\frac{512}{9}S_{4,1}+\frac{512}{9}S_{-2,3}+\frac{16(89N^2+89N+30)S_{3,1}}{27N(N+1)}-\frac{128(10N^2+10N-3)S_{-2,2}}{27N(N+1)}+\frac{512}{9}S_{2,1,-2}$$
$$-\frac{16(3N^2+3N+2)S_{2,1,1}}{3N(N+1)}+\frac{256}{9}S_{3,1,1}+\frac{512(10N^2+10N-3)S_{-2,1,1}}{27N(N+1)}+\frac{512}{9}S_{-2,2,1}-\frac{2048}{9}S_{-2,1,1,1}+\frac{2P_{17}\zeta_3}{9N^2(N+1)^2}\Bigg\}$$
$$+C_AC_FT_F\Bigg\{\frac{P_{46}}{729N^5(N+1)^5}+\frac{S_{-2,1}}{N^2(N+1)}\frac{64}{81}(112N^3+112N^2-39N+18)+L_M^3\left(-\frac{44(3N^2+3N+2)}{27N(N+1)}+\frac{176}{27}S_1\right)$$
$$+L_Q^3\left(-\frac{88(3N^2+3N+2)}{27N(N+1)}+\frac{352}{27}S_1\right)+L_M^2\Bigg(\frac{1}{N^3(N+1)^3}\frac{2}{9}(3N^2+3N+2)P_6-\frac{184}{9}S_1-\frac{32}{3}S_3-\frac{32}{3}S_{-3}$$
$$+\left(\frac{32}{3N(N+1)}-\frac{64}{3}S_1\right)S_{-2}+\frac{64}{3}S_{-2,1}\Bigg)+L_Q^2\Bigg(-\frac{176}{9}S_1^2+\frac{176}{3}S_2-\frac{32}{3}S_{-3}-\frac{32}{3}S_3+\frac{2P_{36}}{27N^3(N+1)^3}$$

$$
\begin{aligned}
&-\frac{16(194N^2+194N-33)S_1}{27N(N+1)}+\frac{64}{3}S_{-2,1}+\Big(\frac{32}{3N(N+1)}-\frac{64}{3}S_1\Big)S_{-2}\Big)+L_Q\Big(-\frac{4P_{45}}{81N^4(N+1)^4(N+2)}+\frac{352}{27}S_1^3\\
&-\frac{S_2}{N(N+1)^2}\frac{16}{9}(230N^3+460N^2+213N-11)+\Big(\frac{4P_{38}}{81N^3(N+1)^3}+32S_3-\frac{32(11N^2+11N+3)S_2}{9N(N+1)}-\frac{128}{3}S_{2,1}\\
&-\frac{256}{3}S_{-2,1}-64\zeta_3\Big)S_1+\Big(\frac{32}{3}S_2+\frac{16(194N^2+194N-33)}{27N(N+1)}\Big)S_1^2-\frac{32}{3}S_2^2+\frac{16(368N^2+440N-45)S_3}{27N(N+1)}-\frac{224}{3}S_4\\
&+\Big(\frac{32P_{23}}{9N^2(N+1)^2(N+2)}+\frac{64(10N^2+22N-9)S_1}{9N(N+1)}+\frac{128}{3}S_1^2-\frac{128}{3}S_2\Big)S_{-2}-32S_{-2}^2\\
&+\Big(\frac{32(10N^2+22N+3)}{9N(N+1)}-\frac{128}{3}S_1\Big)S_{-3}-\frac{224}{3}S_{-4}+\frac{128}{3}S_{-3,1}-\frac{64}{3}S_{3,1}-\frac{64(11N^2+11N-3)S_{2,1}}{9N(N+1)}\\
&-\frac{64(10N^2+22N-9)S_{-2,1}}{9N(N+1)}+64S_{2,1,1}+\frac{256}{3}S_{-2,1,1}+\frac{32(3N^2-N+2)\zeta_3}{N(N+1)}\Big)+L_M\Big(\frac{P_{39}}{81N^4(N+1)^4}+\frac{1792}{27}S_2\\
&+\Big(-\frac{8P_{30}}{81N^3(N+1)^3}+32S_3+\frac{128}{3}S_{-2,1}-64\zeta_3\Big)S_1-\frac{16(31N^2+31N+9)S_3}{9N(N+1)}+\frac{160}{3}S_4\\
&+\Big(\frac{32(16N^2+10N-3)}{9N^2(N+1)^2}-\frac{640}{9}S_1+\frac{64}{3}S_2\Big)S_{-2}-\frac{128}{3}S_{3,1}+\frac{64}{3}S_{-4}+\Big(-\frac{32(10N^2+10N+3)}{9N(N+1)}+\frac{64}{3}S_1\Big)S_{-3}\\
&+\frac{64(10N^2+10N-3)S_{-2,1}}{9N(N+1)}+\frac{64}{3}S_{-2,2}-\frac{256}{3}S_{-2,1,1}+\frac{16(3N^2+3N+2)\zeta_3}{N(N+1)}\Big)\\
&+\Big(\frac{4P_{43}}{729N^4(N+1)^4}-\frac{S_2}{N^2(N+1)^2}\frac{16}{9}(N-1)(2N^3-N^2-N-2)+\frac{112}{9}S_2^2+\frac{80(2N+1)^2S_3}{9N(N+1)}+\frac{64}{3}S_{3,1}-\frac{208}{9}S_4\\
&-\frac{8(9N^2+9N+16)S_{2,1}}{9N(N+1)}+\frac{128(10N^2+10N-3)S_{-2,1}}{27N(N+1)}+\frac{128}{9}S_{-2,2}-32S_{2,1,1}-\frac{512}{9}S_{-2,1,1}-\frac{192\zeta_2^2}{5}\\
&+\frac{4(593N^2+593N+108)\zeta_3}{27N(N+1)}\Big)S_1+\frac{1}{N(N+1)}\frac{48}{5}(3N^2+3N+2)\zeta_2^2+\Big(\frac{4P_{22}}{9N^3(N+1)^3}+\frac{32S_2}{9N(N+1)}-\frac{80}{9}S_3\\
&+\frac{128}{9}S_{2,1}+\frac{128}{9}S_{-2,1}-16\zeta_3\Big)S_1^2+\Big(\frac{4P_{35}}{81N^3(N+1)^3}+\frac{496}{27}S_3-\frac{64}{3}S_{2,1}-\frac{128}{3}S_{-2,1}+16\zeta_3\Big)S_2-\frac{64}{27}S_1^3S_2\\
&-\frac{4(15N^2+15N+14)S_2^2}{9N(N+1)}-\frac{8P_{26}S_3}{81N^2(N+1)^2}+\frac{4(443N^2+443N+78)S_4}{27N(N+1)}-\frac{224}{9}S_5+\Big(\frac{32P_{14}}{81N^3(N+1)^3}+\frac{64S_1^2}{9N(N+1)}\\
&-\frac{64P_{12}S_1}{81N^2(N+1)^2}-\frac{128}{27}S_1^3+\frac{640}{27}S_2-\frac{256}{27}S_3+\frac{256}{9}S_{2,1}\Big)S_{-2}-\Big(\frac{64}{9}S_1^2+\frac{64}{9}S_2+\frac{32}{81}\frac{(112N^3+224N^2+169N+39)}{N(N+1)^2}\\
&-\frac{64(10N^2+10N+3)S_1}{27N(N+1)}\Big)S_{-3}-\frac{128}{9}S_{-5}+\Big(\frac{64(10N^2+10N+3)}{27N(N+1)}-\frac{128}{9}S_1\Big)S_{-4}-\frac{128}{3}S_{2,3}-\frac{8P_{24}S_{2,1}}{9N^2(N+1)^2}\\
&+\frac{256}{9}S_{2,-3}-\frac{8(13N+4)(13N+9)S_{3,1}}{27N(N+1)}+\frac{256}{9}S_{4,1}+\frac{64(10N^2+10N-3)S_{-2,2}}{27N(N+1)}+\frac{64}{3}S_{2,2,1}-\frac{256}{9}S_{-2,3}\\
&+\frac{8(3N^2+3N+2)S_{2,1,1}}{N(N+1)}-\frac{256}{9}S_{2,1,-2}-\frac{256}{9}S_{3,1,1}-\frac{256}{9}S_{-2,2,1}-\frac{256(10N^2+10N-3)S_{-2,1,1}}{27N(N+1)}+\frac{224}{9}S_{2,1,1,1}\\
&+\frac{1024}{9}S_{-2,1,1,1}+\frac{P_{27}\zeta_3}{27N^2(N+1)^2}\Big\}+\hat{c}_{q,2}^{(3)},
\end{aligned}
\tag{3.1}
$$

where $\hat{c}_{q,2}^{(3)}=c_{q,2}^{(3)}(N_F+1)-c_{q,2}^{(3)}(N_F)$ is obtained from the 3-loop massless Wilson coefficient in Ref. [3]. Except for $\hat{c}_{q,2}^{(3)}(N_F)$, the Wilson coefficient is expressed by harmonic sums up to weight $\mathsf{w}=5$. The polynomials in the equation above are defined as follows:

$$P_5 = 7N^4 + 14N^3 + 3N^2 - 4N - 4,$$
$$P_6 = 17N^4 + 34N^3 + 29N^2 + 12N + 24,$$
$$P_7 = 19N^4 + 38N^3 - 9N^2 - 20N + 4,$$
$$P_8 = 28N^4 + 56N^3 + 28N^2 + 2N + 1,$$
$$P_9 = 33N^4 + 38N^3 - 15N^2 - 60N - 28,$$
$$P_{10} = 33N^4 + 66N^3 + 97N^2 + 40N - 12,$$
$$P_{11} = 57N^4 + 72N^3 + 29N^2 - 22N - 24,$$
$$P_{12} = 112N^4 + 224N^3 + 121N^2 + 9N + 9,$$
$$P_{13} = 141N^4 + 198N^3 + 169N^2 - 32N - 84,$$
$$P_{14} = 181N^4 + 266N^3 + 82N^2 - 3N + 18,$$
$$P_{15} = 235N^4 + 596N^3 + 319N^2 + 66N + 72,$$
$$P_{16} = 501N^4 + 750N^3 + 325N^2 - 188N - 204,$$
$$P_{17} = 561N^4 + 1122N^3 + 767N^2 + 302N + 48,$$
$$P_{18} = 1131N^4 + 1926N^3 + 1019N^2 - 64N - 276,$$
$$P_{19} = 1139N^4 + 3286N^3 + 1499N^2 + 504N + 828,$$
$$P_{20} = 1199N^4 + 2398N^3 + 1181N^2 + 18N + 90,$$
$$P_{21} = 1220N^4 + 2251N^3 + 1772N^2 + 303N - 138,$$
$$P_{22} = 3N^5 + 11N^4 + 10N^3 + 19N^2 + 23N + 16,$$
$$P_{23} = 6N^5 - 25N^3 - 45N^2 - 11N + 6,$$
$$P_{24} = 12N^5 + 16N^4 + 18N^3 - 15N^2 - 5N - 8,$$
$$P_{25} = 15N^5 + 39N^4 + 39N^3 - 17N^2 - 32N - 20,$$
$$P_{26} = 27N^5 + 863N^4 + 1573N^3 + 1151N^2 + 144N - 36,$$
$$P_{27} = 648N^5 - 2103N^4 - 4278N^3 - 3505N^2 - 682N - 432,$$
$$P_{28} = -11145N^6 - 30915N^5 - 33923N^4 - 11449N^3 + 1960N^2 - 1032N - 2088,$$
$$P_{29} = -151N^6 - 469N^5 - 181N^4 + 305N^3 + 80N^2 - 88N - 56,$$
$$P_{30} = 155N^6 + 465N^5 + 465N^4 + 155N^3 + 108N^2 + 108N + 54,$$
$$P_{31} = 216N^6 + 459N^5 + 417N^4 - 3N^3 - 125N^2 - 80N + 12,$$
$$P_{32} = 309N^6 + 647N^5 + 293N^4 - 783N^3 - 718N^2 + 68N + 216,$$
$$P_{33} = 525N^6 + 1575N^5 + 1535N^4 + 973N^3 + 536N^2 + 48N - 72,$$
$$P_{34} = 609N^6 + 1029N^5 + 613N^4 - 37N^3 - 74N^2 + 300N + 216,$$
$$P_{35} = 868N^6 + 2469N^5 + 2487N^4 + 940N^3 + 171N^2 + 207N + 144,$$
$$P_{36} = 1407N^6 + 3297N^5 + 2891N^4 + 727N^3 - 514N^2 - 240N + 144,$$
$$P_{37} = 1770N^6 + 4731N^5 + 4483N^4 + 749N^3 + 55N^2 + 1440N + 756,$$
$$P_{38} = 7531N^6 + 26121N^5 + 27447N^4 + 8815N^3 + 1110N^2 + 936N - 324,$$
$$P_{39} = -4785N^8 - 19140N^7 - 18754N^6 + 1320N^5 + 12723N^4 + 6548N^3 + 4080N^2 - 648N - 1728,$$
$$P_{40} = -45N^8 - 138N^7 - 774N^6 - 476N^5 - 881N^4 - 762N^3 - 868N^2 - 88N + 192,$$
$$P_{41} = 3549N^8 + 14196N^7 + 23870N^6 + 25380N^5 + 15165N^4 + 1712N^3 - 2016N^2 + 144N + 432,$$

$$
\begin{aligned}
P_{42} &= -3456B_4N^4(N+1)^4+42591N^8+161388N^7+226848N^6+105790N^5\\
&\quad -26735N^4-28666N^3+3560N^2-3192N-4464,\\
P_{43} &= 1944B_4N^4(N+1)^4-10807N^8-43228N^7-63222N^6-40150N^5-14587N^4-9018N^3\\
&\quad -7452N^2-2376N-324,\\
P_{44} &= 828N^9+3456N^8+4539N^7+2412N^6+1852N^5+5026N^4+4703N^3+2468N^2-324N-576,\\
P_{45} &= 8274N^9+39795N^8+71627N^7+64189N^6+29919N^5+8096N^4+5620N^3+5664N^2-1368N-2160,\\
P_{46} &= -1944B_4N^4(N+1)^4(3N^2+3N+2)+165N^{10}+825N^9+109664N^8+331682N^7\\
&\quad +457641N^6+346145N^5+219290N^4+86724N^3+13608N^2+14256N+10368,\\
P_{47} &= 864B_4N^4(N+1)^4(3N^2+3N+2)-18351N^{10}-87156N^9-198195N^8-244182N^7\\
&\quad -184797N^6-70160N^5-23209N^4-8030N^3-984N^2-2328N-2160.
\end{aligned}
$$

Here the constant B_4 is given by

$$
B_4 = -4\zeta_2\ln^2(2)+\frac{2}{3}\ln^4(2)-\frac{13}{2}\zeta_4+16\mathrm{Li}_4\left(\frac{1}{2}\right). \tag{3.2}
$$

By performing the Mellin inversion to x space, one obtains

$$
\begin{aligned}
L_{q,2}^{W^+-W^-,\mathrm{NS},(3)} &= \delta(1-x)\Bigg\{C_AC_FT_F\Bigg[-\frac{44L_M^3}{9}-\frac{88L_Q^3}{9}+L_M^2\left(\frac{34}{3}-\frac{16\zeta_3}{3}\right)+L_Q^2\left(\frac{938}{9}-\frac{16\zeta_3}{3}\right)\\
&\quad +L_M\left(-\frac{1595}{27}+\frac{136\zeta_2^2}{15}+\frac{272\zeta_3}{9}\right)+L_Q\left(-\frac{11032}{27}-\frac{32\zeta_2}{3}-\frac{392\zeta_2^2}{15}+\frac{1024\zeta_3}{9}\right)+\frac{5248\zeta_2^2}{135}-\frac{10045\zeta_3}{81}\\
&\quad -\frac{16}{9}\zeta_2\zeta_3-\frac{176\zeta_5}{9}+\frac{55}{243}-8B_4\Bigg]+C_F^2T_F\Bigg[6L_M^2L_Q+6L_Q^3+L_M^2\left(\frac{32\zeta_3}{3}-10\right)+2L_ML_Q\\
&\quad +L_Q^2\left(\frac{32\zeta_3}{3}-48\right)-L_M\left(5+\frac{272\zeta_2^2}{15}+\frac{112\zeta_3}{9}\right)+L_Q\left(\frac{368}{3}+\frac{64\zeta_2}{3}+\frac{784\zeta_2^2}{15}-\frac{1616\zeta_3}{9}\right)-\frac{6608\zeta_2^2}{135}\\
&\quad +\frac{13682\zeta_3}{81}+\frac{32\zeta_2\zeta_3}{9}+\frac{352\zeta_5}{9}-\frac{2039}{18}+16B_4\Bigg]+C_FN_FT_F^2\Bigg[\frac{16L_M^3}{9}+\frac{32L_Q^3}{9}-\frac{304L_Q^2}{9}+\frac{700L_M}{27}\\
&\quad +\frac{3248L_Q}{27}-\frac{128\zeta_3}{9}+\frac{4732}{243}\Bigg]+C_FT_F^2\Bigg[\frac{32L_M^3}{9}+\frac{16L_Q^3}{9}+\frac{8L_M^2}{9}-\frac{152L_Q^2}{9}+\frac{496L_M}{27}+\frac{1624L_Q}{27}\\
&\quad +\frac{224\zeta_3}{9}-\frac{3658}{243}\Bigg]\Bigg\}+\Bigg\{\frac{C_FN_FT_F^2}{1-x}\Bigg[\frac{64L_M^3}{27}+\frac{128L_Q^3}{27}-L_Q^2\left(\frac{928}{27}+\frac{256}{9}H_0+\frac{128}{9}H_1\right)\\
&\quad +L_M\left(\frac{2176}{81}-\frac{320}{27}H_0-\frac{32}{9}H_0^2\right)+L_Q\left(\frac{7520}{81}+\frac{4288H_0}{27}+\frac{128}{3}H_0^2+\frac{1856H_1}{27}+\frac{256}{9}H_0H_1\right.\\
&\quad \left.+\frac{128}{9}H_1^2+\frac{256}{9}H_{0,1}-\frac{512\zeta_2}{9}\right)+\frac{128H_0}{81}-\frac{320}{81}H_0^2-\frac{64}{81}H_0^3-\frac{512\zeta_3}{27}+\frac{24064}{729}\Bigg]\\
&\quad +\frac{C_FT_F^2}{1-x}\Bigg[\frac{128L_M^3}{27}+\frac{64L_Q^3}{27}+L_M^2\left(\frac{320}{27}+\frac{64H_0}{9}\right)-L_Q^2\left(\frac{464}{27}+\frac{128}{9}H_0+\frac{64}{9}H_1\right)+\frac{1984L_M}{81}
\end{aligned}
$$

$$+L_Q\left(\frac{3760}{81}+\frac{2144H_0}{27}+\frac{64}{3}H_0^2+\frac{928H_1}{27}+\frac{128}{9}H_0H_1+\frac{64}{9}H_1^2+\frac{128}{9}H_{0,1}-\frac{256\zeta_2}{9}\right)$$
$$+\frac{64H_0}{81}-\frac{160}{81}H_0^2-\frac{32}{81}H_0^3+\frac{896\zeta_3}{27}-\frac{12064}{729}\bigg]$$
$$+\frac{C_AC_FT_F}{(1-x)^2}\left[\frac{32}{9}(x+2)H_{0,1}-\frac{4}{81}(800x-773)H_0^2+\frac{32}{81}(94x-121)\zeta_2\right]$$
$$+\frac{C_AC_FT_F}{1-x}\bigg[-\frac{176L_M^3}{27}-\frac{352L_Q^3}{27}+L_Q^2\left(\frac{3104}{27}+\frac{704H_0}{9}+\frac{16}{3}H_0^2+\frac{352H_1}{9}-\frac{32\zeta_2}{3}\right)+L_M^2\left(\frac{184}{9}+\frac{16}{3}H_0^2-\frac{32\zeta_2}{3}\right)$$
$$+L_Q\bigg(-\frac{30124}{81}-\frac{14144}{27}H_0-\frac{1216}{9}H_0^2-\frac{80}{9}H_0^3-\frac{6208}{27}H_1-\frac{704}{9}H_0H_1-\frac{16}{3}H_0^2H_1-\frac{352}{9}H_1^2+\frac{32}{3}H_0H_1^2-64H_0H_{0,-1}$$
$$-\frac{704}{9}H_{0,1}+\frac{32}{3}H_0H_{0,1}-\frac{128}{3}H_1H_{0,1}+128H_{0,0,-1}-\frac{128}{3}H_{0,0,1}+64H_{0,1,1}+\left(192+\frac{128H_0}{3}+64H_1\right)\zeta_2-\frac{256\zeta_3}{3}\bigg)$$
$$+L_M\left(\frac{1240}{81}+\frac{1792H_0}{27}+\frac{248}{9}H_0^2+\frac{32}{9}H_0^3-16H_0^2H_1+32H_0H_{0,1}-\frac{64}{3}H_{0,0,1}+\left(-\frac{320}{9}-\frac{64}{3}H_0\right)\zeta_2+96\zeta_3\right)$$
$$+\left(-\frac{496}{27}H_0-\frac{112}{9}H_0^2+8H_1-\frac{160}{9}H_0H_1-\frac{128}{9}H_1^2+\frac{32}{9}H_{0,1}\right)\zeta_2+\frac{43228}{729}-\frac{32B_4}{3}+\frac{3256H_0}{81}+\frac{496}{81}H_0^3+\frac{16}{27}H_0^4$$
$$+\frac{32}{3}H_1-\frac{32}{9}H_0H_1-\frac{160}{9}H_0^2H_1-\frac{112}{27}H_0^3H_1+\frac{8}{9}H_0^2H_1^2-\frac{64}{27}H_0H_1^3+\frac{368}{9}H_0H_{0,1}+\frac{16}{3}H_0^2H_{0,1}-8H_1H_{0,1}$$
$$-\frac{128}{9}H_0H_1H_{0,1}+\frac{128}{9}H_1^2H_{0,1}-\frac{32}{9}H_{0,1}^2-\frac{1072}{27}H_{0,0,1}+\frac{32}{9}H_0H_{0,0,1}+\frac{320}{9}H_1H_{0,0,1}+24H_{0,1,1}$$
$$+\frac{160}{9}H_0H_{0,1,1}-32H_1H_{0,1,1}-\frac{224}{9}H_{0,0,1,1}+\frac{224}{9}H_{0,1,1,1}+\frac{592\zeta_2^2}{15}+\left(-\frac{1196}{27}+\frac{160H_0}{9}-\frac{32}{9}H_1\right)\zeta_3\bigg]$$
$$+\frac{C_F^2T_F}{1-x}\bigg[L_M^2L_Q\left(16-\frac{32}{3}H_0-\frac{64}{3}H_1\right)+L_Q^3\left(16-\frac{32}{3}H_0-\frac{64}{3}H_1\right)-L_M^2\bigg(22+16H_0-\frac{16}{3}H_0^2-8H_1-\frac{128}{3}H_0H_1$$
$$-16H_1^2+\frac{64\zeta_2}{3}\bigg)+L_Q^2\left(-\frac{334}{3}+\frac{32H_0}{9}+\frac{80}{3}H_0^2+\frac{856H_1}{9}+\frac{320}{3}H_0H_1+48H_1^2-\frac{256\zeta_2}{3}\right)$$
$$+L_ML_Q\left(\frac{88}{3}-\frac{176}{9}H_0-\frac{32}{3}H_0^2-\frac{640}{9}H_1-\frac{64}{3}H_0H_1+\frac{64\zeta_2}{3}\right)+L_M\bigg(\frac{88}{9}H_0^2-\frac{206}{3}-\frac{112}{3}H_0+\frac{64}{9}H_0^3$$
$$+\frac{152H_1}{3}+\frac{1424}{9}H_0H_1+\frac{160}{3}H_0^2H_1+\frac{160}{3}H_1^2+\frac{32}{3}H_0H_1^2-\frac{64}{3}H_0H_{0,1}-\left(\frac{784}{9}+\frac{128}{3}H_0+\frac{64}{3}H_1\right)\zeta_2-64\zeta_3\bigg)$$
$$+L_Q\bigg(\frac{2360}{9}+\frac{4508H_0}{27}-\frac{160}{3}H_0^2-\frac{224}{9}H_0^3-\frac{4556}{27}H_1-\frac{3680}{9}H_0H_1-128H_0^2H_1-\frac{512}{3}H_1^2-128H_0H_1^2-\frac{320}{9}H_1^3$$
$$+128H_0H_{0,-1}+48H_{0,1}-\frac{64}{3}H_0H_{0,1}+\frac{64}{3}H_1H_{0,1}-256H_{0,0,-1}-64H_{0,1,1}+\left(\frac{2608}{9}+\frac{832H_0}{3}+\frac{448H_1}{3}\right)\zeta_2$$
$$+320\zeta_3\bigg)+\frac{64B_4}{3}-\left(\frac{2488}{27}+\frac{1192}{27}H_0+\frac{80}{9}H_0^2+\frac{160}{9}H_1-\frac{32}{3}H_0H_1-\frac{128}{9}H_1^2+\frac{32}{9}H_{0,1}\right)\zeta_2-\frac{3262}{27}H_0$$
$$+\frac{196}{27}H_0^2+\frac{380}{81}H_0^3+\frac{4}{3}H_0^4+\frac{302H_1}{9}+\frac{13624}{81}H_0H_1+\frac{1628}{27}H_0^2H_1+\frac{304}{27}H_0^3H_1+\frac{448}{9}H_1^2+\frac{80}{9}H_0H_1^2-\frac{8}{9}H_0^2H_1^2$$
$$+\frac{128}{27}H_0H_1^3+\frac{112}{9}H_{0,1}-\frac{1304}{27}H_0H_{0,1}-\frac{32}{3}H_0^2H_{0,1}+\frac{160}{9}H_0H_1H_{0,1}-\frac{128}{9}H_1^2H_{0,1}-\frac{16}{9}H_{0,1}^2+\frac{1184}{27}H_{0,0,1}$$
$$+\frac{128}{9}H_0H_{0,0,1}-\frac{256}{9}H_1H_{0,0,1}-16H_{0,1,1}-\frac{32}{3}H_0H_{0,1,1}+\frac{64}{3}H_1H_{0,1,1}-\frac{128}{9}H_{0,0,0,1}$$
$$+\frac{64}{3}H_{0,0,1,1}-\frac{1328\zeta_2^2}{45}+\left(\frac{3088}{27}-\frac{128}{9}H_0+\frac{160H_1}{9}\right)\zeta_3-\frac{14197}{54}\bigg]\bigg\}_+$$

$$
\begin{aligned}
&+\Bigg\{C_F^2T_F\Bigg[L_M^2L_Q\left(8(x+1)H_0+\frac{32}{3}(x+1)H_1-\frac{8}{3}(x+5)\right)+L_Q^3\left(8(x+1)H_0+\frac{32}{3}(x+1)H_1-\frac{8}{3}(x+5)\right)\\
&+L_ML_Q\left(\frac{4}{9}(19x-85)+\frac{8}{3}(13x+1)H_0+8(x+1)H_0^2+\frac{128}{9}(4x+1)H_1+\frac{32}{3}(x+1)H_0H_1-\frac{32}{3}(x+1)\zeta_2\right)\\
&+L_M^2\Bigg(12(5x-2)-16(2x+1)H_0+\frac{64(x^2+1)H_{-1}H_0}{3(x+1)}-\frac{4(9x^2+10x+9)H_0^2}{3(x+1)}-\frac{16}{3}(3x+2)H_1-\frac{64}{3}(x+1)H_0H_1\\
&-8(x+1)H_1^2-\frac{64(x^2+1)H_{0,-1}}{3(x+1)}-\frac{8}{3}(x+1)H_{0,1}+\frac{8(9x^2+10x+9)\zeta_2}{3(x+1)}\Bigg)+L_Q^2\Bigg(\frac{4}{9}(188x+157)-\frac{88}{3}(3x+1)H_0\\
&+\frac{64(x^2+1)H_{-1}H_0}{3(x+1)}-24(x+1)H_1^2-\frac{16}{9}(59x+26)H_1-\frac{4(21x^2+34x+21)H_0^2}{3(x+1)}-\frac{160}{3}(x+1)H_0H_1\\
&-8(x+1)H_{0,1}-\frac{64(x^2+1)H_{0,-1}}{3(x+1)}+\frac{8(23x^2+38x+23)\zeta_2}{3(x+1)}\Bigg)+L_M\Bigg(\frac{4}{3}(171x-116)\\
&+\left(\frac{8(117x^2+118x+81)}{9(x+1)}+\frac{128(x^2+1)H_{-1}}{3(x+1)}+\frac{16(x+3)(3x+1)H_0}{3(x+1)}+\frac{32}{3}(x+1)H_1\right)\zeta_2-\frac{4}{3}(107x+89)H_0\\
&+\frac{256(4x^2+3x+4)H_{-1}H_0}{9(x+1)}-\frac{4(201x^2+250x+129)H_0^2}{9(x+1)}+\frac{32(x^2+1)H_{-1}H_0^2}{3(x+1)}-\frac{32(3x^2+4x+3)H_0^3}{9(x+1)}\\
&-\frac{4}{9}(327x-73)H_1-\frac{32}{9}(38x+17)H_0H_1-\frac{80}{3}(x+1)H_0^2H_1-\frac{16}{3}(7x+3)H_1^2-\frac{16}{3}(x+1)H_0H_1^2\\
&-\frac{256(4x^2+3x+4)H_{0,-1}}{9(x+1)}+\frac{280}{9}(x+1)H_{0,1}-\frac{128(x^2+1)H_{-1}H_{0,1}}{3(x+1)}+\frac{32}{3}(x+1)H_0H_{0,1}\\
&+\frac{128(x^2+1)H_{0,-1,1}}{3(x+1)}-\frac{64(x^2+1)H_{0,0,-1}}{3(x+1)}+\frac{16(3x^2-2x+3)H_{0,0,1}}{3(x+1)}+\frac{128(x^2+1)H_{0,1,-1}}{3(x+1)}+\frac{16(x^2+14x+1)\zeta_3}{3(x+1)}\Bigg)\\
&+L_Q\Bigg(-\frac{8}{27}(1925x-284)-\frac{64}{9}\frac{(36x^3+61x^2+18x+13)H_{-1}H_0}{x+1}+\frac{64}{9}\frac{(36x^3+61x^2+18x+13)H_{0,-1}}{x+1}\\
&+\frac{4}{9}\frac{(288x^3+801x^2+742x+309)H_0^2}{x+1}+\Bigg(-\frac{32}{9}\frac{(72x^3+199x^2+180x+73)}{x+1}+\frac{64(7x^2+6x+3)H_{-1}}{3(x+1)}\\
&-\frac{16(35x^2+66x+35)H_0}{3(x+1)}+32(x-3)H_1\Bigg)\zeta_2+\frac{8(186x^2+211x+73)H_0}{9(x+1)}+\frac{64(7x^2+6x+3)H_{-1}^2H_0}{3(x+1)}\\
&-\frac{32(11x^2+6x+7)H_{-1}H_0^2}{3(x+1)}+\frac{8}{27}(425x+434)H_1+\frac{32(9x^2+13x+9)H_0^3}{9(x+1)}+\frac{16}{9}(193x+121)H_0H_1\\
&+\frac{32}{3}(x+7)H_0^2H_1+\frac{32}{3}(15x+8)H_1^2+64(x+1)H_0H_1^2+\frac{160}{9}(x+1)H_1^3+\frac{16}{9}(61x+13)H_{0,1}\\
&-\frac{128(7x^2+6x+3)H_{-1}H_{0,-1}}{3(x+1)}-\frac{128x(3x+5)H_0H_{0,-1}}{3(x+1)}+\frac{16}{3}(25x+1)H_0H_{0,1}-\frac{32}{3}(x+1)H_1H_{0,1}\\
&+\frac{128(7x^2+6x+3)H_{0,-1,-1}}{3(x+1)}+\frac{64(23x^2+26x+7)H_{0,0,-1}}{3(x+1)}-\frac{16}{3}(19x-5)H_{0,0,1}+48(x+1)H_{0,1,1}\\
&-\frac{32(21x^2+38x+25)\zeta_3}{3(x+1)}\Bigg)+\Bigg(\frac{4(1619x^2+1338x+1511)}{81(x+1)}+\frac{64(29x^2+18x+29)H_{-1}}{27(x+1)}+\frac{64(x^2+1)H_{-1}^2}{9(x+1)}\\
&+\frac{4(147x^2+298x-9)H_0}{27(x+1)}+\frac{64(x^2+1)H_{-1}H_0}{9(x+1)}+\frac{4(x+5)(5x+1)H_0^2}{9(x+1)}+\frac{16}{9}(x+9)H_1-\frac{16}{3}(x+1)H_0H_1\\
&-\frac{64}{9}(x+1)H_1^2+\frac{16}{9}(x+1)H_{0,1}\Bigg)\zeta_2+\frac{1}{x+1}\frac{8}{45}(131x^2+178x+131)\zeta_2^2-\Bigg(\frac{8(235x^2+404x+409)}{27(x+1)}+\frac{80}{9}(x+1)H_1\\
&+\frac{256(x^2+1)H_{-1}}{9(x+1)}-\frac{8(15x^2+22x+15)H_0}{9(x+1)}\Bigg)\zeta_3-\frac{32}{3}B_4(x+1)+\frac{11}{27}(1093x-378)-\frac{(10159x+8999)}{81}H_0
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{64(199x^2+174x+199)H_{-1}H_0}{81(x+1)}-\frac{64}{9}(x+1)H_{-1}^2H_0+\frac{128(x^2+1)H_{-1}^3H_0}{27(x+1)}-\frac{2(4107x^2+5327x+3012)H_0^2}{81(x+1)}\\
&+\frac{32(19x^2+18x+19)H_{-1}H_0^2}{27(x+1)}-\frac{32(x^2+1)H_{-1}^2H_0^2}{9(x+1)}+\frac{1}{27}(1503-3905x)H_1\\
&-\frac{2(903x^2+1126x+543)H_0^3}{81(x+1)}+\frac{64(x^2+1)H_{-1}H_0^3}{27(x+1)}-\frac{(51x^2+70x+51)H_0^4}{27(x+1)}-\frac{32}{81}(319x+190)H_0H_1\\
&-\frac{40}{27}(31x+16)H_0^2H_1-\frac{152}{27}(x+1)H_0^3H_1-\frac{8}{9}(3x+55)H_1^2-\frac{8}{9}(9x+1)H_0H_1^2+\frac{4}{9}(x+1)H_0^2H_1^2-\frac{64}{27}(x+1)H_0H_1^3\\
&-\frac{64(199x^2+174x+199)H_{0,-1}}{81(x+1)}+\frac{128}{9}(x+1)H_{-1}H_{0,-1}+\frac{4}{27}(311x+467)H_{0,1}-\frac{128(x^2+1)H_{-1}^2H_{0,-1}}{9(x+1)}\\
&-\frac{512(4x^2+3x+4)H_{-1}H_{0,1}}{27(x+1)}+\frac{16}{27}(19x+52)H_0H_{0,1}+\frac{16}{3}(x+1)H_0^2H_{0,1}+\frac{64}{9}(x-1)H_1H_{0,1}-\frac{80}{9}(x+1)H_0H_1H_{0,1}\\
&+\frac{64}{9}(x+1)H_1^2H_{0,1}+\frac{8}{9}(x+1)H_{0,1}^2-\frac{128}{9}(x+1)H_{0,-1,-1}+\frac{256(x^2+1)H_{-1}H_{0,-1,-1}}{9(x+1)}+\frac{512(4x^2+3x+4)H_{0,-1,1}}{27(x+1)}\\
&-\frac{64(19x^2+18x+19)H_{0,0,-1}}{27(x+1)}+\frac{128(x^2+1)H_{-1}H_{0,0,-1}}{9(x+1)}-\frac{64}{9}(x+1)H_0H_{0,0,1}+\frac{4(321x^2+58x+57)H_{0,0,1}}{27(x+1)}\\
&-\frac{128(x^2+1)H_{-1}H_{0,0,1}}{9(x+1)}+\frac{128}{9}(x+1)H_1H_{0,0,1}+\frac{512(4x^2+3x+4)H_{0,1,-1}}{27(x+1)}-\frac{32}{9}(13x+1)H_{0,1,1}\\
&+\frac{256(x^2+1)H_{-1}H_{0,1,1}}{9(x+1)}+\frac{16}{3}(x+1)H_0H_{0,1,1}-\frac{32}{3}(x+1)H_1H_{0,1,1}-\frac{256(x^2+1)H_{0,-1,-1,-1}}{9(x+1)}-\frac{256(x^2+1)H_{0,-1,1,1}}{9(x+1)}\\
&-\frac{128(x^2+1)H_{0,0,-1,-1}}{9(x+1)}+\frac{128(x^2+1)H_{0,0,-1,1}}{9(x+1)}-\frac{128(x^2+1)H_{0,0,0,-1}}{9(x+1)}+\frac{8(21x^2+10x+21)H_{0,0,0,1}}{9(x+1)}\\
&+\frac{128(x^2+1)H_{0,0,1,-1}}{9(x+1)}-\frac{32(7x^2+6x+7)H_{0,0,1,1}}{9(x+1)}-\frac{256(x^2+1)H_{0,1,-1,1}}{9(x+1)}-\frac{256(x^2+1)H_{0,1,1,-1}}{9(x+1)}\Bigg]\\
&+C_AC_FT_F\Bigg[\frac{88}{27}L_M^3(x+1)+\frac{176}{27}L_Q^3(x+1)+L_M^2\Bigg(\frac{32}{3}(x+1)H_0-\frac{4(83x-37)}{9}-\frac{32(x^2+1)H_{-1}H_0}{3(x+1)}-\frac{16xH_0^2}{3(x+1)}\\
&+\frac{32(x^2+1)H_{0,-1}}{3(x+1)}+\frac{32x\zeta_2}{3(x+1)}\Bigg)-L_Q^2\Bigg(\frac{4}{27}(997x+241)+\frac{256}{9}(x+1)H_0+\frac{32(x^2+1)H_{-1}H_0}{3(x+1)}+\frac{16xH_0^2}{3(x+1)}\\
&+\frac{176}{9}(x+1)H_1-\frac{32(x^2+1)H_{0,-1}}{3(x+1)}-\frac{32x\zeta_2}{3(x+1)}\Bigg)+L_M\Bigg(-\frac{4}{81}(4577x-4267)+\Bigg(\frac{16(3x^2+14x-9)}{9(x+1)}\\
&-\frac{64(x^2+1)H_{-1}}{3(x+1)}+\frac{16(3x^2+4x+3)H_0}{3(x+1)}\Bigg)\zeta_2-\frac{16}{27}(29x-109)H_0-\frac{128(4x^2+3x+4)H_{-1}H_0}{9(x+1)}\\
&+\frac{4(19x^2+4x+25)H_0^2}{9(x+1)}-\frac{16(x^2+1)H_{-1}H_0^2}{3(x+1)}-\frac{32xH_0^3}{9(x+1)}+\frac{32}{3}(x-1)H_1+8(x+1)H_0^2H_1\\
&+\frac{128(4x^2+3x+4)H_{0,-1}}{9(x+1)}-\frac{16}{3}(x+1)H_{0,1}+\frac{64(x^2+1)H_{-1}H_{0,1}}{3(x+1)}-16(x+1)H_0H_{0,1}-\frac{64(x^2+1)H_{0,-1,1}}{3(x+1)}\\
&+\frac{32(x^2+1)H_{0,0,-1}}{3(x+1)}+\frac{64xH_{0,0,1}}{3(x+1)}-\frac{64(x^2+1)H_{0,1,-1}}{3(x+1)}-\frac{32(x^2+3x+1)\zeta_3}{x+1}\Bigg)+L_Q\Bigg(\frac{4}{81}(19751x-2371)\\
&+\frac{32}{9}\frac{(36x^3+61x^2+18x+13)H_{-1}H_0}{x+1}-\frac{32}{9}\frac{(36x^3+61x^2+18x+13)H_{0,-1}}{x+1}-\frac{8}{9}\frac{(72x^3+39x^2-89x-36)H_0^2}{x+1}\\
&+\Bigg(\frac{16}{9}\frac{(72x^3+37x^2-84x-29)}{x+1}-\frac{32(7x^2+6x+3)H_{-1}}{3(x+1)}-\frac{16(3x^2+8x+3)H_0}{3(x+1)}-\frac{64}{3}(4x+1)H_1\Bigg)\zeta_2\\
&+\frac{32(401x^2+556x+137)H_0}{27(x+1)}+\frac{80xH_0^3}{9(x+1)}-\frac{32(7x^2+6x+3)H_{-1}^2H_0}{3(x+1)}+\frac{16(11x^2+6x+7)H_{-1}H_0^2}{3(x+1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{8}{27}(1195x+169)H_1+\frac{352}{9}(x+1)H_0H_1+\frac{8}{3}(11x-1)H_0^2H_1+\frac{176}{9}(x+1)H_1^2-\frac{16}{3}(x+1)H_0H_1^2 \\
&+\frac{208}{9}(x+1)H_{0,1}+\frac{64(7x^2+6x+3)H_{-1}H_{0,-1}}{3(x+1)}+\frac{64x(3x+5)H_0H_{0,-1}}{3(x+1)}-\frac{16}{3}(11x-1)H_0H_{0,1} \\
&+\frac{64}{3}(x+1)H_1H_{0,1}-\frac{64(7x^2+6x+3)H_{0,-1,-1}}{3(x+1)}-\frac{32(23x^2+26x+7)H_{0,0,-1}}{3(x+1)}+\frac{32}{3}(7x+1)H_{0,0,1} \\
&-32(x+1)H_{0,1,1}+\frac{64(3x^2+5x+4)\zeta_3}{3(x+1)}\Bigg)+\Bigg(\frac{16(174x^2+209x-189)}{81(x+1)}-\frac{32(29x^2+18x+29)H_{-1}}{27(x+1)} \\
&-\frac{32(x^2+1)H_{-1}^2}{9(x+1)}+\frac{8(63x^2+29x+6)H_0}{27(x+1)}-\frac{32(x^2+1)H_{-1}H_0}{9(x+1)}+\frac{8(3x^2+8x+9)H_0^2}{9(x+1)} \\
&-\frac{8}{9}(3x+14)H_1+\frac{80}{9}(x+1)H_0H_1+\frac{64}{9}(x+1)H_1^2-\frac{16}{9}(7x+1)H_{0,1}\Bigg)\zeta_2 \\
&-\frac{16}{15(x+1)}(36x^2+51x+22)\zeta_2^2+\Bigg(\frac{2(497x^2+1102x+1085)}{27(x+1)}+\frac{16}{9}(x+1)H_1+\frac{128(x^2+1)H_{-1}}{9(x+1)} \\
&+\frac{32(6x^2+4x-3)H_0}{9(x+1)}\Bigg)\zeta_3+\frac{16}{3}B_4(x+1)-\frac{4}{81}(995x-2807)H_0-\frac{32(199x^2+174x+199)H_{-1}H_0}{81(x+1)} \\
&+\frac{32}{9}(x+1)H_{-1}^2H_0-\frac{64(x^2+1)H_{-1}^3H_0}{27(x+1)}+\frac{4(253x^2+391x+586)H_0^2}{81(x+1)}+\frac{16(x^2+1)H_{-1}^2H_0^2}{9(x+1)} \\
&-\frac{16(19x^2+18x+19)H_{-1}H_0^2}{27(x+1)}+\frac{8(22x^2+7x+25)H_0^3}{81(x+1)}-\frac{32(x^2+1)H_{-1}H_0^3}{27(x+1)}-\frac{16xH_0^4}{27(x+1)}-\frac{8}{27}(65x-29)H_1 \\
&+\frac{8}{9}(9x+4)H_0H_1+\frac{8}{9}(14x+3)H_0^2H_1+\frac{56}{27}(x+1)H_0^3H_1-\frac{4}{9}(43x-46)H_1^2-\frac{8}{9}(2x+5)H_0H_1^2-\frac{4}{9}(x+1)H_0^2H_1^2 \\
&+\frac{32}{27}(x+1)H_0H_1^3+\frac{32(199x^2+174x+199)H_{0,-1}}{81(x+1)}-\frac{64}{9}(x+1)H_{-1}H_{0,-1}+\frac{64(x^2+1)H_{-1}^2H_{0,-1}}{9(x+1)} \\
&-\frac{8}{27}(143x+2)H_{0,1}+\frac{256(4x^2+3x+4)H_{-1}H_{0,1}}{27(x+1)}-\frac{16}{9}(13x+6)H_0H_{0,1}-\frac{8}{3}(x+1)H_0^2H_{0,1}+\frac{8}{9}(11x+20)H_1H_{0,1} \\
&+\frac{64}{9}(x+1)H_0H_1H_{0,1}-\frac{64}{9}(x+1)H_1^2H_{0,1}+\frac{16}{9}(7x+1)H_{0,1}^2+\frac{64}{9}(x+1)H_{0,-1,-1}-\frac{128(x^2+1)H_{-1}H_{0,-1,-1}}{9(x+1)} \\
&-\frac{256(4x^2+3x+4)H_{0,-1,1}}{27(x+1)}+\frac{32(19x^2+18x+19)H_{0,0,-1}}{27(x+1)}-\frac{64(x^2+1)H_{-1}H_{0,0,-1}}{9(x+1)} \\
&+\frac{8(9x^2+101x+12)H_{0,0,1}}{27(x+1)}+\frac{64(x^2+1)H_{-1}H_{0,0,1}}{9(x+1)}-\frac{16}{9}(7x+1)H_0H_{0,0,1}-\frac{160}{9}(x+1)H_1H_{0,0,1} \\
&-\frac{256(4x^2+3x+4)H_{0,1,-1}}{27(x+1)}-\frac{16}{9}(x+7)H_{0,1,1}-\frac{128(x^2+1)H_{-1}H_{0,1,1}}{9(x+1)}-\frac{16}{9}(11x+5)H_0H_{0,1,1}+16(x+1)H_1H_{0,1,1} \\
&+\frac{128(x^2+1)H_{0,-1,-1,-1}}{9(x+1)}+\frac{128(x^2+1)H_{0,-1,1,1}}{9(x+1)}+\frac{64(x^2+1)H_{0,0,-1,-1}}{9(x+1)}-\frac{64(x^2+1)H_{0,0,-1,1}}{9(x+1)}+\frac{64(x^2+1)H_{0,0,0,-1}}{9(x+1)} \\
&+\frac{64(5x^2+6x-1)H_{0,0,0,1}}{9(x+1)}-\frac{64(x^2+1)H_{0,0,1,-1}}{9(x+1)}-\frac{16(x^2-2x-11)H_{0,0,1,1}}{9(x+1)}+\frac{128(x^2+1)H_{0,1,-1,1}}{9(x+1)} \\
&+\frac{128(x^2+1)H_{0,1,1,-1}}{9(x+1)}-\frac{112}{9}(x+1)H_{0,1,1,1}-\frac{2}{729}(108295x-86681)\Bigg]
\end{aligned}
$$

$$+C_FT_F^2\Big[-\frac{64}{27}L_M^3(x+1)-\frac{32}{27}L_Q^3(x+1)-L_M^2\Big(\frac{32}{27}(11x-1)+\frac{32}{9}(x+1)H_0\Big)+L_Q^2\Big(\frac{32}{27}(17x+8)+\frac{64}{9}(x+1)H_0$$
$$+\frac{32}{9}(x+1)H_1\Big)-\frac{992}{81}L_M(x+1)+L_Q\Big(-\frac{32}{81}(280x+37)-\frac{64}{27}(34x+19)H_0-\frac{32}{3}(x+1)H_0^2$$
$$-\frac{64}{27}(17x+8)H_1-\frac{64}{9}(x+1)H_0H_1-\frac{32}{9}(x+1)H_1^2-\frac{64}{9}(x+1)H_{0,1}+\frac{128}{9}(x+1)\zeta_2\Big)+\frac{64}{81}(6x-7)H_0$$
$$+\frac{16}{81}(11x-1)H_0^2+\frac{16}{81}(x+1)H_0^3-\frac{448}{27}(x+1)\zeta_3+\frac{16}{729}(431x+323)\Big]+C_FN_FT_F^2\Big[-\frac{32}{27}L_M^3(x+1)$$
$$-\frac{64}{27}L_Q^3(x+1)+L_Q^2\Big(\frac{64}{27}(17x+8)+\frac{128}{9}(x+1)H_0+\frac{64}{9}(x+1)H_1\Big)+L_M\Big(\frac{32}{81}(5x-73)+\frac{32}{27}(11x-1)H_0$$
$$+\frac{16}{9}(x+1)H_0^2\Big)+L_Q\Big(-\frac{64}{81}(280x+37)-\frac{128}{27}(34x+19)H_0-\frac{64}{3}(x+1)H_0^2-\frac{128}{27}(17x+8)H_1$$
$$-\frac{128}{9}(x+1)H_0H_1-\frac{64}{9}(x+1)H_1^2-\frac{128}{9}(x+1)H_{0,1}+\frac{256}{9}(x+1)\zeta_2\Big)+\frac{128}{81}(6x-7)H_0+\frac{32}{81}(11x-1)H_0^2$$
$$+\frac{32}{81}(x+1)H_0^3+\frac{256}{27}(x+1)\zeta_3-\frac{64}{729}(161x+215)\Big]\Big\}+\hat{c}_{q,2}^{(3)}. \tag{3.3}$$

Here the $+$ prescription is defined by

$$\int_0^1 dxg(x)[f(x)]_+=\int_0^1 dx[g(x)-g(1)]f(x). \tag{3.4}$$

The contribution of the massive Wilson coefficient $H_{q,2}$ is found by combining the massless Wilson coefficient $C_{q,2}$ and $L_{q,2}$ by

$$H_{q,2}(x,Q^2)=C_{q,2}(N_F,x,Q^2)+L_{q,2}(x,Q^2). \tag{3.5}$$

Except for $c_{q,2}^{(3)}(N_F)$, the Wilson coefficients are expressed by up to weight $\mathsf{w}=4$ harmonic polylogarithms. Note the emergence of a denominator $1/(1-x)^2$ (cf. [23]), which is properly regularized by its numerator function in the limit $x\to 1$. We note that we have applied the shuffle algebra (cf. [32]), which leads to a reduction of the number of harmonic polylogarithms compared to the linear representation, making the numerical evaluation faster.

IV. NUMERICAL RESULTS

In the following, we illustrate the asymptotic charm corrections up to 3-loop order to the charged current nonsinglet combinations $F_{1,2}^{W^+-W^-}(x,Q^2)$ choosing the renormalization and factorization scales $\mu^2=Q^2$. First we consider the behavior of the corrections at small and large values of the Bjorken variable x. For those of the massless 3-loop Wilson coefficients, see [3]. The limiting behavior for the two contributing functions $L_{q,i}^{W^+-W^-,\mathrm{NS}}(N_F+1)-\hat{C}_{q,i}^{W^+-W^-,\mathrm{NS}}(N_F)$ and $H_{q,i}^{W^+-W^-,\mathrm{NS}}(N_F+1)-C_{q,i}^{W^+-W^-,\mathrm{NS}}(N_F+1)$ is the same; see also [9].

For the 3-loop contributions, yet for general values of μ^2, at low values of x one has

$$L_{q,L}^{W^+-W^-,\mathrm{NS}}(N_F+1)-\hat{C}_{q,L}^{W^+-W^-,\mathrm{NS}}(N_F)\propto a_s^3\frac{8}{3}C_F^2T_F\ln^2(x), \tag{4.1}$$

$$L_{q,2}^{W^+-W^-,\mathrm{NS}}(N_F+1)-\hat{C}_{q,2}^{W^+-W^-,\mathrm{NS}}(N_F)$$
$$\propto a_s^3\Big\{\frac{16}{27}C_AC_FT_F-\frac{5}{9}C_F^2T_F\Big\}\ln^4(x) \tag{4.2}$$

and at large x

$$L_{q,L}^{W^+-W^-,\mathrm{NS}}(N_F+1)-\hat{C}_{q,L}^{W^+-W^-,\mathrm{NS}}(N_F)$$
$$\propto a_s^3C_FT_F\Big\{\frac{128}{3}C_FL_Q\ln^2(1-x)+\Big[C_F\Big(\frac{896}{27}$$
$$+\frac{320}{9}L_M+\frac{32}{3}L_M^2\Big)+32C_FL_Q^2+\Big(-\frac{1088}{9}C_A$$
$$+\frac{80}{9}C_F+\frac{128}{9}T_F+\frac{256}{9}N_FT_F+\frac{128}{3}\zeta_2C_A$$
$$-\frac{256}{3}\zeta_2C_F\Big)L_Q\Big]\ln(1-x)\Big\}, \tag{4.3}$$

$$
\begin{aligned}
&L_{q,2}^{W^+-W^-,\mathrm{NS}}(N_F+1) - \hat{C}_{q,2}^{W^+-W^-,\mathrm{NS}}(N_F) \\
&\quad \propto a_s^3 C_F T_F \Bigg\{ \frac{320}{9} C_F L_Q \left(\frac{\ln^3(1-x)}{1-x}\right)_+ \\
&\qquad + \Bigg[C_F\left(\frac{448}{9} + \frac{160}{3} L_M + 16 L_M^2\right) + 48 C_F L_Q^2 \\
&\qquad + \Bigg(-\frac{352}{9} C_A - \frac{512}{3} C_F + \frac{64}{9} T_F \\
&\qquad + \frac{128}{9} N_F T_F\Bigg) L_Q \Bigg] \left(\frac{\ln^2(1-x)}{1-x}\right)_+ \Bigg\}. \qquad (4.4)
\end{aligned}
$$

Below, we plot the heavy flavor contribution to the structure function $F_1^{W^+-W^-}(x,Q^2)$ for the quark mass $m_c = 1.59$ GeV in the on-shell scheme [33] and the scales $Q^2 = \mu^2 = 10$, 100, and 1000 GeV^2 for the complete structure function, including the massive and massless terms.

In Fig. 1, the scale evolution of the structure function $xF_1^{W^+-W^-}(x,Q^2)$ is shown in the range $Q^2 \in [10,1000]$ GeV^2, including the asymptotic charm quark corrections to 3-loop order.

Here and in the following, we refer to the parton distribution functions [34]. As is typical for nonsinglet contributions, the profile is shifted from larger to smaller values of x with growing values of Q^2. However, the effects are much smaller than in the singlet case. As is well known, the validity of the asymptotic charm quark corrections in the case of $F_L(x,Q^2)$, and therefore for F_2 and in part for xF_1, is setting in at higher scales only due to the F_L contribution; for details, see [19]. We will discuss these aspects in the following figures for xF_1 and F_2.

In Fig. 2, the corrections to $xF_1^{W^+-W^-}(x,Q^2)$ are illustrated for $Q^2 = 10$ GeV^2 by adding the contributions from $O(a_s^0)$ to $O(a_s^3)$, showing an increasing degree of stabilization in the medium x range, while at small and large values of x there are visible differences. Here the 3-loop corrections matter, in particular. We also present the exact heavy flavor corrections to $O(a_s^2)$ [9], showing deviations in the range $x \gtrsim 10^{-2}$, while below there is exact agreement. The latter effect is due to the sufficiently large $W^2 = Q^2(1-x)/x$ values through which the heavy quarks are made effectively massless for this structure function even at this low scale of Q^2. The charm quark corrections for $xF_1^{W^+-W^-}(x,Q^2)$ vary in a range of -8% to

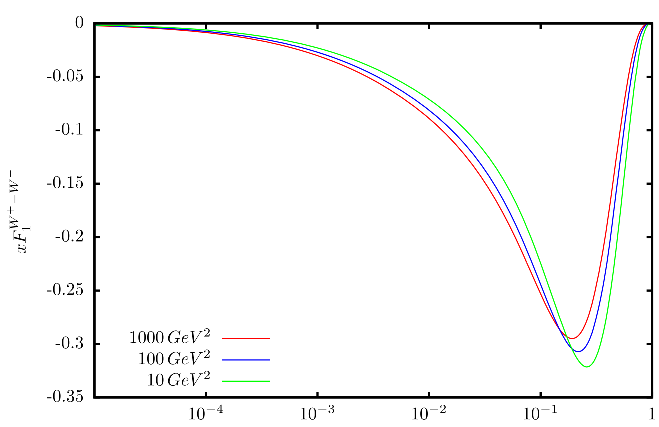

FIG. 1. The structure function $xF_1^{W^+-W^-}(x,Q^2)$, containing the 3-loop corrections including the asymptotic corrections for charm using $m_c^{\mathrm{OMS}} = 1.59$ GeV and the parton distribution functions (PDFs) [34].

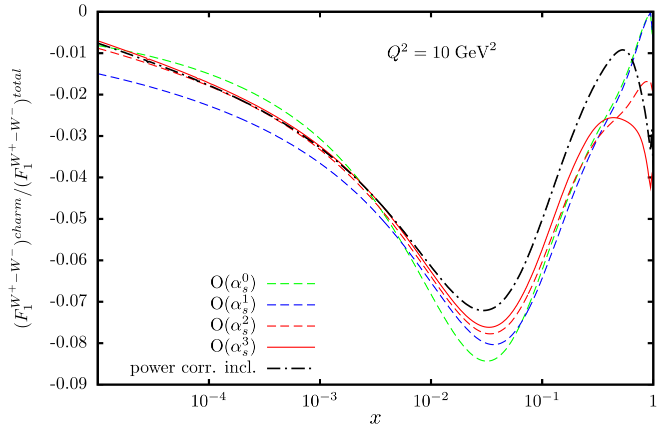

FIG. 2. The ratio of massive contributions to the structure function $xF_1^{W^+-W^-}(x,Q^2)$ over the complete structure function for $Q^2 = 10$ GeV^2, containing the 3-loop corrections including the asymptotic corrections for charm using $m_c^{\mathrm{OMS}} = 1.59$ GeV and the PDFs [34]. For the dash-dotted line, asymptotic corrections at three loops and the complete heavy flavor contributions up to $O(a_s^2)$ [9] are taken into account.

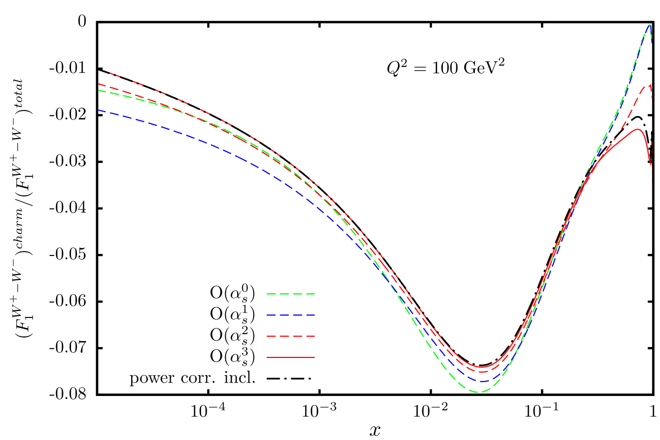

FIG. 3. The ratio of massive contributions to the structure function $xF_1^{W^+-W^-}(x,Q^2)$ over the complete structure function for $Q^2 = 100$ GeV^2, containing the 3-loop corrections including the asymptotic corrections for charm using $m_c^{\mathrm{OMS}} = 1.59$ GeV and the PDFs [34]. For the dash-dotted line, asymptotic corrections at three loops and the complete heavy flavor contributions up to $O(a_s^2)$ [9] are taken into account.

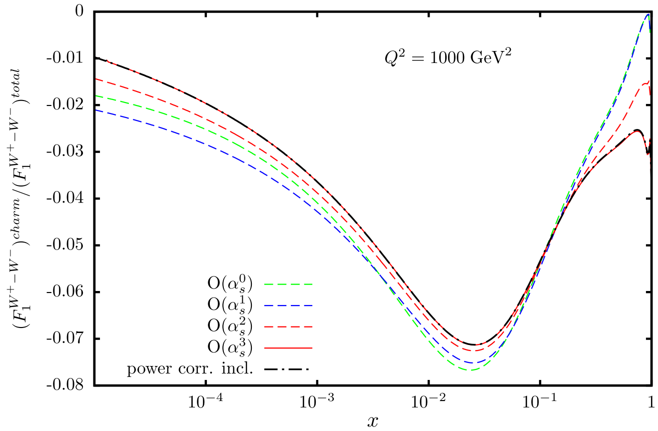

FIG. 4. The ratio of massive contributions to the structure function $xF_1^{W^+-W^-}(x,Q^2)$ over the complete structure function for $Q^2 = 1000$ GeV^2, containing the 3-loop corrections including the asymptotic corrections for charm using $m_c^{\mathrm{OMS}} = 1.59$ GeV and the PDFs [34]. For the dash-dotted line, asymptotic corrections at three loops and the complete heavy flavor contributions up to $O(a_s^2)$ [9] are taken into account.

~0%, depending on x, with a maximal relative contribution around $x \sim 3 \times 10^{-2}$.

Figure 3 shows that at $Q^2 = 100$ GeV^2 the asymptotic corrections agree also in the case where we include the power corrections to larger values of $x \sim 0.3$, and for $Q^2 = 1000$ GeV^2 (Fig. 4) the agreement is obtained in the whole x range.

We turn now to the numerical illustration of the structure function $F_2^{W^+-W^-}(x,Q^2)$. In Fig. 5, we show the scaling violations of $F_2^{W^+-W^-}(x,Q^2)$ in the region $Q^2 \in [10, 1000]$ GeV^2, shifting the profile to lower values of x with growing virtualities Q^2. Figure 6 shows the contributions to $F_2^{W^+-W^-}(x,Q^2)$ at $Q^2 = 10$ GeV^2 for growing order in the strong coupling constant up to

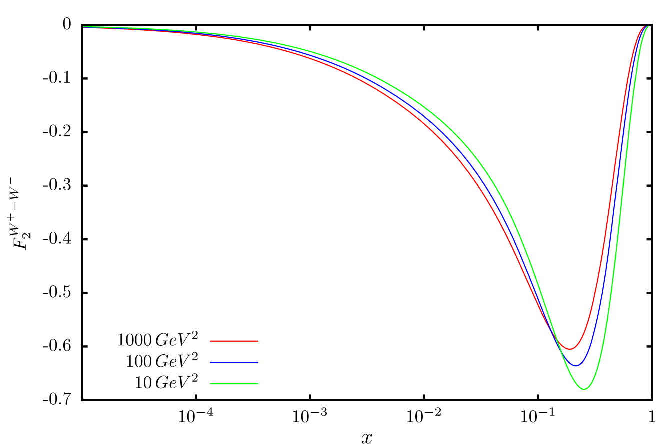

FIG. 5. The structure function $F_2^{W^+-W^-}(x,Q^2)$, containing the 3-loop corrections including the asymptotic corrections for charm using $m_c^{\mathrm{OMS}} = 1.59$ GeV and the PDFs [34].

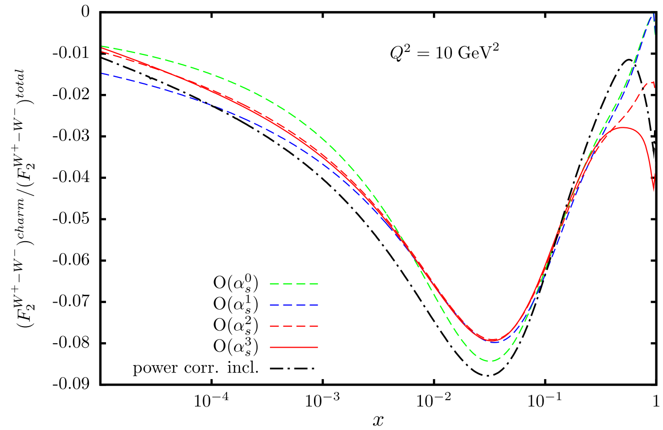

FIG. 6. The ratio of massive contributions to the structure function $F_2^{W^+-W^-}(x,Q^2)$ over the complete structure function for $Q^2 = 10$ GeV^2, containing the 3-loop corrections including the asymptotic corrections for charm using $m_c^{\mathrm{OMS}} = 1.59$ GeV and the PDFs [34]. For the dash-dotted line, asymptotic corrections at three loops and the complete heavy flavor contributions up to $O(a_s^2)$ [9] are taken into account.

3-loop order. Except of very large values of x, power corrections, known up to 2-loop order, do not introduce corrections. At $Q^2 = 10$ GeV^2, by comparing the results for $2xF_1$ and F_2 the effect of $F_L(x,Q^2)$ is clearly visible. The asymptotic expression is not yet valid in the charged current case, as the complete $O(a_s^2)$ charm quark corrections show.

Again, the relative charm quark corrections vary in the range $[-8\%, \sim 0\%]$. As shown in Fig. 7, the asymptotic corrections agree with the case where the power corrections are included, except for a small range at very large x values

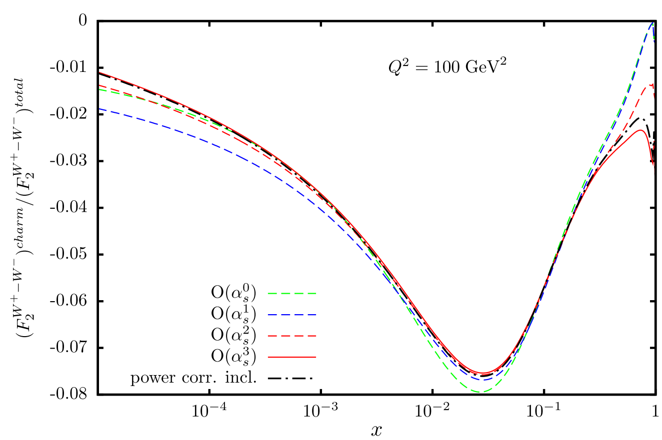

FIG. 7. The ratio of massive contributions to the structure function $F_2^{W^+-W^-}(x,Q^2)$ over the complete structure function for $Q^2 = 100$ GeV^2, containing the 3-loop corrections including the asymptotic corrections for charm using $m_c^{\mathrm{OMS}} = 1.59$ GeV and the PDFs [34]. For the dash-dotted line, asymptotic corrections at three loops and the complete heavy flavor contributions up to $O(a_s^2)$ [9] are taken into account.

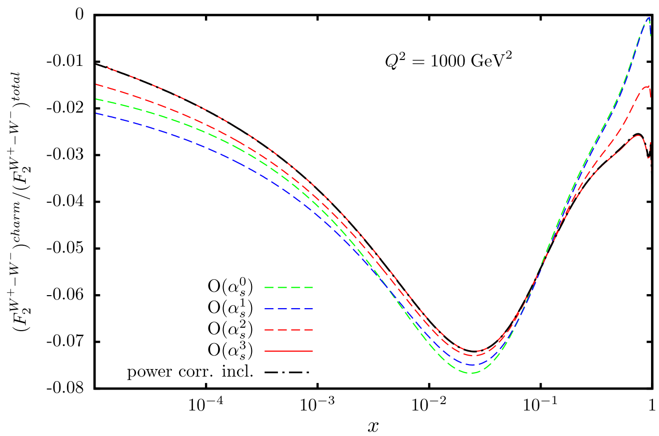

FIG. 8. The ratio of massive contributions to the structure function $F_2^{W^+-W^-}(x,Q^2)$ over the complete structure function for $Q^2 = 1000$ GeV^2, containing the 3-loop corrections including the asymptotic corrections for charm using $m_c^{\rm OMS} = 1.59$ GeV and the PDFs [34]. For the dash-dotted line, asymptotic corrections at three loops and the complete heavy flavor contributions up to $O(a_s^2)$ [9] are taken into account.

at $Q^2 = 100$ GeV^2. Finally, this effect disappears for $Q^2 = 1000$ GeV^2; see Fig. 8.

V. THE SUM RULES

For the combination of the charged current structure functions being considered here, there exist sum rules arising from the lowest Mellin moment. In the case of $F_2^{W^+-W^-}(x,Q^2)$, one obtains the Adler sum rule [2] and for $F_1^{W^+-W^-}(x,Q^2)$ the unpolarized Bjorken sum rule [1], for which also the target mass corrections have to be considered; cf. [9].

The Adler sum rule states

$$\int_0^1 \frac{dx}{x}[F_2^{\bar{\nu}p}(x,Q^2) - F_2^{\nu p}(x,Q^2)] = 2[1+\sin^2(\theta_c)] \tag{5.1}$$

for three massless flavors. Here θ_c denotes the Cabibbo angle [24]. The integral (5.1) receives neither QCD nor quark or target mass corrections [21]; cf. also [35,36]. Up to 2-loop order, the vanishing of the heavy quark corrections has been shown in Ref. [9]. Considering the limit of large scales $Q^2 \gg m^2$, this is confirmed at 3-loop order, since the flavor nonsinglet OMEs vanish for $N = 1$ due to fermion number conservation [23] and the first moment of the corresponding massless Wilson coefficient also vanishes [37].

The unpolarized Bjorken sum rule [1] is given by

$$\int_0^1 dx[F_1^{\bar{\nu}p}(x,Q^2) - F_1^{\nu p}(x,Q^2)] = C_{\rm uBJ}(\hat{a}_s), \tag{5.2}$$

with $\hat{a}_s = \alpha_s/\pi$. The massless 1-loop [10,11,38,39], 2-loop [40], 3-loop [41], and 4-loop [42] QCD corrections have been calculated:

$$\begin{aligned} C_{\rm uBJ}(\hat{a}_s), = 1 &- 0.66667\hat{a}_s + \hat{a}_s^2(-3.83333 + 0.29630N_F) \\ &+ \hat{a}_s^3(-36.1549 + 6.33125N_F - 0.15947N_F^2) \\ &+ \hat{a}_s^4(-436.768 + 111.873N_F - 7.11450N_F^2 \\ &+ 0.10174N_F^3), \end{aligned} \tag{5.3}$$

setting $\mu^2 = Q^2$ for $SU(3)_c$. The massive corrections start at $O(a_s^0)$ with the $s' = (|V_{dc}|^2 d + |V_{sc}|^2 s) \to c$ transitions [12,13] and have been given in complete form in Ref. [9] to 2-loop order. The charm corrections at $O(\hat{a}_s^2)$ are of the same size as the massless $O(\hat{a}_s^4)$ corrections. Reference [9] also contains the target mass corrections. In the asymptotic case, the effect of the heavy flavor corrections reduces to a shift of $N_F \to N_F + 1$ in the massless corrections, since the massive OMEs vanish for $N = 1$ due to fermion conservation, which holds to all orders in the perturbation theory.

VI. CONCLUSIONS

We have derived the massive charm quark 3-loop corrections to the charged current Wilson coefficients for the structure functions $F_{1,2}^{W^+-W^-}(x,Q^2)$ in the asymptotic region $Q^2 \gg m_c^2$ in both Mellin N and x space. The corresponding contributions are composed of two massive Wilson coefficients $L_q^{W^+-W^-,\rm NS}$ and $H_q^{W^+-W^-,\rm NS}$ for which the weak boson couples to either a massless (L) or a massive quark line (H), here in the $s' \to c$ transition. The massless 3-loop Wilson coefficients have been calculated in Ref. [3], and the massive OMEs were presented before in Ref. [23] as part of the present project to compute all massive 3-loop corrections to deep-inelastic scattering at high values of Q^2. The results have a representation in terms of nested harmonic sums and harmonic polylogarithms only. The charm quark corrections in the case of both structure functions amount up to ~8%, depending on x and the 3-loop corrections. Comparing the 3-loop results to those at lower order, there are some ranges with clearly visible differences, pointing to the importance of these corrections, for highly accurate measurements. At low values of Q^2, effects of power corrections are still visible, which we have illustrated using recent complete 2-loop results [9], while for $Q^2 \gtrsim 100$ GeV^2 the asymptotic representation is valid in a rather wide range of x.

We also discussed potential contributions of the present corrections to the Adler and unpolarized Bjorken sum rules. In the former case, in accordance with the expectation, no corrections are obtained. For the Bjorken sum rule, the charm quark contributions lead to a shift of $N_F = 3$ by one unit in the massless result. There are no heavy quark contributions due to fermion number conservation, which is expressed by a vanishing first moment of the operator matrix element in the nonsinglet cases. Therefore, only the massless terms contribute now with $N_F \to N_F + 1$.

The 3-loop charm quark corrections to the structure functions $F_{1,2}^{W^+-W^-}(x, Q^2)$ will improve the analysis of the HERA charged current data and are relevant for precision measurements in deep-inelastic scattering at planned facilities like the EIC [43], LHeC [44], and neutrino factories [45] in the future, which will reach a higher statistical and systematic precision than obtained in present experiments.

ACKNOWLEDGMENTS

We thank J. Ablinger, A. Hasselhuhn, and A. Vogt for discussions, as well A. Vogt for providing us the effective numerical representations of the massless flavor nonsinglet 3-loop Wilson coefficients of Ref. [3]. This work was supported in part by the European Commission through Contract No. PITN-GA-2012-316704 (HIGGSTOOLS) and the Austrian Science Fund (FWF) Grant No. SFB F50 (F5009-N15).

[1] J. D. Bjorken, Phys. Rev. **163**, 1767 (1967).
[2] S. L. Adler, Phys. Rev. **143**, 1144 (1966).
[3] J. Davies, A. Vogt, S. Moch, and J. A. M. Vermaseren, arXiv:1606.08907; Proc. Sci., **DIS2016** (2016) 059.
[4] J. Blümlein, A. Hasselhuhn, and T. Pfoh, Nucl. Phys. **B881**, 1 (2014).
[5] J. Blümlein and N. Kochelev, Nucl. Phys. **B498**, 285 (1997).
[6] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, and C. Schneider, Nucl. Phys. **B890**, 48 (2014).
[7] J. Ablinger, J. Blümlein, C. Raab, C. Schneider, and F. Wißbrock, Nucl. Phys. **B885**, 409 (2014).
[8] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, and C. Schneider, Comput. Phys. Commun. **202**, 33 (2016).
[9] J. Blümlein, G. Falcioni, and A. De Freitas, Nucl. Phys. **B910**, 568 (2016).
[10] W. A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta, Phys. Rev. D **18**, 3998 (1978).
[11] W. Furmanski and R. Petronzio, Z. Phys. C **11**, 293 (1982), and references therein.
[12] J. Blümlein, A. Hasselhuhn, P. Kovacikova, and S. Moch, Phys. Lett. B **700**, 294 (2011).
[13] M. Glück, S. Kretzer, and E. Reya, Phys. Lett. B **398**, 381 (1997); **405**, 392(E) (1997).
[14] T. Gottschalk, Phys. Rev. D **23**, 56 (1981).
[15] M. Buza and W. L. van Neerven, Nucl. Phys. **B500**, 301 (1997).
[16] E. B. Zijlstra and W. L. van Neerven, Phys. Lett. B **297**, 377 (1992).
[17] S. Moch and J. A. M. Vermaseren, Nucl. Phys. **B573**, 853 (2000).
[18] A. Hasselhuhn, Ph.D. thesis, TU Dortmund, 2013 [Report No. DESY-THESIS-2013-050].
[19] M. Buza, Y. Matiounine, J. Smith, R. Migneron, and W. L. van Neerven, Nucl. Phys. **B472**, 611 (1996).
[20] A. Arbuzov, D. Y. Bardin, J. Blümlein, L. Kalinovskaya, and T. Riemann, Comput. Phys. Commun. **94**, 128 (1996); J. Blümlein, M. Klein, T. Naumann, and T. Riemann, Report No. PHE-88-01.
[21] J. Blümlein, Prog. Part. Nucl. Phys. **69**, 28 (2013).
[22] I. Bierenbaum, J. Blümlein, and S. Klein, Nucl. Phys. **B820**, 417 (2009).
[23] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider, and F. Wißbrock, Nucl. Phys. **B886**, 733 (2014).
[24] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963).
[25] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).
[26] K. A. Olive *et al.* (Particle Data Group), Chin. Phys. C **38**, 090001 (2014).
[27] A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, and C. Schneider, Phys. Rev. D **92**, 114005 (2015).
[28] A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, and C. Schneider, Nucl. Phys. **B897**, 612 (2015).
[29] J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider, and F. Wißbrock, *Proc. Sci.*, LL2016 (2016) 065 [arXiv:1609.03397]; J. Blümlein, A. DeFreitas, and C. Schneider, Nucl. Part. Phys. Proc. **261–262**, 185 (2015).
[30] J. A. M. Vermaseren, Int. J. Mod. Phys. A **14**, 2037 (1999); J. Blümlein and S. Kurth, Phys. Rev. D **60**, 014018 (1999).
[31] E. Remiddi and J. A. M. Vermaseren, Int. J. Mod. Phys. A **15**, 725 (2000).
[32] J. Blümlein, Comput. Phys. Commun. **159**; 19 (2004).
[33] S. Alekhin, J. Blümlein, K. Daum, K. Lipka, and S. Moch, Phys. Lett. B **720**, 172 (2013).
[34] S. Alekhin, J. Blümlein, and S. Moch, Phys. Rev. D **89**, 054028 (2014).
[35] V. Ravindran and W. L. van Neerven, Nucl. Phys. **B605**, 517 (2001).
[36] S. L. Adler, arXiv:0905.2923.
[37] S. Moch and M. Rogal, Nucl. Phys. **B782**, 51 (2007).
[38] G. Altarelli, R. K. Ellis, and G. Martinelli, Nucl. Phys. **B143**, 521 (1978); **B146**, 544(E) (1978).
[39] B. Humpert and W. L. van Neerven, Nucl. Phys. **B184**, 225 (1981).
[40] S. G. Gorishnii and S. A. Larin, Phys. Lett. B **172**, 109 (1986).

[41] S. A. Larin, F. V. Tkachov, and J. A. M. Vermaseren, Phys. Rev. Lett. **66**, 862 (1991).

[42] K. G. Chetyrkin, in *Proceedings of Loops and Legs in Quantum Field Theory, Weimar, 2014* (unpublished), https://indico.desy.de/conferenceOtherViews.py?view=standard&confId=8107.

[43] D. Boer *et al.*, arXiv:1108.1713; A. Accardi *et al.*, arXiv:1212.1701.

[44] J. L. Abelleira Fernandez *et al.* (LHeC Study Group Collaboration), J. Phys. G **39**, 075001 (2012).

[45] D. M. Kaplan (MAP and MICE Collaborations), Eur. Phys. J. Web Conf. **95**, 03019 (2015).