PHYSICAL REVIEW D **94,** 106009 (2016)

Protostring scattering amplitudes

Charles B. Thorn*

Institute for Fundamental Theory, Department of Physics, University of Florida, Gainesville, Florida 32611, USA

(Received 22 July 2016; published 15 November 2016)

We calculate some tree-level scattering amplitudes for a generalization of the protostring, which is a novel string model implied by the simplest string bit models. These bit models produce a light-cone world sheet which supports s integer moded Grassmann fields. In the generalization we supplement this Grassmann world-sheet system with $d = 24 - s$ transverse coordinate world-sheet fields. The protostring corresponds to $s = 24$ and the bosonic string to $s = 0$. The interaction vertex is a simple overlap with no operator insertions at the break/join point. Assuming that s is even we calculate the multistring scattering amplitudes by bosonizing the Grassmann fields, each pair equivalent to one compactified bosonic field, and applying Mandelstam's interacting string formalism to a system of $s/2$ compactified and d uncompactified bosonic world-sheet fields. We obtain all amplitudes for open strings with no oscillator excitations and for closed strings with no oscillator excitations and zero winding number. We then study in detail some simple special cases. Multistring processes with maximal helicity violation have much simpler amplitudes. We also specialize to general four-string amplitudes and discuss their high energy behavior. Most of these models are not covariant under the full Lorentz group $O(d+1,1)$. The exceptions are the bosonic string whose Lorentz group is $O(25,1)$ and the protostring whose Lorentz group is $O(1,1)$. The models in between only enjoy an $O(1,1) \times O(d)$ spacetime symmetry.

DOI: 10.1103/PhysRevD.94.106009

I. INTRODUCTION

Recent studies of string bit models [1–3] have motivated the serious consideration of some novel string theories. String bits are hypothetical fundamental constituents of string. However, just which string theories they describe depends on their detailed structure. For example, the superstring bit [3] underlying IIB superstring is created by an operator $\bar{\phi}_{[a_1\cdots a_k]}(\boldsymbol{x})$, where $a_i = 1, \ldots, 8$ are spinor indices, and $k = 0, \ldots, 8$. The vector variable $\boldsymbol{x}$ gives the location of the bit in the eight-dimensional transverse space of light-cone quantized string theory [4,5]. These creation operators are also $N \times N$ matrices in "color," the indices of which we suppress. In the large N limit [6] the color singlet composites of M string bits are closed chains which approximate continuous closed strings for very large bit number M. On these long chains fluctuations in the spinor degrees of freedom lead to the eight Grassmann world-sheet fields $\theta^a_{R,L}(\sigma)$, and fluctuations in the coordinates lead to the eight transverse coordinate world-sheet fields $\boldsymbol{x}(\sigma)$. In this interpretation a longitudinal coordinate x^- arises as the conjugate to a longitudinal momentum identified with bit number $P^+ \equiv Mm$. Each string bit carries one unit m of P^+. In this way holography [7] was realized by these models in its narrowest sense: formulating $d+1$ dimensional physics in one dimension less. These early string bit models, which naturally describe type IIB superstring theory, inspired other matrix model proposals, including the matrix model of M theory [8] and the matrix string model of type IIA superstring theory [9].

However, since the strings are simply large bit number composites, the same string theory can arise from a string bit with considerably less structure. A possibility proposed in [10–13] is to replace each transverse coordinate x^k of a string bit with a two-valued index whose fluctuations on long chains of bits simulate the transverse space. In such string bit models all of space, both longitudinal x^- *and* transverse $\boldsymbol{x}$, is initially absent but emerges dynamically in describing the physics of composites of string bits. But the string bit concept can be realized in more general ways not necessarily tied to the known string models. Nonetheless, certain general constraints should still apply. In particular, the constraint that $1/N$ corrections lead to finite scattering amplitudes typically constrains the size of the world-sheet field system. This is just the critical dimension constraint in bosonic string ($D = 26$) and superstring ($D = 10$) theory. In the latter case supersymmetry further links the spinor dimensionality to the coordinate dimensionality.

In [10–13] we studied, initially as a warmup exercise, the simplest string bit model with only spinor degrees of freedom. The bit creation operator carries only the spinor indices: $\bar{\phi}_{a_1\cdots a_k}$. Here we let $k = 0, \ldots, s$ and $a_i = 1, \ldots, s$, with the integer s to be determined. These operators are bosonic if k is even and fermionic if k is odd. They are also antisymmetric under the interchange of any pair of the a_i. As explained in [13], the large N limit of these models predicts a world-sheet system of s left-right pairs of

*thorn@phys.ufl.edu

Grassmann world-sheet fields. Also, the $1/N$ corrections are described in string language as a simple overlap of three-string states, without the operator prefactor familiar in the superstring vertex. It was found that this overlap amplitude is finite in the continuum limit only if $s = 24$. This is a novel string theory. In a sense it is more like the bosonic string than the superstring because of the absence of prefactors in the three-string vertex. On the other hand, the degree of freedom count of world-sheet fields matches that of the superstring: bosonize 16 of the Grassmann dimensions to match the 8 transverse coordinates, and one is left with the 8 Grassmann dimensions of the superstring. Unlike the bosonic string which has tachyons, this emergent string, which moves in one emergent space dimension x^-, has none: in fact, there is a mass gap. We call this string a protostring since it has the primitive simplicity (but not the instability) of the bosonic string, and it also evokes intimations of superstring. For this reason, we believe that the protostring is well worth serious study in its own right.

The purpose of this article is to explore the protostring's physical properties further by determining its scattering amplitudes. We stress that the calculations we do are pure string theory calculations: the fact that the protostring was the outcome of a simple string bit model will play no role.[1] We only need to faithfully apply Mandelstam's interacting light-cone string formalism [14–16].

The protostring can be succinctly characterized as the string model in which each of the 24 transverse coordinates of the light-cone bosonic string is replaced by a spinor-valued integer moded Grassmann world-sheet field. Here we consider a slight generalization of the protostring in which only s bosonic dimensions are replaced in this way, with the remaining $d = 24 - s$ left as transverse coordinates. In all of these models the string interaction is a simple overlap without operator insertions at the join/break point. The condition $s + d = 24$ ensures the finite continuum limit of the string bit overlap. With the continuum scattering amplitude written in the form $\mathcal{M}\prod_k |p_k^+|^{-1/2}$, this finiteness condition means that $\mathcal{M}$ is invariant under the scale transformation $p_k^+ \to \lambda p_k^+$. In other words, invariance under the subgroup $SO(1,1)$ of the Lorentz group in $d+1$ space dimensions $SO(d+1,1)$ is maintained. For the protostring ($s = 24$ or $d = 0$), this is the entire Lorentz group, but for $s < 24$, with the exception of the bosonic string ($s = 0$), the Lorentz group is broken to $SO(1,1) \times SO(d)$.

In Sec. II we explain the process of bosonization, which is needed in the rest of the paper. Then in Sec. III we apply Mandelstam's interacting string formalism to the bosonized generalized protostring and calculate scattering amplitudes for any number of external strings in states with zero winding number and no oscillator excitations. Section IV discusses amplitudes in special simplifying circumstances. Section V analyzes high energy scattering in the 2-to-2 case. Concluding comments are in Sec. VI. The appendixes review the necessary measure calculations needed for the processes discussed in the main text.

II. BOSONIZATION

Hereafter, we assume that s is even, which allows us to approach the calculation of scattering amplitudes by bosonizing each pair of the s Grassmann dimensions. Then we apply Mandelstam's light-cone interacting string formalism for the bosonic string [14] to calculate the scattering amplitudes. When bosonized, the Grassmann system is equivalent to $s/2$ compactified boson world-sheet fields ϕ^a in which the Kaluza-Klein (KK) momenta $\boldsymbol{\pi}$ assume half-odd-integer multiples of the inverse compactification radius, which is fixed by the nature of the Grassmann system.

The bosonization procedure works only in the continuum limit, $M \to \infty$. At finite M, in the notation of [13],

$$S_k = \frac{B_0}{\sqrt{M}} + \frac{1}{\sqrt{M}} \sum_{n=1}^{M-1} \left(F_n e^{-i\omega_n t} \cos\frac{\pi n}{2M} + \bar{F}_n e^{+i\omega_n t} \sin\frac{\pi n}{2M} \right) e^{2\pi i k n/M}, \tag{1}$$

$$\tilde{S}_k = \frac{B_0}{\sqrt{M}} - i \frac{1}{\sqrt{M}} \sum_{n=1}^{M-1} \left(F_n e^{-i\omega_n t} \sin\frac{\pi n}{2M} - \bar{F}_n e^{+i\omega_n t} \cos\frac{\pi n}{2M} \right) e^{2\pi i k n/M}, \tag{2}$$

$$\{F_n, \bar{F}_m\} = 2\delta_{m+n,M}, \qquad \{B_0, \tilde{B}_0\} = 0,$$
$$B_0^2 = \tilde{B}_0^2 = 1, \qquad \omega_n = \frac{2T_0}{m} \sin\frac{n\pi}{M}. \tag{3}$$

In the continuum limit $M \to \infty$ with $mM = P^+$ fixed, finite energy excitations have either n or $M - n$ finite. The world-sheet coordinate $\sigma \sim km$ and the above formulas read

$$\frac{S_k}{\sqrt{m}} \sim \frac{B_0}{\sqrt{P^+}} + \sqrt{\frac{2}{P^+}} \sum_{n=1}^{\infty} (f_n e^{-2i\pi n(T_0 t - \sigma)/P^+} + f_n^\dagger e^{+2i\pi n(T_0 t - \sigma)/P^+}), \tag{4}$$

$$\frac{\tilde{S}_k}{\sqrt{m}} \sim \frac{\tilde{B}_0}{\sqrt{P^+}} + \sqrt{\frac{2}{P^+}} \sum_{n=1}^{\infty} (\tilde{f}_n e^{-2i\pi n(T_0 t + \sigma)/P^+} + \tilde{f}_n^\dagger e^{+2i\pi n(T_0 t + \sigma)/P^+}), \tag{5}$$

[1] However, the underlying string bit physics may justify the use of analytic continuation to define the integral representations of amplitudes which typically diverge for physical values of the momenta.

$$f_n \equiv \frac{F_n}{\sqrt{2}}, \qquad \tilde{f}_n \equiv -i\frac{F_{M-n}}{\sqrt{2}},$$
$$\{f_n, f_m^\dagger\} = \{\tilde{f}_n, \tilde{f}_m^\dagger\} = \delta_{mn}. \tag{6}$$

We see that in the continuum limit, S describes right-moving and $\tilde{S}$ describes left-moving waves along the string.

Labeling a pair of such Grassmann variables 1, 2, the bosonization formula for right-moving waves (see, for example, Appendix A in [17]) is

$$a_n = \frac{iB_0^1}{\sqrt{2}} f_n^2 + f_n^1 \frac{iB_0^2}{\sqrt{2}} + i\sum_{k=1}^{n-1} f_k^1 f_{n-k}^2 + i\sum_{k=1}^{\infty}(f_k^{1\dagger} f_{n+k}^2 + f_{n+k}^1 f_k^{2\dagger}), \qquad n > 0, \tag{7}$$

$$a_0 = \frac{i}{2} B_0^1 B_0^2 + i\sum_{k=1}^{\infty}(f_k^{1\dagger} f_k^2 - f_k^{2\dagger} f_k^1), \qquad a_{-n} \equiv a_n^\dagger, \tag{8}$$

$$[a_n, a_m] = n\delta_{n,-m} \tag{9}$$

and similar formulas with tildes for the left-moving waves $\tilde{a}_n$. The square of the zero mode part of a_0 or $\tilde{a}_0$ is 1/4, showing that the lowest energy state has values $\pm 1/2$ for a_0 and $\tilde{a}_0$. The sum $a_0 + \tilde{a}_0$ and difference $a_0 - \tilde{a}_0$ have the interpretation of KK momentum and winding number, respectively. We see that they have opposite parity: if one is even, the other is odd and vice versa. We calculate scattering amplitudes for zero winding number strings ($a_0 = \tilde{a}_0$), in which case the KK momentum is an odd integer multiple of some scale.

It is sometimes convenient to express bosonization in terms of the fermion operators of definite helicity, i.e., eigenoperators of a_0:

$$b_0 = \frac{1}{2}(B_0^1 + iB_0^2), \qquad b_n = \frac{1}{\sqrt{2}}(f_n^1 + if_n^2),$$
$$d_n = \frac{1}{\sqrt{2}}(f_n^1 - if_n^2), \tag{10}$$

$$\{b_n, b_n^\dagger\} = \{d_n, d_n^\dagger\} = \{b_0, b_0^\dagger\} = 1, \tag{11}$$

with all other anticommutators vanishing. In terms of these the boson operators are

$$a_n = d_n b_0 + \sum_{k=1}^{n-1} d_k b_{n-k} + b_0^\dagger b_n + \sum_{k=1}^{\infty}(b_k^\dagger b_{n+k} - d_k^\dagger d_{n+k}),$$
$$n > 0, \tag{12}$$

$$a_0 = -\frac{1}{2} + b_0^\dagger b_0 + \sum_{k=1}^{\infty}(b_k^\dagger b_k - d_k^\dagger d_k),$$
$$a_{-n} = a_n^\dagger. \tag{13}$$

In this article we discuss scattering amplitudes for external strings in states with zero winding number and no oscillator excitations, which means they are in states annihilated by $a_n, \tilde{a}_n, n > 0$, but for which $a_0 = \tilde{a}_0$ can have any allowed value. Since $[a_0, a_n] = [a_0, \tilde{a}_n] = 0$, these states are the lowest energy states with the given value of helicity. Define $|0\rangle$ to be the lowest energy helicity $-1/2$ state, which means that $(b_0, b_n, d_n)|0\rangle = 0$. States with helicity $a_0 = k - 1/2$ are obtained by applying n $d^\dagger$'s and m $b^\dagger$'s, with $m - n = k$, to the state $|0\rangle$. Clearly, the lowest energy states of definite helicity are

$$d_k^\dagger d_{k-1}^\dagger \cdots d_1^\dagger |0\rangle, \qquad a_0 = -k - 1/2, \qquad k = 0, 1, 2, \ldots, \tag{14}$$

$$b_{k-1}^\dagger b_{k-2}^\dagger \cdots b_0^\dagger |0\rangle, \qquad a_0 = k - 1/2, \qquad k = 1, 2, \ldots. \tag{15}$$

In Fermi language the energy of a state is the total mode number plus 2/24. The total mode numbers of these lowest energy states in each helicity sector are $\sum_{l=1}^{k} l = k(k+1)/2$ and $\sum_{l=1}^{k-1} l = k(k-1)/2$. In both cases the mode number is $a_0^2/2 - 1/8$. Adding on 2/24 shows that the energy of these states is precisely $a_0^2/2 - 1/24$, as the quantization of the bosonic string requires.

III. SCATTERING AMPLITUDES

A. World-sheet path integral

As we have seen, for strings of even winding number there is a minimum nonzero momentum magnitude which we call γ. Then the KK momenta assume odd integer multiples of γ, $\boldsymbol{\pi} = \pm(2k+1)\boldsymbol{\gamma}$, $k = 0, 1, 2, \ldots$. The world-sheet path integral will be over $n_b = d + s/2 = 24 - s/2$ bosonic fields. In bosonized language, a nonzero three-vertex will require an insertion of the form $e^{\pm i\gamma\phi(\rho)}$ for each ϕ^a at the join/break point. In other words, the KK momentum π^k will not be conserved, with violation of up to $\pm(N-2)\gamma$ for the N point function. Also, for hermiticity reasons, we require a Hermitian linear combination of the insertion factors with $\pm$ in the exponent; for example, we can take the insertion to be $e^{i\gamma\phi} + e^{-i\gamma\phi} = 2\cos(\gamma\phi)$. We call the $d = 24 - s$ coordinate fields x^k, the momenta $\boldsymbol{p}$ of which will be continuous and exactly conserved. Thus, in the evaluation of the scattering amplitude, we allow a different (binary for each of the $s/2$ bosonized Grassmann dimensions) choice at each vertex: we write the insertion at the rth vertex as $\prod_a(e^{i\gamma\phi_a(\rho_r)} + e^{-i\gamma\phi_a(\rho_r)})$. After expanding the vertex factors, one has a collection of terms with a single exponential $e^{i\gamma_r^a\phi(\rho_r)}$ with $\gamma_r^a = \pm\gamma$. Then, the momentum conservation law following from Neumann boundary conditions for each such term takes the form

$$\sum_{k=1}^{N}\boldsymbol{\pi}_k+\sum_{r=1}^{N-2}\boldsymbol{\gamma}_r=0,\qquad \sum_{k=1}^{N}\boldsymbol{p}_k=0, \tag{16}$$

for the bosonized Grassmann fields and regular coordinate fields, respectively.

It is convenient to collect the $\boldsymbol{p}_k$, $\boldsymbol{\pi}_k$ of the kth string participating in the scattering process together in an n_b component transverse momentum $\boldsymbol{P}_k=(\boldsymbol{p}_k,\boldsymbol{\pi}_k)$. We also denote the "Minkowski" (n_b+2)-vector with upper case Roman type, $P_k=(p_k^-,p_k^+,\boldsymbol{P}_k)$, and the Minkowski $(d+2)$-vector with lower case Roman type, $p_k=(p_k^-,p_k^+,\boldsymbol{p}_k)$. The interacting string formalism automatically imposes the mass shell condition for an unexcited string:

$$\alpha' P\cdot P=\begin{cases}\frac{n_b}{24}=1-\frac{s}{48} & \text{Open string}\\ \frac{n_b}{6}=4-\frac{s}{12} & \text{Closed string.}\end{cases} \tag{17}$$

These conditions reduce to the familiar 1 and 4 for the bosonic string ($s=0$). Hereafter, we choose units so that $\alpha'=(2\pi T_0)^{-1}=1$. Note that since $\boldsymbol{\pi}^2\ge s\gamma^2/2$, the "ordinary" momentum p satisfies $p\cdot p\le 1-s(1+24\gamma^2)/48$ or $p\cdot p\le 4-s((1+6\gamma^2)/12$, respectively. As we shall see, $\gamma^2=1/8, 1/2$, respectively, so the theories with $s>12$ have a mass gap.

The contribution of each field ϕ^a to the Boltzmann factor in the world-sheet path integrand for the N string scattering amplitude is

$$B(\phi)=\exp\left\{-\frac{1}{4\pi}\int d^2\rho(\nabla\phi)^2+i\sum_r\gamma_r\phi(\rho(x_r))+i\sum_k\frac{\pi_k}{2\pi|p_k^+|}\int d\sigma_k\phi(\sigma_k,\tau_k)\right\}. \tag{18}$$

The contribution of each component of $\boldsymbol{x}$ is similar, except that there are no γ terms, and the external momenta $\boldsymbol{p}_k$ are continuous. The last term converts the description of the initial and final strings from coordinate space to momentum space. Because we assume no oscillator excitations of the N strings, it suffices to specify a constant momentum density on each string at initial and final times. The σ_k for the kth string ranges over an interval of σ of length $2\pi|p_k^+|$. Here, to conform with Mandelstam's conventions, we have chosen the world-sheet spatial coordinate σ of a single string so that on a given external string $0<\sigma_k-\sigma_k^0<2\pi p_k^+\equiv\pi\alpha_k$. We also introduce the complex world-sheet coordinate $\rho=\tau+i\sigma$. In the complex plane the integral in the last term can be viewed as a closed line integral,

$$i\sum_k\frac{\pi_k}{2\pi|p_k^+|}\int d\sigma_k\phi(\sigma_k,\tau_k)=i\oint dsJ(\rho)\phi(\rho), \tag{19}$$

where $J(\rho)=0$ on any horizontal boundaries and $J(\rho)=\pi_k/(2\pi|p_k^+|)$ on the vertical boundaries where strings enter or leave the diagram. Our convention is that the $\boldsymbol{\pi}_k,\boldsymbol{p}_k$ will always be taken as incoming momenta, and p_k^+,α_k are positive for incoming strings and negative for outgoing strings. With this convention the absolute value signs in $|p_k^+|$ can be removed by reversing the σ_k integral for outgoing external strings as in [14].

We can extract the γ_r, π_k, p_k dependence of the path integral by completing the square in the usual way: shift $\phi\to\phi+c$ and choose c to cancel the linear terms:

$$-\nabla^2c=2\pi i\sum_r\gamma_r\delta(\rho-\rho(x_r)),\qquad \partial_nc|_\partial=2\pi iJ, \tag{20}$$

$$\ln\frac{B(\phi+c)}{B(\phi)|_0}=+\frac{i}{2}\sum_s\gamma_sc(x_s)+\frac{i}{2}\sum_k\frac{\pi_k}{2\pi|p_k^+|}\int d\sigma_kc(\sigma_k,\tau_k). \tag{21}$$

The answer can be expressed in terms of the Neumann function

$$-\nabla^2N(\rho,\rho')=-2\pi\delta(\rho-\rho'),\qquad \partial_nN(\rho,\rho')|_{\rho\in\partial}=f(\rho). \tag{22}$$

Then, applying Green's theorem we have

$$c(\rho')=-i\sum_r\gamma_rN(\rho(x_r),\rho')-i\sum_k\frac{\pi_k}{2\pi|p_k^+|}\int d\sigma_kN(\rho,\rho')+\frac{1}{2\pi}\int d\sigma(cf)|_{\tau_i}^{\tau_f}. \tag{23}$$

The last term, independent of ρ', drops out of $\ln B/B_0$ by momentum conservation:

$$\ln\frac{B(\phi+c)}{B(\phi)|_0}=\frac{1}{2}\sum_{r,s}\gamma_r\gamma_sN(\rho(x_r),\rho(x_s))+\sum_{r,k}\gamma_r\frac{\pi_k}{2\pi|p_k^+|}\int d\sigma_kN(\rho(x_r),\rho_k)+\frac{1}{2}\sum_{kl}\int d\sigma_kd\sigma_l\frac{\pi_k\pi_l}{4\pi^2|p_k^+p_l^+|}N(\rho_k,\rho_l). \tag{24}$$

The Neumann function for the light-cone world sheet ρ can be related to that for the whole or half complex plane by the conformal mapping $\rho(z)$ [14] reviewed in Appendix A. For closed string amplitudes we map from the whole plane, for which the Neumann function is

$$N(z,z')=\ln|z-z'|. \tag{25}$$

In this case

$$\ln\frac{B(\phi+c)}{B(\phi)|_0}=\frac{1}{2}\sum_{r\neq s}\gamma_r\gamma_s\ln|x_r-x_s|+\sum_{r,k}\gamma_r\pi_k\ln|x_r-Z_k|$$
$$+\frac{1}{2}\sum_{k\neq l}\pi_k\pi_l\ln|Z_k-Z_l|+\frac{\gamma^2}{2}\sum_r\ln|x_r-x_r|$$
$$+\frac{1}{2}\sum_k\int d\sigma_k d\sigma'_k\frac{\pi_k^2}{4\pi^2p_k^{+2}}N(\rho_k,\rho'_k),\quad \text{Closed,} \tag{26}$$

where $\rho(x_r)$ are the break/join points of the light-cone diagram and $\rho(Z_k)$ are the locations of the incoming and outgoing strings.

In contrast, for open string amplitudes we need the Neumann function for the upper half-plane,

$$N(z,z')=\ln|z-z'|+\ln|z-z'^*|\to 2\ln|z-z'|, \tag{27}$$

when one or both z's are on the real axis. Then, we find

$$\ln\frac{B(\phi+c)}{B(\phi)|_0}=\sum_{r\neq s}\gamma_r\gamma_s\ln|x_r-x_s|+2\sum_{r,k}\gamma_r\pi_k\ln|x_r-Z_k|$$
$$+\sum_{k\neq l}\pi_k\pi_l\ln|Z_k-Z_l|+\gamma^2\sum_r\ln|x_r-x_r|$$
$$+\frac{1}{2}\sum_k\int d\sigma_k d\sigma'_k\frac{\pi_k^2}{4\pi^2p_k^{+2}}N(\rho_k,\rho'_k),\quad \text{Open.} \tag{28}$$

Note that Z_N, which we have set to ∞, appears on the right side of (26) or (28) in the combination

$$2\pi_N\left(\sum_r\gamma_r+\sum_{k=1}^{N-1}\pi_k\right)\ln Z_N=-2\pi_N^2\ln Z_N, \tag{29}$$

so the terms involving Z_N will enter, just as with the ordinary transverse dimensions, into the terms that implement the mass shell condition and wave function on the Nth leg [14].

B. Self-contractions

The self-contractions on the last line of (26) or (28) need further discussion. Those in the last term are of the same form for all $d+s/2$ bosonic fields and, combined, give the mass shell condition. Consulting the mapping function (A1) or (A23), for z near Z_k, we find

$$z-Z_k\sim e^\rho\prod_{l\neq k}(Z_k-Z_l)^{-\alpha_l/\alpha_k}, \tag{30}$$

$$\ln|z-z'|\sim\ln|e^\rho-e^{\rho'}|+\sum_{l\neq k}\ln|Z_k-Z_l|^{-\alpha_l/\alpha_k}. \tag{31}$$

The contribution of the first term to the ϕ insertion is part of the external wave function that is to be amputated, whereas the second term provides the factors

$$\prod_{k\neq l}|Z_k-Z_l|^{-\pi_k^2\alpha_l/\alpha_k}=\prod_{k<l}|Z_k-Z_l|^{-\pi_k^2\alpha_l/\alpha_k-\pi_l^2\alpha_k/\alpha_l} \tag{32}$$

for the open string, or

$$\prod_{k\neq l}|Z_k-Z_l|^{-\pi_k^2\alpha_l/(2\alpha_k)}=\prod_{k<l}|Z_k-Z_l|^{-\pi_k^2\alpha_l/(2\alpha_k)-\pi_l^2\alpha_k/(2\alpha_l)} \tag{33}$$

for the closed string. The exponents in these factors contribute the π_k-dependent parts of p_k^- in the scalar product,

$$P_k\cdot P_l=\boldsymbol{p}_k\cdot\boldsymbol{p}_l+\boldsymbol{\pi}_k\cdot\boldsymbol{\pi}_l-p_k^+p_l^--p_k^-p_l^+$$
$$=p_k\cdot p_l+\boldsymbol{\pi}_k\cdot\boldsymbol{\pi}_l, \tag{34}$$

which are the eventual powers in the Koba-Nielsen factors $\prod_{k<l}|Z_k-Z_l|^{2P_k\cdot P_l}$ or $\prod_{k<l}|Z_k-Z_l|^{P_k\cdot P_l}$ in the open and closed string integrands, respectively. The light-cone mass shell condition for an open or closed string with no oscillator excitations reads

$$p_k\cdot p_k=P_k\cdot P_k-\boldsymbol{\pi}_k\cdot\pi_k=\boldsymbol{p}_k^2-2p_k^+p_k^-$$
$$=1-\frac{s}{48}-\boldsymbol{\pi}_k^2\leq 1-\frac{s}{48}-\frac{s}{2}\gamma^2,$$
$$p_k\cdot p_k=4-\frac{s}{12}-\boldsymbol{\pi}_k^2\leq 4-\frac{s}{12}-\frac{s}{2}\gamma^2.$$

The left side of this equation is the Minkowski scalar product $p_{k\mu}p_k^\mu$ in $d+2=26-s$ space-time dimensions. The inequality follows because each component of $\boldsymbol{\pi}^k$ is an odd multiple of γ.

Next we give our interpretation of the self-contractions at the interaction points. Infinities in these contractions can be absorbed into the coupling constant, provided they are independent of the geometry of the world sheet. Since the light-cone world sheet is the fundamental starting point, we should set any regulator cutoffs in the ρ coordinate. The same mapping function is used for open and closed strings, except for a factor of 2. Let us examine $\rho(z)$ near $z=x_r$, the break/join point where $d\rho/dz=0$. For the open string case, we expand

$$\rho(z)=\rho(x_r)+\frac{1}{2}\frac{d^2\rho}{dz^2}\bigg|_{z=x_r}(z-x_r)^2+O((z-x_r)^3). \tag{35}$$

Then, use (A3) for the second derivative and solve (35),

$$z-x_r\approx\sqrt{2}\sqrt{\rho-\rho(x_r)}\left[\frac{\prod_k(x_r-z_k)}{-\alpha_N\prod_{s\neq r}(x_r-x_s)}\right]^{1/2}, \tag{36}$$

$$|z(\rho)-z(\rho')|$$
$$\approx|\sqrt{2(\rho-\rho(x_r))}-\sqrt{2(\rho'-\rho(x_r))}|\frac{\prod_k\sqrt{|x_r-z_k|}}{\sqrt{|\alpha_N|}\prod_{s\neq r}\sqrt{|x_r-x_s|}}. \tag{37}$$

We then interpret the self-contraction as

$$\gamma^2\sum_r \ln|x_r - x_r| \to \frac{\gamma^2}{2}\left[(N-2)\ln(2\delta) + \ln\frac{\prod_{r,k}|x_r - Z_k|}{|\alpha_N|^{N-2}\prod_{s\neq r}|x_r - x_s|}\right] \tag{38}$$

$$\to \frac{\gamma^2}{2}\left[(N-2)\ln(2\delta) + \ln\frac{|\alpha_N|}{\prod_{k=1}^{N-1}|\alpha_k|} + \ln\frac{\prod_{r,k}|x_r - Z_k|^2}{\prod_{s\neq r}|x_r - x_s|\prod_{k\neq l}|Z_k - Z_l|}\right], \tag{39}$$

where we have let δ be a measure of the cutoff regularization on the light-cone world sheet. We used the identity (B3) to arrive at the last line, which gives our interpretation of the self-contractions at the interaction points.

C. Amplitudes

Having taken care of the terms involving Z_N and the self-contractions at the external states, what remains of the contribution of the boundary data to the open string path integrand including all $s/2$ compactified bosonic dimensions, and $d = 24 - s$ ordinary uncompactified bosonic dimensions, is

$$I_O = (2\delta)^{s(N-2)\gamma^2/4}\left[\frac{\prod_{r\neq s}|x_r - x_s|^{\gamma_r\cdot\gamma_s - s\gamma^2/4}}{\prod_{r,k<N}|x_r - Z_k|^{-2\boldsymbol{\pi}_k\cdot\gamma_r - s\gamma^2/2}}\right]\left[\frac{|\alpha_N|}{\prod_{k=1}^{N-1}|\alpha_k|}\right]^{s\gamma^2/4} \times \prod_{k<l}|Z_k - Z_l|^{2\boldsymbol{P}_k\cdot\boldsymbol{P}_l - s\gamma^2/2 - \boldsymbol{P}_k^2\alpha_l/\alpha_k - \boldsymbol{P}_l^2\alpha_k/\alpha_l}. \tag{40}$$

For the closed string, the factor of 2 in the mapping function leads to some minor modifications:

$$I_C = (4\delta)^{s(N-2)\gamma^2/8}\frac{\prod_{r\neq s}|x_r - x_s|^{\gamma_r\cdot\gamma_s/2 - s\gamma^2/8}}{\prod_{r,k<N}|x_r - Z_k|^{-\boldsymbol{\pi}_k\cdot\gamma_r - s\gamma^2/4}}\left[\frac{|\alpha_N|}{\prod_{k=1}^{N-1}|\alpha_k|}\right]^{s\gamma^2/8} \times \prod_{k<l}|Z_k - Z_l|^{\boldsymbol{P}_k\cdot\boldsymbol{P}_l - s\gamma^2/4 - \boldsymbol{P}_k^2\alpha_l/(2\alpha_k) - \boldsymbol{P}_l^2\alpha_k/(2\alpha_l)}. \tag{41}$$

The boundary data for one of the $\boldsymbol{x}$ fields contribute the same but with no γ terms.

The Jacobian and determinant factors [see (B4) and (B5) in Appendix B] will have $n_b = d + s/2 = 24 - s/2$. Combining these with the I factors gives, for the open string,

$$\left[I_O\left|\frac{\partial T}{\partial Z}\right|\det{}^{-(24-s/2)/2}(-\nabla^2)\right]_{\text{open}} = \prod_{k=1}^{N}\frac{1}{\sqrt{|\alpha_k|}}\left[\frac{\prod_{k<N}|\alpha_k|}{|\alpha_N|}\right]^{s(1-8\gamma^2)/32}\left[\frac{\prod_{r<t}|x_t - x_r|\prod_{m<l}|Z_l - Z_m|}{\prod_{l,r}|Z_l - x_r|}\right]^{s/48} \times (2\delta)^{s(N-2)\gamma^2/4}\left[\frac{\prod_{r<s}|x_r - x_s|^{2\gamma_r\cdot\gamma_s - s\gamma^2/2}\prod_{k<l<N}|Z_k - Z_l|^{2P_k\cdot P_l - s\gamma^2/2}}{\prod_{r,k<N}|x_r - Z_k|^{-2\boldsymbol{\pi}_k\gamma_r - s\gamma^2/2}}\right]. \tag{42}$$

In string bit models, the factor within the first set of square brackets would scale as M^{N-2} where M is the bit number. So a finite continuum limit requires $\gamma^2 = 1/8$, in which case

$$\left[I_O\left|\frac{\partial T}{\partial Z}\right|\det{}^{-(24-s/2)/2}(-\nabla^2)\right]_{\text{open}} = (2\delta)^{s(N-2)/32}\prod_{k=1}^{N}\frac{1}{\sqrt{|\alpha_k|}}\left[\frac{\prod_{r<s}|x_r - x_s|^{2\gamma_r\cdot\gamma_s - s/24}\prod_{k<l<N}|Z_k - Z_l|^{2P_k\cdot P_l - s/24}}{\prod_{r,k<N}|x_r - Z_k|^{-2\boldsymbol{\pi}_k\gamma_r - s/24}}\right]. \tag{43}$$

Applying parallel considerations to the closed string leads to

$$\left[I_C\left|\frac{\partial T}{\partial Z}\right|^2\det{}^{-(24-s/2)/2}(-\nabla^2)\right]_{\text{closed}} = \prod_{k=1}^{N}\frac{1}{|\alpha_k|}\left[\frac{\prod_{k<N}|\alpha_k|}{|\alpha_N|}\right]^{s/16 - s\gamma^2/8}\left[\frac{\prod_{r<t}|x_t - x_r|\prod_{m<l}|Z_l - Z_m|}{\prod_{l,r}|Z_l - x_r|}\right]^{s/24} \times (4\delta)^{s(N-2)\gamma^2/8}\left[\frac{\prod_{r<s}|x_r - x_s|^{\gamma_r\cdot\gamma_s - s\gamma^2/4}\prod_{k<l<N}|Z_k - Z_l|^{P_k\cdot P_l - s\gamma^2/4}}{\prod_{r,k<N}|x_r - Z_k|^{-\boldsymbol{\pi}_k\cdot\gamma_r - s\gamma^2/4}}\right], \tag{44}$$

and a smooth continuum limit in the closed case requires $\gamma^2 = 1/2$:

$$\left[I_C\left|\frac{\partial T}{\partial Z}\right|^2 \det{}^{-(24-s/2)/2}(-\nabla^2)\right]_{\rm closed} = (4\delta)^{s(N-2)/16}\prod_{k=1}^N \frac{1}{|\alpha_k|}\left[\frac{\prod_{r<s}|x_r-x_s|^{\gamma_r\cdot\gamma_s-s/12}\prod_{k<l<N}|Z_k-Z_l|^{P_k\cdot P_l-s/12}}{\prod_{r,k<N}|x_r-Z_k|^{-\pi_k\cdot\gamma_r-s/12}}\right]. \quad (45)$$

Notice how the power of $|Z_k - Z_l|$ combined in the simplification:

$$2\boldsymbol{P}_k\cdot\boldsymbol{P}_l - s/16 + s/48 - (\boldsymbol{P}_k^2 - n_b/24)\alpha_l/\alpha_k - (\boldsymbol{P}_l^2 - n_b/24)\alpha_k/\alpha_l = 2P_k\cdot P_l - s/24,$$
$$\boldsymbol{P}_k\cdot\boldsymbol{P}_l - s/8 + s/24 - (\boldsymbol{P}_k^2 - n_b/6)\alpha_l/(2\alpha_k) - (\boldsymbol{P}_l^2 - n_b/6)\alpha_k/(2\alpha_l) = P_k\cdot P_l - s/12,$$

for open and closed strings, respectively. In these formulas each component of $\boldsymbol{\gamma}_r$ is $\pm\gamma = \pm 1/(2\sqrt{2})$ for the open string and $\pm\gamma = \pm 1/\sqrt{2}$ for the closed string. And of course each component of $\boldsymbol{\pi}_k$ is an odd integer multiple of γ.

The scattering amplitudes are obtained by integrating the expression (43) or (45) over the unfixed Z_k. In the case of the open string, the Z_k are on the real axis satisfying $Z_1 = 0 < Z_2 < \cdots < Z_{N-2} < Z_{N-1} = 1$. In the case of the closed string the Z_k for $k = 2, \ldots N-2$ are integrated over the whole complex plane. In both cases $Z_1 = 0, Z_{N-1} = 1, Z_N = \infty$. We remind the reader that for physical values of the momenta, the resulting integrals are plagued with divergences. To handle these divergences, one starts with (unphysical) values of the momenta for which the integrals converge, and then one analytically continues to the physical values. For open string amplitudes one can do this, keeping the range of the Z integrations complete. But for closed string amplitudes one is forced to divide the integration region up into cells, with separate analytic continuations in each cell.

IV. SCATTERING AMPLITUDES IN SPECIAL CASES

A. External strings of minimal mass

The scattering amplitudes obtained in the previous section described N external strings, all with zero winding number and no oscillator excitations. However, the compactified momenta $\boldsymbol{\pi}_k$ could have components with any odd multiple of $\pm\gamma$. To specialize to external strings of minimal mass, each component of each $\boldsymbol{\pi}_k$ should be $\pm\gamma$. In that case the mass squared of each string is $-p\cdot p = s/12 - 1$ for the open string, or $s/3 - 4$ for the closed string. There are, altogether, $N + N - 2 = 2(N-1)$ $\boldsymbol{\gamma}$'s and $\boldsymbol{\pi}$'s, so to satisfy the conservation law, $N-1$ should have a value $+\gamma$ and the remaining should have a value $-\gamma$. There are $\binom{2(N-1)}{(N-1)}$ ways to do this for each of the $s/2$ compactified bosonic fields.

A dramatic simplification occurs when there is maximal helicity violation. For instance, choose all components of the first $N-1$ $\boldsymbol{\pi}_k$ to have the value $-\gamma$. Then, necessarily all components of $\boldsymbol{\pi}_N$ and of each $\boldsymbol{\gamma}_r$ have the value $+\gamma$. In this case $\boldsymbol{\gamma}_r\cdot\boldsymbol{\gamma}_s = \boldsymbol{\pi}_k\cdot\boldsymbol{\pi}_l = -\boldsymbol{\gamma}_r\cdot\boldsymbol{\pi}_l = s\gamma^2/2$ for $k, l \neq N$ and $\boldsymbol{\pi}_N$ does not appear in the formula. Then, for the open case ($\gamma^2 = 1/8$) the contribution to the integrand of the scattering amplitude is

$$I_O\left|\frac{\partial T}{\partial Z}\right|\det{}^{-(24-s/2)/2}(-\nabla^2)_{\rm open} \to \epsilon^{s(N-2)/32}\prod_{k=1}^N\frac{1}{\sqrt{|\alpha_k|}}\left[\frac{\prod_{r<s}|x_r-x_s|^2\prod_{k<l<N}|Z_k-Z_l|^2}{\prod_{r,k<N}|x_r-Z_k|^2}\right]^{s/24}|Z_k-Z_l|^{p_k\cdot p_l}, \quad (46)$$

$$\to \epsilon^{s(N-2)/32}\prod_{k=1}^N\frac{1}{\sqrt{|\alpha_k|}}\frac{|\alpha_N|^{s(N-1)/12}}{\prod_{k<N}|\alpha_k|^{s/12}}\left[\frac{\prod_{r<s}|x_r-x_s|^{s/12}}{\prod_{k<l<N}|Z_k-Z_l|^{s/12}}\right]|Z_k-Z_l|^{p_k\cdot p_l}, \quad (47)$$

where we made use of (B3) to arrive at the last line. With this simple choice the scattering amplitude is then

$$A_N^{\rm Open} = g^{N-2}\prod_{k=1}^N\frac{1}{\sqrt{|\alpha_k|}}\frac{|\alpha_N|^{s(N-1)/12}}{\prod_{k<N}|\alpha_k|^{s/12}}\int dZ_2\cdots dZ_{N-2}\prod_{r<s}|x_r-x_s|^{s/12}\prod_{k<l<N}|Z_k-Z_l|^{2p_k\cdot p_l - s/12}. \quad (48)$$

Making the same simplifications for the case of the closed string leads to

$$A_N^{\rm Closed} = g^{2(N-2)} \prod_{k=1}^{N} \frac{1}{|\alpha_k|} \frac{|\alpha_N|^{s(N-1)/6}}{\prod_{k<N}|\alpha_k|^{s/6}} \int d^2Z_2 \cdots d^2Z_{N-2} \times \prod_{r<s} |x_r - x_s|^{s/6} \prod_{k<l<N} |Z_k - Z_l|^{p_k\cdot p_l - s/6}. \tag{49}$$

If desired, one can replace $2p_k \cdot p_l$ by $(p_k + p_l)^2 - 2 + s/6$ in the open case and by $(p_k + p_l)^2 - 8 + 2s/3$ in the closed case. Keep in mind that this simpler expression only applies for a very special choice for the $\boldsymbol{\pi}$'s and $\boldsymbol{\gamma}$'s. In particular, even for a particular set of the $\boldsymbol{\pi}_k$, the full insertion factor at each vertex is $2\cos(\gamma\phi)$, which is to be implemented by summing the amplitudes over each component of each $\boldsymbol{\gamma}_r$ assuming both possible values $\pm\gamma$.

B. Three-string amplitudes

In the case $N = 3$ the three Z's are fixed at $0, 1, \infty$. The relevant conformal map is

$$\rho = \alpha_1 \ln z + \alpha_2 \ln(z-1) \tag{50}$$

which determines $x = \alpha_2/(\alpha_1 + \alpha_2)$ and $1 - x = \alpha_1/(\alpha_1 + \alpha_2)$. Then, the open string amplitude (43) for this case reduces to

$$\frac{1}{\sqrt{|\alpha_1\alpha_2\alpha_{12}|}} \left|\frac{\alpha_2}{\alpha_{12}}\right|^{2\boldsymbol{\pi}_1\cdot\boldsymbol{\gamma}+s/24} \left|\frac{\alpha_1}{\alpha_{12}}\right|^{2\boldsymbol{\pi}_2\cdot\boldsymbol{\gamma}+s/24}. \tag{51}$$

The corresponding three closed string vertex is

$$\frac{1}{|\alpha_1\alpha_2\alpha_{12}|} \left|\frac{\alpha_2}{\alpha_{12}}\right|^{\boldsymbol{\pi}_1\cdot\boldsymbol{\gamma}+s/12} \left|\frac{\alpha_1}{\alpha_{12}}\right|^{\boldsymbol{\pi}_2\cdot\boldsymbol{\gamma}+s/12}. \tag{52}$$

These formulas are valid only if all three strings are on shell. Putting each $p_k^- = (\boldsymbol{p}_k^2 + m_k^2)/\alpha_k$, the on-shell condition is $p_1^- + p_2^- + p_3^- = 0$, which leads to a quadratic equation determining the variable $x = \alpha_2/\alpha_{12}$. The condition that x is real and $0 < x < 1$, which must hold if α_1 and α_2 have the same sign, requires that the mass of particle 3 is greater than the sum of masses 1 and 2 or $m_3^2 \geq (m_1 + m_2)^2$. This is just the requirement that the decay of particle 3 into particles 1 and 2 is energetically allowed.

The squared mass of a string in the unexcited states, which we are considering, is given by $\boldsymbol{\pi}^2 + s/48 - 1$ for the open string and by $\boldsymbol{\pi}^2 + s/12 - 4$ for the closed string. For example, the decay into equal masses requires, in the case of open strings,

$$(\boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 + \boldsymbol{\gamma})^2 + \frac{s}{48} - 1 \geq 4\boldsymbol{\pi}_1^2 + \frac{s}{12} - 4, \tag{53}$$

which is easily satisfied if $\boldsymbol{\gamma}, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ are sufficiently aligned.

C. Four-string amplitudes

First, we specialize the conformal mapping from z to ρ to four external strings,

$$\rho = \alpha_1 \ln z + \alpha_2 \ln(z - Z) + \alpha_3 \ln(z-1). \tag{54}$$

Then, the two interaction points are determined by $d\rho/dz = 0$, which implies the quadratic equation

$$0 = (\alpha_1 + \alpha_2 + \alpha_3)x^2 + (-\alpha_1(Z+1) - \alpha_2 - \alpha_3 Z)x + \alpha_1 Z$$

with solutions

$$x_\pm = \frac{\alpha_1(Z+1) + \alpha_2 + \alpha_3 Z \pm \sqrt{(\alpha_1(Z+1) + \alpha_2 + \alpha_3 Z)^2 + 4\alpha_1\alpha_4 Z}}{2(\alpha_1 + \alpha_2 + \alpha_3)} \tag{55}$$

which lead to

$$\alpha_4^2|x_+ - x_-|^2 = \alpha_{12}^2(1-Z) + \alpha_{23}^2 Z - \alpha_{13}^2 Z(1-Z) \tag{56}$$

where $\alpha_{kl} \equiv \alpha_k + \alpha_l$.

Since the $x_\pm$ enter the integrand of the scattering amplitude, their behaviors as $Z \to 0, 1$, which control the pole locations in the variables $(p_1 + p_2)^2$ and $(p_2 + p_3)^2$, is relevant. We find

$$x_\pm \sim \begin{cases} \begin{pmatrix} -(\alpha_1 + \alpha_2)/\alpha_4 \\ Z\alpha_1/\alpha_{12} \end{pmatrix} & \text{for } Z \sim 0 \\ \begin{pmatrix} -\alpha_1/\alpha_4 \\ 1 - (Z-1)\alpha_3/\alpha_{14} \end{pmatrix} & \text{for } Z \sim 1. \end{cases} \tag{57}$$

As long as α_{12} and α_{23} are both nonzero, we only need to pay attention to the factors x_- and $Z - x_-$ when analyzing $Z \sim 0$, and to the factors $1 - x_-$ and $Z - x_-$ when $Z \sim 1$.

Putting $N = 4$ in (48) we find the four open string amplitude in the special case $\gamma_r = \gamma = 1/\sqrt{8}$, $\pi_k = -\gamma = -1/\sqrt{8}$ for $k < 4$:

$$A_4^{\rm open} = g^2 \prod_{k=1}^{4} \frac{1}{\sqrt{|\alpha_k|}} \int_0^1 dZ \frac{|\alpha_4|^{s/4}|x_2 - x_1|^{s/12}}{|\alpha_1\alpha_2\alpha_3|^{s/12}} \times Z^{(p_1+p_2)^2 - 2 + s/12}(1-Z)^{(p_2+p_3)^2 - 2 + s/12}. \tag{58}$$

It is not hard to check that the pole singularities in $(p_1 + p_2)^2$ and $(p_2 + p_3)^2$ are where they should be as long as α_{12} and α_{23} are nonzero. This is reasonable since excluding these values of the α's guarantees that the dynamical singularities are all due to the long-time propagation of protostring mass eigenstates. If, for example, $\alpha_{23} = 0$, $\alpha_4^2(x_2 - x_1)^2 \sim 4\alpha_1\alpha_2(1-Z)$ as $Z \to 1$, so the poles in $(p_2 + p_3)^2$ are shifted by an amount $s/24$. When $\alpha_{23} = 0$ these singularities are due to the collision of the interaction points on the world sheet and not the long-time propagation of a particle state. This nonuniformity of singularity structure is absent for the bosonic and superstring because the amplitude integrands then turn out to be independent of the x_r.

The four closed string amplitude (49) in the case $N = 4$ and $\gamma_r = \gamma = 1/\sqrt{2}$, $\pi_k = -\gamma = -1/\sqrt{2}$ for $k < 4$ is

$$A_4^{\rm Closed} = g^4 \prod_{k=1}^{4} \frac{1}{|\alpha_k|} \frac{|\alpha_4|^{s/2}}{|\alpha_1\alpha_2\alpha_3|^{s/6}} \times \int d^2Z |x_2 - x_1|^{s/6} |Z|^{(p_1+p_2)^2/2-4+s/6} \times |1-Z|^{(p_2+p_3)^2/2-4+s/6}. \quad (59)$$

D. Four-protostring amplitude

In the case of the protostring ($s = 24$), space is only one dimensional. This puts severe limits on the kinematics: scattering can be only forward ($\alpha_{23} = 0$) or backward ($\alpha_{13} = 0$). To evaluate protostring open string amplitudes, we keep $s < 24$ as a regulator but restrict the kinematics to two space-time dimensions. Thus, we set all the $\boldsymbol{p}_k = 0$, but keep the general s mass values $m_k^2 = (s-12)/12$. Then the Mandelstam invariants are

$$S = -(p_1+p_2)^2 = \frac{\alpha_{12}^2}{\alpha_1\alpha_2}\frac{s-12}{12}, \quad (60)$$

$$t = -(p_2+p_3)^2 = \begin{cases} 0 & \alpha_{23} = 0 \\ \frac{s-12}{3} - S & \alpha_{13} = 0, \end{cases} \quad (61)$$

$$u = -(p_1+p_3)^2 = \begin{cases} 0 & \alpha_{13} = 0 \\ \frac{s-12}{3} - S & \alpha_{23} = 0, \end{cases} \quad (62)$$

and we have

$$\alpha_4^2 |x_+ - x_-|^2 = \begin{cases} \alpha_{12}^2(1-Z)^2 + 4\alpha_1\alpha_2 Z(1-Z) & \alpha_{23} = 0 \\ \alpha_{12}^2 - 4\alpha_1\alpha_2 Z & \alpha_{13} = 0. \end{cases} \quad (63)$$

The maximal helicity-violating four-point function of the preceding section becomes, for forward scattering and by setting $s = 24$,

$$\begin{aligned} A_4^{\alpha_{23}=0} &= g^2 \prod_{k=1}^{4} \frac{1}{\sqrt{|\alpha_k|}} \frac{|\alpha_1|^2}{|\alpha_2|^4} \int dZ [\alpha_{12}^2(1-Z) + 4\alpha_1\alpha_2 Z] Z^{-S}(1-Z)^1 \\ &= g^2 \prod_{k=1}^{4} \frac{1}{\sqrt{|\alpha_k|}} \frac{|\alpha_1|^2}{|\alpha_2|^4} \left[\alpha_{12}^2 \frac{\Gamma(-S+1)\Gamma(3)}{\Gamma(-S+4)} + 4\alpha_1\alpha_2 \frac{\Gamma(-S+2)\Gamma(2)}{\Gamma(-S+4)}\right] \\ &= \frac{g^2}{|\alpha_1\alpha_2|} \frac{|\alpha_1|^2}{|\alpha_2|^4} \left[\frac{2\alpha_{12}^2}{(1-S)(2-S)(3-S)} + \frac{4\alpha_1\alpha_2}{(2-S)(3-S)}\right]. \end{aligned} \quad (64)$$

In doing these integrals, we begin with $S < 0$ so that the integrals converge. One can then continue S to $S > 4$ to reach physical scattering.

For backward scattering, we temporarily keep $s \neq 24$ in the powers of Z and $1 - Z$.

$$\begin{aligned} A_4^{\alpha_{13}=0} &= g^2 \prod_{k=1}^{4} \frac{1}{\sqrt{|\alpha_k|}} \frac{\alpha_2^2}{\alpha_1^4} \int dZ [\alpha_{12}^2 - 4\alpha_1\alpha_2 Z] Z^{-S-2+s/12}(1-Z)^{S-s/4+2} \\ &= \frac{g^2\alpha_2^2}{|\alpha_1\alpha_2|} \left[\frac{\alpha_{12}^2}{\alpha_1^4} \frac{\Gamma(-S-1+s/12)}{\Gamma(2-s/6)} - 4\frac{\alpha_2}{\alpha_1^3}\frac{\Gamma(-S+s/12)}{\Gamma(3-s/6)}\right] \Gamma(S-s/4+3). \end{aligned} \quad (65)$$

In this case there is no value of S for which there is convergence at both $Z = 0$ and $Z = 1$, so we refrained from setting $s = 24$ in the powers as a regulator. When we attempt to set $s = 24$, the denominators of both terms blow up, suggesting that there is no backward scattering in this process. Of course, it is a very special process, corresponding to maximal helicity violation.

Although the kinematics of these special four-string processes do not correspond to fully elastic forward scattering, since the internal state of string 3 is not identical to that of string 2, it does reflect the high energy behavior of forward scattering discussed in the next section for truly elastic scattering process.

V. HIGH ENERGY FOUR-STRING SCATTERING

There is a general argument, based on the light-cone world-sheet description [18], that the forward elastic open string scattering probability amplitude[2] goes to a constant at high energies. In this argument high energy scattering is reached by taking $p_1^+ \to \infty$ at fixed p_2^+ and fixed $p_2^+ + p_3^+ = -p_1^+ - p_4^+$. It is also assumed that the internal state of string 1 is identical to the internal state of string 4,

[2]In Lorentz covariant theories the probability amplitude is obtained by dividing the covariant Feynman amplitude by $\prod_k \sqrt{2p_k^+}$.

so that the “large” string is elastic. Then, $(p_1+p_2)^- \sim m^2/p_2^+$ stays finite. When $p_1^+ \to \infty$ the light-cone world sheet becomes very large, while the effect of the interaction is limited to a region of size of order 1. Since the speed of sound is finite, the amplitude must approach a constant as $p_1^+ \to \infty$. In a Lorentz invariant theory, the amplitude is $\mathcal{M}/\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}$, where $\mathcal{M}$ is the invariant Feynman amplitude. Since $S = -(p_1+p_2)^2 \sim m^2 p_1^+/p_2^+$, it follows that $\mathcal{M} \propto S$ at large S, corresponding to a leading Regge trajectory of intercept 1.

But this result is a consequence of the light-cone parametrization of the world sheet, whether or not Lorentz invariance is met. It is instructive to check this for general s. For the Lorentz invariant bosonic open string scattering amplitude ($s=0$), the limit $-S = (p_1+p_2)^2 \to \infty$ is evaluated by changing variables $Z = e^{-u}$,

$$\mathcal{M} \propto \int_0^\infty du e^{-u(-S-1)}(1-e^{-u})^{-t-2} \sim (-S-1)^{t+1}\Gamma(-t-1),\quad S \to -\infty, \tag{66}$$

where $-t = (p_2+p_3)^2$ is the momentum transfer ($=0$ in the forward direction).

When the Grassmann dimension $s > 0$, the analysis is complicated by the dependence of the integrand on $x_\pm$. The high energy behavior is still controlled by $Z \sim 1$, so consulting (57) we find, holding $\alpha_{23} = -\alpha_{14} \neq 0$,

$$x_+ \sim -\frac{\alpha_1}{\alpha_4}, \quad x_- \sim 1, \quad x_+ - x_- \sim -\frac{1}{\alpha_1}\left[\frac{\alpha_1\alpha_{14}}{\alpha_4}\right], \tag{67}$$

$$1 - x_+ \sim \frac{1}{\alpha_1}\left[\frac{\alpha_1\alpha_{14}}{\alpha_4}\right],$$
$$1 - x_- \sim -(1-Z)\frac{\alpha_3}{\alpha_{14}} \sim -\frac{1}{\alpha_1}\left[u\frac{\alpha_1\alpha_3}{\alpha_{14}}\right], \tag{68}$$

$$Z - x_+ \sim \frac{1}{\alpha_1}\left[\frac{\alpha_1\alpha_{14}}{\alpha_4}\right],$$
$$Z - x_- \sim (1-Z)\frac{\alpha_2}{\alpha_{14}} \sim \frac{1}{\alpha_1}\left[u\frac{\alpha_1\alpha_2}{\alpha_{14}}\right]. \tag{69}$$

The limit $\alpha_{23} \to 0$ is not uniform. Indeed, setting $\alpha_3 = -\alpha_2$ from the beginning, we find

$$x_\pm = 1 + \frac{1}{2\alpha_1}\left[(\alpha_2-\alpha_1)(1-Z) \pm \sqrt{4\alpha_1\alpha_2(1-Z) + (\alpha_1-\alpha_2)^2(1-Z)^2}\right], \tag{70}$$

$$\sim 1 + \frac{1}{2\alpha_1}\left[-\alpha_1 u \pm \sqrt{4\alpha_1\alpha_2 u + \alpha_1^2 u^2}\right], \tag{71}$$

from which we conclude

$$1 - x_\pm \sim \frac{1}{2\alpha_1}\left[\alpha_1 u \mp \sqrt{4\alpha_1\alpha_2 u + \alpha_1^2 u^2}\right], \tag{72}$$

$$Z - x_\pm \sim \frac{1}{2\alpha_1}\left[-\alpha_1 u \mp \sqrt{4\alpha_1\alpha_2 u + \alpha_1^2 u^2}\right], \tag{73}$$

$$x_+ - x_- \sim \frac{1}{\alpha_1}\left[\sqrt{4\alpha_1\alpha_2 u + \alpha_1^2 u^2}\right]. \tag{74}$$

Since the important region of integration at high energy is $1 - Z \approx u = O(S^{-1})$ and we are keeping α_1/S fixed, the factors in square brackets are of order 1, for both cases, in the dominant integration region. We see that in this region $x_\pm \sim O(1)$ and $1-Z$, $1-x_\pm$, $Z-x_\pm$, and $x_+ - x_-$ all scale as α_1^{-1}. Now referring back to (43) for the case $N=4$, we read off the total power of α_1^{-1}:

$$\begin{aligned}
&2\boldsymbol{\gamma}_1\cdot\boldsymbol{\gamma}_2 - s/24 + 2P_2\cdot P_3 - s/24 + 2(\boldsymbol{\pi}_2+\boldsymbol{\pi}_3)\cdot(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2) + s/6\\
&\quad = (\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)^2 - \frac{s}{8} + (P_2+P_3)^2 - 2\left(1-\frac{s}{48}\right) + 2(\boldsymbol{\pi}_2+\boldsymbol{\pi}_3)\cdot(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2) + \frac{s}{12}\\
&\quad = (\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)^2 + (P_2+P_3)^2 - 2 + 2(\boldsymbol{\pi}_2+\boldsymbol{\pi}_3)\cdot(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)\\
&\quad = (p_2+p_3)^2 - 2 + (\boldsymbol{\pi}_1+\boldsymbol{\pi}_4)^2.
\end{aligned} \tag{75}$$

We conclude that the high energy behavior of the scattering amplitude is

$$A_4 \sim \alpha_1^{1+t-(\boldsymbol{\pi}_1+\boldsymbol{\pi}_4)^2}, \quad t \equiv -(p_2+p_3)^2, \tag{76}$$

in accordance with the general argument for $t=0$ provided that $\boldsymbol{\pi}_4 = -\boldsymbol{\pi}_1$, i.e., provided that the internal states of the “long” strings are identical.

We close with a simple example of a fully elastic scattering amplitude. We choose $\boldsymbol{\pi}_4 = -\boldsymbol{\pi}_1$ *and* $\boldsymbol{\pi}_3 = -\boldsymbol{\pi}_2$. Necessarily then, we must have $\boldsymbol{\gamma}_2 = -\boldsymbol{\gamma}_1$. Let $\boldsymbol{\gamma}$ be the $s/2$ vector with each component equal to $\gamma = 1/\sqrt{8}$ for the open string. Then, a simple example of elastic scattering would be

$$\boldsymbol{\pi}_1 = \boldsymbol{\pi}_2 = \boldsymbol{\gamma}, \quad \boldsymbol{\pi}_3 = \boldsymbol{\pi}_4 = -\boldsymbol{\gamma}, \quad \boldsymbol{\gamma}_1 = -\boldsymbol{\gamma}_2 = \pm\boldsymbol{\gamma}. \tag{77}$$

The integrand of the scattering amplitude formula becomes

$$
\begin{aligned}
&|Z|^{-S-2+s/4}|1-Z|^{-t-2}|x_1-x_2|^{-s/6} \\
&\times \begin{cases} |x_1|^{s/6}|x_2|^{-s/12}|1-x_1|^{-s/12}|1-x_2|^{s/6}|Z-x_1|^{s/6}|Z-x_2|^{-s/12} & \text{for } \boldsymbol{\gamma}_1=\boldsymbol{\gamma} \\ |x_1|^{-s/12}|x_2|^{s/6}|1-x_1|^{s/6}|1-x_2|^{-s/12}|Z-x_1|^{-s/12}|Z-x_2|^{s/6} & \text{for } \boldsymbol{\gamma}_1=-\boldsymbol{\gamma}. \end{cases}
\end{aligned} \tag{78}
$$

The second case just interchanges $x_1 \leftrightarrow x_2$.

We first evaluate the high energy limit with $\alpha_{23} \neq 0$, changing integration variables to $Z = e^{-v/\alpha_1}$ and treating $v = O(1)$ to get

$$
\begin{aligned}
\mathcal{M} &\sim \alpha_1^{1+t}\int_0^\infty dv e^{-(-S-1+s/4)v/\alpha_1} v^{-t-2+s/12}|\alpha_{14}|^{-s/6} \\
&\quad \times \begin{cases} |\frac{\alpha_1}{\alpha_4}|^{s/12}|\alpha_3|^{s/6}|\alpha_2|^{-s/12} & \text{for } \boldsymbol{\gamma}_1=\boldsymbol{\gamma} \\ |\frac{\alpha_1}{\alpha_4}|^{-s/6}|\alpha_3|^{-s/12}|\alpha_2|^{s/6} & \text{for } \boldsymbol{\gamma}_1=-\boldsymbol{\gamma} \end{cases} \\
&\sim \alpha_1^{s/12}(-S)^{t+1-s/12}\Gamma(-t-1+s/12)|\alpha_{14}|^{-s/6} \\
&\quad \times \begin{cases} |\frac{\alpha_1}{\alpha_4}|^{s/12}|\alpha_3|^{s/6}|\alpha_2|^{-s/12} & \text{for } \boldsymbol{\gamma}_1=\boldsymbol{\gamma} \\ |\frac{\alpha_1}{\alpha_4}|^{-s/6}|\alpha_3|^{-s/12}|\alpha_2|^{s/6} & \text{for } \boldsymbol{\gamma}_1=-\boldsymbol{\gamma}. \end{cases}
\end{aligned} \tag{79}
$$

Here we see that, with $\alpha_{23} \neq 0$ and fixed, the coefficient of α_1^{t+1} has poles at $t = n + s/12 - 1$ which are the mass-squared eigenvalues of the open protostring. The linear high energy behavior at $t = 0$ is the product of $(-S)^{1-s/12}$ and $\alpha_1^{s/12}$ netting precisely linear growth in the forward direction.

Contrast this with the high energy limit taken with $\alpha_{23} = 0$ from the beginning:

$$
\begin{aligned}
\mathcal{M} &\sim \alpha_1^{1+t}\int_0^\infty dv e^{-(-S-1+s/4)v/\alpha_1} v^{-t-2}[4\alpha_2 v + v^2]^{-s/12} \\
&\quad \times \left|\frac{1}{2}\left[v-\sqrt{4\alpha_2 v+v^2}\right]\right|^{-s/12}\left|\frac{1}{2}\left[v+\sqrt{4\alpha_2 v+v^2}\right]\right|^{s/6} \\
&\quad \times \left|\frac{1}{2}\left[-v-\sqrt{4\alpha_2 v+v^2}\right]\right|^{s/6}\left|\frac{1}{2}\left[-v+\sqrt{4\alpha_2 v+v^2}\right]\right|^{-s/12} \\
&\sim \alpha_1^{1+t}\int_0^\infty dv e^{-(-S-1+s/4)v/\alpha_1} v^{-t-2}[4+v/\alpha_2]^{-s/12}\left|\sqrt{1+\frac{v}{4\alpha_2}}+\sqrt{\frac{v}{4\alpha_2}}\right|^{s/2}.
\end{aligned} \tag{80}
$$

We stress that the limit taken here is $\alpha_1, -S \to \infty$ at fixed ratio. The coefficient of the Regge behavior is a function of t. Its pole locations are *not* those of the particles of the theory: they correspond to a linear Regge trajectory of intercept 1. Because the formula was obtained assuming $\alpha_3 = -\alpha_2$, the high energy behavior comes from the collision of two interaction points on the light-cone world sheet, and not from the long-time propagation of a protostring mass eigenstate as in the first limiting procedure. The mismatch can be allowed because the Lorentz boost symmetry generated by M^{-k} is absent: for the protostring, because there is no transverse space, and for $0 < s < 24$, because this part of the Lorentz symmetry is broken. For the bosonic string ($s = 0$), of course, there is no such mismatch.

VI. CONCLUDING REMARKS

This article is devoted to the calculation of scattering amplitudes for the protostring and a simple generalization thereof. The three-string amplitude with the strings in arbitrary excited states was calculated in [13], where the model was initially proposed. Here the focus has been on general N string amplitudes, but with the strings in states with no oscillator excitations. These amplitudes are analogous to the N tachyon amplitudes of the bosonic string.

The scattering amplitudes are presented as integrals over the Koba-Nielsen variables Z_k. The integrand includes factors $|Z_k - Z_l|$ raised to momentum-dependent powers, familiar from the bosonic string. But in addition, there are factors $|x_r - Z_k|$ and $|x_r - x_s|$ raised to powers dependent on the compactified momentum representing the Grassmann degrees of freedom. Here the $x_r(Z)$ are the locations in the z plane of the break/join points of the light-cone world sheet. If $\rho(z)$ is the conformal map from the z plane to the light-cone world sheet, the x_r are determined by the order $N-2$ polynomial equation $d\rho/dz = 0$. The presence of these other factors complicates the singularity structure of the integrands.

We studied in detail some simple special cases. We found significant simplifications for the maximal helicity (compactified momentum) violating N string amplitudes. But factors involving $|x_r - x_s|$ remain, which become quite unwieldy for $N > 4$. On the other hand, four-string amplitudes are manageable because the x's solve a quadratic equation, $(x_1 - x_2)^2$ being the discriminant. For the protostring case $s = 24$ this contribution is just a quadratic polynomial in Z. So the four-point functions are sums of Euler beta functions. For forward open protostring scattering ($t = 0$), the S, t amplitude[3] is just a sum of a finite number of poles at $S = 1, 2, 3$. Curiously, the S, t amplitude for backward scattering ($t = 4 - S$) seems to vanish[4] for the protostring. This vanishing of backward scattering may be specific to helicity-violating amplitudes. It would be interesting to explore whether other protostring amplitudes vanish.

We analyzed the high energy limit of selected four open string amplitudes, in both helicity-conserving and helicity-violating processes. We confirmed that, in the forward direction, the elastic amplitude is constant at high energy, in line with the general argument in [18]. In a Lorentz covariant string theory in three or more dimensions, such behavior would signal a leading Regge trajectory of unit intercept, which would imply massless gauge particles in the theory. The models studied here are noncovariant for $0 < s < 24$, depending not only on the Lorentz invariants $p_k \cdot p_l$ but also on the $+$ momentum components p_k^+. Instead of the covariant Regge behavior $(p_1 \cdot p_2)^{t+1}/\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}$, we get $(p_1^+)^{s/12}(p_1 \cdot p_2)^{t+1-s/12}/\sqrt{p_1^+ p_2^+ p_3^+ p_4^+}$. Here the Regge trajectory $\alpha(t) = t + 1 - s/12$ reflects the spectrum of the string models of the present paper: for the protostring ($s = 24$) it is $t - 1$, implying a mass gap. The constant high energy behavior when $t = 0$ generally applies [18] when $(p_1 \cdot p_2)/p_1^+$ is fixed as $p_1 \cdot p_2 \to \infty$. This is how constant high energy behavior is consistent with a Regge trajectory intercept less than unity.

There is a lot of work still to be done on these models. The amplitudes involving more than four strings have been obtained, but their physical properties remain to be investigated. One should also be able to calculate these amplitudes in the original unbosonized language and compare results to those of the present paper. This comparison should clarify issues of Fermi statistics, which the process of bosonization has obscured.

The protostring ($s = 24$) moves only in one space dimension. Thus, the entire Lorentz group is $O(1,1)$, and this symmetry is maintained in the construction. As we have said, the protostring is predicted by the simplest of string bit models. In the string limit, the Grassmann dimension s effectively interpolates between the bosonic string $s = 0$ and the protostring ($s = 24$). We have noted that the protostring has the world-sheet field content of the superstring. This suggests that besides being an interesting system in its own right, it may also be a stepping stone to a solid foundation of superstring theory. In its present form, the eight "transverse coordinates" one obtains by bosonizing 16 of the protostring's Grassmann dimensions are all compactified in a way such that the KK momentum is quantized in half-odd integers. Perhaps there is a way to tweak the Hamiltonian of the string bit model underlying the protostring to shift this quantization to integers. An additional tweak would be needed to provide a large compactification radius for at least two of these emergent transverse coordinates. We leave the pursuit of these goals to future research.

ACKNOWLEDGMENTS

I would like to thank Gaoli Chen and Songge Sun for helpful comments on the manuscript. This research was supported in part by the Department of Energy under Grant No. DE-SC0010296.

APPENDIX A: DETERMINANT FOR THE LIGHT-CONE WORLD SHEET

1. Open string

In this appendix we review Mandelstam's calculation of the determinant and Jacobian factors for the bosonic string [16]. The quantities Z_k, with $k = 1 \cdots (N-1)$ and x_r with $r = 1 \cdots (N-2)$ are determined by the map from the upper-half Koba-Nielsen plane (z) to the light-cone world sheet ($\rho = \tau + i\sigma$):

$$\rho = \sum_{k=1}^{N-1} \alpha_k \ln(z - Z_k), \qquad \left.\frac{d\rho}{dz}\right|_{z=x_r} = 0, \tag{A1}$$

$$\frac{d\rho}{dz} = \sum_{k=1}^{N-1} \frac{\alpha_k}{z - Z_k} = \frac{\sum_{k=1}^{N-1} \alpha_k \prod_{l \neq k}(z - Z_l)}{\prod_k (z - Z_k)} = -\alpha_N \frac{\prod_r (z - x_r)}{\prod_k (z - Z_k)}, \tag{A2}$$

[3] With $S = -(p_1 + p_2)^2$ and $t = -(p_2 + p_3)^2$, the S, t amplitude is the one with cyclic ordering 12341. It has poles in the S and t channels.

[4] The argument for vanishing backward scattering relies on an analytic continuation which may be suspect.

$$\left.\frac{d^2\rho}{dz^2}\right|_{z=x_s} = -\alpha_N \frac{\prod_{r\neq s}(x_s - x_r)}{\prod_k (x_s - Z_k)}, \tag{A3}$$

where the last line is true because the factor $(z - x_s)$ in the numerator must be destroyed by the derivative to get a nonzero contribution. The asymptotic strings at $\tau = \pm\infty$ are mapped from the Z_k. In this notation $Z_N = \infty, Z_1 = 0$.

We next determine the world-sheet determinant. We do this by executing a conformal transformation from the Koba-Nielsen plane to the light-cone world sheet. We need

$$\Sigma \equiv \ln\left|\frac{d\rho}{dz}\right| = \ln|\alpha_N| - \sum_{k=1}^{N-1} \ln|z - Z_k| + \sum_{r=1}^{N-2} \ln|z - x_r|. \tag{A4}$$

Clearly, $\partial_y \Sigma = 0$ on the real axis. Since the points $z = Z_k, x_s$ are singular, we deform the boundary near those points into small semicircles, in the upper half-plane, of radii ϵ_k, ϵ_r, respectively. The radius ϵ_k near Z_k can be interpreted in terms of a large time T_k for the asymptotic string k. From the mapping function we find

$$\epsilon_k = e^{T_k/\alpha_k} \prod_{l\neq k} |Z_l - Z_k|^{-\alpha_l/\alpha_k}. \tag{A5}$$

The string N is asymptotic at large z. If R is the radius of a large semicircle, we have from the mapping function

$$T_N \sim -\alpha_N \ln R, \qquad R \sim e^{-T_N/\alpha_N}. \tag{A6}$$

On the other hand, the radius ϵ_s near x_s is a temporary regulator, which maps onto a circular deformation of the boundary near the corresponding interaction point on the light-cone world sheet. From the mapping function we see that the radius of this regulating circle on the world sheet is given by

$$\delta_s = \frac{1}{2}\epsilon_s^2 \left|\frac{d^2\rho}{dz^2}\right|_{z=x_s} = \frac{1}{2}\epsilon_s^2 |\alpha_N| \frac{\prod_{r\neq s}|x_s - x_r|}{\prod_k |x_s - Z_k|}, \tag{A7}$$

$$\epsilon_s = \sqrt{\frac{2\delta_s}{|\alpha_N|}} \frac{\prod_k |x_s - Z_k|^{1/2}}{\prod_{r\neq s}|x_s - x_r|^{1/2}}, \tag{A8}$$

$$\prod_s \epsilon_s = |\alpha_N|^{-N+3/2} \prod_k |\alpha_k|^{1/2} \prod_s \sqrt{2\delta_s} \frac{\prod_{l\neq k}|Z_l - Z_k|^{1/2}}{\prod_{r\neq s}|x_s - x_r|^{1/2}}. \tag{A9}$$

To calculate the determinant for the light-cone world sheet, we start with the determinant for the region in the upper-half z plane bounded by the real axis, the large radius R semicircle, and the small radius ϵ_k semicircles. Then, we apply the generalized Kac-McKean-Singer formula [19–21] to transform to the determinant for the world sheet.

In this case the boundary conditions are either Dirichlet everywhere or Neumann everywhere. Then, in the limit of large R and small ϵ, factorization implies that the z-plane figure determinant has the behavior

$$-\frac{1}{2}\mathrm{Tr}\ln(-\nabla^2)_z \sim \frac{5}{24}\ln R + \frac{1}{24}\sum_k \ln\epsilon_k + \text{const}, \tag{A10}$$

where the constant term, representing the determinant for the upper half-plane with the same boundary conditions everywhere, has nothing to depend on.

Next, we develop the transformation of the determinant from this z-plane figure to the light-cone world sheet using (A4). Clearly, $\partial_y\Sigma = -\partial_n\Sigma = 0$ on the real axis. Thus, the change formula receives contributions from the corners and semicircles only. For z near Z_k, we put $z = Z_k + re^{i\varphi}$ and approximate

$$\Sigma \approx \ln|\alpha_N| - \ln r - \sum_{l\neq k}^{N-1} \ln|Z_l - Z_k| + \sum_{r=1}^{N-2} \ln|Z_k - x_r|,$$

$$\partial_n\Sigma \approx \frac{1}{r}. \tag{A11}$$

Then,

$$\Delta_{\epsilon_k} = \left[\frac{1}{24} - \frac{1}{12} + \frac{1}{8}\right]\Sigma$$
$$= \frac{1}{12}\ln\left(\frac{|\alpha_N|}{\epsilon_k}\frac{\prod_r |Z_k - x_r|}{\prod_{l\neq k}|Z_k - Z_l|}\right) = \ln\left(\frac{|\alpha_k|}{\epsilon_k}\right)^{1/12}. \tag{A12}$$

The three terms in square brackets are the $\int dl\Sigma\partial_n\Sigma$ term, the extrinsic curvature term (negative here), and the two corners at this semicircle, respectively.

For $z = x_s + re^{i\varphi}$, on the other hand, we have

$$\Sigma \approx \ln|\alpha_N| + \ln r - \sum_l^{N-1} \ln|Z_l - x_s| + \sum_{r\neq s} \ln|x_s - x_r|,$$

$$\partial_n\Sigma \approx -\frac{1}{r}. \tag{A13}$$

Then,

$$\Delta_{\epsilon_s} = \left[-\frac{1}{24}\right]\Sigma = -\frac{1}{24}\ln\frac{2\delta_s}{\epsilon_s}. \tag{A14}$$

Note that in this case, only the $\int \Sigma\partial_n\Sigma$ term contributes since there is no singularity in the initial surface at $z = x_s$: the singularity comes only in Σ, which is determined by the mapping function.

Finally, for the large semicircle, $\Sigma \approx -\ln(r/|\alpha_N|)$, $\partial_n\Sigma \approx -1/r$, and

$$\Delta_R = \left[-\frac{1}{24} + \frac{1}{12} + \frac{1}{8}\right]\Sigma = -\frac{1}{6}\ln\frac{R}{|\alpha_N|}. \tag{A15}$$

Combining all the contributions, we have

$$\det{}^{-1/2}(-\nabla^2)_\rho = \det{}^{-1/2}(-\nabla^2)_z\left(\frac{|\alpha_N|}{R}\right)^{1/6}\prod_k\left(\frac{|\alpha_k|}{\epsilon_k}\right)^{1/12}\prod_r\left(\frac{\epsilon_r}{2\delta_r}\right)^{1/24}$$
$$= C|\alpha_N|^{1/6}\exp\left\{-\sum_{k=1}^N\frac{T_k}{24\alpha_k}\right\}\prod_{k\neq l}|Z_k-Z_l|^{\alpha_k/24\alpha_l}\prod_k|\alpha_k|^{1/12}\left[\frac{\prod_r(2\delta_r)}{|\alpha_N|^{N-2}}\frac{\prod_k\prod_r|x_r-Z_k|}{\prod_{r\neq s}|x_s-x_r|}\right]^{1/48}\prod_r(2\delta_r)^{-1/24}$$
$$= C\prod_r(2\delta_r)^{-1/48}\prod_{k=1}^N\frac{|\alpha_k|^{1/8}}{|\alpha_k|^{1/48}}\prod_{k\neq l}|Z_k-Z_l|^{\alpha_k/24\alpha_l}\left[\frac{1}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l-Z_k|}{\prod_{r<s}|x_s-x_r|}\right]^{1/24}\exp\left\{-\sum_{k=1}^N\frac{T_k}{24\alpha_k}\right\}. \quad (A16)$$

If there are n_b bosonic world-sheet dimensions, this entire factor should be raised to the power n_b.

The world-sheet path integral is this determinant factor times a factor e^{iW_c} which arises from removing boundary data in the path integral by shifting the $\boldsymbol{x}$ by the classical solution that satisfies those boundary data, as shown in the text. Among other things, e^{iW_c} includes factors $R^{-\boldsymbol{p}^2}\prod_k\epsilon_k^{\boldsymbol{p}_k^2}$ in the limit that the $-T_k/\alpha_k$ get large. If $W_c = \sum_{kl}p_kN(\rho_k,\rho_l)p_l/2$ is expressed in terms of a Neumann function, these factors arise from the diagonal $l=k$ terms. The rest of these diagonal terms, combined with the factors $|\alpha_k|^{1/8}$, provide a factor of the ground-string wave function for each external string. The N ground string scattering amplitude is obtained by amputating these ground-state wave functions together with the factors $e^{\sum_k(\boldsymbol{p}_k^2-d/24)T_k/\alpha_k} = e^{\sum_k T_kP_k^-}$ from the path integral and integrating over the interaction times $\int d\tau_1\cdots d\tau_{N-2}$, where $\rho_r=\tau_r+i\sigma_r$ are the locations of the $N-2$ interaction points on the world sheet. By translational invariance in x^+ the integrand after amputation will acquire a factor $e^{a\sum_k P_k^-}$ if all the τ_r are translated by a. This means that integrating over one of the τ_r simply produces a P^--conserving delta function. The coefficient of this delta function is just the integral over only $N-3$ of the τ_r. Note that $\sum_k\alpha_k=0$ by the light-cone world-sheet construction, and $\sum_k P_k=0$ when Neumann conditions are chosen for the $\boldsymbol{x}$ integrals, as explained in Sec. III.

$$\mathcal{A} = \int d\tau_2\cdots d\tau_{N-2}[\det{}^{-n_b/2}(-\nabla^2)_\rho e^{iW_c}]_{\rm amputated}, \quad (A17)$$

where we have set $\tau_1=0$ and understand that $\sum_k P_k^-=0$.

The final result for $[e^{iW_c}]_{\rm amputated}$ includes the off-diagonal terms in its Neumann function representation, together with the parts of ϵ_k that remain after amputating $e^{\sum_k T_kP_k^-}$: for the ordinary bosonic coordinates the result is

$$[e^{iW_c}]_{\rm amputated} = \prod_{k<l}|Z_l-Z_k|^{2\boldsymbol{p}_k\cdot\boldsymbol{p}_l}\left(\prod_{k\neq l}|Z_k-Z_l|\right)^{-\alpha_l\boldsymbol{p}_k^2/\alpha_k}. \quad (A18)$$

The amputated determinant drops the exponential dependence on T_k and the factor $\prod_k|\alpha_k|^{n_b/8}$:

$$[\det{}^{-n_b/2}(-\nabla^2)_\rho]_{\rm amputated} = C\prod_r(2\delta_r)^{-n_b/48}\prod_{k=1}^N\frac{1}{|\alpha_k|^{n_b/48}}\prod_{k\neq l}|Z_k-Z_l|^{n_b\alpha_k/24\alpha_l} \times\left[\frac{1}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l-Z_k|}{\prod_{r<s}|x_s-x_r|}\right]^{n_b/24}. \quad (A19)$$

Notice that when the two are combined, the net power of $|Z_k-Z_l|$ simplifies nicely:

$$p_k\cdot p_l = \boldsymbol{p}_k\cdot\boldsymbol{p}_l - p_k^+p_l^- - p_k^-p_l^+$$
$$= \boldsymbol{p}_k\cdot\boldsymbol{p}_l - \alpha_k(\boldsymbol{p}_l^2-n_b/24)/2\alpha_l - \alpha_l(\boldsymbol{p}_k^2-n_b/24)/2\alpha_k.$$

It is convenient to change integration variables from the τ's to the Z's. Mandelstam's result for the Jacobian is (taking $Z_1, Z_{N-1}, Z_N = 0, 1, \infty$, respectively)

$$\mathcal{J} = \frac{\partial(\tau_2,\ldots,\tau_{N-2})}{\partial(Z_2,\ldots,Z_{N-2})} = \left[\frac{1}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l-Z_k|}{\prod_{r<s}|x_s-x_r|}\right]^{-1}, \quad (A20)$$

$$\mathcal{J}[\det{}^{-n_b/2}(-\nabla^2)_\rho]_{\rm amputated} = C\prod_r(2\delta_r)^{-n_b/48}\prod_{k=1}^N\frac{1}{|\alpha_k|^{n_b/48}}\prod_{k\neq l}|Z_k-Z_l|^{n_b\alpha_k/24\alpha_l} \times\left[\frac{1}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l-Z_k|}{\prod_{r<s}|x_s-x_r|}\right]^{(n_b-24)/24} \quad (A21)$$

so the scattering amplitude for the purely bosonic string ($n_b=d$) becomes

$$\mathcal{A} = C\prod_r(2\delta_r)^{-n_b/48}\prod_{k=1}^N\frac{1}{|\alpha_k|^{n_b/48}}\int dZ_2\cdots dZ_{N-2} \times\prod_{k<l}|Z_k-Z_l|^{2p_k\cdot p_l} \times\left[\frac{1}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l-Z_k|}{\prod_{r<s}|x_s-x_r|}\right]^{(d-24)/24}. \quad (A22)$$

The factor raised to the power $d-24$ depends on the Lorentz frames, so the critical dimension $D=26$ is necessary for Lorentz invariance [5], in which case $\mathcal{A}$ is proportional to the N particle dual resonance amplitude. Of course, factorization implies that $C = g^{N-2}$ and $\delta_r = \delta$, independent of r. Then, $\prod_r(2\delta_r) = (2\delta)^{N-2}$, so δ can be absorbed in the coupling constant.

2. Closed string

For the closed string the map from the whole Koba-Nielsen plane (z) to the light-cone world sheet ($\rho = \tau + i\sigma$) is nearly identical to that for the open string.

$$\rho = \frac{1}{2}\sum_{k=1}^{N-1}\alpha_k \ln(z - Z_k), \qquad \frac{d\rho}{dz}\bigg|_{z=x_r} = 0, \tag{A23}$$

$$\begin{aligned}\frac{d\rho}{dz} &= \frac{1}{2}\sum_{k=1}^{N-1}\frac{\alpha_k}{z-Z_k} = \frac{\sum_{k=1}^{N-1}\alpha_k\prod_{l\neq k}(z-Z_l)}{2\prod_k(z-Z_k)}\\ &= -\frac{\alpha_N}{2}\frac{\prod_r(z-x_r)}{\prod_k(z-Z_k)},\end{aligned} \tag{A24}$$

$$\frac{d^2\rho}{dz^2}\bigg|_{z=x_s} = -\frac{\alpha_N}{2}\frac{\prod_{r\neq s}(x_s-x_r)}{\prod_k(x_s-Z_k)}. \tag{A25}$$

The quantities Z_k, with $k = 1\cdots(N-1)$, and x_r, with $r = 1\cdots(N-2)$, can now be anywhere in the complex plane. The factors of $1/2$ on the right are to normalize the range of σ on string k to $\pi\alpha_k$, since the phase of $z - Z_k$ advances by 2π as z encircles Z_k. The asymptotic strings at $\tau = \pm\infty$ are mapped from the Z_k. In this notation $Z_N = \infty, Z_1 = 0$.

We next turn to the transformation of the determinant.

$$\Sigma = \ln\left|\frac{\alpha_N}{2}\right| - \sum_{k=1}^{N-1}\ln|z-Z_k| + \sum_{r=1}^{N-2}\ln|z-x_r|. \tag{A26}$$

Since the points $z = Z_k, x_s$ are singular, we cut out small circular disks of radii ϵ_k, ϵ_r about each of those points, respectively. The radius ϵ_k near Z_k can be interpreted in terms of a large time T_k for the asymptotic string k. From the mapping function we find

$$\epsilon_k = e^{2T_k/\alpha_k}\prod_{l\neq k}|Z_l - Z_k|^{-\alpha_l/\alpha_k}. \tag{A27}$$

The string N is asymptotic at large z, so we cut out the region $|Z| > R$. We take R large, and referring to the mapping function,

$$T_N \sim -\frac{\alpha_N}{2}\ln R, \qquad R \sim e^{-2T_N/\alpha_N}. \tag{A28}$$

On the other hand, the radius ϵ_s near x_s is a temporary regulator, which maps onto a circular deformation of the boundary near the corresponding interaction point on the light-cone world sheet. From the mapping function we see that the radius of this regulating circle on the world sheet is given by

$$\delta_s = \frac{1}{2}\epsilon_s^2\left|\frac{d^2\rho}{dz^2}\right|_{z=x_s} = \frac{1}{4}\epsilon_s^2|\alpha_N|\frac{\prod_{r\neq s}|x_s-x_r|}{\prod_k|x_s-Z_k|}, \tag{A29}$$

$$\epsilon_s = \sqrt{\frac{4\delta_s}{|\alpha_N|}}\frac{\prod_k|x_s-Z_k|^{1/2}}{\prod_{r\neq s}|x_s-x_r|^{1/2}}, \tag{A30}$$

$$\prod_s\epsilon_s = |\alpha_N|^{-N+3/2}\prod_k|\alpha_k|^{1/2}\prod_s\sqrt{4\delta_s}\frac{\prod_{l\neq k}|Z_l-Z_k|^{1/2}}{\prod_{r\neq s}|x_s-x_r|^{1/2}}. \tag{A31}$$

To calculate the determinant for the light-cone world sheet, we start with the determinant for the region in the z plane with disks about the Z_k removed and bounded by the large radius R circle. Then, we apply the generalized Kac-McKean-Singer formula [19–21] to transform to the determinant for the world sheet.

We take boundary conditions on the circles to be Dirichlet or Neumann. Then, in the limit of large R and small ϵ, factorization implies that the z-plane figure determinant has the behavior

$$-\frac{1}{2}\mathrm{Tr}\ln(-\nabla^2)_z \sim \frac{1}{6}\ln R - \frac{1}{6}\sum_k\ln\epsilon_k + \mathrm{const}, \tag{A32}$$

where the constant term, representing the determinant for the whole plane with the same boundary conditions everywhere, has nothing to depend on.

Next, we develop the transformation of the determinant from this z-plane figure to the light-cone world sheet: the change formula receives contributions from the circular boundaries only. For z near Z_k, we put $z = Z_k + re^{i\varphi}$ and approximate

$$\Sigma \approx \ln|\alpha_N| - \ln r - \sum_{l\neq k}^{N-1}\ln|Z_l-Z_k| + \sum_{r=1}^{N-2}\ln|Z_k-x_r|,$$

$$\partial_n\Sigma \approx \frac{1}{r}. \tag{A33}$$

Then,

$$\begin{aligned}\Delta_{\epsilon_k} &= \left[\frac{1}{12}-\frac{1}{6}\right]\Sigma = -\frac{1}{12}\ln\left(\frac{|\alpha_N|}{\epsilon_k}\frac{\prod_r|Z_k-x_r|}{\prod_{l\neq k}|Z_k-Z_l|}\right)\\ &= \ln\left(\frac{|\alpha_k|}{\epsilon_k}\right)^{-1/12}.\end{aligned} \tag{A34}$$

The terms in square brackets are the $\int dl\Sigma\partial_n\Sigma$ term and the extrinsic curvature term (negative here).

For $z = x_s + re^{i\varphi}$, on the other hand, we have

$$\Sigma \approx \ln|\alpha_N| + \ln r - \sum_l^{N-1}\ln|Z_l - x_s| + \sum_{r\neq s}\ln|x_s-x_r|,$$

$$\partial_n\Sigma \approx -\frac{1}{r}. \tag{A35}$$

Then,

$$\Delta_{\epsilon_s} = \left[-\frac{1}{12}\right]\Sigma = -\frac{1}{12}\ln\left(\epsilon_s|\alpha_N|\frac{\prod_{r\neq s}|x_s-x_r|}{\prod_l|Z_l-x_s|}\right)$$
$$= -\frac{1}{12}\ln\frac{4\delta_s}{\epsilon_s}. \tag{A36}$$

Finally, for the large circle, $\Sigma \approx -\ln(r/|\alpha_N|)$, $\partial_n\Sigma \approx -1/r$, and

$$\Delta_R = \left[-\frac{1}{12}+\frac{1}{6}\right]\Sigma = \frac{1}{12}\Sigma = -\frac{1}{12}\ln\frac{R}{|\alpha_N|}. \tag{A37}$$

Combining all the contributions, we have

$$\det{}^{-1/2}(-\nabla^2)_\rho = \det{}^{-1/2}(-\nabla^2)_z\left(\frac{|\alpha_N|}{R}\right)^{1/12}$$
$$\times\prod_k\left(\frac{|\alpha_k|}{\epsilon_k}\right)^{-1/12}\prod_s\left(\frac{4\delta_s}{\epsilon_s}\right)^{-1/12}$$
$$= K|\alpha_N|^{1/12}R^{1/12}\prod_k\epsilon_k^{-1/12}\prod_r\epsilon_r^{1/12}$$
$$\times\prod_k|\alpha_k|^{-1/12}\prod_s(4\delta_s)^{-1/12}. \tag{A38}$$

We see that the R, ϵ_k, ϵ_s, and δ_s dependence of this result is just the square of the corresponding dependence of the open string determinant with $\delta_s \to 2\delta_s$:

$$\frac{\det^{-1/2}(-\nabla^2)_{\rm closed}}{\det^{-1}(-\nabla^2)_{\rm open}(\delta_s\to 2\delta_s)}$$
$$= \frac{K}{C^2}\frac{|\alpha_N|^{1/12}\prod_k|\alpha_k|^{-1/12}\prod_s(\delta_s)^{-1/12}}{|\alpha_N|^{1/3}\prod_k|\alpha_k|^{1/6}\prod_r(\delta_r)^{-1/12}}$$
$$= \frac{K}{C^2}|\alpha_N|^{-1/4}\prod_{k<N}|\alpha_k|^{-1/4} = \frac{K}{C^2}\prod_{k\leq N}|\alpha_k|^{-1/4}.$$

From this we read off from (A16), sending $T_k, \delta_r \to 2T_k, 2\delta_r$ in the square of the open string determinant,

$$\det{}^{-1/2}(-\nabla^2)_\rho$$
$$= K\prod_s(4\delta_s)^{-1/24}\prod_{k=1}^N\frac{1}{|\alpha_k|^{1/24}}\times\prod_{k\neq l}|Z_k-Z_l|^{\alpha_k/12\alpha_l}$$
$$\times\left[\frac{1}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l-Z_k|}{\prod_{r<s}|x_s-x_r|}\right]^{1/12}\exp\left\{-\sum_{k=1}^N\frac{T_k}{6\alpha_k}\right\}. \tag{A39}$$

If there are n_b transverse bosonic dimensions, this entire factor should be raised to the power n_b.

The construction of scattering amplitudes follows the same steps as for the open string: one amputates external string wave functions and exponential T_k factors. In addition to integrating over interaction times, one also integrates over the break/join location on the string, so the amplitude takes the form

$$\mathcal{A} = \int d\tau_2 d\sigma_2\cdots d\tau_{N-2}d\sigma_{N-2}[\det{}^{-n_b/2}(-\nabla^2)_\rho e^{iW_c}]_{\rm amputated}, \tag{A40}$$

where we have set $\tau_1 = 0$ and understand that $\sum_k P_k^- = 0$.

For ordinary bosonic coordinates the relevant expressions are

$$[e^{iW_c}]_{\rm amputated} = \prod_{k<l}|Z_l-Z_k|^{\boldsymbol{p}_k\cdot\boldsymbol{p}_l}\left(\prod_{k\neq l}|Z_k-Z_l|\right)^{-\alpha_l\boldsymbol{p}_k^2/(2\alpha_k)},$$
$$[\det{}^{-n_b/2}(-\nabla^2)_\rho]_{\rm amputated} = K\prod_r(4\delta_r)^{-n_b/24}\prod_{k=1}^N\frac{1}{|\alpha_k|^{n_b/24}}\prod_{k\neq l}|Z_k-Z_l|^{n_b\alpha_k/12\alpha_l}\left[\frac{1}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l-Z_k|}{\prod_{r<s}|x_s-x_r|}\right]^{n_b/12}. \tag{A41}$$

To change integration variables from the τ's to the Z's, Mandelstam's result for the Jacobian in the closed string case [16] is (taking $Z_1, Z_{N-1}, Z_N = 0, 1, \infty$, respectively)

$$|\mathcal{J}|^2 = \left|\frac{\partial(\rho_2,\ldots,\rho_{N-2})}{\partial(Z_2,\ldots,Z_{N-2})}\right|^2 = \left[\frac{1}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l-Z_k|}{\prod_{r<s}|x_s-x_r|}\right]^{-2}, \tag{A42}$$

$$|\mathcal{J}|^2[\det{}^{-n_b/2}(-\nabla^2)_\rho]_{\rm amputated} = K\prod_r(4\delta_r)^{-n_b/24}\prod_{k=1}^N\frac{1}{|\alpha_k|^{n_b/24}}\prod_{k\neq l}|Z_k-Z_l|^{n_b\alpha_k/12\alpha_l}\left[\frac{1}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l-Z_k|}{\prod_{r<s}|x_s-x_r|}\right]^{(n_b-24)/12}, \tag{A43}$$

so the bosonic string scattering amplitude becomes

$$\mathcal{A} = K\prod_r (4\delta_r)^{-n_b/24}\prod_{k=1}^{N}\frac{1}{|\alpha_k|^{n_b/24}}\int d^2Z_2\cdots d^2Z_{N-2}\prod_{k<l}|Z_k - Z_l|^{p_k\cdot p_l}\left[\frac{1}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l - Z_k|}{\prod_{r<s}|x_s - x_r|}\right]^{(n_b-24)/12}. \quad \text{(A44)}$$

The factor raised to the power $n_b - 24$ depends on the Lorentz frames, so the critical dimension $n_b = 24$ is necessary for Lorentz invariance [5].

APPENDIX B: ALTERNATIVE FORM FOR THE MEASURE

Inspecting $d\rho/dz$ [see (A2) and (A24)] we have the identity

$$-\alpha_N\prod_r(z - x_r) = \sum_{k=1}^{N-1}\alpha_k\prod_{l\neq k}(z - Z_l), \quad \text{(B1)}$$

Putting $z = Z_m$ in this equation, we find

$$-\alpha_N\prod_r(Z_m - x_r) = \sum_{k=1}^{N-1}\alpha_k\prod_{l\neq k}(Z_m - Z_l) = \alpha_m\prod_{l\neq m}(Z_m - Z_l), \quad \text{(B2)}$$

$$|\alpha_N|^N\prod_{m,r}|Z_m - x_r| = \prod_{m=1}^{N}|\alpha_m|\prod_{l\neq k}|Z_k - Z_l|. \quad \text{(B3)}$$

Using this last equation, the measures can be put in a form more useful for the protostring scattering amplitudes:

$$\left[\left|\frac{\partial T}{\partial Z}\right|\det{}^{-n_b/2}(-\nabla^2)\right]_{\text{open}} = \prod_{k\neq l}|Z_k - Z_l|^{n_b\alpha_k/24\alpha_l}\prod_{k=1}^{N}\frac{1}{\sqrt{|\alpha_k|}}\left[\frac{\prod_{k<N}|\alpha_k|}{|\alpha_N|}\right]^{(24-n_b)/16} \times\left[\frac{\prod_{r<t}|x_t - x_r|\prod_{m<l}|Z_l - Z_m|}{\prod_{l,r}|Z_l - x_r|}\right]^{(24-n_b)/24} \quad \text{(B4)}$$

and

$$\left[\left|\frac{\partial T}{\partial Z}\right|^2\det{}^{-n_b/2}(-\nabla^2)\right]_{\text{closed}} = \prod_{k\neq l}|Z_k - Z_l|^{n_b\alpha_k/12\alpha_l}\prod_{k=1}^{N}\frac{1}{|\alpha_k|}\left[\frac{\prod_{k<N}|\alpha_k|}{|\alpha_N|}\right]^{(24-n_b)/8} \times\left[\frac{\prod_{r<t}|x_t - x_r|\prod_{m<l}|Z_l - Z_m|}{\prod_{l,r}|Z_l - x_r|}\right]^{(24-n_b)/12}. \quad \text{(B5)}$$

[1] R. Giles and C. B. Thorn, Lattice approach to string theory, Phys. Rev. D **16**, 366 (1977).

[2] C. B. Thorn, Reformulating string theory with the $1/N$ expansion, arXiv:hep-th/9405069.

[3] O. Bergman and C. B. Thorn, String bit models for superstring, Phys. Rev. D **52**, 5980 (1995).

[4] P. Goddard, C. Rebbi, and C. B. Thorn, Lorentz covariance and the physical states in dual resonance models, Nuovo Cimento A **12**, 425 (1972).

[5] P. Goddard, J. Goldstone, C. Rebbi, and C. B. Thorn, Quantum dynamics of a massless relativistic string, Nucl. Phys. **B56**, 109 (1973).

[6] G. 't Hooft, A planar diagram theory for strong interactions, Nucl. Phys. **B72**, 461 (1974).

[7] G. 't Hooft, Quantization of discrete deterministic theories by Hilbert space extension, Nucl. Phys. **B342**, 471 (1990); On the Quantization of Space and Time, *Proceedings of the 4th Seminar on Quantum Gravity, Moscow, 1987*, edited by M. A. Markov (World Scientific Press, Singapore, 1988); Dimensional reduction in quantum gravity, arXiv:gr-qc/9310026.

[8] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, M theory as a matrix model: A conjecture, Phys. Rev. D **55**, 5112 (1997).

[9] R. Dijkgraaf, E. P. Verlinde, and H. L. Verlinde, Matrix string theory, Nucl. Phys. **B500**, 43 (1997).

[10] S. Sun and C. B. Thorn, Stable string bit models, Phys. Rev. D **89**, 105002 (2014).

[11] G. Chen and S. Sun, Numerical study of the simplest string bit model, Phys. Rev. D **93**, 106004 (2016).

[12] C. B. Thorn, Space from string bits, J. High Energy Phys. 11 (2014) 110.

[13] C. B. Thorn, $1/N$ perturbations in superstring bit models, Phys. Rev. D **93**, 066003 (2016).

[14] S. Mandelstam, Interacting string picture of dual resonance models, Nucl. Phys. **B64**, 205 (1973).

[15] S. Mandelstam, Interacting string picture of the Neveu-Schwarz-Ramond model, Nucl. Phys. **B69**, 77 (1974).

[16] S. Mandelstam, The Interacting String Picture And Functional Integration, published in *Unified String Theories*, edited by M. Green and D. Gross (World Scientific, Singapore, 1986).

[17] C. B. Thorn, String field theory, Phys. Rep. **175**, 1 (1989).
[18] A. H. Mueller, A geometrical interpretation of the Reggeon and Pomeron intercepts in the string model, Nucl. Phys. **B118**, 253 (1977).
[19] M. Kac, Can one hear the shape of a drum?, Am. Math. Mon. **73**, 1 (1966).
[20] H. P. McKean and I. M. Singer, Curvature and eigenvalues of the Laplacian, J. Diff. Geom. **1**, 43 (1967).
[21] C. B. Thorn, Determinants for the lightcone worldsheet, Phys. Rev. D **86**, 066010 (2012).