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Basis tensor gauge theory: Reformulating gauge theories with basis tensor fields

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We reformulate gauge theories in analogy with the vierbein formalism of general relativity. More specifically, we reformulate gauge theories such that their gauge dynamical degrees of freedom are local fields that transform linearly under the dual representation of the charged matter field. These local fields, which naively have the interpretation of nonlocal operators similar to Wilson lines, satisfy constraint equations. A set of basis tensor fields is used to solve these constraint equations, and their field theory is constructed. A new local symmetry in terms of the basis tensor fields is used to make this field theory local and maintain a Hamiltonian that is bounded from below. The field theory of the basis tensor fields is what we call the basis tensor gauge theory.

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I. INTRODUCTION

Gauge theories (see e.g. Refs. [1–8]) are extremely robust and successful in describing fundamental interactions of nature such as in the Standard Model (SM) of particle physics [8–17]. In the usual gauge theoretic formulation, the gauge field is a connection on principal bundles (see e.g. Refs. [18,19]). In the usual formulation of general relativity, Christoffel symbols are connections on the tangent bundle and can be expressed nonlinearly in terms of the metric. Another widely used formulation of general relativity that is particularly useful when spinors need to be defined in curved spacetime is the vierbein formalism. In this formalism, N basis vector fields are introduced as a way of taking the square root of the metric, in which N is the dimension of spacetime. However, in the case of gauge theories, there is no widely known analogous vierbein formulation, presumably because there is no obvious nontrivial metric analog that carries the gauge field information. In this work, we construct a vierbeinlike field theory of a $U(1)$ gauge theory coupled to complex scalars.

Our approach is to construct an explicit representation of the matter field direction in the group representation space as a *local* Lorentz tensor field that is subject to constraints that arise from matching to the ordinary gauge field connection. This tensor transforms as a dual to the matter representation, and the constraint equation is reminiscent of the relationship between the spacetime vierbein field and the Christoffel symbol. In this sense, this tensor field is the analog of the general relativistic vierbein for our construction. We then solve this constraint equation by decomposing the log of this tensor in terms of N fields that we call basis tensor fields. These fields effectively span the Lie algebra that generates the tensor field. The field theory of these basis tensor fields is local and has new local symmetries that allow this theory to perturbatively match to ordinary gauge theories.

More explicitly, the vierbeinlike field is taken to be a Lorentz tensor $G(x)$ that satisfies a constraint equation. Since $G(x)$ transforms in the gauge group representation space as a dual to the matter field, if a matter field ϕ is charged under $U(1)$ with charge 1, $G(x)$ transforms with a charge -1, and the object $\phi(x)G(x)$ is gauge invariant. We show that the minimal Lorentz tensor rank of $G(x)$ that has this desired dual property and can accommodate the local gauge field degrees of freedom is 2: i.e., $G^{\alpha}{}_{\beta}(x)$. The constraint equation of $G^{\alpha}{}_{\beta}(x)$ can be solved in terms of another set of unconstrained fields $\{\theta^a(x)(H^a)^{\mu}{}_{\nu}|a \in \{0,\ldots N-1\}\}$ (similarly in spirit to sigma model constructions), which are the basis tensor fields. The field theory of $\theta^a(x)$ is what we will call *basis tensor gauge theory* (BTGT) and is an alternate to the gauge theory description in terms of $A_\mu(x)$. It is the theory of θ^a that will exhibit a *new local symmetry* to maintain the (perturbative) isomorphism between the usual gauge theory and BTGT.

Giving a vierbein expression of gauge fields in this work makes gauge theories look more like general relativity, which in some sense is similar in philosophy to Kaluza-Klein theories [20], but the approach here is different in that we try to minimize the disturbance to the theory. More precisely, instead of trying to unify the gauge theory with spacetime dynamics, the theory is merely rewritten such that the gauge fields more closely resemble the matter fields. In the usual model building description of gauge theories, the gauge fields are put on a different footing than the matter fields in that the gauge fields do not form a linear representation of the gauge group while the matter fields typically do. In our approach, the $G^{\alpha}{}_{\beta}(x)$ fields, which have the same information as the gauge fields, form a linear

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representation. The most interesting result arising from this is the emergence of a local symmetry that is independent of the ordinary gauge symmetry.

To our knowledge, the previous work that most closely resembles our approach is that of Mandelstam [21], in which the group space linear representation is given as an object similar to a Wilson line (for several examples of the vast literature on this topic, see e.g. Refs. [22–28] and references therein). In some sense, this object can be viewed as the analog of ${G^\alpha}_\beta$.[1] However, in addition to the fact that Wilson lines are manifestly nonlocal, the purpose of Mandelstam's work was to formulate gauge theories without any gauge fields. In contrast, the purpose of our work is to explicitly construct a gauge group representation direction as a local spacetime Lorentz tensor field, not to hide the group representation space.

The order of presentation is as follows. In Sec. II, we derive the relationship between ${G^\alpha}_\beta(x)$ and $A_\mu(x)$ using an ansatz analogous to the equivalence principle. This relationship serves as a constraint equation. We then solve this constraint equation using the basis tensors $\theta^a H^a$. In Sec. IV, we show how the integral over A_μ is related to the θ^a field. The naive nonlocality will be eliminated by the symmetries imposed when defining the partition function in Sec. V. In Sec. VI, we go through the exercise of constructing a BTGT model based on the recipe in Sec. V. We give Feynman rules and apply them to a a simple scattering computation. Section VII lists some of the peculiarities of the model: a) each charged elementary field has its own group direction field (that are all related to each other through the same θ^a), and the covariant derivative can be written as a peculiar divergence of a composite field; b) the Hamiltonian is bounded from below despite the fact that the θ^a theory is a higher derivative theory; c) BTGT gives a novel way of computing nonlocal correlators. We conclude by speculating on future research directions. The Appendixes present explanations of the minimal rank of the Lorentz tensor for BTGT as well as the relationship of the new local symmetry to translational symmetry. The last Appendix section explicitly displays the analogy between ${G^\alpha}_\beta(x)$ and the general relativistic vierbein.

II. GROUP SPACE MATTER DIRECTION FIELD

The purpose of this work is to construct an alternate description to the usual gauge field that puts matter fields and the gauge fields on a more similar mathematical categorization. Because relativistic quantum field theory naturally partitions into relativistic tensor field degrees of freedom, any alternate local description of the gauge field has a natural description in terms of Lorentz tensors. We therefore define a *local Lorentz tensor* field in the dual representation of the matter field which describes the "direction" of the matter field in the group representation space. For simplicity, we focus here on the $U(1)$ group, although we foresee no insurmountable obstacles to generalize this to non-Abelian theories.

Given a field ϕ that is a complex scalar charged under $U(1)$ as

$$\phi(x) \to e^{i\theta(x)}\phi(x), \tag{1}$$

we wish to construct a Lorentz tensor object $G_{\alpha\beta}$ and its field theory that exhibits the $U(1)$ gauge group transformation property

$${G^\alpha}_\beta(x) \to {G^\alpha}_\beta(x) e^{-i\theta(x)}, \tag{2}$$

such that $\phi {G^\alpha}_\beta$ is gauge invariant. We note that we can view ${G^\alpha}_\beta$ as the direction in gauge group linear representation space. We discuss in Appendix A that a rank 2 Lorentz tensor is the smallest rank for which such a local description alternate to the gauge field is possible. We also show in Appendix C how ${G^\alpha}_\beta(x)$ is analogous to the general relativistic vierbein. To construct the theory of ${G^\alpha}_\beta$, we will match to the known A_μ gauge theory. To this end, we need to find a relationship between ${G^\alpha}_\beta$ and the ordinary gauge field A_μ.

Some degree of rigidity in the construction can be attained, and the spirit of making gauge theories look more like general relativity can be followed if we use an analog of the equivalence principle approach (see e.g. Ref. [29]) of making a general coordinate transformation away from the freely falling frame of the matter to define the Christoffel symbol (the connection on the tangent bundle).[2] Here, the analog of the freely falling frame can be defined to be the frame in which the $U(1)$ connection $A_\mu(x)$ vanishes at a spacetime point x_1, since A_μ enters without any derivatives in the matter Lagrangian:

$$\mathcal{L}_\phi = (\partial_\mu + iA_\mu)\phi(\partial^\mu - iA^\mu)\phi^*. \tag{3}$$

(Note that this definition is in contrast with the gravitational equivalence principle which relies on the equation of motion rather than the Lagrangian.) In this frame, the Lagrangian at point x_1 looks like there is no gauge field (just as locally, the Christoffel symbol vanishes in the freely falling frame):

$$\mathcal{L}_\phi(x_1) = \partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi}^*(x_1). \tag{4}$$

We demand in this special gauge frame that the vierbeinlike tensor field has the following value at point x_1:

[1] A Wilson line transforms as a nonlocal adjoint. If one views one end of the Wilson line to be at infinity and demands that the gauge transformations vanish there, then it looks as if the Wilson line transforms as a fundamental.

[2] Of course, this is simply an ansatz for defining the representation since there is no universality of charge to mass ratio in gauge theories.

$$\tilde{G}_{\alpha\beta}(x_1) = S_{\alpha\beta}(x_1). \tag{5}$$

Upon making a gauge transformation to move to the general frame, we have

$$\phi(x) = \tilde{\phi}(x)e^{i\theta(x)}, \tag{6}$$

which gives

$$\mathcal{L}_\phi(x_1) = (\partial_\mu - i\partial_\mu\theta)\phi(\partial^\mu + i\partial^\mu\theta)\phi^*. \tag{7}$$

Comparing this to the usual definition of the connection covariant derivative

$$D_\mu = \partial_\mu + iA_\mu \tag{8}$$

that appears in Eq. (3), we identify the connection as

$$A_\mu(x_1) = -\partial_\mu\theta(x_1). \tag{9}$$

Note that this does not mean $A_\mu(x)$ is a pure gauge configuration everywhere, as this equation applies at only one point x_1. Since $G_{\alpha\beta}$ is *defined* to obey the transformation rule of Eq. (2), we have

$$G_{\alpha\beta}(x_1) = S_{\alpha\beta}(x_1)e^{-i\theta(x_1)}. \tag{10}$$

Because of Eq. (9), we want to solve for $\partial_\mu\theta(x_1)$ in terms of G evaluated at x_1. To achieve this, we take derivatives of the general gauge-transformed object

$$e^{i\theta}(G^{-1})^{\alpha\beta}\partial_\alpha(G_{\beta\mu}e^{-i\theta}) = (G^{-1})^{\alpha\beta}\partial_\alpha G_{\beta\mu}(x) - i\partial_\mu\theta(x) \tag{11}$$

and evaluate this general expression at x_1 in the special gauge frame:

$$e^{i\theta}(\tilde{G}^{-1})^{\alpha\beta}\partial_\alpha(\tilde{G}_{\beta\mu}e^{-i\theta})|_{x_1} = (\tilde{G}^{-1})^{\alpha\beta}\partial_\alpha\tilde{G}_{\beta\mu}(x_1) - i\partial_\mu\theta(x_1). \tag{12}$$

Because of Eq. (9), we conclude

$$A_\mu(x_1) = -i[(G^{-1})^{\alpha\beta}(\partial_\alpha G_{\beta\mu})|_{x_1} - (\tilde{G}^{-1})^{\alpha\beta}(\partial_\alpha\tilde{G}_{\beta\mu})|_{x_1}], \tag{13}$$

in which

$$G_{\alpha\beta}(x) \equiv \tilde{G}_{\alpha\beta}(x)e^{-i\theta(x)} \tag{14}$$

is the general gauge field.

We can now simplify Eq. (13) further by noting that Eq. (13) has an additional set of $\tilde{U}(1)$ symmetry transformations,

$$G_{\alpha\beta} \to G_{\alpha\beta}e^{-i\Lambda_\beta(x)} \qquad \tilde{G}_{\alpha\beta} \to \tilde{G}_{\alpha\beta}e^{-i\Lambda_\beta(x)}, \tag{15}$$

that leaves Eq. (13) invariant. This means we can use it to choose $\partial_\alpha\tilde{G}_{\beta\mu} = 0$ as follows. First, we execute a $\tilde{U}(1)$ transform to go to the barred frame,

$$(G^{-1})^{\alpha\beta}(\partial_\alpha G_{\beta\mu}) = (\overline{G}^{-1})^{\alpha\beta}(\partial_\alpha\overline{G}_{\beta\mu}) - i\partial_\mu\Lambda_\mu \quad \text{no sum over } \mu \tag{16}$$

$$(\tilde{G}^{-1})^{\alpha\beta}(\partial_\alpha\tilde{G}_{\beta\mu}) = (\overline{\tilde{G}}^{-1})^{\alpha\beta}(\partial_\alpha\overline{\tilde{G}}_{\beta\mu}) - i\partial_\mu\Lambda_\mu \quad \text{no sum over } \mu, \tag{17}$$

where the yet-to-be-determined $\Lambda_\mu(x)$ parametrizes the transformation to the barred frame. We can then impose the condition

$$(\overline{\tilde{G}}^{-1})^{\alpha\beta}(\partial_\alpha\overline{\tilde{G}}_{\beta\mu}) = 0 \tag{18}$$

to solve for Λ_μ. This implies that

$$(\overline{G}^{-1})^{\alpha\beta}(\partial_\alpha\overline{G}_{\beta\mu}) = (G^{-1})^{\alpha\beta}(\partial_\alpha G_{\beta\mu}) - (\tilde{G}^{-1})^{\alpha\beta}(\partial_\alpha\tilde{G}_{\beta\mu}). \tag{19}$$

In this $\tilde{U}(1)$ fixed system, we have

$$A_\mu(x) = -i(\overline{G}^{-1})^{\alpha\beta}(\partial_\alpha\overline{G}_{\beta\mu}), \tag{20}$$

in which the bar indicates that we have fixed the $\tilde{U}(1)$ gauge through Eq. (18). For notational convenience, we can simply drop the bar: i.e., we then have

$$\boxed{A_\mu = -i(G^{-1})^\alpha{}_\beta\partial_\alpha G^\beta{}_\mu}. \tag{21}$$

This equation gives the relationship between the vierbein-like field $G^\alpha{}_\beta$ and the gauge field A_μ. We note that, because of the way the Lorentz tensor indices are contracted, this is not a pure gauge configuration. As explained in Appendix A, this is in contrast with the situation with lower rank tensors. It is also here that we see how Eq. (21) is reminiscent of the relationship between the Christoffel symbol and the vierbein. In the next section, we will introduce new basis fields to decompose $G^\alpha{}_\beta$.

III. DECOMPOSING THE VIERBEIN

In this section, we will show that demanding (a) the reality condition implied by Eq. (21), (b) that $G^\alpha{}_\beta$ transform like a $(1 \quad 1)$ Lorentz tensor, and (c) $G^\alpha{}_\beta \to \eta^\alpha{}_\beta$ in the vacuum limit "uniquely" fixes

$$\boxed{G^\alpha{}_\beta \in \bigoplus_{n=1}^{N} U(1)}, \tag{22}$$

where $N = 4$ for four spacetime dimensions and each $U(1)$ in the sum means a one-dimensional representation. Each of N phase fields are what we will call $\theta^a H^a$, which are the basis tensors.

Since the $G^\alpha{}_\beta$ constrained by Eq. (21) are difficult to work with, we will solve this constraint equation here in terms of the unconstrained fields. Consider the representation (just as in sigma model constructions)

$$G^\beta{}_\mu = (e^{i\theta^a(x)H^a})^\beta{}_\mu \tag{23}$$

$$(G^{-1})^\alpha{}_\beta = (e^{-i\theta^a(x)H^a})^\alpha{}_\beta, \tag{24}$$

in which θ^a is real without loss of generality, H^a is a general set of constant matrices (maximally $2N^2$ such matrices exist where $N = 4$ for four spacetime dimensions), and the repeated indices here are summed. We note that Eq. (23) contains an assumption about going to a manifestly Lorentz-invariant vacuum in the limit of $\theta(x) \to 0$; i.e., in the limit $\theta(x) \to 0$, $G^\alpha{}_\beta$ becomes an identity matrix, which is Lorentz invariant. To satisfy Eq. (21), we expand for small θ:

$$A_\mu = \partial_\alpha \theta^a (H^a)^\alpha{}_\mu + O(\theta^2). \tag{25}$$

This says that H^a should be a *real* matrix.

If we keep the entire power series, we have

$$G^\beta{}_\mu = \left(\sum_{n=0}^\infty \frac{1}{n!}[i\theta^a(x)H^a]^n\right)^\beta{}_\mu. \tag{26}$$

We can take any generic m power term in this series as follows:

$$\theta^{a_1}\theta^{a_2}\ldots\theta^{a_m}H^{a_1}H^{a_2}\ldots H^{a_m}. \tag{27}$$

We define each H^a to transform like a rank 2 tensor under Lorentz transformations. Hence, each such term transforms as

$$\theta^{a_1}\theta^{a_2}\ldots\theta^{a_m}\Lambda H^{a_1}\Lambda^{-1}\Lambda H^{a_2}\Lambda^{-1}\ldots\Lambda H^{a_m}\Lambda^{-1}, \tag{28}$$

which means that the matrix ansatz Eq. (23) does transform like a $(1\quad 1)$ tensor under Lorentz transformations.

Let us now consider the reality condition on the rest of the terms in the power series. First, we use the Baker-Campbell-Hausdorff formula to express the gauge field in terms of a parametric integral:

$$\partial_\alpha G^\beta{}_\mu = i\partial_\alpha\theta^f \int_0^1 dt[e^{i(1-t)\theta^a H^a}H^f e^{it\theta^a H^a}]^\beta{}_\mu \tag{29}$$

$$A_\mu = \partial_\alpha\theta^f \int_0^1 dt[e^{-it\theta^a H^a}H^f e^{it\theta^a H^a}]^\alpha{}_\mu. \tag{30}$$

Taking the complex conjugate of this yields

$$\left(\partial_\alpha\theta^f \int_0^1 dt[e^{-it\theta^a H^a}H^f e^{it\theta^a H^a}]^\alpha{}_\mu\right)^* = \partial_\alpha\theta^f \int_0^1 dt[e^{it\theta^a H^a}H^f e^{-it\theta^a H^a}]^\alpha{}_\mu. \tag{31}$$

Next, using the identity

$$e^A B e^{-A} = \sum_{n=0}^\infty \frac{1}{n!}[A,[A,[\ldots[A,B]]\ldots]], \tag{32}$$

we split even and odd powers

$$\begin{aligned}\partial_\alpha\theta^f &\int_0^1 dt[e^{it\theta^a H^a}H^f e^{-it\theta^a H^a}]^\alpha{}_\mu \\ &= \partial_\alpha\theta^f \int_0^1 dt \sum_{n=\mathrm{odd}}^\infty \frac{1}{n!}[it\theta^{a_n}H^{a_n},[\ldots[it\theta^{a_1}H^{a_1},H^f]\ldots]]^\alpha{}_\mu \\ &\quad + \partial_\alpha\theta^f \int_0^1 dt \sum_{n=\mathrm{even}}^\infty \frac{1}{n!}[it\theta^{a_n}H^{a_n},[\ldots[it\theta^{a_1}H^{a_1},H^f]\ldots]]^\alpha{}_\mu\end{aligned} \tag{33}$$

to separate the sign dependence. Although the even power terms do not depend on the sign in front of $it\theta^a H^a$, the odd power terms are odd under the sign change. Since $\theta^a(x)$ and $\partial_\alpha\theta^f(x)$ can have any value, we conclude that the only representation for which A_μ can be represented this way is

$$\partial_\alpha\theta^f[\theta^{a_{2m+1}}H^{a_{2m+1}},[\ldots[\theta^{a_1}H^{a_1},H^f]]\ldots]^\alpha{}_\mu = 0 \tag{34}$$

for every integer $m \geq 0$. Hence, we conclude that the only matrices H^f that can satisfy this are $(1\quad 1)$ Lorentz tensors that satisfy

$$\boxed{[H^a, H^b] = 0}. \tag{35}$$

These form a reducible representation of $U(1)$ given by Eq. (22).

One explicit representation of H^a is furnished by the following real polarization vectors,

$$(H^a)^\mu{}_\nu = \psi^\mu_{(a)}\psi_{(a)\nu}, \tag{36}$$

in which

$$\psi^\mu_{(a)} = \Lambda^\mu{}_a \tag{37}$$

are components of the Lorentz transformation matrix Λ [the fundamental representation of $SO(N-1,1)$]. The N fields

$$\theta^a(x)(H^a)^\mu{}_\nu \quad \text{no sum on } a \tag{38}$$

appearing in Eq. (23) span the spacetime tensor space and can be used to expand the vierbeinlike field $G^{\alpha}{}_{\beta}(x)$. On the other hand, they span the Lie algebra of the gauge group instead of the group representation itself. This makes them more like gauge fields. The fact that H^a is a complete basis is manifest in the identities

$$\sum_a H^a = \mathbb{I} \tag{39}$$

$$\mathrm{Tr}(H^a H^b) = \delta^{ab}. \tag{40}$$

We can summarize this section with the statement that the vierbeinlike field which transforms as a dual of the $U(1)$ matter representation is given by Eq. (23), in which the H^a are real, commuting $N \times N$ matrices that transform like a $(1 \quad 1)$ Lorentz tensor.

IV. θ^a AS AN INTEGRAL OVER A_μ

To gain intuition regarding the variable θ^a, it is instructive to express θ^a in terms of A_μ. Since the H^a are commuting matrices, Eq. (21) gives

$$\boxed{A_\mu = \sum_a \partial_\alpha \theta^a (H^a)^\alpha{}_\mu}. \tag{41}$$

This equation can be solved for θ^a,

$$\boxed{\theta^a(y) = \int_{Y_0}^{y} dz^\mu (H^a)^\lambda{}_\mu A_\lambda(x(z,y)) + Z^a(y)} \tag{42}$$

$$x^\lambda(z,y) \equiv (H^a)^\lambda{}_\mu z^\mu + \sum_{b \neq a} (H^b)^\lambda{}_\nu y^\nu, \tag{43}$$

in which the dz^μ integral is over a straight path connecting Y_0 and y and the $Z^a(x)$ are the zero modes of the derivative operator in Eq. (41) and satisfy

$$(H^a)^\alpha{}_\mu \frac{\partial}{\partial x^\alpha} Z^a(x) = 0 \quad \text{no sum over } a. \tag{44}$$

This means Z^a is a function that depends on a three-dimensional subspace of the four-dimensional space. Another way of saying this is that $Z^a(x)$ is translationally invariant,

$$Z^a(y + T_a \psi_{(a)}) = Z^a(y), \tag{45}$$

for any constant T_a. Hence, $Z^a(x)$ occupies a similar amount of functional volume as the residual $U(1)$ gauge symmetry associated with Lorentz gauge fixing: $\partial_\mu A^\mu = 0$. Equation (41) states that the theory of the local field $\theta^a(x)$ is related to the theory of a nonlocal operator if viewed from the $A_\mu(x)$ perspective. On the other hand, the exact nature of the relationship depends on how the data $\{Z^a(y), Y_0\}$ are handled in the partition function. This will be discussed in Sec. V.

The substitution of Eq. (42) into Eq. (23) gives us explicitly the relationship between A_μ and $G^\alpha{}_\gamma$:

$$G^\alpha{}_\beta(y) = \exp\left[i \sum_{a=1}^{N} \left(\int_{Y_0}^{y} dz^\mu (H^a)^\lambda{}_\mu A_\lambda(x(z,y)) + Z^a(y) \right) H^a \right]^\alpha{}_\beta. \tag{46}$$

Hence, when expressed in terms of $A_\mu(x)$, this theory looks manifestly like a nonlocal theory just as in the case of the Wilson line field. However, when expressed in terms of $\theta^a(y)$ without reference to A_μ, the theory is manifestly local. The two seemingly conflicting viewpoints will be reconciled later, where we will see that symmetries of the theory in terms of $\theta^a(y)$ will cause the theory to be insensitive to Y_0 and Z^a, eliminating most of the nonlocality. At the same time, it is interesting that field operators formed out of $\theta^a(y)$ exist which are multilocal at a *finite* number of discrete points do not depend on Y_0 or Z^a but represent a sum of an *infinite* number of A_μ operators (i.e., an integral over A_μ):

$$\mathcal{O}_a(y, T_a) \equiv \int_{y}^{y+T_a\psi_{(a)}} dz^\mu (H^a)^\lambda{}_\mu A_\lambda(x(z, y + T_a \psi_{(a)})) \tag{47}$$

$$= \theta^a(y + T_a \psi_{(a)}) - \theta^a(y) \quad \text{no sum over } a. \tag{48}$$

Hence, it is interesting that BTGT allows us to collapse an integral of local fields into evaluation of local fields at two points. It is beyond the scope of this paper to see if this feature lends itself to an interesting description of holography (see e.g. Refs. [30,31]).

V. PARTITION FUNCTION

Now that we have identified the field that we wish to use to describe the gauge theory, we need to construct the partition function. What we can do to construct the partition function is to start with the A_μ theory and make a change of variables to the θ^a theory. After the construction, we can eliminate the starting point of the A_μ and give the path integral construction rules just in terms of θ^a. However, we will see that we need to impose a new symmetry to carry out this program.

The procedure to start from the A_μ theory is as follows:

(1) Start with an ordinary gauge theory functional measure and ordinary ξ-gauge fixing,

$$\mathcal{Z}_1 = N_\xi \int Dg |\det \Box| \int DA D\phi D\phi^* e^{i(S+S_{gf})}, \tag{49}$$

in which

$$S_{gf} = \frac{-1}{2\xi}\int d^4x(\partial^\mu A_\mu)^2 \tag{50}$$

and S contains the matter field and ordinary gauge-invariant combination of A_μ.

(2) Make a change of variables using Eq. (41),

$$\mathcal{Z}_1 = N_\xi \int Dg|\det\Box|\mathcal{J}\int D\theta_{nz}D\phi D\phi^* e^{i(S+S_{gf})}, \tag{51}$$

in which

$$\mathcal{J} = \left|\det\left[\frac{\delta A_\mu(x)}{\delta\theta^a_{nz}(y)}\right]\right| \tag{52}$$

$$= \left|\det\left[(H^a)^\alpha{}_\mu \frac{\partial}{\partial x^\alpha}\delta^{(4)}(x-y)\right]\right| \tag{53}$$

and θ^a_{nz} stands for functions which are not annihilated by

$$(H^a)^\alpha{}_\mu \frac{\partial}{\partial x^\alpha}. \tag{54}$$

(Note that if we do not separate the zero modes out, then we would obtain $\mathcal{J}=0$.) However, it is difficult to restrict the integration to θ_{nz}, and it is worthwhile to find a way to include the zero modes of Eq. (54). One way to do this is to multiply by $D\theta_z$, which integrates over zero modes:

$$\mathcal{Z}_2 = \int D\theta_z \mathcal{Z}_1 \tag{55}$$

$$= \mathcal{N}\int D\theta D\phi D\phi^* e^{i(S[\theta,\phi,\phi^*]+S_{gf}[\theta])} \tag{56}$$

$$\mathcal{N} \equiv N_\xi \int Dg|\det\Box|\mathcal{J}. \tag{57}$$

This should be as harmless as multiplying by the residual gauge degrees of freedom in the Feynman gauge. This is the main difference between the ordinary gauge theory and the BTGT theory, and it most likely will not show up in perturbative computations, just as the residual gauge degree of freedom in Feynman gauge does not destroy perturbation theory.

Hence, we now have the partition function $\mathcal{Z}_2$ describing the theory of θ^a and ϕ.

At this point, we can forget that we started with the A_μ theory and construct the theory of θ^a and ϕ using the following procedure:

(1) Define the partition function in ξ-gauge as

$$\boxed{\mathcal{Z}_3 = \int D\theta D\phi D\phi^* \exp\left[i\left(S[\theta,\phi,\phi^*] - \frac{1}{2\xi}\int d^4x\left[\sum\nolimits_a (H^a)^\alpha{}_\mu \partial^\mu\partial_\alpha\theta^a\right]^2\right)\right]}. \tag{58}$$

(2) Choose S such that it is invariant under the usual Lorentz-invariant local field theory symmetry and the following two additional symmetries:

(a) Gauge invariant under the $U(1)$ transformations:

$$\theta^a(x) \to \theta^a(x) - \Lambda(x) \tag{59}$$

$$\phi(x) \to e^{i\Lambda(x)}\phi(x) \qquad \phi^*(x) \to e^{-i\Lambda(x)}\phi^*(x) \tag{60}$$

(b) Invariant under a lower-dimensional functional shift transformation,

$$\boxed{\theta^a(x) \to \theta^a(x) + Z^a(x)}, \tag{61}$$

where

$$(H^a)^\alpha{}_\mu \frac{\partial}{\partial x^\alpha} Z^a(x) = 0 \quad \text{no sum over } a. \tag{62}$$

This is manifestly a local symmetry without gauge fields.

The gauge symmetry conditions Eqs. (59) and (60) in item 2 lead to the usual gauge couplings (but in terms of θ^a) once one is guaranteed that θ^a only comes in the package of $A_\mu(\theta^a(y), y)$ [i.e., through Eq. (41)]. As we explicitly check in the next section, this packaging is partly enforced by the local symmetry Eq. (61). Furthermore, this local symmetry is very important in that it eliminates gauge theory destabilizing terms $\Delta\mathcal{L}_1$ of the form

$$\Delta\mathcal{L}_1 = \frac{\mu^2}{16}|\phi|^2(\mathrm{tr}G)(\mathrm{tr}G^{-1}) \tag{63}$$

$$\approx \mu^2|\phi|^2\left(1 - \frac{3}{16}\sum_a(\theta^a)^2 + \frac{1}{16}\sum_{b\neq c}\theta^b\theta^c + O(\theta^4)\right), \tag{64}$$

which is gauge invariant in the sense of Eqs. (59) and (60) but not Eq. (61). Note that this local symmetry also forbids global charge violating terms such as

$$\Delta\mathcal{L}_2 = \frac{\mu^2}{16}[\phi^2(\mathrm{tr}G)^2 + \mathrm{H.c.}], \tag{65}$$

which means that the theory inherits the global charge conservation as an accidental symmetry just as in ordinary gauge theories once the ordinary gauge symmetry condition is imposed. We note that, as long as the measure is chosen such that $D\theta$ is integrated over an unrestricted function space, Eq. (61) is not anomalous, at least in flat space.

Before closing this section, it is important to emphasize that Eq. (61) is a symmetry that is new and intrinsic to BTGT. This symmetry's origin is in the derivative operator appearing in Eq. (41), which does not have an analog in ordinary gauge theories. As alluded to in Eq. (45), this symmetry is the main reason why the integration origin Y_0 and the arbitrary function $Z^a(y)$ appearing in Eq. (42) are not meaningful. (More discussion of this in terms of translational invariance is given in Appendix B). This in turn means that, even though naively $G^\alpha{}_\beta(x)$ when expressed in terms of the gauge field [i.e., Eq. (42)] seems to be just as nonlocal as a Wilson line operator, it is not. At the same time, as shown in Eq. (48), $\theta^a(x)$ has a different degree of locality when compared to the gauge field $A_\mu(x)$, since two points are effectively mapped to an integral of $A_\mu(x)$ (i.e., a sum over an infinite number of points). Incidentally, we call the shift function $Z^a(x)$ a lower-dimensional function because Eq. (62) implies Eq. (45).

One naive downside of this construction is that power counting is more difficult because θ^a is a dimensionless variable. Unlike a sigma model parametrization where the kinetic term for the analog of θ^a is of the form $(\partial_\mu\theta)^2$ which would allow θ to acquire dimension upon canonical normalization, the θ^a kinetic term is quartic in derivatives. However, due to the new local symmetry Eq. (61), θ^a always enters with derivatives. Hence, there does not seem to be real harm done to bottom up model constructions by the loss of power counting. Incidentally, we show in Sec. VII B that, even though the higher derivative nature of the theory might seem to imply that we should worry about the stability of the theory (Ostrogradsky instability [32]), the theory is stable as the Hamiltonian is bounded from below. This stability is related to the fact that the additional local symmetry of Eq. (61) makes the Hamiltonian identical to ordinary gauge theories.

VI. ELEMENTARY COMPUTATION

Let us consider a simple example theory and compute a simple scattering process as a basic check of the formalism. Consider a scalar field ϕ charged under a $U(1)$ gauge charge e. The quadratic term for the ϕ field that is invariant under the global $U(1)$ subgroup is

$$\mathcal{L}_{k1} = |\partial\phi|^2 - m^2|\phi|^2. \tag{66}$$

(We can of course add quartic self-interactions at the marginal operator level, but we will omit it since we will not be using it.) As noted in Eqs. (67) and (68), we have to impose a separate gauge invariance given by

$$e\theta^a(x) \to e\theta^a(x) - e\Lambda(x) \tag{67}$$

$$\phi(x) \to e^{ie\Lambda(x)}\phi(x) \qquad \phi^*(x) \to e^{-ie\Lambda(x)}\phi^*(x) \tag{68}$$

as well as the new local symmetry [Eq. (61)]

$$e\theta^a(x) \to e\theta^a(x) + eZ^a(x). \tag{69}$$

To consider the ramifications of Eq. (69) a bit more explicitly, consider the θ^a dependent terms in the Lagrangian to be a Lorentz-invariant function combination $\mathcal{F}(\theta, \partial_\mu\theta, \partial_\mu\partial_\nu\theta, \ldots)$, where we can truncate the "..." at a finite derivative order due to power counting, and restrict the new local gauge invariance to imply the invariance of the Lagrangian instead of the action. The variation in the Lagrangian due to Eq. (69) is

$$\delta\mathcal{F}(\theta, \partial_\mu\theta, \partial_\mu\partial_\nu\theta, \ldots) = Z^a(x)\frac{\partial\mathcal{F}}{\partial\theta^a} + \partial_\mu Z^a(x)\frac{\partial\mathcal{F}}{\partial\partial_\mu\theta^a} + \partial_\mu\partial_\nu Z^a(x)\frac{\partial\mathcal{F}}{\partial\partial_\mu\partial_\nu\theta^a} + \cdots, \tag{70}$$

where the sum over a is implied. Since there is an infinite number of constraints imposed on the finite number of terms, each of these terms must vanish independently. This implies

$$\frac{\partial\mathcal{F}}{\partial\theta^a} = 0. \tag{71}$$

The condition that the next term vanishes,

$$\partial_\mu Z^a(x)\frac{\partial\mathcal{F}}{\partial\partial_\mu\theta^a(x)} = 0, \tag{72}$$

can be solved by

$$\frac{\partial\mathcal{F}}{\partial\partial_\mu\theta^a} = (H^a)^\mu{}_\delta\mathcal{V}^\delta, \tag{73}$$

in which $\mathcal{V}^\delta$ is a $(1\quad 0)$ Lorentz tensor. This means that every $\partial_\mu\theta^a$ dependence in $\mathcal{F}$ must come in the form with $(H^a)^\mu{}_\delta$ attached since, if there were any other solutions, then Z^a would have to satisfy other independent constraints.

Now, suppose the next term $\partial_\mu\partial_\nu Z^a(x)\frac{\partial\mathcal{F}}{\partial\partial_\mu\partial_\nu\theta^a}$ vanishes without $\frac{\partial\mathcal{F}}{\partial\partial_\mu\partial_\nu\theta^a}$ being proportional to $(H^a)^\mu{}_\delta$ or $(H^a)^\nu{}_\delta$. Then, we must impose a new constraint on Z^a,

$$\mathcal{F}^{\mu\nu}_{(b)}\partial_\mu\partial_\nu Z^b(x) = 0 \quad \text{no sum over } b, \tag{74}$$

where $\mathcal{F}^{\mu\nu}_{(b)}$ is a tensor. Since we do not want to contradict the fact that the only constraint on Z^a is Eq. (62) and it is otherwise arbitrary, Eq. (74) can be possible if $\mathcal{F}^{\mu\nu}_{(b)}$ is antisymmetric. However, that would imply

$$\frac{\partial \mathcal{F}}{\partial\partial_\mu\partial_\nu\theta^a} \tag{75}$$

is antisymmetric in $\mu \leftrightarrow \nu$, which is impossible for the smooth θ^a relevant for perturbation theory. Similar arguments apply for higher derivatives.

Hence, we conclude we can only write θ^a in the combination of Eq. (41) for the Lorentz-invariant local Lagrangian satisfying the invariance of Eq. (69). The renormalizable dimension coupling between θ^a and ϕ that obeys Eq. (69) is

$$\mathcal{L}_I = -ig_1\phi^*\partial^\mu\phi\partial_\alpha\theta^a(H^a)^\alpha{}_\mu + \text{H.c.} \\ + g_2|\phi|^2\partial_\alpha\theta^a(H^a)^\alpha{}_\mu\partial_\beta\theta^b(H^b)^{\beta\mu}. \tag{76}$$

In addition, the renormalizable kinetic terms would be

$$\mathcal{L}_{k2} = c_2(\partial_\alpha\theta^a(H^a)^\alpha{}_\mu)(\partial_\beta\theta^b(H^b)^{\beta\mu}) \\ + c_{41}\partial_\mu(\partial_\alpha\theta^a(H^a)^\alpha{}_\nu)\partial^\mu(\partial_\beta\theta^b(H^b)^{\beta\nu}) \\ + c_{42}\partial_\mu(\partial_\alpha\theta^a(H^a)^\alpha{}_\nu)\partial^\nu(\partial_\beta\theta^b(H^b)^{\beta\mu}), \tag{77}$$

in which the repeated indices are summed. Imposing Eqs. (67) and (68) on $\mathcal{L} = \mathcal{L}_{k1} + \mathcal{L}_{k2} + \mathcal{L}_I$ results in setting $c_2 = 0$, $c_{41} = -c_{42} = 2c$ (where c is a constant determined by Coulomb scattering), $g_1 = e$, and $g_2 = e^2$. We note that, after imposing the invariance of Eq. (69), the rest of the invariances fixing these coefficients are identical to ordinary gauge invariance.

To simplify the computations, it is useful to go to a Lorentz frame in which $\theta^a H^a$ is diagonal:

$$\sum_a \theta^a(H^a)^\mu{}_\nu = \sum_a \bar{\theta}^a\delta^\mu_{(a)}\delta_{(a)\nu}. \tag{78}$$

In this gauge, the $\langle\theta^\beta\theta^\lambda\rangle$ analog of the $\langle A_\mu A_\nu\rangle$ propagator giving the Feynman rule $i\eta_{\mu\nu}/(4ck^2) = -i\eta_{\mu\nu}/k^2$ [where $\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$] is

$$\int d^4x e^{ik\cdot(x-y)}\langle\bar{\theta}^\beta(x)\bar{\theta}^\lambda(y)\rangle = \frac{i\frac{\eta^{\beta\lambda}}{k_\beta k_\lambda}}{4ck^2} \quad \text{no sum}, \tag{79}$$

where one can count the minus signs as $(i)^2$ coming from $k^\beta k^\lambda$ and an extra minus sign from integrating by parts one of the factors $\partial_\delta\bar{\theta}^\delta$ to obtain the quartic differential operator to invert. The cubic and quartic vertices are

$$\frac{\partial}{\partial\phi}\frac{\partial}{\partial\phi^*}\frac{\partial}{\partial\bar{\theta}^\gamma}i\mathcal{L}_I| = [p+k]^\gamma q_\gamma e \quad \text{no sum} \tag{80}$$

$$\frac{\partial}{\partial\phi}\frac{\partial}{\partial\phi^*}\frac{\partial}{\partial\bar{\theta}^\gamma}\frac{\partial}{\partial\bar{\theta}^\lambda}i\mathcal{L}_I| = -2iq_\lambda r^\lambda\delta^\lambda{}_\gamma e^2 \quad \text{no sum} \tag{81}$$

according to the usual prescription. The noncovariant notation here comes from having made a frame choice that H^a are diagonal matrices. For example, a more manifestly covariant tree-level propagator is

$$\int d^4x e^{ik\cdot(x-y)}\langle\theta^b(x)\theta^a(y)\rangle = \frac{i}{4c}\frac{\delta^b{}_a}{(H^b)^\mu{}_\delta k_\mu k_\nu(H^a)^\nu{}_\gamma\eta^{\delta\gamma}}\frac{1}{k^2}, \tag{82}$$

which reverts to Eq. (79) when

$$(H^a)^\alpha{}_\beta = \delta^\alpha_{(a)}\delta_{(a)\beta}. \quad \text{no sum over } a. \tag{83}$$

The t-channel tree-level Coulomb scattering gives the amplitude

$$i\mathcal{M} = -ie^2\frac{1}{4c(k_1-p_1)^2}[p_1+k_1]\cdot[p_2+k_2], \tag{84}$$

which matches the usual scalar field theory result with $c = -1/4$ as expected.

VII. PECULIARITIES OF THE FORMALISM

A. Charge dependent axes

It is interesting to note that we can rewrite the covariant derivative as an ordinary divergence acting on a composite field consisting of $G^\alpha{}_\beta$ and a matter field ϕ_1,

$$\left(\frac{\partial}{\partial x^\mu} + iq_1A_\mu(x)\right)\phi_1(x) = \frac{\partial}{\partial y^\alpha_{(q_1)}}\Psi^\alpha_{(q_1)\mu}(x), \tag{85}$$

in which

$$dy^\alpha_{(q_1)} = G^\alpha_{(q_1)\mu}(x)dx^\mu \tag{86}$$

and

$$\Psi^\alpha_{(q_1)\ \mu}(x) \equiv \phi_1(x)G^\alpha_{(q_1)\ \mu}(x), \tag{87}$$

where there is a mismatch between the derivative variable $y_{(q_1)}$ on the right-hand side of Eq. (85) and the argument of $\Psi^\lambda_{(q_1)\delta}(x)$. We note that, since $\Psi^\lambda_{(q_1)\delta}$ is a covariant tensor, the tensor components in the y coordinate system is different from that in the x coordinate system. Furthermore, unlike before, we have displayed the charge assignment of the gauge group explicitly. Hence, if the G tensor is treated as a spacetime axis, then there are as many axes in spacetime as there are number of different charges. On the other hand, there is only one set of basis tensor fields θ^a that decomposes all of the axes, at least when matching to standard gauge theories.

B. Hamiltonian is bounded from below

It is well known that higher derivative theories generally exhibit an instability associated with the Hamiltonian being unbounded from below (for a review, see e.g. Refs. [33–37]). This instability is sometimes referred to as the Ostrogradsky instability. Here, we will show that, although BTGT is a higher derivative theory, it has a Hamiltonian that is bounded from below. This can be partially explained by the novel local symmetry Eq. (61) which effectively eliminates the θ^a degree of freedom from the action in favor of $(H^a)^\mu{}_\nu \partial_\mu \theta^a$.

The energy density for a gauged massive scalar field is

$$T_{00} = T_{00}^{(\phi)} + T_{00}^{(\theta)}, \tag{88}$$

where

$$T_{00}^{(\phi)} = |\partial_0 \phi + iA^0(\theta)\phi|^2 + \sum_{i=1}^{3} |\partial_i \phi - iA^i(\theta)\phi|^2 + m^2|\phi|^2 \tag{89}$$

$$\begin{aligned} T_{00}^{(\theta)} &= \frac{1}{2}\sum_{i=1}^{3} (\partial_0 A^i(\theta) + \partial_i A^0(\theta))^2 \\ &\quad + \frac{1}{2}\sum_{l=1}^{3} \left(\sum_{m,n=1}^{3} \epsilon_{lmn} \partial_m A^n(\theta) \right)^2 \end{aligned} \tag{90}$$

$$A_\delta(\theta) = \sum_a \partial_\mu \theta^a (H^a)^\mu{}_\delta, \tag{91}$$

which is positive definite. Hence, we do not expect the Ostrogradsky instability to arise in this theory. Again, this protection partly comes from the novel local symmetry Eq. (61). As discussed around Eq. (70), other ingredients include locality and Lorentz invariance, which all play a role in having θ^a come in the form of Eq. (91).

C. Computing nonlocal correlators

We can in principle use the new formalism to compute nonlocal correlators in novel ways. For example, consider the correlator

$$\mathcal{G}_{a_1 a_2} \equiv \langle \mathcal{O}_{a_1}(x_1, T_{a_1}) \mathcal{O}_{a_2}(x_2, T_{a_2}) \rangle, \tag{92}$$

in which $\mathcal{O}_a$ are the operators defined in Eq. (48). Note that $\mathcal{G}_{a_1 a_2}$ is invariant under the local transformations of Eq. (61). This correlator is easy to compute in the BTGT formalism. At tree level, it is given by

$$\begin{aligned} \mathcal{G}_{a_1 a_2} = -i \int \frac{d^4 k}{(2\pi)^4} & \frac{e^{-ik\cdot(x_1 - x_2)} \delta_{a_1 a_2}}{(k^2 + i\epsilon)(H^{a_1})^\mu{}_\delta k_\mu k_\nu (H^{a_2})^\nu{}_\gamma \eta^{\delta\gamma}} \\ & \times (e^{iT_{a_1} k\cdot\psi_{(a_1)}} - 1)(e^{-iT_{a_2} k\cdot\psi_{(a_2)}} - 1), \end{aligned} \tag{93}$$

in which we have used Eq. (82). This result in the usual A_μ formalism corresponds to

$$\begin{aligned} \mathcal{G}_{a_1 a_2} = & \int_{x_1}^{x_1 + T_{a_1}\psi_{(a_1)}} dz_1^\mu (H^{a_1})^\lambda{}_\mu \int_{x_2}^{x_2 + T_{a_2}\psi_{(a_2)}} dz_2^\nu (H^{a_2})^\beta{}_\nu \\ & \times \langle A_\lambda(x(z_1, x_1 + T_{a_1}\psi_{(a_1)})) \\ & \times A_\beta(x(z_2, x_2 + T_{a_2}\psi_{(a_2)})) \rangle. \end{aligned} \tag{94}$$

Hence, this offers a novel way to compute correlators. In the limit $T_{a_1} = T_{a_2} = T \to 0$, Eq. (93) becomes

$$\mathcal{G}_{a_1 a_2} = -iT^2 \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik\cdot(x_1 - x_2)} \eta^{a_1 a_2}}{k^2 + i\epsilon}, \tag{95}$$

recovering the photon propagator information. Hence, $\mathcal{G}_{a_1 a_2}$ is a nonlocal object that in the local limit gives back the photon propagator. It is interesting that the local limit of the fundamental nonlocal Green's function[3] of BTGT is the ordinary photon Green's function.

VIII. CONCLUSIONS

In this paper, we have constructed a novel formulation for gauge theories based on analogies with the vierbein formulation of general relativity. For simplicity, we have focused on a simple $U(1)$ theory in this work. This has led us to introduce a vierbeinlike field $G^\alpha{}_\beta(x)$ (indicating the direction in the gauge group representation space) that can be further decomposed (to solve constraint equations) in terms of another set of basis tensor fields $\theta^a(x)(H^a)^\mu{}_\nu$. Unlike the Wilson line, $\theta^a(x)$ is a local field. The basis tensor field $\theta^a(x)$ has new local symmetries given by Eq. (61) that are important for preserving translational invariance as discussed in Appendix B and maintaining stability as discussed in Sec. VII B. Intuitively, the field theory of θ^a contains the gauge theory information by way of Eq. (94).

There are many future research directions that are suggested by this work. Perhaps most obviously, BTGT should be generalizable to non-Abelian theories.[4] It would also be interesting to find practical applications for this theory in computing nonlocal correlators similar to Eq. (93). The novelty in part is related to the different

[3] It is fundamental since it is invariant under the new local symmetry of Eq. (61) defining BTGT.

[4] There are also certain technical details of the construction in this paper that can be improved. For example, although the argument surrounding Eq. (70) is sufficient for constructing an action only in terms of $A_\mu(\theta)$, it does not address the possibility of the action having variations of a total derivative term.

degree of locality due to the higher derivative nature of this theory as noted around Eq. (48). Loop corrections, BRST invariance, and Ward identities associated with the new local symmetry of Eq. (61) may be interesting to explore. Instantons, sphalerons, and other nonperturbative excitations in BTGT may be a bit different in ordinary gauge theories since the gauge theory has been nonperturbatively modified through the measure [see Eq. (55)]. This formalism should also be tested by embedding it into curved spacetime.

It is interesting that matter fields and gauge fields in this formalism can be packaged in the same category of mutually dual objects in group representation space. However, one satisfies a constraint equation, and the other does not. If there can be a way to spontaneously generate this asymmetry starting from an even more symmetric framework, that would open up new avenues for constructing physics beyond the SM.

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APPENDIX A: LOWER RANK TENSOR

Instead of a rank 2 tensor as in Eq. (10), suppose we postulated a Lorentz scalar transforming under $U(1)$ as a matter dual field representing the matter direction in representation space. There are then not enough local functional degrees of freedom to replace an N-vector field.[5]

The next-smallest rank to consider is 1. Suppose we choose

$$G_\gamma(x_1) = S_\gamma e^{-i\theta(x_1)}. \qquad \text{(A1)}$$

Because of Eq. (9), we want to solve for $\partial_\mu\theta(x_1)$ in terms of G evaluated at x_1. To this end, we can take derivatives of the general gauge-transformed object,

$$e^{i\theta}(G^{-1})^\gamma \partial_\mu (G_\gamma e^{-i\theta}) = (G^{-1})^\gamma \partial_\mu G_\gamma - i\partial_\mu\theta, \qquad \text{(A2)}$$

in which $(G^{-1})^\gamma G_\gamma \equiv 1$ defines the inverse. Evaluating this general expression at x_1 in the special gauge frame yields

$$(G^{-1})^\gamma \partial_\mu G_\gamma|_{x_1} = (\tilde{G}^{-1})^\gamma \partial_\mu \tilde{G}_\gamma(x_1) - i\partial_\mu\theta(x_1). \qquad \text{(A3)}$$

Because of Eq. (9), we conclude

$$A_\mu(x_1) = -i[(G^{-1})^\gamma(\partial_\mu G_\gamma)|_{x_1} - (\tilde{G}^{-1})^\gamma(\partial_\mu\tilde{G}_\gamma)|_{x_1}], \qquad \text{(A4)}$$

in which

$$G_\alpha(x) \equiv \tilde{G}_\alpha(x)e^{-i\theta(x)} \qquad \text{(A5)}$$

is the general gauge field.

We can now simplify Eq. (A4) further by noting that Eq. (A4) has an additional $\tilde{U}(1)$ symmetry transformation,

$$G_\gamma \to G_\gamma e^{-i\theta_2(x)} \qquad \text{(A6)}$$

$$\tilde{G}_\gamma \to \tilde{G}_\gamma e^{-i\theta_2(x)}, \qquad \text{(A7)}$$

that leaves Eq. (A4) invariant. This means we can use it to choose $\partial_\mu\tilde{G}_\gamma = 0$ as follows. First, we execute a $\tilde{U}(1)$ transform,

$$(G^{-1})^\gamma(\partial_\mu G_\gamma) = (\overline{G}^{-1})^\gamma(\partial_\mu\overline{G}_\gamma) - i\partial_\mu\theta_2 \quad \text{no sum} \qquad \text{(A8)}$$

$$(\tilde{G}^{-1})^{\mu\gamma}(\partial_\mu\tilde{G}_{\gamma\delta}) = (\overline{\tilde{G}}^{-1})^{\mu\gamma}(\partial_\mu\overline{\tilde{G}}_{\gamma\delta}) - i\partial_\mu\theta_2 \quad \text{no sum}, \qquad \text{(A9)}$$

parametrized by a yet-to-be-determined θ_2. We then impose the condition

$$(\overline{\tilde{G}}^{-1})^\gamma(\partial_\mu\overline{\tilde{G}}_\gamma) = 0 \qquad \text{(A10)}$$

to solve for θ_2. This implies

$$(\overline{G}^{-1})^\gamma(\partial_\mu\overline{G}_\gamma) = (G^{-1})^\gamma(\partial_\mu G_\gamma) - (\tilde{G}^{-1})^\gamma(\partial_\mu\tilde{G}_\gamma). \qquad \text{(A11)}$$

In this $\tilde{U}(1)$ gauge fixed system, we have

$$A_\mu(x) = -i(\overline{G}^{-1})^\gamma(\partial_\mu\overline{G}_\gamma), \qquad \text{(A12)}$$

where the bar indicates that we have fixed the $\tilde{U}(1)$ gauge through Eq. (A10). For notational convenience, we can simply drop the bar; i.e.,

$$A_\mu = -i(G^{-1})^\gamma\partial_\mu G_\gamma \qquad \text{(A13)}$$

$$= -i\frac{G^\gamma}{G^\gamma G_\gamma}\partial_\mu G_\gamma \qquad \text{(A14)}$$

$$= \frac{-i}{2}\partial_\mu \ln G^\gamma G_\gamma, \qquad \text{(A15)}$$

which is a pure gauge configuration.

Hence, we must go to higher rank tensors for a basis tensor. The next rank tensor is rank 2, and this is what we present in this work.

[5]We note that the approach of Ref. [21] effectively has a nonlocal function that is a scalar: i.e., $h_P(x,x_i) = \int_{x_i,P}^{x} dX^\mu A_\mu(X)$ where P is a path. The manifest nature of the nonlocality can be seen by the fact that it is a path dependent functional and the field strength is derived from h_P through $h_{P+\delta P}(x,x_i) - h_P(x,x_i) = F_{\mu\nu}\sigma^{\mu\nu}$, where $\sigma^{\mu\nu}$ represents the area of the path difference δP.

APPENDIX B: SPACETIME TRANSLATION SYMMETRY

Here, we discuss one way to motivate the requirement of local symmetry as given in Eq. (61). Suppose we start with a theory $S[A]$ of local $A_\mu(x)$, and in view of making a change of variables to θ^a starting from $S[A]$, suppose we add a nonlocal interaction ΔS involving $\theta^a(A_\mu)$ in the form of Eq. (42),

$$\Delta S = \Delta S(\theta^a(A_\mu)), \tag{B1}$$

which is ordinary $U(1)$ gauge invariant [e.g. see Eq. (63)] but not invariant under Eq. (61). This means that the partition function

$$\mathcal{Z}_0 = \int DA_\mu D\phi D\phi^* e^{i(S+\Delta S)} \tag{B2}$$

is sensitive to Y_0 in Eq. (42). However, this breaks spacetime translational invariance, since the interactions have a preferred point. Hence, one way to eliminate ΔS from the theory is to impose the local symmetry Eq. (61).

One cannot for example try to use $\mathcal{O}_a(y, T_a)$ defined in Eq. (48) as a substitute for the $\theta^a(x)$ field in making a change of variables from $A_\mu(x)$ to obtain a local field theory because $\mathcal{O}_a(y, T_a)$ is manifestly nonlocal (although translationally invariant in y). The local symmetry Eq. (61) also has the advantage of helping to protect against the Ostrogradsky instability.

APPENDIX C: VIERBEIN ANALOGY

In this Appendix, we explicitly list the analogy between $G^\alpha{}_\beta(x)$ formalism and the general relativistic vierbein $(e_a)_\mu$ formalism, where the index a is the fictitious Minkowski space index and μ is the spacetime coordinate index. As a start, the vierbeinlike field correspondence is

$$G^\alpha{}_\beta \leftrightarrow (e_a)_\mu, \tag{C1}$$

where effectively the real and imaginary elements of $G^\alpha{}_\beta$ [i.e., the real and imaginary elements of $U(1)$ maps to $SO(2)$] are analogs of the label μ and (α, β) labels are the analogs of a. The analogy of the constraint equation is

$$\begin{aligned} A_\lambda &= -i(G^{-1})^\alpha{}_\beta \partial_\alpha G^\beta{}_\lambda \leftrightarrow \Gamma^\gamma_{\lambda\beta} \\ &= (e^a)^\gamma \partial_{(\lambda}(e_a)_{\beta)} + g^{\epsilon\gamma}(e^c)_{(\beta}\partial_{\lambda)}(e_c)_\epsilon \\ &\quad - g^{\epsilon\gamma}(e^c)_{(\beta}\partial_{|\epsilon|}(e_c)_{\lambda)} \end{aligned} \tag{C2}$$

$$g_{\alpha\beta} \equiv (e_a)_\alpha (e_b)_\beta \eta^{ab}. \tag{C3}$$

The reason why $G^\alpha{}_\beta(x)$ cannot be considered to be analogous to an ordinary dual basis element such as a coordinate basis object ∂_μ is because such objects do not carry metric information by themselves.

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