PHYSICAL REVIEW D **94,** 105012 (2016)

Light adjoint quarks in the instanton-dyon liquid model. IV.

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(Received 21 June 2016; published 14 November 2016)

We discuss the instanton-dyon liquid model with N_f Majorana quark flavors in the adjoint representation of color $SU_c(2)$ at finite temperature. We briefly recall the index theorem on $S^1 \times R^3$ for twisted adjoint fermions in a Bogomolny-Prasad-Sommerfeld (BPS) dyon background of arbitrary holonomy and use the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction to derive the adjoint antiperiodic zero modes. We use these results to derive the partition function of an interacting instanton-dyon ensemble with N_f light and antiperiodic adjoint quarks. We develop the model in details by mapping the theory on a three-dimensional quantum effective theory with adjoint quarks with manifest $SU(N_f) \times Z_{4N_f}$ symmetry. Using a mean-field analysis at weak coupling and strong screening, we show that center symmetry requires the spontaneous breaking of chiral symmetry, which is shown to only take place for $N_f = 1$. For a sufficiently dense liquid, we find that the ground state is center symmetric and breaks spontaneously flavor symmetry through $SU(N_f) \times Z_{4N_f} \to O(N_f)$. As the liquid dilutes with increasing temperature, center symmetry and chiral symmetry are restored. We present numerical and analytical estimates for the transition temperatures.

DOI: 10.1103/PhysRevD.94.105012

I. INTRODUCTION

This work is a continuation of our earlier studies [1–3] of the gauge topology in the confining phase of a theory with the simplest gauge group $SU(2)$. We suggested that the confining phase below the transition temperature is an "instanton dyon" (and antidyon) plasma which is dense enough to generate strong screening. The dense plasma is amenable to standard mean field methods.

The key ingredients in the instanton-dyon liquid model are the so-called Kraan-van-Baal-Lee-Lu (KvBLL) instantons threaded by finite holonomies [4] split into their constituents, the instanton dyons. Diakonov and Petrov [5,6] have shown that the KvBLL instantons dissociate in the confined phase and recombine in the deconfined phase, using solely the BPS protected moduli space. The inclusion of the non-BPS-induced interactions, through the so-called streamline set of configuration, is important numerically, but it does not alter this observation [7]. The dissociation of instantons into constituents was advocated originally by Zhitnitsky and others [8].

Unsal and collaborators [9] proposed a specially tuned setting in which instanton constituents (they call instanton monopoles) induced confinement even at exponentially small densities, at which the semiclassical approximations is parametrically accurate. The key feature of this setting is the cancellation of the perturbative Gross-Pisarski-Yaffe holonomy potential. More specifically, in Ref. [9], the nontrivial center-symmetric phase emerges in the dilute vacuum at weak coupling for periodic boundary conditions of adjoint quarks where the instanton dyons combine into pairs of "bions." However, in the present work as we detail below and also in the work presented in Refs. [5,6], the nontrivial center-symmetric phase at low temperatures emerges for antiperiodic boundary conditions for adjoint quarks where the instanton dyons form a dense liquid.

The KvBLL instantons fractionate into constituents with fractional topological charge $1/N_c$. Their fermionic zero modes do not fractionate but rather migrate between various constituents [10]. This interplay between the zero modes and the constituents is captured precisely by the Nye-Singer index theorem [11]. For fundamental fermions, we have recently shown in the mean-field approximation that the center symmetry and chiral symmetry breaking are intertwined in this model [2]. The broken and restored chiral symmetry correspond to a center-symmetric or center-asymmetric phases, respectively. Similar studies were developed earlier in Refs. [12–14].

In this work, we would like to address this interplay between confinement and chiral symmetry breaking using N_f massless quarks in the *adjoint* representation of color $SU_c(2)$. We will detail the nature of the flavor symmetry group of the effective action induced by dissociated KvBLL calorons in the confined phase and investigate its change into an asymmetric phase at increasing temperature. Throughout, we will use the words "center-symmetric phase" and "confining phase" interchangeably, although their meanings convey different requirements. The former is a weaker form of confinement as it requires only that the the vacuum expectation value (vev) of the Polyakov line is zero. Whenever used below, these words would mostly refer to the former.

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Lattice simulations with adjoint quarks [15] have shown that the deconfinement and restoration of center symmetry occurs well before the restoration of chiral symmetry. These lattice results show that the ratio of the chiral to deconfinement temperatures is large and decreases with the number of adjoint flavors. More recent lattice simulations have suggested instead a rapid transition to a conformal phase [16]. Effective Polyakov-Nambu-Jona-Lasinio models with adjoint fermions have also been discussed recently [17,18].

The organization of the paper is as follows. In Sec. II, we briefly review the index theorem on $S^1 \times R^3$ for an adjoint fermion with twisted boundary condition. In Sec. III, we detail the ADHM construction and use it to derive the antiperiodic adjoint fermion in self-dual BPS dyons. In Sec. IV, we develop the partition function of an instanton-dyon ensemble with one light quark in the adjoint representation of $SU_c(2)$. By using a series of fermionization and bosonization techniques, we construct the three-dimensional effective action, accommodating light adjoint quarks with explicit $SU(N_f) \times Z_{4N_f}$ flavor symmetry. In Sec. V, we discuss the nature of the confinement-deconfinement in the quenched sector ($N_f = 0$) of the induced effective action. In Sec. VI, we show that for a sufficiently dense instanton-dyon liquid with light adjoint quarks the three-dimensional ground state is still center symmetric and spontaneously breaks $SU(N_f) \times Z_{4N_f} \to O(N_f)$ flavor symmetry. Center symmetry is broken, and chiral symmetry is restored only in a more dilute instanton-dyon liquid, corresponding to higher temperatures. Our conclusions are summarized in Sec. VII. In Appendix A, we check that our ADHM construct reproduces the expected periodic zero modes for BPS dyons. In Appendix B, we derive the pertinent equations for the antiperiodic adjoint fermions in a BPS monopole without using the ADHM method. In Appendix C, we detail the ADHM construction for the antiperiodic zero modes in a KvBLL caloron. In Appendix D, we detail the Fock correction to the mean-field analysis. In Appendix E, we briefly outline the one-loop analysis for completeness. In Appendix F, we quote the general result for the one-loop contribution to the holonomy potential with N_f adjoint massless quarks.

II. INDEX THEOREM FOR TWISTED QUARKS

In this section, we revisit the general Nye-Singer index theorem for fermions on a finite temperature Euclidean manifold $S^1 \times R^3$ for a general fermion representation. For periodic fermions, a very transparent analysis was provided by Popitz and Unsal [19]. We will extend it to fermions with arbitrary "twist" (phase), which is the used for our case of antiperiodic fermions in the adjoint representation.

A. Index

Consider chiral Dirac fermions on $S^1 \times R^3$ interacting with an anti-self-dual gauge field A through

$$(D \equiv \gamma_\mu D_\mu \equiv \gamma_\mu(\partial_\mu + igT^a A^a_\mu))\Psi(x) = 0 \tag{1}$$

with twisted fermion boundary conditions ($\beta = 1/T$)

$$\Psi(x_4 + \beta, \mathbf{x}) = e^{i\varphi}\Psi(x_4, \mathbf{x}). \tag{2}$$

Here, D satisfies

$$D^\dagger D = -D_\mu D_\mu + 2\sigma^m B_m = DD^\dagger + 2\sigma^m B_m. \tag{3}$$

For monopoles, the difference between the zero modes of different chiralities in arbitrary R representation is captured by the Calias index [20]

$$\mathbb{I}_R = \lim_{M\to 0} M\mathrm{Tr}\langle\Psi^\dagger\gamma_5\Psi\rangle = \lim_{M\to 0}\mathrm{Tr}\left(\gamma_5\frac{M}{-D+M}\right) \tag{4}$$

with the trace carried over spin-color-flavor and space-time. Using the local chiral anomaly condition for the isosinglet axial current $J^5_\mu = \Psi^\dagger\gamma_5\gamma_\mu\Psi$ in Euclidean four-dimensional space

$$\partial_\mu J^5_\mu = -2M\Psi^\dagger\gamma_5\Psi - \frac{T_R}{8\pi^2}F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}, \tag{5}$$

we can rewrite the index in the form

$$\mathbb{I}_R = -\frac{1}{2}\int_{S^1\times S^2} d\sigma^2_k\langle J^5_k\rangle - \frac{T_R}{16\pi^2}\int_{S^1\times R^3} F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} \tag{6}$$

with T_R the Casimir operator in the R representation. The second contribution ($\mathbb{I}_2$) in (6) depends only on the gauge field, but the first contribution ($\mathbb{I}_1$) in Eq. (6) depends on the nature of the fermion field.

B. L, M Dyons

To evaluate (6) for twisted $SU_c(2)$ adjoint fermions in the background of an anti-self-dual or $\bar{M}$ dyon, we follow Popitz and Unsal [19] and write

$$\begin{aligned}\langle J^5_k\rangle &\equiv \mathrm{Tr}\langle x|\gamma^k\gamma_5 D\frac{1}{-D^2+M^2}|x\rangle \\ &= \mathrm{Tr}\langle x|i\sigma^k D_4\left(\frac{1}{-D^2+M^2+2\sigma\cdot B} - \frac{1}{-D^2+M^2}\right)|x\rangle.\end{aligned} \tag{7}$$

In the $\bar{M}$ antidyon background, we have at asymptotic spatial infinity

$$\begin{aligned}-D^2 &\to -\nabla^2 + \left(\langle A_4\rangle + \frac{\pi(2p+\frac{\varphi}{\pi})}{\beta}\right)^2 \\ B_m &\to -\frac{r_m}{r^3}.\end{aligned} \tag{8}$$

The compact character of A_4 on S^1 breaks $SU_c(2) \to \mathbf{Ab}(SU_c(2))$. After expanding the ratio with B in (7), only the first term carries a nonvanishing net flux in (6) on S^2 thanks to the asymptotic in (8). If we recall that the trace now carries a summation over the windings along S^1 labeled by p and use the identity

$$\sum_{p=-\infty}^{\infty} \mathrm{sgn}(x+p) = 1 - 2x + 2[x], \tag{9}$$

we have

$$\begin{aligned}\mathbb{I}_1 &= -\sum_{m=-1}^{m=1} m \sum_{p=-\infty}^{\infty} \mathrm{sgn}\left(-\frac{2\pi\nu}{\beta}m + \frac{\pi(2p+\frac{\varphi}{\pi})}{\beta}\right) \\ &= -4\nu + 2\left[\nu + \frac{\varphi}{2\pi}\right] - 2\left[-\nu + \frac{\varphi}{2\pi}\right].\end{aligned} \tag{10}$$

For color $SU_c(2)$, $T_R = 1/2$ in the fundamental representation, and $T_R = 2$ in the adjoint representation. For the latter,

$$\mathbb{I}_2 = -\frac{2}{16\pi^2}\int_{S^1\times R^3} F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = 4\nu, \tag{11}$$

it follows that

$$\mathbb{I}_M = \mathbb{I}_1 + \mathbb{I}_2 = 2\left[\nu + \frac{\varphi}{2\pi}\right] - 2\left[-\nu + \frac{\varphi}{2\pi}\right]. \tag{12}$$

We note that Eq. (12) was originally derived in Ref. [18].

For the L dyon, we note that the surface contributions satisfy $\mathbb{I}_{1L} = -\mathbb{I}_{1M}$ since the asymptotics at spatial infinity have the same A_4 with B_m of opposite sign. Therefore, we obtain

$$\mathbb{I}_L = 4 - \mathbb{I}_M \tag{13}$$

whatever the twist φ as expected. For antiperiodic fermions with $\varphi = \pi$, we find that for $\nu < \frac{1}{2}$ the L dyon carries four antiperiodic zero modes, and the M dyon carries no zero mode. For $\frac{1}{2} < \nu < 1$, the M dyon carries four zero modes, and the L dyon carries no zero mode. The confining holonomy with $\nu = \frac{1}{2}$ is special as the zero modes are shared equally between the L and M dyons, two on each.

III. ADHM CONSTRUCTION OF ADJOINT ZERO MODES

In this section, we first remind the reader of the general framework for the ADHM [4,6,21,22] construction for adjoint fermions and then apply it to the special case of adjoint fermions in the background field of BPS dyons. A concise presentation of this approach can be found in Refs. [6,22], the notations of which we will use below. Throughout this section, we will set the circle circumference $\beta = 1/T \to 1$, unless specified otherwise. We note that our construction is similar in spirit to the one presented in Ref. [23] for adjoint fermions in calorons but is different in some details. In particular, it does not rely on the replica trick and therefore does not double the size of the ADHM data.

A. ADHM construction

The basic building block in the ADHM construction is the asymmetric matrix of data $\Delta(x)$ of dimension $[N+2k]\times[2k]$ for an $SU(N)$ gauge configuration of topological charge k. The null vectors of $\Delta(x)$ can be assembled into a matrix-valued complex matrix $U(x)$ of dimension $[N+2k]\times[N]$, satisfying $\bar{\Delta}U = 0$ or specifically

$$\bar{\Delta}_i^{\dot\alpha\lambda} U_{\lambda u} = 0 \tag{14}$$

with the ADHM label $\lambda = u + i\alpha$ running over $1 \le u \le N$, $0 \le i \le k$ and α, $\dot\alpha = 1, 2$ referring to the Weyl-Dirac indices which are raised by ϵ_2. They are orthonormalized by $\bar{U}U = \mathbf{1}_N$. In terms of Eq. (14), the classical ADHM gauge field A_m with $1 \le m \le 4$ reads

$$A_m = \bar{U} i\partial_m U. \tag{15}$$

For $k = 0$, it is a pure gauge transformation with a field strength A_{mn} that satisfies the self-duality condition $A_{mn} = {}^*A_{mn}$. For $k \ne 0$, it still satisfies the self-duality condition provided that [22]

$$\bar{\Delta}_i^{\dot\beta\lambda}\Delta_{\lambda j\dot\alpha} = \delta^{\dot\beta}_{\dot\alpha} f^{-1}_{ij} \tag{16}$$

with $f^\dagger = f$ a positive matrix of dimension $[k]\times[k]$. The matrix of data is taken to be linear in the space-time variable x_n,

$$\begin{aligned}\Delta_{\lambda i\dot\alpha} &= a_{\lambda i\dot\alpha} + b^\alpha_{\lambda i} x_{\alpha\dot\alpha} \\ \bar{\Delta}_i^{\dot\alpha\lambda} &= \bar{a}_i^{\dot\alpha\lambda} + \bar{x}^{\dot\alpha\alpha}\bar{b}^\lambda_{\alpha i},\end{aligned} \tag{17}$$

with the quaternionic notation $x_{\alpha\dot\alpha} = x_n(\sigma_n)_{\alpha\dot\alpha}$ and $\sigma_n = (\mathbf{1}_2, i\vec{\sigma})$.

B. Antiperiodic adjoint fermion in general

Given the matrix of ADHM data as detailed above, the adjoint fermion zero mode in a self-dual gauge configuration of topological charge k reads [22]

$$\lambda_\alpha = \bar{U}Mf\bar{b}_\alpha U - \bar{U}b_\alpha f\bar{M}U, \tag{18}$$

which can be checked to satisfy the Weyl-Dirac equation provided that the Gassmanian matrix $M \equiv M_{\lambda i}$ of dimension $[N+2k]\times[k]$ satisfies the constraint condition

$$\bar{\Delta}^{\dot{\alpha}} M + \bar{M}\Delta^{\dot{\alpha}} = 0. \quad (19)$$

To unravel the constraints (16) and (19), it is convenient to rewrite the ADHM matrix of data $\Delta(x)$ in quaternionic blocks through a pertinent choice of the complex matrices a, b, i.e.,

$$\Delta(x) = \begin{pmatrix} \xi \\ B - x\mathbf{1}_2 \end{pmatrix}, \quad (20)$$

with

$$\xi \equiv \xi_{ui\dot{\alpha}} \equiv (\xi_{\dot{\alpha}})_{ui}$$
$$B \equiv (B_{\alpha\dot{\alpha}})_{ij}. \quad (21)$$

In quaternionic blocks, the null vectors (14) are

$$U(x) \equiv \frac{1}{\sqrt{\phi(x)}} \begin{pmatrix} -\mathbf{1}_2 \\ u(x) \end{pmatrix} = \frac{1}{\sqrt{\phi(x)}} \begin{pmatrix} -\mathbf{1}_2 \\ (B^\dagger - x^\dagger \mathbf{1}_2)^{-1}\xi^\dagger \end{pmatrix} \quad (22)$$

with the normalization $\phi(x) = 1 + u^\dagger(x)u(x)$. To solve the constraint condition (19), we also define

$$M \equiv \begin{pmatrix} c_{uj} \\ M_{\alpha ij} \end{pmatrix} \quad (23)$$

and its conjugate $\bar{M} \equiv (\bar{c}_{ju}, \bar{M}^{\alpha}_{ji})$. Therefore, the solution to (19) satisfies $M^\alpha = \bar{M}^\alpha$ and the new constraint between the Grassmanians

$$[M^\alpha, B_{\alpha\dot{\alpha}}] + \bar{c}\xi_{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}} c = 0. \quad (24)$$

Finally, for periodic gauge configurations on $S^1 \times R^3$ such as the KvBLL calorons or BPS dyons, the index k is extended to all charges in $\mathbb{Z}$. It is then more convenient to use the Fourier representations

$$f(z) = \sum_{k=-\infty}^{\infty} f_k e^{i2\pi kz}$$
$$B(z,z') = \sum_{k,l=-\infty}^{\infty} B_{kl} e^{2\pi i(kz - lz')}, \quad (25)$$

which are z periodic of period 1.

C. Antiperiodic adjoint fermion in a BPS dyon

For BPS dyons, the previous arguments apply [24]. In particular, for the $SU(2)$ M dyon on $S^1 \times R^3$, the preceding construct is simplified. In particular, the quaternion blocks in the ADHM matrix of data in Eq. (20) are simply

$$\xi = 0$$
$$B(z,z') = \delta(z - z')\frac{1}{2\pi i\nu}\frac{\partial}{\partial z}. \quad (26)$$

The normalized null vector is readily found in the form

$$U = \begin{pmatrix} 0 \\ u(x,z) \end{pmatrix} \quad (27)$$

with

$$u(x,z) = \left(\frac{2\pi v r}{\sinh(2\pi v r)}\right)^{\frac{1}{2}} e^{i2\pi z v(x_4 - i\sigma\cdot x)} \quad (28)$$

with the vev $v = \nu/\beta$.

The constraint (16) following from the self-duality condition translates to the equation for the resolvent

$$\left(i\frac{\partial}{\partial z} + 2\pi\nu x_4\right)^2 f(z,z') + (2\pi\nu r)^2 f(z,z') = \delta(z - z'). \quad (29)$$

The solution is

$$f(z,z') = -\frac{e^{2\pi i v x_4(z-z')}}{8\pi v r}(\sinh(2\pi v r|z - z'|) + \coth(\pi v r)\sinh(2\pi v r z)\sinh(2\pi v r z') - \tanh(\pi v r)\cosh(2\pi v r z)\cosh(2\pi v z')). \quad (30)$$

We have explicitly checked that Eq. (30) satisfies the identities used in the ADHM construction as noted in Ref. [22]. In our case, these identities read

$$2\int_{-\frac{1}{2}}^{\frac{1}{2}} dz_1 \tilde{f}(z,z_1)\left(\frac{\partial}{\partial z_1}\right)\tilde{f}(z_1,z') = -(z - z')\tilde{f}(z,z')$$
$$-\frac{\partial}{\partial x_i}\tilde{f}(z,z') = 2x_i(2\pi\nu)^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} dz_1 \tilde{f}(z,z_1)\tilde{f}(z_1,z') \quad (31)$$

with the definition $f/\tilde{f} = e^{2\pi i x_4(z-z')}$, and

$$\delta(z - z') - \frac{\partial^2}{\partial z\partial z'} f(z,z') - 2\pi\nu r\sigma\cdot\hat{r}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z'}\right) f(z,z') - (2\pi\nu r)^2 f(z,z') = \frac{2\nu\pi r}{\sinh(2\pi\nu r)}(\cosh(2\pi\nu r(z + z')) + \sigma\cdot\hat{r}\sinh(2\pi\nu r(z + z'))). \quad (32)$$

We note that the periodicity on S^1 translates to the quasiperiodicities

$$u(x_4+\beta,\vec{x},z)=e^{2\pi i\nu z}u(x_4,\vec{x},z)$$
$$f(x_4+\beta,\vec{x},z,z')=e^{2\pi i\nu(z-z')}f(x_4,\vec{x},z,z'). \tag{33}$$

For the adjoint fermion zero mode, the Grassmanian matrix also simplifies

$$M(z-z')=M(z')\delta\left(z-z'\pm\frac{1}{2\nu}\right). \tag{34}$$

Inserting Eq. (34) in the constraint equation (24) and noting that now $\xi=0$ yield

$$\frac{d}{dz}M(z)=0\rightarrow M(z)=M^{\pm} \tag{35}$$

with normalized constant spinors $M^{\pm}$. This allows us to rewrite Eq. (34) in the explicit form

$$M(z-z')=M^{+}\delta\left(z-z'+\frac{1}{2\nu}\right)+M^{-}\delta\left(z-z'-\frac{1}{2\nu}\right). \tag{36}$$

With the above in mind, the adjoint zero-mode solution (18) in the $SU_c(2)$ BPS M dyon is simplified to

$$\lambda^{\pm}_{\alpha}(x)=-\int_{-\frac{1}{2}}^{+\frac{1}{2}}dzdz'u^{\dagger}(x,z)\epsilon M^{\pm}f\left(z\mp\frac{1}{2\nu},z'\right)u_{\alpha}(x,z')$$
$$-\int_{-\frac{1}{2}}^{+\frac{1}{2}}dzdz'u^{\dagger}_{\alpha}(x,z)f(z,z')M^{\pm,T}u\left(x,z'\mp\frac{1}{2\nu}\right). \tag{37}$$

For $\nu>1/2$, the integrations can be undone. For that, we translate the vectors in Eq. (37) to spinors using the quaternionic form $\lambda^{\pm}_{m}=\lambda^{\pm}_{\alpha ab}\sigma_{mba}$ and make Eq. (37) more explicit. The result is

$$\lambda^{\pm}_{m}(x)=(f_1(r)\sigma_m+f_2(r)\sigma\cdot\hat{r}\sigma_m\sigma\cdot\hat{r}$$
$$\pm f_3(r)\sigma_m\sigma\cdot\hat{r}\pm f_4(r)\sigma\cdot\hat{r}\sigma_m)\chi \tag{38}$$

with $f_{1,2,3,4}$ defined as

$$\begin{aligned}
-16s^2\sinh(s)\cosh(s/2)\sinh(s/2)f_1 &= s^2(-(x^2-1))\cosh(s(x-1))+s^2x^2\cosh(s(x+1))\\
&\quad+2s^2x\cosh(sx)-2s^2\cosh(sx)-2s^2x\cosh(s(x+1))+s^2\cosh(s(x+1))\\
&\quad-sx\sinh(s(x-2))+2s\sinh(s(x-1))+sx\sinh(sx)-2s\sinh(sx)\\
&\quad+\cosh(s(x-2))-\cosh(sx)\\
16s^2\sinh(s)\cosh(s/2)\sinh(s/2)f_2 &= (1-2s^2(x-1))\cosh(sx)+s(-s(x^2-1)\cosh(s-sx)\\
&\quad+x\sinh(s(x-2))-x\sinh(sx)+2\sinh(sx)\\
&\quad-2\sinh(s-sx)+s(x-1)^2\cosh(s(x+1)))-\cosh(s(x-2))\\
-16s\sinh(s)\cosh(s/2)\sinh(s/2)f_3 &= x\cosh(s(x-2))+x(2s(x-1)\sinh(s)-1)\cosh(sx)\\
&\quad-2s(x-1)(\cosh(s)-1)\sinh(sx)\\
8s\sinh(s)\cosh(s/2)\sinh(s/2)f_4 &= \sinh(sx)(s(-x)+\cosh(s)(s(-x)+x\sinh(s)+s)+s)\\
&\quad-x\sinh(s)(s(-x)+s+\sinh(s))\cosh(sx),
\end{aligned} \tag{39}$$

where we have set $s=2\nu\omega_0 r$ and $x=1/2\nu$. Asymptotically, the zero modes (39) are simplified to

$$f_1\approx-f_2\approx f_3\approx-f_4\rightarrow(2\nu-1)^2e^{\omega_0(1-2\nu)r}, \tag{40}$$

and therefore Eq. (38) is asymptotically ($r\rightarrow\infty$)

$$\lambda^{\pm}_{m}(x)\approx(1\mp\sigma\cdot r)\sigma_m(1\pm\sigma\cdot r)e^{\omega_0(1-2\nu)r}\chi. \tag{41}$$

We will use this approximation to carry out explicitly the analysis below. The four zero modes (41) are localized on the M dyon for $\nu>1/2$ and by duality on the L-dyon for $\nu<1/2$, in agreement with the index theorem reviewed above. For $\nu<1/2$, the integration vanishes with $\lambda^{+}\equiv0$.

For $\nu>1/2$, we note that the four adjoint zero modes are normalizable as they fall asymptotically with $e^{\omega_0(1-2\nu)r}$. Equation (37) can be explicitly checked to be normalized as

$$\int_{R^3}d^3x\mathrm{Tr}(\lambda^{\pm}_{\alpha}\lambda'^{\mp}_{\beta}\epsilon_{\alpha\beta})=\frac{1}{8(2\nu\omega_0^2)}\int_{S^2}d\vec{S}\cdot\vec{\nabla}\mathrm{Tr}_z(\bar{M}(P+1)$$
$$\times M'f+\bar{M}'(P+1)Mf)$$
$$=\frac{\pi^2(1-\frac{1}{2\nu})}{2(\nu\omega_0)^3}\bar{M}'\epsilon\bar{M}. \tag{42}$$

We note that at $\nu=1/2$ the normalization vanishes. This is precisely where the zero modes reorganize equally between the L- and M-instanton dyons, a pair on each.

D. Antiperiodic adjoint fermion in a BPS dyon with $\nu = \frac{1}{2}$

The case $\nu = 1/2$ for the adjoint zero mode is more subtle. The preceding arguments show that the asymptotic is 1, i.e.,: $e^{\omega_0(1-1)r} = 1$. In this limit, the index theorem states that two zero modes are localized on the M dyon and two zero modes are localized on the L dyon. In this section, we show that the reduction of the result (37) for $\nu = 1/2$ simplifies. Specifically,

$$(\lambda_\alpha^\pm)_{ab}(r) = \frac{1}{\sinh(\omega_0 r)}\left(\left(\cosh\left(\frac{\omega_0 r}{2}\right) \pm \sigma\cdot\hat{r}\sinh\left(\frac{\omega_0 r}{2}\right)\right)_{a\beta}(\epsilon M)_\beta(f(\omega_0 r) \pm g(\omega_0 r)\sigma\cdot\hat{r})_{\alpha b} - \epsilon_{\alpha\beta}(f(\omega_0 r) \mp g(\omega_0 r)\sigma\cdot\hat{r})_{a\beta}M_\gamma\left(\cosh\left(\frac{\omega_0 r}{2}\right) \mp \sigma\cdot\hat{r}\sinh\left(\frac{\omega_0 r}{2}\right)\right)_{\gamma b}\right) \quad (43)$$

with

$$f(\omega_0 r) = \frac{1}{4\cosh(\frac{\omega_0 r}{2})}(-\omega_0 r - \sinh(\omega_0 r))$$
$$g(\omega_0 r) = \frac{1}{4\sinh(\frac{\omega_0 r}{2})}(-\omega_0 r + \sinh(\omega_0 r)). \quad (44)$$

Equation (43) can be written in a more concise form by translating the vectors to spinors using the quaternionic form

$$\lambda_m^\pm = \lambda_{\alpha ab}^\pm \sigma_{mba} \quad (45)$$

with

$$(\lambda_\alpha^\pm)(r) = \frac{1}{\sinh(\omega_0 r)}(f(\omega_0 r) \pm g(\omega_0 r)\sigma\cdot\hat{r})\sigma_m \times \left(\cosh\left(\frac{\omega_0 r}{2}\right) \pm \sigma\cdot\hat{r}\sinh\left(\frac{\omega_0 r}{2}\right)\right)\epsilon M - \epsilon\left(M^T\left(\cosh\left(\frac{\omega_0 r}{2}\right) \mp \sigma\cdot\hat{r}\sinh\left(\frac{\omega_0 r}{2}\right)\right) \times \sigma_m(f(\omega_0 r) \pm g(\omega_0 r)\sigma\cdot\hat{r})\right)^T. \quad (46)$$

Using $\sigma_m^T = \epsilon\sigma_m\epsilon$, the transpose of the second term in Eq. (46) can be reduced. The result is

$$(\lambda_\alpha^\pm)(r) = \frac{2}{\sinh(\omega_0 r)}(f(\omega_0 r) \pm g(\omega_0 r)\sigma\cdot\hat{r})\sigma_m \times \left(\cosh\left(\frac{\omega_0 r}{2}\right) \pm \sigma\cdot\hat{r}\sinh\left(\frac{\omega_0 r}{2}\right)\right)\chi \quad (47)$$

with the identified spinor $\chi = \epsilon M$. The (color-)invariant group norm of Eq. (47) is finite. Specifically, if we set $\lambda_{m,\alpha}^\pm = B_{\alpha\beta}^{m\pm}\chi_\beta$, then

$$\mathrm{Tr}(\lambda_\alpha^\pm\lambda_\beta^\pm\epsilon_{\alpha\beta}) = \chi^T\sum_m B^{mT}\epsilon B^m\chi = -3\chi^T\epsilon\chi\frac{(f^2(\omega_0 r) - g^2(\omega_0 r))}{\mathrm{sh}^2(\omega_0 r)}, \quad (48)$$

which is convergent in R^3. Note the difference between the Matsubara arrangements in Eqs. (48) and (42). For completeness, we note that Eq. (48) is the analog of the gluino condensate using the antiperiodic zero modes. The periodic zero modes are briefly discussed in Appendix A using the same ADHM construct. In Appendix B, we verify explicitly that the ADHM zero modes are consistent with a direct reduction of the Dirac equation. For completeness, we detail in Appendix C the ADHM construct for the zero modes around KvBLL instantons.

IV. PARTITION FUNCTION WITH ADJOINT FERMIONS

In this section, we will use the adjoint zero modes made explicit in Eqs. (37)–(41) to construct the partition function for an ensemble of interacting dyons and antidyons with adjoint fermions. We will show that the partition function is amenable to a three-dimensional effective theory. The derivation will be for the nonsymmetric case with $\nu > 1/2$, where all four adjoint zero modes are localized on the M dyon (antidyon). The nonsymmetric case with $\nu < 1/2$ with the adjoint zero modes localized on the L dyon (antidyon) is equivalent and follows by duality $L \leftrightarrow M$ and $\nu \to \bar{\nu} = 1 - \nu$. The symmetric case with each L and M dyon carrying two of the four adjoint zero modes will be understood in the limit $\nu \to 1/2$.

A. Partition function

In the semiclassical approximation, the Yang-Mills partition function is assumed to be dominated by an interacting ensemble of instanton dyons (antidyons). They are constituents of KvBLL instantons (anti-instantons) with fixed holonomy [4]. The $SU_c(2)$ grand-partition function with N_f adjoint Majorana quarks is

$$\mathcal{Z}_1[T] \equiv \sum_{[K]} \prod_{i_L=1}^{K_L} \prod_{i_M=1}^{K_M} \prod_{i_{\bar{L}}=1}^{K_{\bar{L}}} \prod_{i_{\bar{M}}=1}^{K_{\bar{M}}} \times \int \frac{f_L d^3 x_{Li_L}}{K_L!} \frac{f_M d^3 x_{Mi_M}}{K_M!} \frac{f_L d^3 y_{\bar{L}i_{\bar{L}}}}{K_{\bar{L}}!} \frac{f_M d^3 y_{\bar{M}i_{\bar{M}}}}{K_{\bar{M}}!} \times \det(G[x]) \det(G[y]) |\det \tilde{\mathbf{T}}(x,y)|^{\frac{N_f}{2}} \times e^{-V_{D\bar{D}}(x-y)} e^{-V_L(x-y)} e^{-V_M(x-y)}. \tag{49}$$

Here, x_{mi} and y_{nj} are the three-dimensional coordinate of the i dyon of m kind and j antidyon of n kind. Here, $G[x]$ is a $(K_L + K_M)^2$ matrix, and $G[y]$ is a $(K_{\bar{L}} + K_{\bar{M}})^2$ matrix, of which the explicit forms are given in Refs. [5,6]. The fugacities f_i are related to the overall dyon plus antidyon density n_D [25].

$V_{D\bar{D}}$ is the streamline interaction between $D = L$, M dyons and $\bar{D} = \bar{L}, \bar{M}$ antidyons as numerically discussed in Refs. [1,7]. For the $SU(2)$ case, it is Coulombic asymptotically [1],

$$V_{D\bar{D}}(x-y) \to -\frac{C_D}{\alpha_s T}\left(\frac{1}{|x_M - y_{\bar{M}}|} + \frac{1}{|x_L - y_{\bar{L}}|} - \frac{1}{|x_M - y_{\bar{L}}|} - \frac{1}{|x_L - y_{\bar{M}}|}\right). \tag{50}$$

The strength of the Coulomb interaction in Eq. (50) is $C_D = 2$. Following Ref. [12], we define the core interactions $V_{L,M}(x-y)$ between $L\bar{L}$ and $M\bar{M}$, respectively, which we assume to be step functions of height V_0 and range x_0,

$$V_M(x-y) = TV_0 \theta(x_0 - 2\omega_0 \nu |x-y|)$$
$$V_L(x-y) = TV_0 \theta(x_0 - 2\omega_0 \bar{\nu} |x-y|), \tag{51}$$

with $x_0/2$ normalized to the dimensionless unit volume

$$\left(\frac{x_0}{2}\right)^3 = \frac{4\pi}{3}. \tag{52}$$

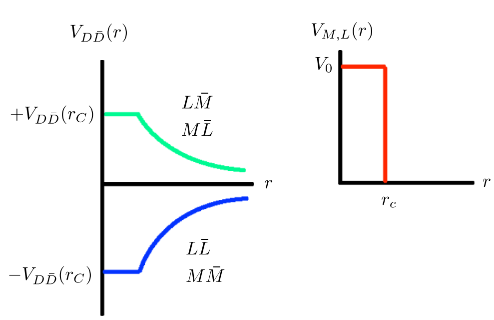

FIG. 1. Schematic description for the streamline (left) and core (right) potentials between a pair of an $SU_c(2)$ instanton dyon and anti-instanton dyon.

We recall that the $L\bar{M}$ and $M\bar{L}$ channels are repulsive. A sketch of the interaction potentials is given in Fig. 1. Below the core value of $a_{D\bar{D}}$, the streamline configuration annihilates into perturbative gluons.

B. Determinant of the adjoint fermions

The fermionic determinant in Eq. (49) is composed of all the hoppings between the dyons and antidyons through the adjoint fermionic zero modes. To make the hopping explicit, we consider in details the case $N_f = 1$ and only quote at the end the generalization to arbitrary N_f. To explicit the hopping for $N_f = 1$, we define

$$\Psi(x) \equiv \sum_{I,\pm} \Psi^{\pm}(x - x_I)\chi_I^{\pm}$$
$$\bar{\Psi}(x) \equiv \sum_{\bar{J},\pm} \bar{\Psi}^{\pm}(x - x_{\bar{J}})\bar{\chi}_{\bar{J}}^{\pm}, \tag{53}$$

with the sum running over all dyons and antidyons and the two Matsubara frequencies $\pm\omega_0$ subsumed in the zero modes. The adjoint dyon and antidyon zero modes are labelled by

$$\lambda_D^{\pm}(x) \equiv \Psi^{\pm}(x - x_D)\chi_D^{\pm}. \tag{54}$$

Here, $\chi_D^{\pm}$ is a two-component Grassmanian spinor, and $\Psi^{\pm}$ is a 2×2 valued matrix, both of which refer to a D dyon (antidyon). From Eqs. (37)–(41), the Fourier transforms of $\Psi^{\pm}$ read

$$\tilde{\nu}^{-\frac{3}{2}}\Psi_m^{+}(p) = f_1(p)\sigma_m + if_2(p)[\sigma_m, \sigma\cdot\hat{p}] + f_3(p)\hat{p}_m \sigma\cdot\hat{p} \tag{55}$$

with

$$f_1(p) = \frac{\tilde{\nu}}{(p^2+\tilde{\nu}^2)^2} + \frac{1}{p^3}\left(\tilde{\nu}p\frac{(2p^2+\tilde{\nu}^2)}{(p^2+\tilde{\nu}^2)^2} - \tan^{-1}\left(\frac{p}{\tilde{\nu}}\right)\right)$$
$$f_2(p) = \frac{p}{(p^2+\tilde{\nu}^2)^2}$$
$$f_3(p) = -\frac{1}{p^3}\left(\frac{p\tilde{\nu}(5\tilde{p}^2 + 3\tilde{\nu}^2)}{(p^2+\tilde{\nu}^2)^2} - 3\tan^{-1}\left(\frac{p}{\tilde{\nu}}\right)\right). \tag{56}$$

Here, $p = |\vec{p}|$ and $\tilde{\nu} = (2\nu - 1)\omega_0$.

In terms of Eqs. (53)–(54), the hopping action for massive adjoint quarks takes the explicit form

$$i\int d^4x (\Psi^T, \bar{\Psi}^T)\begin{pmatrix} m & \epsilon\sigma\cdot\partial \\ -\epsilon\bar{\sigma}\cdot\partial & m \end{pmatrix}\begin{pmatrix}\Psi \\ \bar{\Psi}\end{pmatrix} = \sum_{\pm}(\chi_I^{T\pm}, \bar{\chi}_J^{T\mp})\begin{pmatrix} im\tilde{\mathbf{K}}(x_{II'}) & \mathbf{T}^{\pm}(x_{IJ}) \\ -\mathbf{T}^{T\pm}(x_{IJ}) & -im\tilde{\mathbf{K}}(x_{JJ'}) \end{pmatrix}\begin{pmatrix}\chi_I^{\pm} \\ \bar{\chi}_J^{\mp}\end{pmatrix} \tag{57}$$

with $x_{IJ} \equiv x_I - x_J$. We note that the matrix entries in Eq. (57) are 2×2 valued or quaternionic and that the matrix overall is antisymmetric under transposition. This observation is consistent with the observations made in Ref. [26]. The matrix entries in Eq. (57) satisfy

$$\begin{aligned} \mathbf{T}^{\pm}(x_{IJ}) &= -\epsilon\tilde{\mathbf{T}}^{\pm}(x_{IJ}) \\ \mathbf{T}^{T\pm}(x_{IJ}) &= -\epsilon\tilde{\mathbf{T}}^{\mp\dagger}(x_{IJ}) \end{aligned} \tag{58}$$

$$\tilde{\mathbf{K}}(x_{II'}) = -\epsilon\mathbf{K}(x_{II'}). \tag{59}$$

Using Eq. (58), we can rewrite Eq. (49) for massive fermions in the basis $(\chi^+,\chi^-,\bar\chi^+,\bar\chi^-)^T$ as

$$|\det\tilde{\mathbf{T}}(x,y)|^{\frac{1}{2}} \equiv \left|\det\begin{pmatrix} 0 & -\epsilon m\mathbf{K}_{ii'} & 0 & i\epsilon\tilde{\mathbf{T}}^{+}_{ij} \\ -m\epsilon\mathbf{K}_{ii'} & 0 & -i\epsilon\tilde{\mathbf{T}}^{+\dagger}_{ji} & 0 \\ 0 & -i\tilde{\mathbf{T}}^{+\star}_{ij}\epsilon & 0 & m\epsilon\mathbf{K}_{ii'} \\ i\tilde{\mathbf{T}}^{+T}_{ji}\epsilon & 0 & m\epsilon\mathbf{K}_{ii'} & 0 \end{pmatrix}\right|^{\frac{1}{2}}, \tag{60}$$

with dimensionality $4(K_I + K_{\bar I})^2$. Each of the quaternionic entries in $\tilde{\mathbf{T}}^{+}_{ij}$ is a "hopping amplitude" for a fermion between an instanton dyon and an instanton antidyon. Each of the quaternion entries in $\mathbf{K}_{ii'}$ is an overlap between two instanton dyons or two anti-instanton anti-lyons.

C. Hopping amplitudes

In momentum space, the quaternionic entries are given by

$$\mathbf{T}^{\pm}(p) = \Psi^{T\pm}(-p)\epsilon\sigma\cdot p_{\pm}\bar\Psi^{\mp}(p) \tag{61}$$

with again $p_\pm = (\pm\omega_0, \vec{p})$. Since

$$\Psi^T(p) = \epsilon\Psi(p)\epsilon, \tag{62}$$

we also have the identities

$$\begin{aligned} \mathbf{T}^{\pm}(p) &= -\epsilon\Psi^{\pm}(-p)(\pm\omega_0 + i\sigma\cdot p)\bar\Psi^{\mp}(p) \\ \mathbf{T}^{T\pm}(p) &= -\epsilon\bar\Psi^{\mp}(p)(\mp\omega_0 + i\sigma\cdot p)\Psi^{\pm}(-p). \end{aligned} \tag{63}$$

We note the relations

$$\begin{aligned} \Psi^{\pm}(p) &= \bar\Psi^{\mp}(p) \\ (\Psi^{\pm})^\dagger(-p) &= \Psi^{\mp}(p), \end{aligned} \tag{64}$$

and therefore we have the additional identities

$$\begin{aligned} \mathbf{T}^{\pm}(p) &= -\epsilon\tilde{\mathbf{T}}^{\pm}(p) \\ \mathbf{T}^{T\pm}(p) &= -\epsilon\tilde{\mathbf{T}}^{\mp^\dagger}(-p). \end{aligned} \tag{65}$$

Here, we have

$$\tilde{\mathbf{T}}^{+}(p) = \Psi^{+}(-p)(\omega_0 + i\sigma\cdot p)\Psi^{+}(p) \tag{66}$$

or more explicitly

$$\begin{aligned} \omega_0^3\tilde\nu^{-3}\tilde{\mathbf{T}}^{+}(p) = {} & \left(3f_1^2 + f_3^2 + 2f_1f_3 - 8f_2^2 + 8f_1f_2\frac{p}{\omega_0}\right)\omega_0 \\ & + i\sigma\cdot p\left(-f_1^2 + f_3^2 + 2f_1f_3 + 8f_2^2 \right. \\ & \left. + 8f_1f_2\frac{\omega_0}{p}\right). \end{aligned} \tag{67}$$

We also have

$$\begin{aligned} \mathbf{K}(p) &= \Psi^{-}(-p)\Psi^{+}(p) = \Psi^{+\dagger}(p)\Psi^{+}(p) \\ &= \omega_0^{-3}\tilde\nu^3(3f_1^2 + f_3^2 + 2f_1f_3 + 8f_2^2). \end{aligned} \tag{68}$$

V. EFFECTIVE ACTION WITHOUT ADJOINT FERMIONS

In this section, we will derive the three-dimensional effective action in the case without the adjoint fermions, to be referred to as the $N_f = 0$ case below. We will analyze it in the limit of weak coupling and large densities across the transition region. We will explicitly derive the induced effective potential for the $SU_c(2)$ holonomies $\nu, \bar\nu$ and show that for a critical density the ground state of the three-dimensional effective theory is confined.

A. Bosonic fields

Following Refs. [1,2,5], the moduli determinants in Eq. (49) can be fermionized using four pairs of ghost fields $\chi^\dagger_{L,M}, \chi_{L,M}$ for the dyons and four pairs of ghost fields $\chi^\dagger_{\bar L,\bar M}, \chi_{\bar L,\bar M}$ for the antidyons. The ensuing Coulomb factors from the determinants are then bosonized using four boson fields $v_{L,M}$, $w_{L,M}$ for the dyons and similarly for the antidyons. The result is

$$\begin{aligned} S_{1F}[\chi, v, w] = {} & -\frac{T}{4\pi}\int d^3x(|\nabla\chi_L|^2 + |\nabla\chi_M|^2 + \nabla v_L\cdot\nabla w_L \\ & + \nabla v_M\cdot\nabla w_M) + (|\nabla\chi_{\bar L}|^2 + |\nabla\chi_{\bar M}|^2 \\ & + \nabla v_{\bar L}\cdot\nabla w_{\bar L} + \nabla v_{\bar M}\cdot\nabla w_{\bar M}). \end{aligned} \tag{69}$$

For the interaction part $V_{D\bar D}$, we note that the pair Coulomb interaction in Eq. (49) between the dyons and antidyons can also be bosonized using standard methods [27–29] in terms of σ and b fields. As a result, each dyon species acquires additional fugacity factors such that

$$M:\ e^{-b-i\sigma}\quad L:\ e^{b+i\sigma}\quad \bar{M}:\ e^{-b+i\sigma}\quad \bar{L}:\ e^{b-i\sigma} \tag{70}$$

with an additional contribution to the free part (69),

$$S_{2F}[\sigma,b]=T\int d^3xd^3y(b(x)V^{-1}(x-y)b(y)+\sigma(x)V^{-1}(x-y)\sigma(y)). \tag{71}$$

The streamline interaction is asymptotically Coulombic and attractive in the $L\bar{L}$ and $M\bar{M}$ channels with

$$V(r)\approx-\frac{C_D}{\alpha_s}\frac{1}{\mathrm{Tr}}=-\frac{2}{\alpha_s}\frac{1}{\mathrm{Tr}} \tag{72}$$

and repulsive in the $\bar{L}M$ and $L\bar{M}$ channels as illustrated in Fig. 1. At short distances, these four channels reduce to perturbative gluons that should be subtracted. We follow Ref. [14] and introduce a core interaction as illustrated in Fig. 1 to achieve that. Specifically, for the core interactions $V_{L,M}(r)$, we have

$$S_{3F}[\phi_1,\phi_2]=\int d^3x(\phi_1^\dagger V_M^{-1}\phi_1+\phi_2^\dagger V_L^{-1}\phi_2), \tag{73}$$

and the interaction part is now

$$\begin{aligned}S_I[v,w,b,\sigma,\chi]=-\int d^3xf_M(4\pi v_m+|\chi_M-\chi_L|^2+v_M-v_L)e^{-b+i\sigma+i\phi_1^\dagger}e^{w_M-w_L}+f_L(4\pi v_l+|\chi_L-\chi_M|^2+v_L-v_M)\\ \times e^{+b-i\sigma+i\phi_2^\dagger}e^{w_L-w_M}+f_{\bar{M}}(4\pi v_{\bar{m}}+|\chi_{\bar{M}}-\chi_{\bar{L}}|^2+v_{\bar{M}}-v_{\bar{L}})e^{-b-i\sigma+i\phi_1}e^{w_{\bar{M}}-w_{\bar{L}}}\\ +f_{\bar{L}}(4\pi v_{\bar{l}}+|\chi_{\bar{L}}-\chi_{\bar{M}}|^2+v_{\bar{L}}-v_{\bar{M}})e^{-b-i\sigma+i\phi_2}e^{w_{\bar{L}}-w_{\bar{M}}}\end{aligned} \tag{74}$$

without the fermions. The minimal modifications to Eq. (74) due to the hopping fermions in the adjoint representation will be detailed below.

In terms of Eqs. (69)–(74), the instanton-dyon partition function (49) can be exactly rewritten as an interacting effective field theory in three dimensions,

$$\mathcal{Z}_{D\bar{D}}[T]\equiv\int D[\chi]D[v]D[w]D[\sigma]D[b]D[\phi]\times e^{-S_{1F}-S_{2F}-S_{3F}-S_I}. \tag{75}$$

In the absence of the fields σ, b, ϕ, Eq. (75) describes a three-dimensional effective field theory discussed in Ref. [5], which was found to be integrable. In the presence of σ, b, ϕ, the integrability is lost as the dyon-antidyon screening upsets the hyper-Kahler nature of the moduli space. Since the effective action in Eq. (75) is linear in the $v_{M,L,\bar{M},\bar{L}}$, the latter are auxiliary fields that integrate into delta-function constraints. However, and for convenience, it is best to shift away the b, σ fields from Eq. (74) through

$$\begin{aligned}w_M-b+i\sigma&\rightarrow w_M\\ w_{\bar{M}}-b-i\sigma&\rightarrow w_{\bar{M}},\end{aligned} \tag{76}$$

which carries unit Jacobian and no anomalies, and recover them in the pertinent arguments of the delta function constraints as

$$\begin{aligned}&-\frac{T}{4\pi}\nabla^2w_M+fe^{i\phi_1^\dagger}e^w-fe^{i\phi_2^\dagger}e^{-w}\\ &=\frac{T}{4\pi}\nabla^2(b-i\sigma)-\frac{T}{4\pi}\nabla^2w_L+fe^{i\phi_2^\dagger}e^{-w}-fe^w\\ &=0\end{aligned} \tag{77}$$

with $w\equiv w_M-w_L$, $f\equiv\sqrt{f_Mf_L}$, and similarly for the antidyons.

B. Effective action with $N_f=0$

In Ref. [5], it was observed that the classical solutions to Eq. (77) can be used to integrate the w's in Eq. (75) to one loop. The resulting bosonic determinant was shown to cancel against the fermionic determinant after also integrating over the χ's in Eq. (75). This holds for our case as well. However, the presence of σ, b, ϕ makes the additional parts of Eq. (75) still very involved in three dimensions. To proceed further, we solve the constraint (77)

$$\begin{aligned}b-i\sigma&=w+\frac{8\pi f}{T(-\nabla^2+M_D^2)}(e^{i\phi_1^\dagger}e^w-e^{i\phi_2^\dagger}e^{-w})\\ b+i\sigma&=\bar{w}+\frac{8\pi f}{T(-\nabla^2+M_D^2)}(e^{i\phi_1}e^{\bar{w}}-e^{i\phi_2}e^{-\bar{w}})\end{aligned} \tag{78}$$

with a screening mass M_D to be fixed variationally. In terms of Eq. (78), the effective action without the fermionic contributions ($N_f=0$) is

$$S = S_\phi + T\bar{w}V^{-1}w + \left(-4\pi f\nu(e^w e^{i\phi_1^\dagger} + e^{\bar{w}}e^{i\phi_1}) + 8\pi f\left(e^w e^{i\phi_1^\dagger}\frac{V^{-1}}{M_D^2+\nabla^2}\bar{w} + e^{i\phi_1}e^{\bar{w}}\frac{V^{-1}}{-\nabla^2+M_D^2}w\right)\right)$$
$$+ (-4\pi f\bar{\nu}(e^{-w}e^{i\phi_2^\dagger} + e^{-\bar{w}}e^{i\phi_2}) - 8\pi f\left(e^{-w}e^{i\phi_2^\dagger}\frac{V^{-1}}{-\nabla^2+M_D^2}\bar{w} + e^{-\bar{w}}e^{i\phi_2}\frac{V^{-1}}{M_D^2-\nabla^2}w\right)$$
$$+ \frac{(8\pi f)^2}{T}(e^{i\phi_1^\dagger}e^w - e^{i\phi_2^\dagger}e^{-w})\frac{1}{M_D^2-\nabla^2}V^{-1}\frac{1}{M_D^2-\nabla^2}(e^{i\phi_1}e^{\bar{w}} - e^{i\phi_2}e^{-\bar{w}})$$
$$+ \mathrm{Tr}\ln\left(1+\frac{8\pi f}{T(M_D^2-\nabla^2)}(e^{i\phi_1^\dagger}e^w + e^{i\phi_2^\dagger}e^{-w})\right) + \mathrm{Tr}\ln\left(1+\frac{8\pi f}{T(M_D^2-\nabla^2)}(e^{i\phi_1}e^{\bar{w}} + e^{i\phi_2 e^{-\bar{w}}})\right), \tag{79}$$

with $v_l = v_{\bar{l}} = \nu$ and $v_m = v_{\bar{m}} = \bar{\nu} = 1-\nu$. Thus, for constant w, we have

$$S = S_\phi + V_3 C_D\alpha_s\bar{w}wM_D^2 + \int 4\pi f(-\nu(e^{i\phi_1^\dagger}e^w + e^{i\phi_1}e^{\bar{w}}) + +2C_D\alpha_s(we^{i\phi_1}e^{\bar{w}} + \bar{w}e^{i\phi_1^\dagger}e^w))$$
$$+ \int 4\pi f(-\bar{\nu}(e^{i\phi_2^\dagger}e^{-w} + e^{i\phi_2}e^{-\bar{w}}) - 2C_D\alpha_s(we^{i\phi_2}e^{-\bar{w}} + \bar{w}e^{i\phi_2^\dagger}e^{-w})) + \frac{(8\pi f)^2}{T}(e^{i\phi_1^\dagger}e^w - e^{i\phi_2^\dagger}e^{-w})\frac{1}{M_D^2-\nabla^2}V^{-1}$$
$$\times\frac{1}{M_D^2-\nabla^2}(e^{i\phi_1}e^{\bar{w}} - e^{i\phi_2}e^{-\bar{w}}) + \mathrm{Tr}\ln\left(1+\frac{8\pi f}{T(M_D^2-\nabla^2)}(e^{i\phi_1^\dagger}e^w + e^{i\phi_2^\dagger}e^{-w})\right)$$
$$+ \mathrm{Tr}\ln\left(1+\frac{8\pi f}{T(M_D^2-\nabla^2)}(e^{i\phi_1}e^{\bar{w}} + e^{i\phi_2}e^{-\bar{w}})\right). \tag{80}$$

To proceed further, we will treat the core interaction using the cumulant expansion. In leading order, only the second cumulant is retained, and the result is

$$\frac{\ln Z}{V_3} \approx +T\alpha_s C_D M_D^2\left(\bar{w}+\frac{16\pi f}{TM_D^2}\sinh\bar{w}\right)\left(w+\frac{16\pi f}{TM_D^2}\sinh w\right) - 4\pi f(\nu(e^w+e^{\bar{w}}) + \bar{\nu}(e^{-w}+e^{-\bar{w}}))$$
$$+ \int d^3r(e^{-V_1}-1)F_1 + \int d^3r(e^{-V_2}-1)F_2 + \int\frac{d^3p}{(2\pi)^3}\ln\left(1+\frac{8\pi f}{T}\frac{e^w+e^{-w}}{M_D^2+p^2}\right)$$
$$+ \int\frac{d^3p}{(2\pi)^3}\ln\left(1+\frac{8\pi f}{T}\frac{e^{\bar{w}}+e^{-\bar{w}}}{M_D^2+p^2}\right) \tag{81}$$

with $F_2 = F_1(w\to -w)$ and

$$F_1 = 16\pi^2 f^2 e^{w+\bar{w}}\left|-\nu + 2C_D\alpha_s\bar{w} + \int\frac{2}{T}\frac{d^3p}{(2\pi)^3}\frac{1}{p^2+M_D^2}\right|^2 + \frac{(8\pi f)^2}{T}e^{w+\bar{w}}$$
$$\times\int d^3r_1 d^3r_2 G_{M_D}(r-r_1)V^{-1}(r_1-r_2)G_{M_D}(r_2). \tag{82}$$

C. Effective potential with $N_f = 0$

For small α_s and strong screening, we may neglect the terms proportional to α_s and drop the screening contributions. Since $\bar{w} = w^\dagger$, the effective potential associated to Eq. (81) and including the one-loop perturbative contribution for finite holonomy is

$$-\frac{\mathcal{P}_D}{8\pi f} = -\cos\sigma(\nu e^{\mathbf{b}} + \bar{\nu}e^{-\mathbf{b}}) + n\left(\frac{e^{2\mathbf{b}}}{\nu}+\frac{e^{-2\mathbf{b}}}{\bar{\nu}}\right) + \frac{4\pi^2}{3}\frac{T^3}{8\pi f}\nu^2\bar{\nu}^2, \tag{83}$$

with $\mathbf{b} = \mathrm{Re}w$ and

$$n = \frac{2\pi f(1-e^{-V_0})}{(2\pi T/x_0)^3} \equiv \frac{2\pi f}{T^3}\frac{32}{3\pi^2}(1-e^{-V_0}). \tag{84}$$

The extremum in $\sigma \equiv \mathrm{Im}w$ in Eq. (83) occurs at $\sigma = 0$. The minimum with respect to $\mathbf{b}$ is fixed by the quartic equation for $e^{\mathbf{b}}$,

$$2n\left(\frac{e^{2\mathbf{b}(\nu)}}{\nu} - \frac{e^{-2\mathbf{b}(\nu)}}{\bar{\nu}}\right) = (\nu e^{\mathbf{b}(\nu)} - \bar{\nu}e^{-\mathbf{b}(\nu)}), \quad (85)$$

with $\mathbf{b}(\nu)$ as a solution. Equation (85) admits always the symmetric solution $\mathbf{b}(1/2) = 0$ as an explicit solution for large n. The quenched effective potential for the holonomy with $N_f = 0$ follows in the form

$$-\frac{\mathcal{P}_D}{8\pi f} \to -(\nu e^{\mathbf{b}(\nu)} + \bar{\nu}e^{-\mathbf{b}(\nu)}) + n\left(\frac{e^{2\mathbf{b}(\nu)}}{\nu} + \frac{e^{-2\mathbf{b}(\nu)}}{\bar{\nu}}\right) + \frac{4\pi^2}{3}\frac{T^3}{8\pi f}\nu^2\bar{\nu}^2. \quad (86)$$

We note that Eq. (86) is similar but not identical to the effective potential discussed in Ref. [12] using an excluded volume approach. Equation (86) admits a critical instanton-dyon density n_C, above which the minimum of the quenched potential (86) occurs for $\nu = 1/2$ or in the confined phase and below which two minima develop, moving away from $\nu = 1/2$ toward the $\nu = 0$, 1 or deconfined phase. To proceed further, we fix $V_0 = \ln 2$ with $n \approx \pi f/T^3$. Equation (86) reduces to

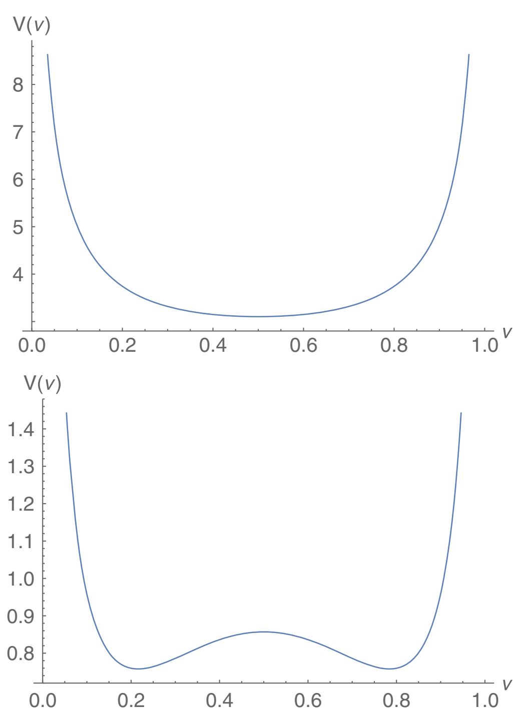

FIG. 2. The holonomy potential (87) for the density $n = 1$, in a "symmetric phase" (above), compared to its shape at smaller density $n = 0.4$, in an "asymmetric phase" (below).

$$-\frac{\mathcal{P}_D}{8\pi f} \to n\left(\frac{e^{2\mathbf{b}(\nu)}}{\nu} + \frac{e^{-2\mathbf{b}(\nu)}}{\bar{\nu}}\right) - (\nu e^{\mathbf{b}(\nu)} + \bar{\nu}e^{-\mathbf{b}(\nu)}) + \frac{\pi^2}{6n}\nu^2\bar{\nu}^2, \quad (87)$$

as shown in the upper part ($n = 1$) and the lower part ($n = 0.4$) of Fig. 2. The critical density is found numerically to be $n_D \approx 0.56$ or $8\pi f/T^3 \approx 4.48$. For $n < n_C$, Eq. (87) displays two minima at $\nu_1 < 1/2$ and $\nu_2 = 1 - \nu_1$. For $n > n_C$, we have a single minimum at $\nu = 1/2$. The alternative choice of the core $V_0 \to \nu V_0$ yields a finite effective potential at $\nu = 0$, 1. For $\nu V_0 = 2\nu$, the critical density occurs at a larger density with $n_C \approx 3.7$ and a minimum at $\mathbf{b} = 0$ for $n > n_C$,

$$-\frac{\mathcal{P}_{D\,\mathrm{min}}}{8\pi f} = 4n - 1 + \frac{\pi^2}{96n}. \quad (88)$$

D. Electric and magnetic masses with $N_f = 0$

In the center-symmetric phase with $\nu = 1/2$ with $N_f = 0$, we may define a class of electric and magnetic masses as the curvatures of the induced potential $-\mathcal{P}_D$ [12]. Specifically, we have

$$Tm_E^2 = \frac{1}{2C_D\alpha_s}\frac{\partial^2(-\mathcal{P}_D)}{\partial^2\mathbf{b}} = \frac{4nT^3}{\alpha_s C_D}(8n - 1)$$
$$Tm_M^2 = \frac{1}{2C_D\alpha_s}\frac{\partial^2(-\mathcal{P}_D)}{\partial^2\sigma} = \frac{4nT^3}{\alpha_s C_D}. \quad (89)$$

We note that $M_E^2/M_M^2 = 8n - 1 > 1$ in the symmetric phase since $n_D \approx 0.56 > 1/4$. These masses are distinct from the electric and magnetic screening masses $M_{E,M}$ following from the decorrelation of the electric and magnetic fields in the instanton-dyon liquid as discussed in Ref. [1]. The latter are spacelike poles in suitably defined propagators.

VI. EFFECTIVE ACTION WITH ADJOINT FERMIONS

A. Fermionic fields with $N_f = 1$

To fermionize the determinant (60) and for simplicity, consider first the case of $N_f = 1$ flavor and the lowest two Matsubara frequencies $\pm\omega_0$. As we noted earlier, the quaternionic matrix in Eq. (60) is real and antisymmetric of dimensionality $4(K_I + K_{\bar{I}})^2$. Its fermionization will only require the use of a single species of Grassmanians with no need for their conjugate. Specifically, we have

$$|\det \tilde{\mathbf{T}}|^{\frac{1}{2}} = \int D[\chi]e^{\chi^T\tilde{\mathbf{T}}\chi} \quad (90)$$

with $\chi = (\chi^+, \chi^-, \bar{\chi}^+, \bar{\chi}^-)$. This is the analog of the Majorana-like representation for our hopping matrix in

Euclidean $S^1 \times R^3$. We can rearrange the exponent in Eq. (90) by defining a Grassmanian source $\mathbb{J}(x) = (\mathbb{J}^+(x), \mathbb{J}^-(x), \bar{\mathbb{J}}^+(x), \bar{\mathbb{J}}^-(x))^T$ with

$$\mathbb{J}^+_\alpha(x) = \sum_{I=1}^{K_L} \chi_\alpha^{+I} \delta^3(x - x_I)$$
$$\bar{\mathbb{J}}^{+\dot\beta}(x) = \sum_{J=1}^{K_{\bar{I}}} \bar{\chi}_{2J}^{+\dot\beta} \delta^3(x - y_J) \tag{91}$$

and by introducing two additional fermionic fields $\psi(x) = (\psi_+(x), \psi_-(x), \bar{\psi}_+, \bar{\psi}_-)^T$. Thus,

$$e^{\chi^T \tilde{\mathbf{T}} \chi} = \frac{\int D[\psi] \exp(-\int \psi^T \tilde{\mathbf{G}} \psi + 2 \int \mathbb{J}^T \psi)}{\int dD[\psi] \exp(-\int \psi^T \tilde{\mathbf{G}} \psi)} \tag{92}$$

with $\tilde{\mathbf{G}}$ a 4×4 chiral block matrix defined by

$$\tilde{\mathbf{G}}\, \tilde{\mathbf{T}} = \mathbf{1}. \tag{93}$$

For massless adjoint quarks, we have the explicit form

$$\begin{pmatrix} 0 & 0 & 0 & i\epsilon \mathbf{G}^T(y-x) \\ 0 & 0 & -i\epsilon \mathbf{G}^\star(x-y) & 0 \\ 0 & -i\mathbf{G}^\dagger(y-x)\epsilon & 0 & 0 \\ i\mathbf{G}(x-y)\epsilon & 0 & 0 & 0 \end{pmatrix} \tag{94}$$

with entries $\mathbf{TG} = \mathbf{1}$. The Grassmanian source contributions in Eq. (92) generate a string of independent exponents for the instanton dyons and instanton antidyons,

$$\prod_{I=1}^{K_I} e^{2\chi_I^{+T}\psi_+(x_I) + 2\chi_I^{-T}\psi_-(x_I)} \prod_{J=1}^{K_{\bar{I}}} e^{2\bar{\chi}_J^{+T}\bar{\psi}_+(y_J) + 2\bar{\chi}_J^{-T}\bar{\psi}_-(y_J)}. \tag{95}$$

The Grassmanian integration over the χ_i in each factor in Eq. (95) is now readily done to yield

$$\prod_I [\psi_+^T \epsilon \psi_+ \psi_-^T \epsilon \psi_-] \prod_J [\bar{\psi}_+^T \epsilon \bar{\psi}_+ \bar{\psi}_-^T \epsilon \bar{\psi}_-]$$
$$= \prod_I [\psi_+^T \epsilon \psi_- \psi_+^T \epsilon \psi_-] \prod_J [\bar{\psi}_+^T \epsilon \bar{\psi}_- \bar{\psi}_+^T \epsilon \bar{\psi}_-] \tag{96}$$

for the instanton dyons and instanton antidyons. The net effect of the additional fermionic determinant in Eq. (49) is to shift the dyon and antidyon fugacities in Eq. (74) through

$$f_I \to f_I \psi_+^T \epsilon \psi_-(x_I) \psi_+^T \epsilon \psi_-(x_I)$$
$$f_{\bar{I}} \to f_{\bar{I}} \bar{\psi}_+^T \epsilon \bar{\psi}_-(x_{\bar{I}}) \bar{\psi}_+^T \epsilon \bar{\psi}_-(x_{\bar{I}}). \tag{97}$$

B. Resolving the constraints

In terms of Eqs. (69)–(74) and the substitution (97), the dyon-antidyon partition function (49) for finite N_f can be exactly rewritten as an interacting effective field theory in three dimensions,

$$\mathcal{Z}_1[T] \equiv \int D[\psi] D[\chi] D[v] D[w] D[\sigma] D[b] D[\phi_1] D[\phi_2]$$
$$\times e^{-S_{1F} - S_{2F} - S_I - S_\psi - S_\phi}, \tag{98}$$

with the additional $N_f = 1$ chiral fermionic contribution $S_\psi = \psi^T \tilde{\mathbf{G}} \psi$. Since the effective action in Eq. (98) is linear in $v_{M,L,\bar{M},\bar{L}}$, the latter integrate to give the constraints

$$-\frac{T}{4\pi}\nabla^2 w_M + (\psi_+^T \epsilon \psi_-)^2 f_M e^{w_M - w_L + i\phi_1^\dagger}$$
$$- f_L e^{w_L - w_M + i\phi_2^\dagger} = \frac{T}{4\pi}\nabla^2(b - i\sigma)$$
$$-\frac{T}{4\pi}\nabla^2 w_L - (\bar{\psi}_+^T \epsilon \bar{\psi}_-)^2 f_M e^{w_M - w_L + i\phi_1^\dagger}$$
$$+ f_L e^{w_L - w_M + i\phi_2^\dagger} = 0 \tag{99}$$

and similarly for the antidyons with $M, L, \psi \to \bar{M}, \bar{L}, \bar{\psi}$. To proceed further, the formal classical solutions to the constraint equations or $w_{M,L}[\sigma, b]$ should be inserted back into the three-dimensional effective action. The result is

$$\mathcal{Z}_1[T] = \int D[\psi] D[\sigma] D[b] D[\phi] e^{-S} \tag{100}$$

with the three-dimensional effective action

$$S = S_F[\sigma, b] + S[\phi] + \int d^3x \psi^T \tilde{\mathbf{G}} \psi \tag{101}$$

$$- 4\pi f_M v_m \int d^3x (\psi_+^T \epsilon \psi_-)^2 e^{w_M - w_L + i\phi_1^\dagger}$$
$$- 4\pi f_M v_m \int d^3x (\bar{\psi}_+^T \epsilon \bar{\psi}_-)^2 e^{w_{\bar{M}} - w_{\bar{L}} + i\phi_1}$$
$$- 4\pi f_L v_l \int d^3x (e^{w_L - w_M + i\phi_2^\dagger} + e^{w_{\bar{L}} - w_{\bar{M}} + i\phi_2}). \tag{102}$$

Here, S_F is S_{2F} in Eq. (72) plus additional contributions resulting from the $w_{M,L}(\sigma, b)$ solutions to the constraint equations (99) after their insertion back. The fermionic contributions in Eq. (102) are Z_4 symmetric.

C. Ground state with $N_f = 1$

We first consider the massless case with $m = 0$. The uniform ground state of the three-dimensional effective theory described by Eqs. (98)–(102) corresponds to b, σ, w constant, with a finite condensate with

$$\langle \psi_+^T \epsilon \psi_- \rangle = \langle \psi_-^T \epsilon \psi_+ \rangle = \Sigma$$
$$\langle \bar{\psi}_+^T \epsilon \bar{\psi}_- \rangle = \langle \bar{\psi}_-^T \epsilon \bar{\psi}_+ \rangle = \Sigma \tag{103}$$

that breaks the Z_4 symmetry of Eq. (102). This is the mechanism by which the instanton-dyon liquid enforces the anomalous $U_A(1)$ breaking with adjoint fermions. The fermionic quadrilinears in Eq. (102) can be reduced by introducing pertinent Lagrange multipliers Λ's through the identity as detailed in Ref. [2]. Assuming parity symmetry, in the mean field or Hartree approximation, Eq. (102) becomes

$$S \to S + \int d^3x \psi^T \tilde{\mathbf{G}} \psi + \sum_{\pm} \int d^3x \Lambda_1(x)(\psi_\pm^T \epsilon \psi_\mp - \Sigma)$$
$$+ \sum_{\pm} \int d^3x \Lambda_2(x)(\bar{\psi}_\pm^T \epsilon \bar{\psi}_\mp - \Sigma). \tag{104}$$

We observe that the mean-field constraints in Eq. (104) enforce the substitution $\psi^T \epsilon \psi \to \Sigma$, and therefore the shift for $\Sigma \neq 0$

$$e^{w_M - w_L} \to \sqrt{\frac{f_L}{f_M}} |\Sigma| e^{w_M - w_L}$$
$$e^{w_L - w_M} \to \sqrt{\frac{f_M}{f_L}} \frac{1}{|\Sigma|} e^{w_L - w_M}. \tag{105}$$

For completeness, the exchange or Fock correction to the mean-field approximation (103) is detailed in Appendix D. Also, a one-loop alternative approximation is presented in Appendix E.

To insure a smooth limit for $\nu \to 1/2$, we will redefine the magnetic fugacity $f_M(2\nu - 1)^6 \to f_M$ throughout. As half the zero modes jump when $\nu = 1/2$, the hopping is singular in the ensemble made of constituent instanton dyons and instanton antidyons. This singularity does not appear if the constituents are jumping within the KvBLL caloron as all infrared tails are tamed, as we have shown in Appendix C. But again, because of the fact that the delocalization of the zero modes makes use of the hopping between instanton dyons and instanton antidyons with opposite chirality, it is necessary to unlock the constituents from their respective KvBLL calorons and anticalorons as we have detailed.

With the above in mind, a repeat of the quenched arguments shows that the unquenched pressure $\mathcal{P}_D = -\mathcal{V}/V_3$ with adjoint and massless fermions is now

$$\frac{\mathcal{P}_{D+F}}{T^3} = -\frac{\tilde{n}_\Sigma^2}{8}\left(\frac{e^{2\mathbf{b}}}{\nu} + \frac{e^{-2\mathbf{b}}}{\bar{\nu}}\right) + \tilde{n}_\Sigma(\nu e^{\mathbf{b}} + \bar{\nu} e^{-\mathbf{b}}) - 4\tilde{\Sigma}\tilde{\Lambda}$$
$$+ \pi \int \tilde{p}^2 d\tilde{p} \ln(1 + \tilde{\Lambda}^2 \mathbb{F}) + \frac{4\pi^2}{3}\nu^2 \bar{\nu}^2 \tag{106}$$

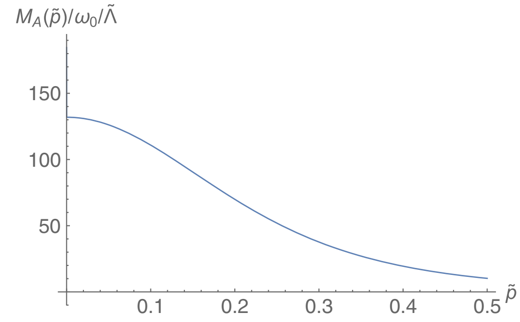

FIG. 3. Adjoint constituent mass for $\nu = 0.7$.

with $\tilde{n}_\Sigma = 8\pi f \Sigma / T^3$ and $\tilde{\Lambda} = \Lambda / T^2$. We have defined

$$\pi^4 \mathbb{F}(\tilde{p}, 2\nu - 1)$$
$$= (3f_1^2 + f_3^2 + 2f_1 f_3 - 8f_2^2 + 8f_1 f_2 \tilde{p})^2$$
$$+ \tilde{p}^2 \left(-f_1^2 + f_3^2 + 2f_1 f_3 + 8f_2^2 + 8f_1 f_2 \frac{1}{\tilde{p}}\right)^2. \tag{107}$$

The f_i are given in Eq. (56) after replacing $p \to \tilde{p} = p/\omega_0$ and $\tilde{\nu} \to \tilde{\nu}/\omega_0$, all of which are now dimensionless. We have numerically checked that the momentum integration in Eq. (106) does not change much if we were to simplify the f_i in Eq. (56) to

$$f_1 \approx -\frac{f_3}{3} \to -\frac{1}{\tilde{p}^3} \tan^{-1}\left(\frac{\tilde{p}}{2\nu - 1}\right)$$
$$f_2 \to \frac{\tilde{p}}{(\tilde{p}^2 + (2\nu - 1)^2)^2} \tag{108}$$

so that

$$\mathbb{F}(\tilde{p}, 2\nu - 1) \approx \frac{1}{\pi^4}(6f_1^2 - 8f_2^2 - 8f_1 f_2 \tilde{p})^2$$
$$+ \tilde{p}^2 \left(2f_1^2 + 8f_2^2 - 8f_1 f_2 \frac{1}{\tilde{p}}\right)^2. \tag{109}$$

The integral contribution in Eq. (106) is that of a constituent adjoint quark, with a momentum-dependent mass $M_A(\tilde{p})$ given by

$$\frac{M_A(\tilde{p})}{\omega_0 \tilde{\Lambda}} = ((1 + \tilde{p}^2)\mathbb{F})^{\frac{1}{2}}, \tag{110}$$

as shown in Fig. 3 for $\nu = 0.7$.

D. Confining symmetric phase

The center-symmetric state with $\mathbf{b} = 0$ and $\nu = 1/2$ is an extremum of Eq. (106), provided that $\Sigma \neq 0$. This means that the spontaneous breaking of chiral symmetry is a necessary (but not sufficient) condition for center symmetry to take place in the instanton-dyon liquid model with massless adjoint quarks. This is similar to the observation made in Ref. [2] for massless fundamental quarks. For fixed $\tilde{\Lambda}$, the fermionic contribution in Eq. (106) is maximal for $\nu = 1/2$. The additional extremum with respect to Σ yields the condition

$$4\tilde{\Lambda}\,\tilde{\Sigma} = \tilde{n}_\Sigma(\nu e^{\mathbf{b}} + \bar{\nu}e^{-\mathbf{b}}) - \frac{\tilde{n}_\Sigma^2}{4}\left(\frac{e^{\mathbf{b}}}{\nu} + \frac{e^{-\mathbf{b}}}{\bar{\nu}}\right) \tag{111}$$

with $\tilde{n}_\Sigma = n_\Sigma/T^3$. Equation (111) requires $\tilde{n}_\Sigma < 1$ so that $\tilde{\Lambda} \neq 0$ and is therefore a final quark condensate. We recall that for $N_f = 0$, $\tilde{n}_\Sigma > \tilde{n}_D = 0.56$ is required for a center-symmetric state. With this in mind, and for $0.56 < \tilde{n}_\Sigma < 1$, the extremum in the $\tilde{\Lambda}$ direction gives the gap equation

$$\tilde{n}_\Sigma - \tilde{n}_\Sigma^2 = 2\pi \int \tilde{p}^2 d\tilde{p}\, \frac{\tilde{\Lambda}^2 \mathbb{F}}{1 + \tilde{\Lambda}^2 \mathbb{F}}. \tag{112}$$

Equation (112) yields a finite $\tilde{\Lambda}$ and thus a finite chiral condensate. We note that a core strength $V_0 \to 0$ amounts to a vanishingly small $\tilde{n}_\Sigma^2 \to 0$ contribution. Note that in the center-symmetric phase phase with $\tilde{n}_D \approx 1/2$ the core correction is about 50% of the free instanton-dyon contribution. It decreases substantially in the center-asymmetric phase as the instanton-dyon liquid is diluted.

More explicitlly, for small $\tilde{\Lambda}$, the dominant contributions from the hopping fermions stem from the small momentum sector of the p integrals in Eqs. (106) and (112) with

$$\mathbb{F}(p \to 0, 0) \approx \frac{0.47}{\tilde{p}^{12}}. \tag{113}$$

Inserting Eq. (113) into Eq. (112) allows for an explicit solution to the gap equation in the form

$$\tilde{\Lambda} \approx \left(\frac{\tilde{n}_\Sigma - \tilde{n}_\Sigma^2}{1.92}\right)^2. \tag{114}$$

E. Magnitude of the chiral condensate

For massive adjoint quarks, the fermionic part of Eq. (106) is

$$\pi \int \tilde{p}^2 d\tilde{p} \ln\left((1 + \tilde{m}\mathbf{t}\tilde{\Lambda})^2 + \tilde{\Lambda}^2 \mathbb{F}\right), \tag{115}$$

where all contributions are dimensionless. We have defined

$$\mathbf{t}(p) = \frac{\omega_0^3}{\pi^2}\mathbf{K}(p)$$
$$\tilde{m} = \frac{m}{\omega_0}. \tag{116}$$

The chiral condensate for massless adjoint fermions follows from the general relation

$$\langle i\mathrm{Tr}(\lambda\lambda)\rangle = \frac{1}{TV_3}\left(\frac{\partial \ln Z}{\partial m}\right)_{m=0} = T^3 \int \tilde{p}^2 d\tilde{p}\, \frac{2\mathbf{t}\tilde{\Lambda}}{1 + \tilde{\Lambda}^2 \mathbb{F}}. \tag{117}$$

Again, the integration in Eq. (117) is dominated by small momenta for small $\tilde{\Lambda}$. In the confined state with $\nu = 1/2$, we can use Eq. (113) and the small momentum limit of Eq. (116),

$$\mathbf{t}(p \to 0) \approx \frac{2.31}{\tilde{p}^6}, \tag{118}$$

to obtain

$$\frac{\langle i\mathrm{Tr}(\lambda\lambda)\rangle}{T^3} \approx 2\sqrt{\tilde{\Lambda}} \approx (\tilde{n}_\Sigma - \tilde{n}_\Sigma^2). \tag{119}$$

Again, we note that for a vanishingly small core with $V_0 \to 0$ the contribution $n_\Sigma^2 \to 0$ in Eq. (114) with a chiral condensate for adjoint fermions of order $\tilde{n}$, which is the rescaled instanton-dyon density. This result is totally consistent with the result derived in Ref. [2] for massless fundamental quarks with no core. The transition from a symmetric state with $\nu = 1/2$ to an asymmetric state with $\nu < 1/2$ takes place $n_\Sigma < n_D$ as the instanton-dyon liquid is diluted, and the chiral condensate (119) also vanishes (see below).

Finally, we note that the case of $N_f = 1$ adjoint quarks at zero temperature corresponds to $\mathcal{N} = 1$ supersymmetric theory with a nonvanishing gluino condensate [30]. While our finite-temperature analysis of $N_f = 1$ breaks supersymmetry explicitly, Eq. (119) can be viewed as the remnant of the gluino condensate at finite temperature. Since Eq. (119) was derived under the condition that $0.56 < \tilde{n}_\Sigma < 1$, the zero-temperature limit cannot be reached in our case.

F. General case with $N_f \geq 1$

The preceding analysis generalizes to $N_c = 2$ and $N_f \geq 1$ adjoint fermions through the substitution

$$\psi_+^T \epsilon \psi_- \to \frac{1}{N_f!} \det{}_{fg} \psi_{+f}^T \epsilon \psi_{-g} \tag{120}$$

in Eq. (102) with all other labels unchanged. As a result, the fermionic terms are $SU(N_f) \times Z_{4N_f}$ flavor symmetric. The $U_A(1)$ symmetry for adjoint QCD is explicitly broken by the instanton-dyon liquid model. The flavor symmetry is further broken spontaneously through $SU(N_f) \times Z_{4N_f} \to O(N_f)$ with the appearance of a condensate,

$$\langle \psi_{+f}^T \epsilon \psi_{-g}\rangle = \Sigma \delta_{fg}, \tag{121}$$

the dual of the chiral condensate. Equation (121) is explicitly symmetric under the transformations $\psi_{\pm f} \to O_{fg}\psi_{\pm g}$ and $\bar{\psi}_{\pm f} \to \bar{\psi}_{\pm g} O^T_{gf}$.

A rerun of the preceding arguments yields the instanton dyon plus adjoint fermions pressure for arbitrary N_f,

$$\mathcal{P}_{D+F} = -\frac{8\pi^2 f^2 \Sigma^{2N_f}}{T^3}\left(\frac{e^{2\mathbf{b}}}{\nu} + \frac{e^{-2\mathbf{b}}}{\bar{\nu}}\right) + 8\pi f \Sigma^{N_f}(\nu e^{\mathbf{b}} + \bar{\nu} e^{-\mathbf{b}}) - 4N_f \Lambda\Sigma + N_f \int \frac{d^3p}{(2\pi)^3} \ln(1 + \Lambda^2 \tilde{\mathbf{T}}^{+2}) + P_{\rm loop}(N_f). \quad (122)$$

The last contribution is briefly detailed in Appendix F and is seen to be dominated by the first term in the expansion. If we were to define $\tilde{n}_{\Sigma f} = 8\pi f \Sigma^{N_f}/T^3$, then the results from Eq. (122) for arbitrary N_f map onto those from Eq. (106) for $N_f = 1$, now with

$$\frac{\mathcal{P}_{D+F}}{T^3} = -\frac{\tilde{n}^2_{\Sigma f}}{8}\left(\frac{e^{2\mathbf{b}}}{\nu} + \frac{e^{-2\mathbf{b}}}{\bar{\nu}}\right) + \tilde{n}_{\Sigma f}(\nu e^{\mathbf{b}} + \bar{\nu} e^{-\mathbf{b}}) - 4N_f \tilde{\Sigma}\,\tilde{\Lambda} + \pi N_f \int \tilde{p}^2 d\tilde{p} \ln(1 + \tilde{\Lambda}^2 \mathbb{F}) - \frac{4\pi^2}{3}(1 + N_f)\nu^2\bar{\nu}^2. \quad (123)$$

The ground state is center symmetric for a sufficiently dense instanton-dyon liquid, provided that chiral symmetry is spontaneously broken with $\Sigma \neq 0$, and symmetric in the dilute limit. Here, $\tilde{\Lambda}$ and $\tilde{\Sigma}$ follow from the extrema of Eq. (123) as coupled gap equations,

$$\tilde{\Sigma} = \frac{\pi}{2}\int \tilde{p}^2 d\tilde{p} \frac{\tilde{\Lambda}\mathbb{F}}{1 + \tilde{\Lambda}^2 \mathbb{F}}$$
$$\tilde{\Lambda}\,\tilde{\Sigma} = -\frac{\tilde{n}^2_{\Sigma f}}{16}\left(\frac{e^{2\mathbf{b}}}{\nu} + \frac{e^{-2\mathbf{b}}}{\bar{\nu}}\right) + \frac{\tilde{n}_{\Sigma f}}{4}(\nu e^{\mathbf{b}} + \bar{\nu} e^{-\mathbf{b}}). \quad (124)$$

The solutions $\tilde{\Sigma}(\mathbf{b}, \nu)$ and $\tilde{\Lambda}(\mathbf{b}, \nu)$ to Eq. (124) should be inserted back in Eq. (123) to maximize numerically the pressure in the parameter space ν, $\mathbf{b}$.

In Fig. 4, we show the numerical results for the dimensionless pressure (dotted middle line), Polyakov line (solid line), and chiral condensate (dotted upper line) with increasing $8\pi f/T^2$ (decreasing temperature), for $N_f = 1$ in the symmetric phase. The breaking of chiral symmetry is lost for $8\pi f/T^2 < 2.6$, which causes all topological effects to vanish in the chiral limit. For $N_f > 2$, Eqs. (123) and (124) do not support a solution that breaks chiral symmetry.

Finally, the restoration of chiral symmetry can be estimated analytically from Eqs. (123) and (124), by dropping the first or core contribution and noting that the resulting expression maps onto the one derived for fundamental quarks in Ref. [2] [see Eq. (80) there] with $N_c N_f$. This mapping shows that Eqs. (123) and (124) do not sustain a chiral condensate for $N_c N_f / N_c \geq 2$, or $N_f \geq 2$ Majorana quarks.

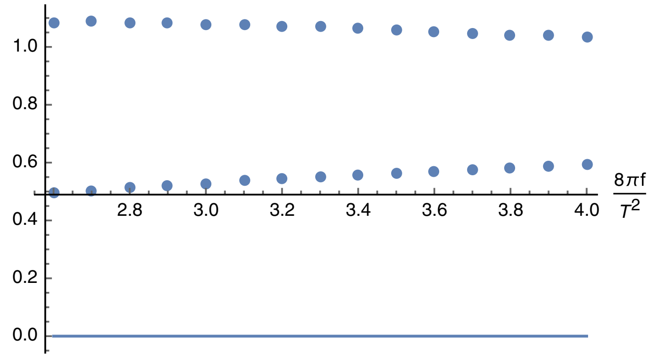

FIG. 4. Dimensionless pressure (middle dotted line), Polyakov line (solid line), and chiral condensate (upper dotted line) vs $8\pi f/T^2$ (decreasing temperature) for $N_f = 1$.

G. Critical temperature estimates

For general N_f, we can estimate the critical temperature for the restoration of center symmetry T_D, by neglecting both the core and fermionic contributions in Eq. (123), i.e.,

$$\frac{\mathcal{P}_{D+F}}{T^3} \to \tilde{n}_{\Sigma f}(\nu e^{\mathbf{b}} + \bar{\nu} e^{-\mathbf{b}}) - \frac{4\pi^2}{3}(1 + N_f)\nu^2\bar{\nu}^2. \quad (125)$$

An estimate of the deconfining temperature T_D follows by balancing the first contribution in the center-symmetric phase with $\mathbf{b} = 0$ and $\nu = \bar{\nu} = 1/2$ against the last one-loop contribution stemming from the adjoint free gluons and quarks. The result is

$$\frac{n_{\Sigma f}}{T_D^3} \approx \frac{\pi^2}{12}(1 + N_f). \quad (126)$$

In the presence of adjoint quarks, the fundamental string tension does not vanish, $\sigma/T^2 = n_{\Sigma f}/T^3$. For $N_c = 2$ QCD with N_f adjoint Majorana quarks, the ratio of the critical temperature for center-symmetry loss normalized by the fundamental string tension decreases with N_f as

$$\frac{T_D}{\sqrt{\sigma}} \approx \frac{2}{\pi}\left(\frac{3}{1 + N_f}\right)^{\frac{1}{2}}. \quad (127)$$

It would be useful to check Eq. (127) against current lattice simulations with adjoint quarks.

The estimate of the chiral symmetry restoration temperature for the chirally broken phase with $N_f < 2$ is more subtle. For that, we recall that the delocalization of the adjoint zero modes generates the so-called zero-mode zone with a finite eigenvalue density $\rho(\lambda)$ normalized to the 4-volume V_3/T. The details of the interactions in the small virtuality λ limit do not matter [31], as the distribution follows a Wigner semicircle,

$$\rho(\lambda) = \frac{4n_{\Sigma f}}{(\lambda_{\max}(T)/T)}\left(1 - \frac{\lambda^2}{\lambda^2_{\max}(T)}\right)^{\frac{1}{2}}. \tag{128}$$

The normalization is fixed by the overall number of zero modes in the instanton-dyon liquid. Here, $2\lambda_{\max}(T)$ is the size of the zero-mode zone at finite T. Combining Eq. (119) with the Banks-Casher relation [32], we have

$$|\tilde{n}_{\Sigma f} - \tilde{n}^2_{\Sigma f}| \approx \pi\rho(0), \tag{129}$$

which fixes $x(T) = \lambda_{\max}(T)/(\pi T)$ as

$$\tilde{n}_{\Sigma f} \approx 1 - \frac{2}{x(T)}. \tag{130}$$

The chiral transition temperature T_C is fixed by the quarks turning massless or $\Sigma \to 0$, which implies that the instanton-dyon density $\tilde{n}_{\Sigma f} \to 0$, as all topological contributions are suppressed. From Eq. (130), this occurs when

$$T_C = \frac{\lambda_{\max}(T_C)}{2\pi}. \tag{131}$$

We now note that at the chiral transition temperature the quark hopping stalls into topologically neutral molecules. As a result, $\tilde{\mathbf{T}}$ in Eq. (49) becomes banded, and $\lambda_+(T_C)$ is comparable to the strength of the nearest neighbor hopping (67)

$$\lambda_{\max}(T_C) = |\mathbf{T}^+(x_{IJ} = 0)| = \left|\int \frac{d^3p}{(2\pi)^3}\mathbf{T}^+(p)\right| = \kappa\pi T_C|2\nu_C - 1|, \tag{132}$$

with $\kappa = 0.557$. Using Eqs. (131) and (132), it follows that chiral restoration occurs when the holonomy reaches $\nu_C = 1/2 + 1/\kappa = 0.3$ (mod 1), and in general $T_C > T_D$.

Using the quenched effective potential discussed earlier for an estimate, this corresponds to an instanton-dyon density for chiral restoration $\tilde{n}_C = 0.48$, which is surprisingly close to the quenched instanton-dyon density for the breaking of center symmetry $\tilde{n}_D = 0.56$. Using the instanton-dyon density for the $N_c = 2$ and $N_f = 1$ Majorana quark

$$\tilde{n}(T) \approx Ce^{-\pi/\alpha_s(T)} \approx C\left(\frac{0.36T_D}{T}\right)^{\frac{21}{6}}, \tag{133}$$

we find that

$$\left(\frac{T_C}{T_D}\right) \approx \left(\frac{0.56}{0.48}\right)^{\frac{6}{21}} \approx 1, \tag{134}$$

which is much smaller than the ratio reported in lattice simulations [15].

VII. CONCLUSIONS

We have presented a mean-field analysis of key characteristics of the instanton-dyon liquid with adjoint light quarks. The index theorem on $S^1 \times R^3$ shows that dissociated instanton dyons support four antiperiodic zero modes that localize on the M-instanton dyon in the center-asymmetric phase with $\nu > 1/2$, or alternatively on the L-instanton dyon for $\nu < 1/2$. These two cases are dual to each other, so only one can be considered. In the symmetric phase, the four antiperiodic zero modes are shared equally (two on each) by the L- and M-instanton dyons. We have used the ADHM construction to derive the explicit form of these zero modes.

We have detailed the construction of the partition function for the dissociated KvBLL calorons with N_f light adjoint quarks, including the classical streamline interactions and the quantum Coulomb interactions induced by the coset manifold. We have retained a core interaction between the like instanton dyon and antidyons to distinguish them from perturbative fluctuations. By a series of fermionization and bosonization techniques, we have mapped this interacting many-body system on a three-dimensional effective theory. We have presented a mean-field analysis of the dense phase that exhibits both confinement with center symmetry and spontaneously broken chiral symmetry.

We have shown that in such an approximation the deconfinement with breaking of center symmetry and the restoration of chiral symmetry occur about simultaneously. Furthermore, the latter is always unbroken for $N_f \geq 2$. In contrast, exploratory lattice simulations [15] have shown that $SU_c(2)$ gauge theory with $N_f = 0$, 1, 4 adjoint Majorana fermions still supports chiral symmetry, which may point to a major shortcoming of the mean-field analysis. A numerical simulation of the dyon-liquid model would be welcome.

The mean-field analysis we have presented also has a major shortcoming as the instanton-dyon liquid is diluted. It does not account for the molecular pairing of the instanton dyon-antidyon configurations through light adjoint pairs. We have presented a qualitative argument for the chiral transition using the assumption of pairing, but a more reliable analysis is likely numerical as the analysis goes beyond the mean-field results presented here.

ACKNOWLEDGMENTS

This work was supported by the U.S. Department of Energy under Contract No. DE-FG-88ER40388.

APPENDIX A: PERIODIC ZERO MODES

In this Appendix, we briefly detail the ADHM construct as applied to the periodic adjoint zero modes. This is partly a check on our general ADHM construction. For that, we note that the Grassmanian matrix for periodic adjoint fermions simplifies to

$$M(z,z') = \delta(z - z')M. \tag{A1}$$

A rerun of the preceding arguments yields the periodic zero modes

$$\lambda_m(r) = \frac{1}{\mathrm{sh}(\omega_0 \mathrm{r})}(a(\omega_0 r)\sigma_m + b(\omega_0 r)\sigma\cdot\hat{r}\sigma_m\sigma\cdot\hat{r})\epsilon M \\ -\epsilon(M^T a(\omega_0 r)\sigma_m + M^T b(\omega_0 r)\sigma\cdot\hat{r}\sigma_m\sigma\cdot\hat{r})^T). \tag{A2}$$

For $\omega_0 r \to \infty$, we have $a \approx b \approx -\sinh(\omega_0 r)/(\omega_0 r)^2$ so that

$$\lambda_m(r\to\infty) = \frac{1}{r^2}(\sigma_m + \sigma\cdot\hat{r}\sigma_m\sigma\cdot\hat{r})\chi \\ = \frac{2}{r^2} r_m \sigma\cdot\hat{r}\chi, \tag{A3}$$

with $\chi = \epsilon M$. Equations (A3) are in agreement with the known periodic zero modes in the hedgehog gauge [6,22].

APPENDIX B: ZERO MODES IN A BPS DYON WITHOUT ADHM

In this Appendix, we explicitly derive the Dirac equation for antiperiodic adjoint fermions in the state of lowest total angular momentum, without using the ADHM construction. We will use the equations to investigate the nature of the fermionic zero mode at the origin and asymptotically. Without the ADHM construct, the equations are only solvable numerically.

Without loss of generality, we will consider the $\overline{M}$-dyon gauge configuration given by

$$(A_4^a, A_i^a) = (\hat{r}^a\phi(r), \epsilon_{aij}\hat{r}_j A(r)) \tag{B1}$$

with the boundary values

$$A(r\to 0) = 0 \qquad A(r\to\infty) = -\frac{1}{r} \\ \phi(r\to 0) = 0 \qquad \phi(r\to\infty) = 2\pi T\nu. \tag{B2}$$

In the adjoint representation of $SU_c(2)$, the color matrices are $T^a_{mn} = i\epsilon_{amn}$. In the chiral basis, the adjoint Dirac fermions will be sought in the form

$$\Psi \equiv \begin{pmatrix} \Psi_m^+ \\ \Psi_m^- \end{pmatrix}. \tag{B3}$$

The Dirac equation (1) for the two lowest Matsubara frequencies $\pm\omega_0$ is given by

$$(i\sigma\cdot\nabla\delta_{nm} + i(\sigma_n\hat{r}_m - \sigma_m\hat{r}_n)A(r) \\ \pm\epsilon_{nam}\hat{r}_a\phi(r))\Psi_m^\pm = i\omega_0\Psi_n^\pm. \tag{B4}$$

To solve Eq. (B4) explicitly, we decompose the vector-valued chiral components in Eq. (B3) using the independent vector basis [33]

$$(1, \vec{\sigma}\cdot\hat{r})(\hat{r}, (\vec{r}\times\vec{p}), (\vec{r}\times\vec{p})\times\hat{r}), \tag{B5}$$

which is seen to commute with the total angular momentum $\vec{J} = \vec{l} + \vec{s}$. We seek the zero modes in the state of zero orbital angular momentum or $J = 1/2$. Therefore,

$$\Psi_m^\pm(\vec{r}) \equiv \hat{r}_m\Theta_3^\pm + (\vec{r}\times\vec{p})_m(\sigma\cdot\hat{r})\Theta_4^\pm \\ + \hat{r}_m\sigma\cdot\hat{r}\Theta_1^\pm + i((\vec{r}\times\vec{p})\times\hat{r})_m(\sigma\cdot\hat{r})\Theta_2^\pm \tag{B6}$$

with the scalar radial spinor functions

$$\Theta_i^\pm \equiv \sum_{s=\pm} F_i^\pm(r,s)|s\rangle. \tag{B7}$$

Inserting Eqs. (B6) and (B7) into Eq. (B4) yields

$$\left(\frac{d}{dr} + \frac{2}{r}\right)F_1^\pm - 2\rho F_2^\pm = \omega_0 F_3^\pm \\ \left(\frac{d}{dr} + \frac{1}{r} \pm \phi\right)F_2^\pm - \rho F_1^\pm = \omega_0 F_4^\pm \\ \frac{d}{dr}F_3^\pm + 2\rho F_4^\pm = \omega_0 F_1^\pm \\ \left(\frac{d}{dr} + \frac{1}{r} \pm \phi\right)F_4^\pm + \rho F_3^\pm = \omega_0 F_2^\pm. \tag{B8}$$

Here, $\rho \equiv \langle A_4\rangle + 1/r$, with the label s subsumed. Using the asymptotics, it is readily found at infinity that

$$F_{1,3}^\pm(r\to\infty) = c_1 e^{-\omega_0 r} + c_2 e^{+\omega_0 r} \\ F_{2,4}^\pm(r\to\infty) = c_3 e^{-\omega_0(1\pm 2\nu)r} + c_4 e^{+\omega_0(1\mp 2\nu)r}, \tag{B9}$$

while at the origin, we have

$$F_{3,4}^\pm(r\to 0) = b_3 r + b_4\frac{1}{r^2} \to b_3 r \\ F_{1,2}^\pm(r\to 0) = b_1 + b_2\frac{1}{r^3} + b_3 r^2 \\ + b_4\frac{1}{r} \to b_1 + b_3 r^2. \tag{B10}$$

For F^+ with fixed $s = \pm$, we always have two ($b_{1,3}$) out of four ($b_{1,2,3,4}$) total dimensions of solutions, which are normalizable at zero. We have two ($c_{1,3}$) out of four ($c_{1,2,3,4}$) total dimensions of solutions, which are normalizable at infinity for $\nu \leq \frac{1}{2}$ and 3 ($c_{1,3,4}$) for $\nu > 1/2$. We conclude that for $\nu > \frac{1}{2}$ there exists at least one zero mode. For $\nu < \frac{1}{2}$, the existence cannot be proven on general grounds, and a numerical analysis is required. However, their existence is supported by the index theorem reviewed earlier. For $\nu > 1/2$, the dominant contribution at large distances stems from the asymptotic in Eq. (B9) or $c_4 e^{-(2\nu-1)\omega_0 r}$. As $\nu \to 1/2$, it asymptotes a constant which is not square integrable. This analysis for $\nu = 1/2$ requires more care, as we discussed earlier in the ADHM construction.

APPENDIX C: ADJOINT FERMIONS IN A KVBLL CALORON

The adjoint fermions in the classical background of KvBLL calorons can be constructed using the general ADHM construct presented above. For an alternative derivation using the replica trick for adjoint fermions in calorons, we refer to Ref. [23]. We recall that the BPS dyon results follow by taking various limits. The matrix of ADHM data is more involved in a KvBLL caloron. For the $SU(2)$ KvBLL caloron with a holonomy $P_\infty = e^{i2\pi\omega\cdot\sigma}$ and $\omega = \nu/2\beta \to \nu/2$, we have for the quaternionic blocks

$$\begin{aligned}\lambda(z) &= (P_+\delta(z-\omega) + P_-\delta(z+\omega))q\\ B(z,z') &= \delta(z-z')\left(\frac{1}{2\pi i}\frac{\partial}{\partial z'} + A(z')\right),\end{aligned} \tag{C1}$$

with $P_\pm$ as projectors and

$$A(z) = \chi_{[-\omega,\omega]}(z) + \bar{q}\omega\cdot\sigma q(\chi_{[-\omega,\omega]}(z) - 2\omega). \tag{C2}$$

The periodicity of the gauge field $A_m(x_4+\beta) = A_m(x_4)$ (modulo a gauge transformation) and the antiperiodicity of the adjoint fermions yield

$$\begin{aligned}c_m &= -e^{2\pi i\omega\cdot\sigma}c_{m-1}\\ \bar{c}_m &= -\bar{c}_m e^{-2\pi i\omega\cdot\sigma}\\ M_{mn} &= -M_{m-1,n-1}.\end{aligned} \tag{C3}$$

Their Fourier transforms are

$$\begin{aligned}c(z) &= \left(P_+\delta\left(z-\omega+\frac{1}{2}\right)+P_-\delta\left(z+\omega+\frac{1}{2}\right)\right)c\\ \bar{c}(z) &= \bar{c}\left(P_+\delta\left(-z-\omega+\frac{1}{2}\right)+P_-\delta\left(-z+\omega+\frac{1}{2}\right)\right)\\ M(z,z') &= \delta\left(z-z'+\frac{1}{2}\right)M(z').\end{aligned} \tag{C4}$$

Inserting Eq. (C4) in the adjoint zero mode constraint gives

$$\begin{aligned}&\frac{1}{2\pi i}\frac{d}{dz}M(z) + \left(A^T(z) - A^T\left(z+\frac{1}{2}\right)\right)M(z)\\ &-\epsilon_2\bar{q}P_+c\delta\left(z+\omega+\frac{1}{2}\right) - \epsilon_2\bar{q}P_-c\delta\left(z-\omega+\frac{1}{2}\right)\\ &-q^TP_+^T\bar{c}^T\delta(z+\omega) - q^TP_-^T\bar{c}^T\delta(z-\omega) = 0.\end{aligned} \tag{C5}$$

The explicit form of the zero modes is

$$\begin{aligned}(\lambda_\alpha)_{ab}\phi(x) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} dz dz'((&-c_a(-z) + u^\dagger_{\alpha\beta}(z+1/2)\\ &\times(\epsilon M)_\beta(z))f(z,z')u_{\alpha b}(z')\\ &-u^\dagger_{\alpha\beta}(z)\epsilon_{\alpha\beta}f(z,z')(-\bar{c}_b(z'+1/2)\\ &+M_\gamma(z')u_{\gamma b}(z')))\end{aligned} \tag{C6}$$

$$\begin{aligned}\lambda_m\phi(x) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} dz dz' \times ((&u(z')f^\star(z',z)\sigma_m(-c(-z)\\ &+u^\dagger(z+1/2)(\epsilon M)(z)) - \epsilon((-\bar{c}(z'+1/2)\\ &+M^T(z')u(z'))\sigma_m f(z,z')u^\dagger(z))^T),\end{aligned} \tag{C7}$$

with $\phi(x) = 1 + u^\dagger(x)u(x)$. Here, the m summation and z integration are subsumed. The x argument in $u(x,z)$ has been omitted for convenience.

1. Special case $\nu = \frac{1}{2}$

For the center-symmetric case with $\omega = 1/2\nu = 1/4$, we set $\omega\cdot\sigma = \tau_3/4$ and $q = i\rho\tau_3$ and identify the coordinates of the constituents M, L of the KvBLL caloron as

$$\begin{aligned}\mathbf{r} &= x\cdot\sigma + \pi\rho^2\tau_3/2\\ \mathbf{s} &= x\cdot\sigma - \pi\rho^2\tau_3/2,\end{aligned} \tag{C8}$$

in terms of which

$$\begin{aligned}A(z) - x &= -i\mathbf{s}\chi_{[-1/4,1/4]}(z) - i\mathbf{r}\chi_{[1/4,3/4]}(z)\\ &\equiv -i\mathbf{R}(z).\end{aligned} \tag{C9}$$

In this case, the equation for M simplifies,

$$\begin{aligned}\epsilon M &= e^{\pi\rho^2\tau_3 z}M_0 \qquad -1/4 < z < 1/4\\ \epsilon M &= e^{-\pi\rho^2(z-1/2)}M_0 \qquad +1/4 < z < 3/4,\end{aligned} \tag{C10}$$

and $\bar{c}^T = -\epsilon c$. The C zero mode and M zero mode decouple, with, respectively,

$$\begin{aligned}\lambda^C_m\phi(x) = &\int_{-\frac{1}{2}}^{+\frac{1}{2}} dz f(3/4,z)u(x,z)\sigma_m P_+c\\ &+\int_{-\frac{1}{2}}^{+\frac{1}{2}} dz f(1/4,z)u(x,z)\sigma_m P_-c\end{aligned} \tag{C11}$$

and

$$\lambda_m^M \phi(x) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} f(z_1, z_2) u(x, z_2) \sigma_m u^\dagger(x, z_1 + 1/2) \times \epsilon(M(z_1) + M(-z_1 - 1/2)) dz_1 dz_2. \quad \text{(C12)}$$

Here, $u(z)$ is the solution to the inhomogeneous and linear differential equation with piecewise potential

$$\left(\frac{1}{2\pi i}\frac{\partial}{\partial z} + i\mathbf{R}(z) - x_4\right) u(x, z) = -i\tau_3 \rho(P_+ \delta(-z + 1/4) + P_- \delta(-z - 1/4)), \quad \text{(C13)}$$

with the projectors $P_\pm = (1 \pm \tau^3)/2$. The explicit solutions are

$$u(x,z) = e^{2\pi i x_4 z} e^{2\pi \mathbf{s} z} B_1(x) \quad -1/4 < z < 1/4$$
$$u(x,z) = e^{2\pi i x_4 (z-1/2)} e^{2\pi \mathbf{r}(z-1/2)} B_2(x) \quad +1/4 < z < 3/4 \quad \text{(C14)}$$

and satisfy the completeness relations

$$e^{-\pi i x_4/2} e^{-\pi \mathbf{r}/4} B_2(x) - e^{\pi i x_4/2} e^{\pi \mathbf{s}/4} B_1(x) = +2\pi\rho P_+$$
$$e^{-\pi i x_4/2} e^{-\pi \mathbf{s}/4} B_1(x) - e^{\pi i x_4/2} e^{\pi \mathbf{r}/4} B_2(x) = -2\pi\rho P_-. \quad \text{(C15)}$$

Here, $B_{1,2}(x)$ are defined in Appendix C. The solutions obey the quasiperiodicity conditions

$$u(x_4 + 1, \mathbf{x}, z) = e^{2\pi i z} u(x_4, \mathbf{x}, z) e^{-\pi\tau_3/2}$$
$$B_1(x_4 + 1, \mathbf{x}) = B_1(x_4, \mathbf{x}) e^{-\pi\tau_3/2}$$
$$B_2(x_4 + 1, \mathbf{x}) = -B_2(x_4, \mathbf{x}) e^{-\pi\tau_3/2}. \quad \text{(C16)}$$

With the above in mind, the explicit form of the C zero mode is

$$\lambda_m^C \phi(x) = (f_1 + \hat{s}\cdot\sigma f_2) B_1 \sigma_m P_+ c + (\tilde{f}_1 + \hat{s}\cdot\sigma \tilde{f}_2) B_1 \sigma_m P_- c + (g_1 + \hat{r}\cdot\sigma g_2) B_2 \sigma_m P_+ c + (\tilde{g}_1 + \hat{r}\cdot\sigma \tilde{g}_2) B_2 \sigma_m P_- c, \quad \text{(C17)}$$

where we have set $s \equiv \omega_0 |\vec{s}|$ and $r = \omega_0 |\vec{r}|$. Also, we have

$$sr\psi(s, r, x_4) f_1(x_4, r, s) = \frac{e^{-\frac{1}{2}i\pi x_4}}{4s}(s + \sinh(s)) \times \left(\sinh\left(\frac{s}{2}\right)(d\sinh(r) + re^{2i\pi x_4} + r\cosh(r)) + s\sinh(r)\cosh\left(\frac{s}{2}\right)\right), \quad \text{(C18)}$$

with ψ given below, $d = \pi\rho^2$, and

$$sr\psi(s, r, x_4) f_2(x_4, r, s) = -\frac{e^{-\frac{1}{2}i\pi x_4}}{4s}(s - \sinh(s)) \times \left(-\cosh\left(\frac{s}{2}\right)(d\sinh(r) + r(-e^{2i\pi x_4}) + r\cosh(r)) - s\sinh(r)\sinh\left(\frac{s}{2}\right)\right), \quad \text{(C19)}$$

with the following identities among the $f, \tilde{f}, g, \tilde{g}$ functions:

$$\tilde{f}_1 \equiv f_1(-x_4, \mathbf{x}), \qquad \tilde{f}_2 \equiv -f_2(-x_4, \mathbf{x})$$
$$g_1 \equiv \tilde{f}_1(x_4, s, r), \qquad g_2 \equiv \tilde{f}_2(x_4, s, r)$$
$$\tilde{g}_1 \equiv g_1(-x_4, \mathbf{x}), \qquad \tilde{g}_2 \equiv -g_2(-x_4, \mathbf{x}). \quad \text{(C20)}$$

2. Adjoint zero mode for Dyon from KvBLL caloron

To isolate the adjoint zero modes on the constituents of the KvBLL caloron, we take the limit $d, |\vec{r}| \to \infty$ but with fixed s, which means that $r \to \infty$ as shown in Fig. 5. Most of the expressions simplify. Specifically, we have

$$f_1(x) = \frac{e^{-\frac{1}{2}i\pi x_4}}{2s^2} \frac{(s + \sinh(s))\sinh(\frac{s}{2})}{(\cosh(s) + \cos\theta\sinh(s))}$$
$$f_2(x) = \frac{e^{-\frac{1}{2}i\pi x_4}}{2s^2} \frac{(s - \sinh(s))\cosh(\frac{s}{2})}{(\cosh(s) + \cos\theta\sinh(s))}, \quad \text{(C21)}$$

with $s \equiv \omega_0 |\vec{s}|$, $\cos\theta = \vec{s}\cdot\hat{z}$, and

$$B_1 = 4\pi\rho(-\cos(\pi x_4))\left(\cosh\left(\frac{s}{2}\right)\tau_3 + \sinh\left(\frac{s}{2}\right)\hat{s}\right) + ie^{\pi i x_4 \tau_3 - \frac{i\pi}{2}\tau_3 x_4}\sin(\pi x_4)\frac{(\cosh(\frac{s}{2}) + \sinh(\frac{s}{2})\hat{s}\tau_3)}{\cosh(s) + \cos\theta\sinh(s)}$$
$$B_2 \to 0, \quad \text{(C22)}$$

with also

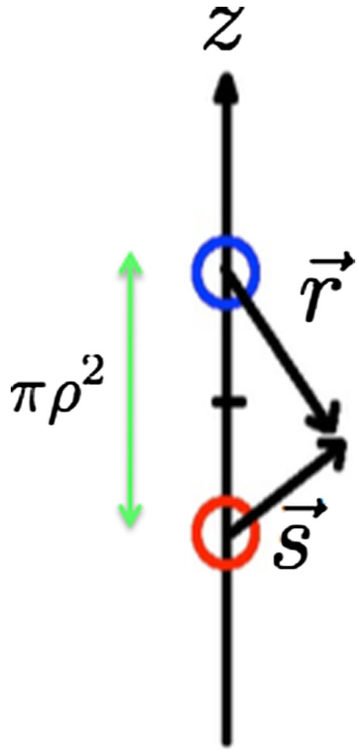

FIG. 5. *L-M* dyon at a distance $d = \pi\rho^2$ in a KvBLL caloron.

$$\psi = e^r(\cosh(s) + \cos\theta \sinh(s))$$
$$\phi = \frac{2d\cosh(s)}{s(\cosh(s) + \cos\theta\sinh(s))}. \qquad \text{(C23)}$$

Inserting Eqs. (C21)–(C23) into Eq. (C17) yields the asymptotic zero mode on the localized instanton dyon,

$$\begin{aligned} &s\cosh(s)(\cosh(s) + \cos\theta\sinh(s))\lambda_m^C \\ &= e^{-\frac{i\pi x_4}{2}}(sB_+ + \sinh(s)B_-)e^{-\frac{\pi i \tau_3 x_4}{2}}B\sigma_m P_+ c \\ &\quad + e^{\frac{i\pi x_4}{2}}(sB_- + \sinh(s)B_+)e^{-\frac{\pi i\tau_3 x_4}{2}}B\sigma_m P_+ c, \end{aligned} \qquad \text{(C24)}$$

with

$$B_\pm = \sinh\left(\frac{s}{2}\right) \pm \hat{s}\cdot\sigma\cosh\left(\frac{s}{2}\right)$$
$$B = \cosh\left(\frac{s}{2}\right)\tau_3 + \sinh\left(\frac{s}{2}\right)\hat{s}\cdot\sigma. \qquad \text{(C25)}$$

3. String gauge

The dyon reduced zero mode from the KvBLL caloron (C24) carries a θ dependence contrary to Eq. (47). Equation (C24) is expressed in the quasistring gauge, while Eq. (47) is in the hedgehog gauge. To express Eq. (C24) in the string gauge, we first gauge transform it using $g = e^{i2\pi\omega\cdot\tau}$ to obtain

$$\begin{aligned} &s\sinh(s)(\cosh(s) + \cos\theta\sinh(s))\lambda_b \\ &= e^{-i\omega_0 x_4}(P_+c)_a(sB_+B + \sinh(s)B_-B)_{\alpha b} \\ &\quad + e^{i\omega_0 x_4}(P_-c)_a(sB_-B + \sinh(s)B_+B)_{\alpha b}. \end{aligned} \qquad \text{(C26)}$$

In the same gauge, the dyon gauge field reads

$$\begin{aligned} A_4 &= \tau_3\partial_3\ln\kappa + \kappa\tau_\perp\cdot\partial_\perp\zeta + 2\omega\tau_3 \\ A_i &= \tau_3\epsilon_{ij3}\partial_3\ln\kappa + \kappa\tau_\perp\cdot\epsilon_{\perp ij}\partial_j\zeta \\ &\quad + 4\pi\omega\theta\kappa(\delta_{i1}\tau_2 - \delta_{i2}\tau_1) \end{aligned} \qquad \text{(C27)}$$

with

$$\zeta = \frac{4\pi\omega r}{\sinh(4\pi\omega r)}$$
$$\zeta\kappa = \frac{1}{\cosh(4\pi\omega r) + \cos(\theta)\sinh(4\pi\omega r)}, \qquad \text{(C28)}$$

which is still not in the string gauge. To bring the configuration (94) to the string gauge, we make use of

$$\mathbf{U} = \frac{\cosh(s/2)\tau_3 + \sinh(s/2)\sigma\cdot s}{\sqrt{\cosh(s) + \cos(\theta)\sinh(s)}}, \qquad \text{(C29)}$$

which is unitary.

4. Definitions

The matrices $B_{1,2}$ and the function ψ are in agreement with those used in Ref. [6]. We quote them here for completeness. Specifically,

$$B_1 = b_{12}b_{11}e^{-i2\pi x + 4\omega\tau_3}\mathbb{U}^\dagger/\psi$$
$$B_1 = b_{22}b_{21}e^{-i2\pi x + 4\omega\tau_3}\mathbb{U}^\dagger/\psi, \qquad \text{(C30)}$$

with $\mathbb{U}$ a unitary color rotation and

$$b_{11} = i2\pi\rho\left(\overline{\cosh_{\frac{1}{2}}} + \hat{r}\tau_3\overline{\sinh_{\frac{1}{2}}}\right)e^{i\pi x_4\tau_3}$$
$$b_{21} = i2\pi\rho\left(\cosh_{\frac{1}{2}} + \hat{s}\tau_3\sinh_{\frac{1}{2}}\right)e^{i\pi x_4\tau_3}$$
$$\begin{aligned} b_{12} = \Big(&-\cos(\pi x_4)\left(\cosh_{\frac{1}{2}}\overline{\sinh_{\frac{1}{2}}}\,\hat{r} + \cosh_{\frac{1}{2}}\overline{\sinh_{\frac{1}{2}}}\,\hat{r}\,s\right) \\ &+ i\sin(\pi x_4)\left(\cosh_{\frac{1}{2}}\overline{\cosh_{\frac{1}{2}}} + \hat{s}\,\hat{r}\sinh_{\frac{1}{2}}\overline{\sinh_{\frac{1}{2}}}\right)\Big) \end{aligned}$$
$$\begin{aligned} b_{22} = \Big(&-\cos(\pi x_4)\left(\cosh_{\frac{1}{2}}\overline{\sinh_{\frac{1}{2}}}\,\hat{r} + \cosh_{\frac{1}{2}}\overline{\sinh_{\frac{1}{2}}}\,\hat{r}\,s\right) \\ &+ i\sin(\pi x_4)\left(\cosh_{\frac{1}{2}}\overline{\cosh_{\frac{1}{2}}} + \hat{r}\,\hat{s}\sinh_{\frac{1}{2}}\overline{\sinh_{\frac{1}{2}}}\right)\Big) \end{aligned} \qquad \text{(C31)}$$

and

$$\psi \equiv -\cos(2\pi x_4) + \cosh\overline{\cosh} + \frac{\vec{s}\cdot\vec{r}}{sr}\sinh\overline{\sinh} \tag{C32}$$

with the short notation

$$\begin{aligned}
\sinh_{\frac{1}{2}} &= \sinh(\omega_0 \nu s)\\
\cosh_{\frac{1}{2}} &= \cosh(\omega_0 \nu s)\\
\overline{\sinh}_{\frac{1}{2}} &= \sinh(\omega_0(1-\nu)r)\\
\overline{\cosh}_{\frac{1}{2}} &= \cosh(\omega_0(1-\nu)r).
\end{aligned} \tag{C33}$$

APPENDIX D: FOCK CONTRIBUTION

In the main text, the mean-field analysis was presented using the so-called Hartree approximation. Here, we show how the Fock or exchange terms can be included. We first omit the cross interaction in

$$\langle \psi^T \epsilon \psi(x) \bar{\psi}^T \epsilon \bar{\psi}(y)\rangle e^{i\phi_1^\dagger(x) + i\phi_1(y)} \tag{D1}$$

can be retained by defining the 2×2 propagator

$$\langle (\psi(x), \bar{\psi}(x))(\epsilon\psi^T(y), -\epsilon\bar{\psi}^T(y))^T\rangle = \mathbf{S}(x-y) \tag{D2}$$

in terms of which the effective action $\mathbb{S}$ is a functional of (D2)

$$\begin{aligned}
-\mathbb{S}[\mathbf{S},\mathbf{b},\nu] &= \mathrm{Tr}(\mathbf{S}_0^{-1}\mathbf{S}) - \mathrm{Tr}\ln\mathbf{S}\\
&\quad + 8\pi f_M \left(\frac{\mathrm{Tr}\mathbf{S}}{2}\right)^2 \nu e^{\mathbf{b}} + 8\pi f_L \bar{\nu} e^{-\mathbf{b}}\\
&\quad - \frac{16\pi^2 f_M^2}{T^3}\left(\frac{\mathrm{Tr}\mathbf{S}}{2}\right)^4 \frac{1-e^{-V_0}}{\nu} e^{\mathbf{2b}}\\
&\quad - \frac{16\pi^2 f_L^2}{T^3}\frac{1-e^{-V_0}}{\bar{\nu}} e^{-\mathbf{2b}} + \frac{16\pi^2 f_M^2}{T^3} e^{-V_0} e^{\mathbf{2b}}\\
&\quad \times \int_0^{\frac{x_0}{2\omega_0\nu}} d^3x \mathrm{Tr}(\mathbf{S}_{12}^+(x)\mathbf{S}_{21}^+(-x))\\
&\quad \times \mathrm{Tr}(\mathbf{S}_{12}^-(x)\mathbf{S}_{21}^-(-x)).
\end{aligned} \tag{D3}$$

Here, S_{ij} are the pertinent entries in Eq. (D2). The two gap equations are now extrema of $\delta\mathbb{S}/\delta\mathbf{S}_{ij} = 0$. If we were to approximate the term $\mathrm{Tr}(\mathbf{SS})$ with free propagators, then the gap equations simplify, and we have for the dyonic part of the pressure

$$\begin{aligned}
\mathcal{P}_D &\to 8\pi f_M \nu \Sigma^2 e^{\mathbf{2b}} + 8\pi f_L \bar{\nu} e^{-\mathbf{b}}\\
&\quad - \frac{16\pi^2 f_M^2}{T^3}\Sigma^4 \frac{1-e^{-V_0}}{\nu} e^{\mathbf{2b}} - \frac{16\pi^2 f_L^2}{T^3}\frac{1-e^{-V_0}}{\bar{\nu}} e^{-\mathbf{2b}}\\
&\quad + \frac{16\pi^2 f_M^2}{T^3} e^{-V_0} e^{\mathbf{2b}} \int_0^{\frac{x_0}{2\omega_0\nu}} d^3r \mathrm{Tr}(\mathbf{T}(r)\mathbf{T}(-r))\\
&\quad - 4\Lambda\Sigma.
\end{aligned} \tag{D4}$$

APPENDIX E: ONE-LOOP APPROXIMATION

An alternative to the mean-field analysis is based on the use of the one-loop fermionic contribution only. The one-loop result is then used to compute the contractions induced by the second cumulant contribution stemming from the core. The result for the constraint equation is

$$\Lambda(\mathbf{b},\nu) = 2\pi\sqrt{f_L f_M}(\nu e^{\mathbf{b}} + \bar{\nu} e^{-\mathbf{b}}), \tag{E1}$$

and the gap equation is

$$2\tilde{\Sigma}(\Lambda) = \pi \int \tilde{p}^2 d\tilde{p} \frac{\tilde{\Lambda}\mathbb{F}}{1+\tilde{\Lambda}^2\mathbb{F}}. \tag{E2}$$

To one loop, the dressed fermionic propagator is

$$S^{-1} = \tilde{\mathbf{G}}^{-1} + \Lambda(\mathbf{b},\nu)\epsilon \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{E3}$$

Equations (E1)–(E3) can be used to reduce the contractions stemming from the second cumulant of the core, as we detailed in Sec. V B. The result is an effective action solely dependent on $\mathbf{b}$, ν, that is readily analyzed in the weak coupling and strong screening limits. The results of this analysis will be reported elsewhere.

APPENDIX F: HOLONOMY POTENTIAL

For completeness, the instanton-dyon pressure with hopping fermions has to be supplemented with the one-loop perturbative contributions from the adjoint periodic gluons and antiperiodic fermions for a finite holonomy ν [17]. The result for N_f massless adjoint quarks is

$$\begin{aligned}
\mathcal{P}_{1\mathrm{loop}}(N_f) &= \frac{4T^3}{\pi^2}\sum_{n=1}^{\infty}(1-N_f(-1)^n)\frac{\mathrm{Tr}_A L^n}{n^4}\\
\mathcal{P}_{1\mathrm{loop}}(1) &= \frac{16T^3}{\pi^2}\sum_{n=0}\frac{\cos(4n+2)\pi\nu}{(2n+1)^4},
\end{aligned} \tag{F1}$$

with $L = e^{i2\pi\nu T_3}$. The first contribution is from the adjoint gluons, while the second contribution is from the antiperiodic adjoint fermions. The perturbative minima of Eq. (F1) at $\nu = 0$, 1 yields a finite Polyakov line or an asymmetric (nonconfining) ground state. Note that for $N_f = 1$ periodic adjoint fermions $(-1)^n \to 1$ in Eq. (F1), and the bosonic and fermionic contributions cancel out. This result is expected from supersymmetry.

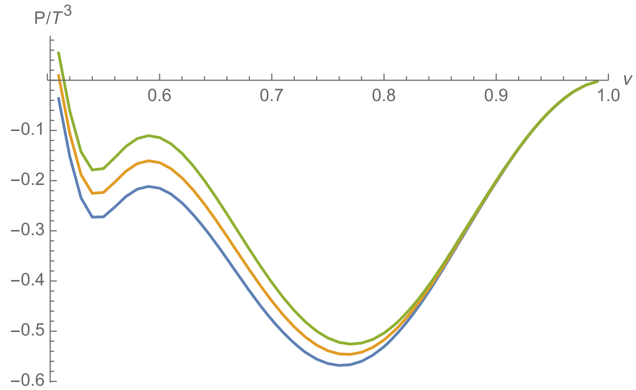

FIG. 6. Pressure (G1) vs ν for $\mathbf{n} = 0.50, 0.53, 0.56$ for lower-blue, middle-orange, and upper-green curves, respectively.

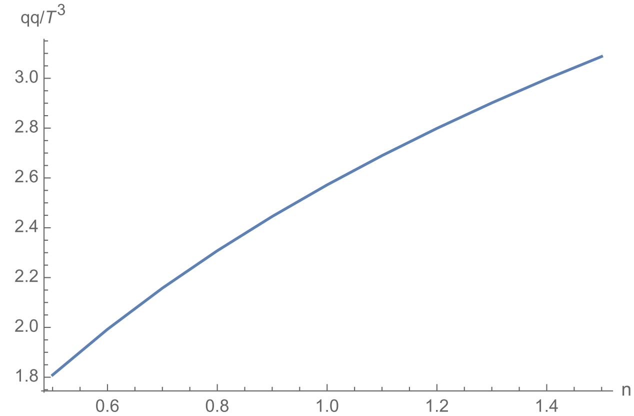

FIG. 7. Chiral condensate vs $\mathbf{n} = 2\pi f/T^2$ for $N_f = 1$.

APPENDIX G: CORE INTERACTION REVISITED

All of our analyses so far were carried out using the core interactions $V_{M,L}$ in Eq. (51). If we were to remove them by setting $\mathbf{b} = 0$ and consider only the induced repulsive interactions from the determinantal interactions in Eq. (49), a rerun of our preceding arguments yields Eq. (123) in the form

$$\begin{aligned}\frac{\mathcal{P}_{D+F}}{T^3} &= +(1-N_f)\tilde{\Lambda}\left(\frac{\tilde{\Lambda}}{\mathbf{n}(\nu\bar{\nu})^{\frac{1}{2}}}\right)^{\frac{1}{N_f-1}} \\ &\quad + \pi N_f \int \tilde{p}^2 d\tilde{p} \ln(1+\tilde{\Lambda}^2\mathbb{F}) \\ &\quad - \frac{4\pi^2}{3}(1+N_f)\nu^2\bar{\nu}^2, \end{aligned} \tag{G1}$$

with $\mathbf{n} = 2\pi f T^{N_f}/T^3$. We note that in deriving Eq. (G1) we have enforced the constraints (99) only after eliminating the w' s by variation. For $N_f = 1$, the first contribution in Eq. (G1) is absent, and $\tilde{\Lambda} = \mathbf{n}(\nu\bar{\nu})^{\frac{1}{2}}$. For $N_f > 1$, $\tilde{\Lambda}$ is fixed by the extremum of Eq. (G1).

In Fig. 6, we display Eq. (G1) for $N_f = 1$, which shows a first-order transition from a center symmetric for $\mathbf{n} > 0.5$ (low temperature) to a center asymmetric for $\mathbf{n} < 0.5$ (high temperature). The center-symmetric phase spontaneously breaks chiral symmetry with the chiral condensate shown in Fig. 7. Chiral symmetry is restored when center symmetry is lost. We have checked that this behavior persists for all $N_f > 1$, in contrast to the case with the core interaction discussed above, which does not support a chiral condensate for $N_f > 1$.

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