Radiative processes of two entangled atoms outside a Schwarzschild black hole

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We consider radiative processes of a quantum system composed by two identical two-level atoms in a black-hole background. We assume that these identical two-level atoms are placed at fixed radial distances outside a Schwarzschild black hole and interacting with a quantum electromagnetic field prepared in one of the usual vacuum states, namely, the Boulware, Unruh, or Hartle-Hawking vacuum states. We study the structure of the rate of variation of the atomic energy. The intention is to identify in a quantitative way the contributions of vacuum fluctuations and the radiation reaction to the entanglement generation between the atoms as well as the degradation of entangled states in the presence of an event horizon. We find that for a finite observation time the atoms can become entangled for the case of the field in the Boulware vacuum state, even if they are initially prepared in a separable state. In addition, the rate of variation of atomic energy is not well behaved at the event horizon due to the behavior of the proper accelerations of the atoms. We show that the thermal nature of the Hartle-Hawking and Unruh vacuum state allows the atoms to get entangled even if they were initially prepared in the separable ground state.

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I. INTRODUCTION

Quantum entanglement is the essential feature underlying quantum information, cryptography, and quantum computation [1,2]. Systems of two-level atoms interacting with a bosonic field have been one of the leading prototypes in the investigations concerning entangled states [3–7]. In turn, radiative processes of entangled states have been substantially considered in the literature [8,9]. Here we quote Ref. [10], in which the authors investigate the properties of emission from two entangled atoms coupled with an electromagnetic field in unbounded space. In addition, in Ref. [11], the authors demonstrate for spontaneous emission processes how nonlocal disentanglement times can be shorter than local decoherence times for arbitrary entangled states. See also Ref. [12].

The field of relativistic quantum information has emerged in recent years as an active research program connecting concepts from gravitational physics and quantum computing. In this respect, several important works were developed [13–18]. We also quote Refs. [19–25], which establish important results concerning entanglement generation between two localized causally disconnected atoms. On the other hand, many investigations were also implemented on a curved background. For instance, it was shown in Ref. [26] that an expanding space-time acts as a decohering agent which forces the entanglement of the vacuum to greatly decrease due to the effects of the Gibbons-Hawking temperature [27]. Another example, of immense current interest, is related to investigations of quantum entanglement in a Schwarzschild space-time which was undertaken in Ref. [28]. This was also the subject of study by the authors in Ref. [29]. In such a reference, entanglement was considered in the framework of open quantum systems. In the presence of a weak gravitational field, the authors in Ref. [30] have given manifest evidence that the amount of entanglement that Unruh-DeWitt detectors can extract from the vacuum can be increased. Similar results were found in Refs. [31–33]. For a review of recent results regarding entanglement in curved space-times, we refer the reader the work in Ref. [34] and references cited therein. The point that should be emphasized is that many such studies seem to imply the importance of considering the observer-dependency property of quantum entanglement [35]. Hence, a detailed understanding of such phenomena is mandatory in investigations concerning quantum information processes in the presence of a gravitational field or, more specifically, near an event horizon.

The aim of the present paper is to contribute to the investigations of relativistic quantum information theory in the light of an alternative perspective. Most of the investigations aforementioned were implemented in a framework of open quantum systems or by employing a time-dependent perturbation theory or similar techniques. The heuristic picture raised in such methods is that the generation (or degradation) of entanglement between two-level atoms is triggered by the vacuum fluctuations of the quantum field. In this respect, in a recent work, radiative processes of entangled atoms interacting with a massless scalar field prepared in the vacuum state in the presence of boundaries were considered [36]. Nevertheless, when discussing stimulated emission and absorption, which have equal Einstein B

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coefficients, it is not clear whether vacuum fluctuations always act as the only source of (or degradation of) entanglement. This is a consequence of the fact that it is possible to interpret spontaneous decay as a radiation-reaction effect [37]. As carefully demonstrated by Milonni, both effects, vacuum fluctuations and the radiation reaction, depend on the ordering chosen for commuting atomic and field operators [38]. Following such debates recently, quantum entanglement between inertial atoms [39] and uniformly accelerated atoms [40] coupled with an electromagnetic field was discussed in the framework developed by Dalibard, Dupont-Roc, and Cohen-Tannoudji (DDC) [41,42]. The results for uniformly accelerated atoms compare with the situation in which two atoms at rest are coupled individually to two spatially separated cavities at different temperatures, recovering, in some sense, the outcomes described in Ref. [11]. In addition, for equal accelerations it was obtained that one of the maximally entangled antisymmetric Bell state is a decoherence-free state.

We remark that the DDC formalism was also successfully implemented in many interesting physical situations [43–47], including quantum fields in a curved space-time [48–50]. For uniformly accelerated atoms, such a method quantitatively motivates the scenario presented in Ref. [51]. On the other hand, in the investigations concerning quantum entanglement, the DDC formalism has proved to be a pivotal treatment in order to better understand the structure responsible for supporting entanglement in radiative processes involving atoms, as demonstrated in Refs. [39,40]. Specifically, in such references it was shown how the rate of variation of atomic energy evaluated within the DDC approach can be a useful quantity in order to signalize the emergence of quantum entanglement. This is the idea we intend to continue to explore further in this work by considering the resonant interaction between atoms in a Schwarzschild space-time.

Even though close to the event horizon the Schwarzschild metric takes the form of the Rindler line element, there are important distinctions between an event horizon in Schwarzschild space-time and an acceleration horizon in Rindler space-time. In the present paper, we propose to generalize the results of Refs. [39,40] for the case of identical two-level atoms in a Schwarzschild space-time. Being more specific, we intend to investigate these atoms coupled with quantum electromagnetic fluctuations in the Boulware, Unruh, and Hartle-Hawking vacuum states. We use the approach above discussed, which allows an easy comparison of quantum mechanical and classical concepts. The organization of the paper is as follows. In Sec. II, we discuss the identification of vacuum fluctuations and the radiation-reaction effect in the situation of interest. In Sec. III, we calculate the rates of variation of atomic energy with finite observation time intervals for atoms placed at fixed radial distances outside a Schwarzschild black hole. Conclusions and final remarks are given in Sec. IV. In the Appendixes, we briefly digress on the correlation functions of an electromagnetic field in a Schwarzschild space-time. We also discuss the asymptotic evaluation of mode sums which will be important in what follows. In this paper, we use units such that $\hbar = c = G = k_B = 1$.

II. THE COUPLING OF ATOMS WITH ELECTROMAGNETIC FIELDS IN BLACK-HOLE SPACE-TIME

Let us suppose the case of two identical two-level atoms interacting with a common electromagnetic field. In this paper, we work in the multipolar coupling scheme, which means that all interactions are realized through the quantum electromagnetic fields. This formalism is suitable for describing retarded dipole-dipole interactions between the atoms. In general, the atoms will be moving along different world lines, so there will be two different proper times parameterizing each of these curves. We are working in a four-dimensional Schwarzschild space-time, which is described by the line element:

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

which is the vacuum solution to the Einstein field equations that describes the gravitational field outside a spherically symmetric body of mass M (we employ the convention that the Minkowski metric is given by $\eta_{\alpha\beta} = -1$, $\alpha = \beta = 1, 2, 3$, $\eta_{\alpha\beta} = 1$, $\alpha = \beta = 0$, and $\eta_{\alpha\beta} = 0$, $\alpha \neq \beta$). The collapse of an electrically neutral star endowed with spherical symmetry produces a spherical black hole of mass M with an external gravitational field described by the Schwarzschild line element (1). The surface of the black hole, i.e., the event horizon, is located at $r = 2M$, the position where the Schwarzschild coordinates become singular. Only the region on and outside the black hole's surface, $r \geq 2M$, is relevant to external observers. Events inside the horizon can never influence the exterior, at least in the classical regime. An interesting discussion on Schwarzschild's original solution can be found in Ref. [52].

Here we are interested in the situation of black-hole background geometry. Being more specific, we are prompted to dispense entirely with the spherically symmetric body and examine the quantum field theory for the electromagnetic field on the maximally extended manifold which is everywhere a solution of the vacuum Einstein equation. This is obtained from Eq. (1) by replacing the coordinates (t, r) by the so-called Kruskal-Szekeres coordinates (v, u). For an extensive discussion, see Refs. [53,54]. The Schwarzschild geometry consists of four different regions; see Fig. (31.3) of Ref. [53]. Regions I and III portray two distinct asymptotically flat universes with $r > 2M$; in fact, in region III the coordinate time t runs backwards with respect to region I. Regions II and IV are also time-reversed regions in which physical singularities ($r = 0$) evolve. In the Kruskal-Szekeres coordinates, one can show that the metric is perfectly well defined and nonsingular at the event horizon. In addition, such transformations and the metric (1) make clear that near the event horizon the line element approaches the form of the Rindler line element. Therefore, for $r \approx 2M$ the Schwarzschild coordinates t and r behave as Rindler space-time coordinates.

In this paper, we propose to identify quantitatively the contributions of quantum field vacuum fluctuations and the radiation reaction to the entanglement dynamics of atoms in black-hole space-time. With this respect, one must consider the Heisenberg picture. We consider both atoms moving along different stationary trajectories $x^\mu(\tau_i) = (t(\tau_i), \mathbf{x}(\tau_i))$, where τ_i denotes the proper time of the atom i. Because of this fact, in what follows, we describe the time evolution with respect to the Schwarzschild coordinate time t, which, because of (1), has a functional relation with each of the proper times of the atoms.

We suppose that the two-level atoms are placed at fixed radial distances outside the black hole. The stationary trajectory condition guarantees the existence of stationary states. Within the multipolar-coupling scheme, the purely atomic part of the total Hamiltonian describes the free atomic Hamiltonian. A brief and important comment is in order. It is known that the presence of gravitational fields affects the Coulomb interaction between charges within the atoms as well as dipole energies [55,56]. In addition, van der Waals forces are modified by gravity [57]. As a first approximation, we shall consider that the coupling between the atoms and the gravitational field is sufficiently weak. Hence, we take the free atomic Hamiltonian as having the same functional form as in the absence of gravitation. In this context, the Hamiltonian of this atomic system can then be written as

$$H_A(t) = \frac{\omega_0}{2}\left[(\sigma_1^z(\tau_1(t)) \otimes \hat{1})\frac{d\tau_1}{dt} + (\hat{1} \otimes \sigma_2^z(\tau_2(t)))\frac{d\tau_2}{dt}\right], \tag{2}$$

where $d\tau/dt = \sqrt{g_{00}} = (1-2M/r)^{1/2}$ and $\sigma_a^z = |e_a\rangle\langle e_a| - |g_a\rangle\langle g_a|, a = 1, 2$. Here $|g_1\rangle, |g_2\rangle$ and $|e_1\rangle, |e_2\rangle$ denote the ground and excited states of isolated atoms, respectively. One has that the space of the two-atom system is spanned by four product states with respective eigenenergies

$$\begin{aligned} E_{gg} &= -\omega_0, \qquad |gg\rangle = |g_1\rangle|g_2\rangle, \\ E_{ge} &= 0, \qquad |ge\rangle = |g_1\rangle|e_2\rangle, \\ E_{eg} &= 0, \qquad |eg\rangle = |e_1\rangle|g_2\rangle, \\ E_{ee} &= \omega_0, \qquad |ee\rangle = |e_1\rangle|e_2\rangle. \end{aligned} \tag{3}$$

Here we consider that the two-atom system is coupled with an electromagnetic field. The Hamiltonian $H_F(t)$ of the free electromagnetic field can be obtained in the usual way from Eq. (A1); see Appendix A. In this way, one has that

$$H_F(t) = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger(t) a_{\mathbf{k}}(t), \tag{4}$$

where $a_{\mathbf{k},\lambda}^\dagger$ and $a_{\mathbf{k},\lambda}$ are the usual creation and annihilation operators, respectively, of the electromagnetic field and we have neglected the zero-point energy. In addition, $\mathbf{k}$ labels the wave vector and polarization of the field modes. Furthermore, we also assume that the presence of a gravitational field does not affect substantially the physical consequences in considering the interaction between the atoms and the fields. Hence, in the multipolar coupling scheme and using the electric-dipole approximation, one has that the Hamiltonian which describes the interaction between the atoms and the field is given by

$$H_I(t) = -\boldsymbol{\mu}_1(\tau_1(t)) \cdot \mathbf{E}(x_1(\tau_1(t)))\frac{d\tau_1}{dt} - \boldsymbol{\mu}_2(\tau_2(t)) \cdot \mathbf{E}(x_2(\tau_2(t)))\frac{d\tau_2}{dt}, \tag{5}$$

where $\boldsymbol{\mu}_i$ ($i = 1, 2$) is the electric dipole moment operator for the ith atom. The electric field above is the measured electric field defined through the measured force it exerts on the atoms. The dipole moment operator is given by

$$\boldsymbol{\mu}_i(\tau_i) = \boldsymbol{\mu}[\sigma_i^+(\tau_i) + \sigma_i^-(\tau_i)], \tag{6}$$

where we have assumed that the dipole matrix elements $\langle g_i|\boldsymbol{\mu}_i|e_i\rangle$ are real and we denote them by $\boldsymbol{\mu}$, since they are independent of the index i (identical and similarly oriented atoms). In the above, we have defined the raising and lowering operators as $\sigma_i^+ = |e_i\rangle\langle g_i|$ and $\sigma_i^- = |g_i\rangle\langle e_i|$, respectively. Incidentally, suppose that our atoms are spinless one-electron systems. Hence, $\boldsymbol{\mu}_a = e\hat{\mathbf{r}}_a$, where e is the electron charge and $\hat{\mathbf{r}}_a$ is the position operator of the atom a.

The Heisenberg equations of motion for the dynamical variables of the atom and the field with respect to t can be derived from the total Hamiltonian $H(t) = H_A(t) + H_F(t) + H_I(t)$. After establishing the equations of motion, in order to solve them, one usually separates the solutions in two parts, namely, the free part, which is independent of the presence of a coupling between atoms and fields, and the source part, which is caused by the interaction between atoms and fields. That is, for atomic and field operators, respectively, $\sigma_a^z(\tau_a(t)) = \sigma_a^{z,f}(\tau_a(t)) + \sigma_a^{z,s}(\tau_a(t))$ and also $a_{\mathbf{k}}(t) = a_{\mathbf{k}}^f(t) + a_{\mathbf{k}}^s(t)$. Since one can construct from the annihilation and creation field operators the free and source part of the quantum electric field, one can also write $\mathbf{E}(t) = \mathbf{E}^f(t) + \mathbf{E}^s(t)$. As extensively discussed in Refs. [41–43], this calculation produces an ambiguity of operator ordering. In summary, this implies that one must choose an operator ordering when discussing the effects of $\mathbf{E}^f$ and $\mathbf{E}^s$ separately. This is the root of the feature already discussed in the introduction by which the effects of vacuum fluctuations (which are caused by $\mathbf{E}^f$) and the radiation reaction (which is originated from $\mathbf{E}^s$) depend on the ordering chosen for commuting atomic and field operators. Nonetheless, here we adopt a particular prescription which enables to interpret the effects of such phenomena as independent physical processes [41–43]. This is essentially the DDC formalism mentioned above.

We do not intend to give a thorough treatment of the DDC formalism here, since this approach has been analyzed in detail in many works. The reader may benefit from reading the several expositions we have already quoted above, especially Ref. [43], which, to the best of this author's knowledge, was one of the first works to discuss the Unruh effect [54,58,59] within such a framework. Therefore, we expound only the main results. The idea is to evaluate dH_A/dt, where H_A is given by Eq. (2), and consider only the part which is due to the interaction with the field; afterwards, one extracts from the remaining quantity the contributions of vacuum fluctuations and the radiation reaction, and then one takes the expectation value of the resulting quantities. The latter consists of two different operations: First, we consider an averaging over the field degrees of freedom (obtained by taking vacuum expectation values); subsequently, one takes the expectation value of the associated expressions in an atomic state $|\nu\rangle$, with energy ν. Such a state is usually one of the product states given by Eq. (3), but it can be any given state. For the purposes of studying entanglement, one can conveniently take $|\nu\rangle$ as a generic entangled state. For instance, consider the entangled states

$$|\Omega^{\pm}\rangle = c_1|g_1\rangle|e_2\rangle \pm c_2|e_1\rangle|g_2\rangle, \qquad (7)$$

where c_1 and c_2 are complex numbers. Note that $|\Omega^{\pm}\rangle$ are eigenstates of the atomic Hamiltonian H_A. Here we will be particularly interested in the situation where $c_1 = c_2 = 1/\sqrt{2}$. Such states constitute familiar examples of maximally entangled Bell states. The other Bell states are given by

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|g_1\rangle|g_2\rangle \pm |e_1\rangle|e_2\rangle). \qquad (8)$$

The Bell states form an alternative basis of the two-qubit Hilbert space. They play a fundamental role in Bell measurements, and they are also known as the four maximally entangled two-qubit Bell states.

Coming back to our problem, let us present the contributions of vacuum fluctuations and the radiation reaction in the evolution of the atoms' energies. Proceeding with a usual perturbative expansion and taking into account only terms up to order μ^2, the vacuum-fluctuation contribution reads

$$\left\langle \frac{dH_A}{dt} \right\rangle_{VF} = \frac{i}{2}\int_{t_0}^{t} dt' \sum_{a,b=1}^{2} \frac{d\tau_a}{dt}\frac{d\tau'_b}{dt'} D_{ij}(x_a(\tau_a(t)), x_b(\tau'_b(t')))\frac{\partial}{\partial \tau_a}\Delta^{ij}_{ab}(\tau_a(t), \tau'_b(t')), \qquad (9)$$

where the notation $\langle(\ldots)\rangle = \langle 0,\nu|(\ldots)|0,\nu\rangle$ has been employed ($|0\rangle$ is the vacuum state of the field, to be discussed below). In the above,

$$\Delta^{ij}_{ab}(\tau_a(t), \tau'_b(t')) = \langle\nu|[\mu^{i,f}_a(\tau_a(t)), \mu^{j,f}_b(\tau'_b(t'))]|\nu\rangle, \qquad a, b = 1, 2, \qquad (10)$$

is the linear susceptibility of the two-atom system in the state $|\nu\rangle$ and

$$D_{ij}(x_a(\tau_a(t)), x_b(\tau'_b(t'))) = \langle 0|\{E^f_i(x_a(\tau_a(t))), E^f_j(x_b(\tau'_b(t')))\}|0\rangle, \qquad (11)$$

a, $b = 1$, 2, is Hadamard's elementary function. On the other hand, for the radiation-reaction contribution, one has

$$\left\langle \frac{dH_A}{dt} \right\rangle_{RR} = \frac{i}{2}\int_{t_0}^{t} dt' \sum_{a,b=1}^{2} \frac{d\tau_a}{dt}\frac{d\tau'_b}{dt'} \Delta_{ij}(x_a(\tau_a(t)), x_b(\tau'_b(t')))\frac{\partial}{\partial \tau_a}D^{ij}_{ab}(\tau_a(t), \tau'_b(t')), \qquad (12)$$

where

$$D^{ij}_{ab}(\tau_a(t), \tau'_b(t')) = \langle\nu|\{\mu^{i,f}_a(\tau_a(t)), \mu^{j,f}_b(\tau'_b(t'))\}|\nu\rangle, \qquad a, b = 1, 2, \qquad (13)$$

is the symmetric correlation function of the two-atom system in the state $|\nu\rangle$ and

$$\Delta_{ij}(x_a(\tau_a(t)), x_b(\tau'_b(t'))) = \langle 0|[E^f_i(x_a(\tau_a(t))), E^f_j(x_b(\tau'_b(t')))]|0\rangle, \qquad (14)$$

a, $b = 1$, 2, is the Pauli-Jordan function. We see from Eqs. (9) and (12) that one can identify two distinct contributions. One is due to the existence itself of the atoms, and it is independent of any interaction whatsoever. The other is related with the emergence of cross-correlations between the atoms mediated by the field. Likewise, observe that such a formalism enables one to discuss the interplay between vacuum fluctuations and the radiation reaction in the generation or degradation of entanglement between atoms.

As emphasized in many texts, Δ_{ab}^{ij} and D_{ab}^{ij} characterize only the two-atom system itself. The explicit forms of such quantities are given by

$$\Delta_{ab}^{ij}(t,t') = \sum_{\nu'} [\mathcal{U}_{ab}^{ij}(\nu,\nu') e^{i\Delta\nu(\tau_a(t)-\tau_b(t'))} - \mathcal{U}_{ba}^{ji}(\nu,\nu') e^{-i\Delta\nu(\tau_a(t)-\tau_b(t'))}] \tag{15}$$

and

$$D_{ab}^{ij}(t,t') = \sum_{\nu'} [\mathcal{U}_{ab}^{ij}(\nu,\nu') e^{i\Delta\nu(\tau_a(t)-\tau_b(t'))} + \mathcal{U}_{ba}^{ji}(\nu,\nu') e^{-i\Delta\nu(\tau_a(t)-\tau_b(t'))}], \tag{16}$$

where $\Delta\nu = \nu - \nu'$ and we have conveniently introduced a suitable generalized atomic transition dipole moment $\mathcal{U}_{ab}^{ij}(\nu,\nu')$ defined as

$$\mathcal{U}_{ab}^{ij}(\nu,\nu') = \langle \nu | \mu_a^{i,f}(0) | \nu' \rangle \langle \nu' | \mu_b^{j,f}(0) | \nu \rangle. \tag{17}$$

Finally, observe that from (1) one can easily perform a change of variables in Eqs. (9) and (12) in order to describe the time evolution in terms of one of the proper times of the atoms. In fact, the use of the proper time is the customary procedure, since it is the quantity directly measurable by the clocks of the observers. However, as remarked above, here we adopt an alternative method in which we use the Schwarzschild coordinate time as the parameter that describes the time evolution of the system.

Now we are ready to characterize the entanglement generation (or degradation) between atoms as transitions between particular stationary states of the atomic Hamiltonian. The rate of variation of the atomic energy clearly identifies the permissible transitions between states, and, depending on the nature of the initial and final states, one may plainly perceive the constitution (or destruction) of an entangled state. In particular, as discussed above, within the DDC formalism, we can study how the interplay between vacuum fluctuations and the radiation reaction significantly influences the occurrence of these phenomena. Hence, we propose to investigate the creation of entanglement as well as how entangled states reduce to separable states. For instance, assume that the atoms were initially prepared in an entangled state, that is, $|\nu\rangle = |\Omega^{\pm}\rangle$. Hence, the only allowed transitions are $|\Omega^{\pm}\rangle \to |gg\rangle$, with $\Delta\nu = \nu - \nu' = \omega_0 > 0$, and $|\Omega^{\pm}\rangle \to |ee\rangle$, with $\Delta\nu = \nu - \nu' = -\omega_0 < 0$. In other words, the rate of variation of atomic energy should indicate the probability for the transitions $|\Omega^{\pm}\rangle \to |gg\rangle$ or $|\Omega^{\pm}\rangle \to |ee\rangle$ by displaying a nonzero value. On the other hand, suppose that the atoms were initially prepared in the atomic ground state ($|\nu\rangle = |gg\rangle$). The transition rates to one of the entangled states $|\Omega^{\pm}\rangle$ are nonvanishing, with the energy gap $\Delta\nu = -\omega_0 < 0$. In all such transitions, the nonzero matrix elements are given by, with $c_1 = c_2 = 1/\sqrt{2}$,

$$\mathcal{U}_{11}^{ij}(\nu,\nu') = \frac{\mu^i\mu^j}{2},$$
$$\mathcal{U}_{22}^{ij}(\nu,\nu') = \frac{\mu^i\mu^j}{2},$$
$$\mathcal{U}_{12}^{ij}(\nu,\nu') = \mathcal{U}_{21}(\nu,\nu') = \pm\frac{\mu^i\mu^j}{2}, \tag{18}$$

where ν stands for the ground state $|gg\rangle$ (or the excited state $|ee\rangle$) and ν' stands for the entangled states $|\Omega^{\pm}\rangle$, or vice versa.

In the next section, we will consider in detail the rate of variation of atomic energy for atoms at rest in various important physical situations.

III. RATE OF VARIATION OF THE ATOMIC ENERGY IN VACUUM

As discussed, we consider our two-atom system in a situation where the atoms are placed at fixed radial distances with the world lines given, respectively, by $x^{\mu}(\tau_i) = (\tau_i/\sqrt{g_{00(r_i)}}, r_i, \theta_i, \phi_i)$, $i = 1$, 2, and $g_{00}(r) = 1 - 2M/r$. Let us investigate the rate of change of the atomic energy of the two-atom system for each one of the possible vacua discussed in Appendix A. We consider the transitions discussed at the end of the previous section. For simplicity, we assume that the atoms are polarized along the radial direction defined by their positions relative to the black-hole space-time rotational Killing vector fields. This means that we do not need to calculate the contributions associated with the polarizations in the θ and ϕ directions, and the only field correlation functions that we should evaluate are the ones associated with the radial component of the electric field. This is extensively discussed in Appendix A. In the course of the calculations, one will typically deal with asymptotic estimation of mode sums. In the cases of interest, this is substantially discussed in Appendix B.

A. The Boulware vacuum

The Boulware vacuum has a close similarity to the innermost concept of an empty state at large radii, but it has pathological behavior at the horizon: The renormalized expectation value of the stress tensor, in a freely falling frame, diverges as $r \to 2M$ [51]. The Boulware vacuum is the appropriate choice of vacuum state for quantum fields in the vicinity of an isolated, cold neutron star.

We now proceed to calculate the rate of variation of atomic energy in the Boulware vacuum state. From the results derived in Appendix A, one may compute all the relevant correlation functions of the electric field which appears in Eqs. (9) and (12). The associated Hadamard's elementary functions are given by

$$D^B_{rr}(x_i(t),x_j(t')) = \frac{1}{16\pi^2}\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}_i\cdot\hat{r}_j)\int_0^\infty d\omega\omega\{e^{-i\omega(t-t')}[\vec{R}^{(1)}_{\omega l}(r_i)\vec{R}^{(1*)}_{\omega l}(r_j)+\overleftarrow{R}^{(1)}_{\omega l}(r_i)\overleftarrow{R}^{(1*)}_{\omega l}(r_j)] + e^{i\omega(t-t')}[\vec{R}^{(1)}_{\omega l}(r_j)\vec{R}^{(1*)}_{\omega l}(r_i)+\overleftarrow{R}^{(1)}_{\omega l}(r_j)\overleftarrow{R}^{(1*)}_{\omega l}(r_i)]\}, \tag{19}$$

i, $j = 1, 2$, where we have employed the addition theorem for the spherical harmonics [60]

$$\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') = P_l(\hat{r}\cdot\hat{r}'),$$

where $\hat{r}$ and $\hat{r}'$ are two unit vectors with spherical coordinates (θ,ϕ) and (θ',ϕ'), respectively, and P_l is the Legendre polynomial of degree l [61]. On the other hand, the Pauli-Jordan functions are given by

$$\Delta^B_{rr}(x_i(t),x_j(t')) = \frac{1}{16\pi^2}\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}_i\cdot\hat{r}_j)\int_0^\infty d\omega\omega\{e^{-i\omega(t-t')}[\vec{R}^{(1)}_{\omega l}(r_i)\vec{R}^{(1*)}_{\omega l}(r_j)+\overleftarrow{R}^{(1)}_{\omega l}(r_i)\overleftarrow{R}^{(1*)}_{\omega l}(r_j)] - e^{i\omega(t-t')}[\vec{R}^{(1)}_{\omega l}(r_j)\vec{R}^{(1*)}_{\omega l}(r_i)+\overleftarrow{R}^{(1)}_{\omega l}(r_j)\overleftarrow{R}^{(1*)}_{\omega l}(r_i)]\}. \tag{20}$$

The contributions (9) and (12) to the rate of variation of atomic energy can be evaluated by inserting in such expressions the statistical functions of the two-atom system, given by Eqs. (15) and (16), and the electromagnetic-field statistical functions given by (19) and (20). Initially, let us present the contributions coming from the vacuum fluctuations. Performing a simple change of variable $u = t - t'$, these can be expressed as, with $\Delta t = t - t_0$,

$$\left\langle\frac{dH_A}{dt}\right\rangle_{VF} = -\frac{1}{32\pi^2}\sum_{\nu'}\sum_{k,j=1}^{2}\mathcal{U}^{rr}_{kj}(\nu,\nu')\sqrt{g_{00}(r_k)g_{00}(r_j)}\exp[i(\widetilde{\Delta\nu}_k-\widetilde{\Delta\nu}_j)t]\Delta\nu \times\int_0^\infty d\omega\omega\left\{C_B(\omega,r_k,r_j)\int_0^{\Delta t}du[e^{i(\widetilde{\Delta\nu}_j-\omega)u}+e^{-i(\widetilde{\Delta\nu}_k-\omega)u}] + C_B(\omega,r_j,r_k)\int_0^{\Delta t}du[e^{i(\widetilde{\Delta\nu}_j+\omega)u}+e^{-i(\widetilde{\Delta\nu}_k+\omega)u}]\right\}, \tag{21}$$

where $\widetilde{\Delta\nu}_i = \sqrt{g_{00}(r_i)}\Delta\nu$ (this comes from the usual gravitational redshift effect) and the generalized atomic transition dipole moment $\mathcal{U}^{ij}_{ab}(\nu,\nu')$ is given by Eq. (18). Also, we have defined

$$C_B(\omega,r,r') = \vec{C}_B(\omega,r,r') + \overleftarrow{C}_B(\omega,r,r'), \tag{22}$$

with

$$\vec{C}_B(\omega,r,r') = \sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\vec{R}^{(1)}_{\omega l}(r)\vec{R}^{(1*)}_{\omega l}(r'),$$
$$\overleftarrow{C}_B(\omega,r,r') = \sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\overleftarrow{R}^{(1)}_{\omega l}(r)\overleftarrow{R}^{(1*)}_{\omega l}(r'). \tag{23}$$

The above integrals can be easily solved, and the result is

$$\left\langle\frac{dH_A}{dt}\right\rangle_{VF} = -\frac{1}{16\pi^2}\sum_{\nu'}\sum_{k,j=1}^{2}\mathcal{U}^{rr}_{kj}(\nu,\nu')\sqrt{g_{00}(r_k)g_{00}(r_j)}\exp[i(\widetilde{\Delta\nu}_k-\widetilde{\Delta\nu}_j)t]\Delta\nu$$
$$\times\int_0^\infty d\omega\omega\Bigg\{C_B(\omega,r_k,r_j)\left[\frac{e^{i\Delta t/2(\widetilde{\Delta\nu}_j-\omega)}\sin(\Delta t/2(\widetilde{\Delta\nu}_j-\omega))}{\widetilde{\Delta\nu}_j-\omega}+\frac{e^{-i\Delta t/2(\widetilde{\Delta\nu}_k-\omega)}\sin(\Delta t/2(\widetilde{\Delta\nu}_k-\omega))}{\widetilde{\Delta\nu}_k-\omega}\right]$$
$$+C_B(\omega,r_j,r_k)\left[\frac{e^{i\Delta t/2(\widetilde{\Delta\nu}_j+\omega)}(\sin(\Delta t/2(\widetilde{\Delta\nu}_j+\omega)))}{\widetilde{\Delta\nu}_j+\omega}+\frac{e^{-i\Delta t/2(\widetilde{\Delta\nu}_k+\omega)}\sin(\Delta t/2(\widetilde{\Delta\nu}_k+\omega))}{\widetilde{\Delta\nu}_k+\omega}\right]\Bigg\}. \tag{24}$$

For sufficiently large Δt, $\sin(\Delta tx)/x\to\pi\delta(x)$ and the integral over ω can be explicitly solved. In this limit, it becomes clear to note that vacuum fluctuations tend to excite $[\widetilde{\Delta\nu}_i<0\Rightarrow\langle dH_A/dt\rangle_{VF}>0]$ as well as deexcite $[\widetilde{\Delta\nu}_i>0\Rightarrow\langle dH_A/dt\rangle_{VF}<0]$ the atomic system. In the present context, this means that the atoms disentangle and can also entangle in a finite observation time due to vacuum fluctuations of the electromagnetic field. The creation of entanglement due to vacuum fluctuations persists even at late times, as this result plainly shows.

Now let us present the radiation-reaction contributions. Performing similar calculations as above, one gets

$$\left\langle\frac{dH_A}{dt}\right\rangle_{RR} = -\frac{1}{16\pi^2}\sum_{\nu'}\sum_{k,j=1}^{2}\mathcal{U}^{rr}_{kj}(\nu,\nu')\sqrt{g_{00}(r_k)g_{00}(r_j)}\exp[i(\widetilde{\Delta\nu}_k-\widetilde{\Delta\nu}_j)t]\Delta\nu$$
$$\times\int_0^\infty d\omega\omega\Bigg\{C_B(\omega,r_k,r_j)\left[\frac{e^{i\Delta t/2(\widetilde{\Delta\nu}_j-\omega)}\sin(\Delta t/2(\widetilde{\Delta\nu}_j-\omega))}{\widetilde{\Delta\nu}_j-\omega}+\frac{e^{-i\Delta t/2(\widetilde{\Delta\nu}_k-\omega)}\sin(\Delta t/2(\widetilde{\Delta\nu}_k-\omega))}{\widetilde{\Delta\nu}_k-\omega}\right]$$
$$-C_B(\omega,r_j,r_k)\left[\frac{e^{i\Delta t/2(\widetilde{\Delta\nu}_j+\omega)}(\sin(\Delta t/2(\widetilde{\Delta\nu}_j+\omega)))}{\widetilde{\Delta\nu}_j+\omega}+\frac{e^{-i\Delta t/2(\widetilde{\Delta\nu}_k+\omega)}\sin(\Delta t/2(\widetilde{\Delta\nu}_k+\omega))}{\widetilde{\Delta\nu}_k+\omega}\right]\Bigg\}. \tag{25}$$

Observe that the effect of the radiation reaction always leads to a loss of atomic energy $\langle dH_A/dt\rangle_{VF}<0$ independent of how the atomic system were initially prepared. In other words, with respect to absorption processes, the radiation reaction does not contribute to the generation of entanglement between the atoms; in this case, it always tends to disentangle an entangled state via spontaneous emission processes. This is reminiscent of the fact that classical noise coupled to an entangled quantum two-level system will generally lead to decoherence and disentanglement processes.

For completeness, let us present the total rate of change of the atomic energy. This is obtained by adding the contributions of vacuum fluctuations and the radiation reaction. One gets

$$\left\langle\frac{dH_A}{dt}\right\rangle_{\mathrm{tot}} = -\frac{1}{8\pi^2}\sum_{\nu'}\sum_{k,j=1}^{2}\mathcal{U}^{rr}_{kj}(\nu,\nu')\sqrt{g_{00}(r_k)g_{00}(r_j)}\exp[i(\widetilde{\Delta\nu}_k-\widetilde{\Delta\nu}_j)t]\Delta\nu$$
$$\times\int_0^\infty d\omega\omega C_B(\omega,r_k,r_j)\left[\frac{e^{i\Delta t/2(\widetilde{\Delta\nu}_j-\omega)}\sin(\Delta t/2(\widetilde{\Delta\nu}_j-\omega))}{\widetilde{\Delta\nu}_j-\omega}+\frac{e^{-i\Delta t/2(\widetilde{\Delta\nu}_k-\omega)}\sin(\Delta t/2(\widetilde{\Delta\nu}_k-\omega))}{\widetilde{\Delta\nu}_k-\omega}\right]. \tag{26}$$

The result clearly shows that it is possible to generate entanglement between the atoms via the absorption process for a finite observation time in the case of a quantum electromagnetic field in the Boulware vacuum state. However, once entanglement is created, it lasts only a finite duration and always disappears at late observation times Δt (see the remark above concerning the ω integral for large Δt). This is a similar result as the one found in Ref. [25] yet in an entirely different scenario. Typically, this entangled state lasts a duration of the order of $\sim 1/(\widetilde{\Delta\nu}_i-\omega)$, which corresponds roughly to the greatest width of the peaks of the functions $\sin x/x$ in the above integrand. In other words, one gets a finite result only for $\Delta\nu>0$ for asymptotic Δt: The balance between vacuum fluctuations and the radiation reaction prevents the atoms from getting entangled via an absorption process. In addition, for a finite observation time, notice that the situation in which $\Delta t<|\Delta\mathbf{x}|$, $|\Delta\mathbf{x}|$ being the distance between the atoms, is allowed. This does not bring any controversial issues regarding causality, since it is widely known that entangled quantum states produce nonlocal correlations [62]. Furthermore, note from Eq. (26) with large Δt that, as the atoms approach each other, one gets $\langle dH_A/dt\rangle\to 0$ for atoms initially prepared in the entangled state $|\Omega^-\rangle$ with $c_1=c_2=1/\sqrt{2}$, which means that such a state is stable with respect to radiative processes. Thus, we recover the well-known result which states that, for atoms confined into a region much smaller than the optical wavelength, the antisymmetric entangled state $|\Omega^-\rangle$ with $c_1 = c_2 = 1/\sqrt{2}$ can be regarded as a decoherence-free state [9].

Let us briefly discuss the rate of change of the atomic energy for the asymptotic regions of interest. All the relevant calculations are presented in detail in Appendix B. For simplicity, assume a large enough Δt so that one could approximate $\sin x/x$ as delta functions. First, let us consider the asymptotic region $r_1, r_2 \to \infty$. From the results derived in Appendix B, one gets

$$C_B(\omega, r_i, r_j) \approx H_+(\omega, r_i, r_j) + F(\omega, \mathbf{x}_i, \mathbf{x}_j), \tag{27}$$

where the function $F(\omega, \mathbf{x}, \mathbf{x}')$ is given by Eq. (B11) and we have defined

$$H_\pm(\omega, r, r') = \sum_{l=1}^{\infty} l(l+1)(2l+1)P_l(\hat{r}\cdot\hat{r}')|\mathcal{B}_l(\omega)|^2 \times \frac{e^{\pm i\omega(r_*-r'_*)}}{\omega^2 r^2 r'^2}, \tag{28}$$

where $\mathcal{B}_l(\omega)$ is the usual transmission coefficient defined through Eqs. (B1) and $r_* = r + 2M\ln(r/2M - 1)$ is the Regge-Wheeler tortoise coordinate. For estimation of the sum on Eq. (28), one can study the gravitational capture cross section of test particles whose trajectories terminates in the black hole [63]. One finds that, if the impact parameter b of an ultrarelativistic particle coming in from infinity is less than the critical value $\sqrt{27}M$, such a particle gets captured by the black hole. Employing the relation $l = \omega b$, one rewrites the capture condition as $l < \sqrt{27}M\omega$. Hence, assuming that all modes obeying such a relation are absorbed by the black hole, one can suitably approximate the transmission coefficient by $|\mathcal{B}_l(\omega)|^2 \sim \theta(\sqrt{27}M\omega - l)$, where $\theta(z)$ is the usual Heaviside step function. This is sometimes called the DeWitt approximation [64], but it is essentially a geometrical optics approximation for all wavelengths. Hence, one gets

$$H_\pm(\omega, r, r') \approx \sum_{l=1}^{\infty} \frac{l(l+1)(2l+1)P_l(\hat{r}\cdot\hat{r}')\theta(\sqrt{27}M\omega - l)e^{\pm i\omega(r_*-r'_*)}}{\omega^2 r^2 r'^2} \approx \frac{2e^{\pm i\omega(r_*-r'_*)}}{\omega^2 r^2 r'^2}\int_0^{\sqrt{27}M\omega} dl l^3 J_0(l\gamma), \tag{29}$$

where $\cos\gamma = \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'$ and we have employed the asymptotic result: $P_\nu[\cos(x/\nu)] \approx J_0(x) + \mathcal{O}(\nu^{-1})$, with $J_\mu(x)$ being the Bessel function of the first kind [61]. We distinguish two separate cases. For $\mathbf{r} = \mathbf{r}'$ one gets

$$H_\pm(\omega, r, r) \approx \frac{729M^4\omega^2}{2r^4}, \tag{30}$$

and for $\mathbf{r} \neq \mathbf{r}'$ one gets

$$H_\pm(\omega, r, r') \approx \frac{54M^2 e^{\pm i\omega(r_*-r'_*)}}{\gamma^2 r^2 r'^2} \times [2J_2(z(\omega)) - z(\omega)J_3(z(\omega))], \tag{31}$$

where we have defined the function $z(\omega) = \sqrt{27}M\gamma\omega$ and the following integral was used [65]:

$$\int_0^x dy y^3 J_0(y) = x^2[2J_2(x) - xJ_3(x)].$$

It is easy to see that, at infinity, $H_\pm$ gives vanishingly small contributions, so that the rate of change reduces essentially to that of inertial atoms in the Minkowski vacuum in flat space-times with no boundaries. In this way, the results of Ref. [39] are reproduced. Note also that $F(\omega, \mathbf{x}, \mathbf{x}') \sim 0$ for a large distance between the atoms, but it is finite for $r_1 = r_2 \to \infty$; see Eq. (B13). This means that for a large asymptotic separation between the atoms the cross-correlations arising in $\langle dH_A/dt\rangle$ vanish, and one is left with terms corresponding to isolated atoms.

The other important region is when $r_1, r_2 \to 2M$. One has that

$$C_B(\omega, r_i, r_j) \approx H_-(\omega, r_i, r_j) + \frac{16a^2(r_i)\sinh(\pi\xi(\omega))}{g_{00}(r_i)\pi\xi(\omega)} \times [A_{i\xi(\omega)}(\gamma, g_{00}(r_j), g_{00}(r_i)) + A_{-i\xi(\omega)}(\gamma, g_{00}(r_j), g_{00}(r_i))], \tag{32}$$

where $a(r) = M/(r^2\sqrt{g_{00}(r)})$ is the proper acceleration of the static atom at r, $\xi(\omega) = 4M\omega$, and $A_{i\xi}(\gamma, r, r') = A_{i\xi}(\gamma, g_{00}(r), g_{00}(r'))$ is properly defined in Appendix B. For finite Δt, as in the previous case, one gets a finite result regardless of the sign of $\Delta\nu$. In addition, note that $g_{00}(r)$ vanishes as the event horizon is approached; hence, the rate of change of the atomic energy diverges.

As a last analysis concerning the Boulware vacuum we take, say, $r_2 \to 2M$, whereas r_1 is kept arbitrary. One gets

$$C_B(\omega, r_1, r_2) \approx \sum_{l=1}^{\infty} \frac{l(l+1)(2l+1)P_l(\hat{r}_1\cdot\hat{r}_2)}{(r_2)^2(r_1)^2\omega^2} \times \left[\overleftarrow{\mathcal{R}}^{(1)}_{\omega l}(r_1)\mathcal{B}^*_l(\omega)e^{i\omega r_{2*}} + \frac{2e^{-i\omega/2\kappa}}{\Gamma(i\omega/\kappa)}\vec{\mathcal{R}}^{(1)}_{\omega l}(r_1)e^{i\omega\ln l/\kappa}K_{i\omega/\kappa}(2l\sqrt{g_{00}(r_2)})\right], \tag{33}$$

where $\kappa = 1/4M$, and a similar result for $\mathcal{C}_B(\omega, r_2, r_1)$; see Eq. (B27). From the results found in Appendix B, as $r_1 \to \infty$ the cross terms vanish, and again we are left only with terms corresponding to isolated atoms. On the other hand, as r_1 approaches $2M$, $\langle dH_A/dt \rangle$ diverges, and we recover the previous results discussed. Observe the general result: As the atoms approach $r = 2M$, the rate of variation of atomic energy grows rapidly and violently. For large Δt, this implies a vastly fast degradation of entanglement between the atoms initially prepared in one of the entangled states $|\Omega^{\pm}\rangle$.

B. The Hartle-Hawking vacuum

The Hartle-Hawking vacuum state is not empty at infinity, corresponding to a thermal distribution of quanta at the black-hole temperature. In other words, the Hartle-Hawking vacuum describes the physical situation in which the black hole is in equilibrium with an infinite sea of blackbody radiation, such as would be observed by constraining the black hole to the interior of a perfectly reflecting cavity. The renormalized expectation value of the stress tensor is well behaved in a freely falling frame on the horizon [51].

We now proceed to calculate the rate of variation of atomic energy in the Hartle-Hawking vacuum state. From the results derived in Appendix A, one may compute all the relevant correlation functions of the electric field which appears in Eqs. (9) and (12). The associated Hadamard's elementary functions are given by

$$D^H_{rr}(x_i(t), x_j(t')) = \frac{1}{16\pi^2}\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}_i\cdot\hat{r}_j)\int_{-\infty}^{\infty} d\omega\omega\Bigg\{e^{-i\omega(t-t')}\left[\frac{\vec{R}^{(1)}_{\omega l}(r_i)\vec{R}^{(1*)}_{\omega l}(r_j)}{1-e^{-2\pi\omega/\kappa}} + \frac{\overleftarrow{R}^{(1)}_{\omega l}(r_i)\overleftarrow{R}^{(1*)}_{\omega l}(r_j)}{e^{2\pi\omega/\kappa}-1}\right] + e^{i\omega(t-t')}\left[\frac{\vec{R}^{(1)}_{\omega l}(r_j)\vec{R}^{(1*)}_{\omega l}(r_i)}{1-e^{-2\pi\omega/\kappa}} + \frac{\overleftarrow{R}^{(1)}_{\omega l}(r_j)\overleftarrow{R}^{(1*)}_{\omega l}(r_i)}{e^{2\pi\omega/\kappa}-1}\right]\Bigg\}, \tag{34}$$

$i, j = 1, 2$, where we have employed the addition theorem for the spherical harmonics quoted above. On the other hand, the Pauli-Jordan functions are given by

$$\Delta^H_{rr}(x_1 i(t), x_j(t')) = \frac{1}{16\pi^2}\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}_i\cdot\hat{r}_j)\int_{-\infty}^{\infty} d\omega\omega\Bigg\{e^{-i\omega(t-t')}\left[\frac{\vec{R}^{(1)}_{\omega l}(r_i)\vec{R}^{(1*)}_{\omega l}(r_j)}{1-e^{-2\pi\omega/\kappa}} - \frac{\overleftarrow{R}^{(1)}_{\omega l}(r_i)\overleftarrow{R}^{(1*)}_{\omega l}(r_j)}{e^{2\pi\omega/\kappa}-1}\right] - e^{i\omega(t-t')}\left[\frac{\vec{R}^{(1)}_{\omega l}(r_j)\vec{R}^{(1*)}_{\omega l}(r_i)}{1-e^{-2\pi\omega/\kappa}} - \frac{\overleftarrow{R}^{(1)}_{\omega l}(r_j)\overleftarrow{R}^{(1*)}_{\omega l}(r_i)}{e^{2\pi\omega/\kappa}-1}\right]\Bigg\}. \tag{35}$$

Now such expressions as well as the statistical functions of the two-atom system, given by Eqs. (15) and (16), should be inserted in Eqs. (9) and (12). As above, we begin with the contributions coming from the vacuum fluctuations. Performing a simple change of variable $u = t - t'$, these can be expressed as, with $\Delta t = t - t_0$,

$$\left\langle\frac{dH_A}{dt}\right\rangle_{VF} = -\frac{1}{32\pi^2}\sum_{\nu'}\sum_{k,j=1}^{2}\mathcal{U}^{rr}_{kj}(\nu,\nu')\sqrt{g_{00}(r_k)g_{00}(r_j)}\exp[i(\widetilde{\Delta\nu}_k - \widetilde{\Delta\nu}_j)t]\Delta\nu \times\int_{-\infty}^{\infty} d\omega\omega\Bigg\{C^+_H(\omega, r_k, r_j)\int_0^{\Delta t} du[e^{i(\widetilde{\Delta\nu}_j-\omega)u} + e^{-i(\widetilde{\Delta\nu}_k-\omega)u}] + C^+_H(\omega, r_j, r_k)\int_0^{\Delta t} du[e^{i(\widetilde{\Delta\nu}_j+\omega)u} + e^{-i(\widetilde{\Delta\nu}_k+\omega)u}]\Bigg\}, \tag{36}$$

where we have defined

$$C^{\pm}_H(\omega, r, r') = \vec{C}_H(\omega, r, r') \pm \overleftarrow{C}_H(\omega, r, r'), \tag{37}$$

with

$$\vec{C}_H(\omega, r, r') = \sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\vec{R}^{(1)}_{\omega l}(r)\vec{R}^{(1*)}_{\omega l}(r')\left(1+\frac{1}{e^{2\pi\omega/\kappa}-1}\right),$$
$$\overleftarrow{C}_H(\omega, r, r') = \sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\frac{\overleftarrow{R}^{(1)}_{\omega l}(r)\overleftarrow{R}^{(1*)}_{\omega l}(r')}{e^{2\pi\omega/\kappa}-1}. \tag{38}$$

Solving the above integrals leads us to the following result:

$$\left\langle \frac{dH_A}{dt} \right\rangle_{VF} = -\frac{1}{16\pi^2} \sum_{\nu'} \sum_{k,j=1}^{2} \mathcal{U}_{kj}^{rr}(\nu,\nu') \sqrt{g_{00}(r_k) g_{00}(r_j)} \exp[i(\widetilde{\Delta\nu}_k - \widetilde{\Delta\nu}_j)t]\Delta\nu$$
$$\times \int_{-\infty}^{\infty} d\omega\omega \left\{ C_H^+(\omega, r_k, r_j) \left[\frac{e^{i\Delta t/2(\widetilde{\Delta\nu}_j-\omega)} \sin(\Delta t/2(\widetilde{\Delta\nu}_j - \omega))}{\widetilde{\Delta\nu}_j - \omega} + \frac{e^{-i\Delta t/2(\widetilde{\Delta\nu}_k-\omega)} \sin(\Delta t/2(\widetilde{\Delta\nu}_k - \omega))}{\widetilde{\Delta\nu}_k - \omega} \right] \right.$$
$$\left. + C_H^+(\omega, r_j, r_k) \left[\frac{e^{i\Delta t/2(\widetilde{\Delta\nu}_j+\omega)} (\sin(\Delta t/2(\widetilde{\Delta\nu}_j + \omega)))}{\widetilde{\Delta\nu}_j + \omega} + \frac{e^{-i\Delta t/2(\widetilde{\Delta\nu}_k+\omega)} \sin(\Delta t/2(\widetilde{\Delta\nu}_k + \omega))}{\widetilde{\Delta\nu}_k + \omega} \right] \right\}. \tag{39}$$

Observe the appearance of the thermal terms coming from the function C_H^+. This is most readily seen by letting Δt approach asymptotic values, which leads to delta functions in the above expressions, and again the integral over ω can be explicitly solved. As for the Boulware vacuum state, vacuum fluctuations tend to generate entanglement between atoms initially prepared in the ground state which is sustained over late periods of observational time. Similarly, vacuum fluctuations may destroy initially entangled atoms. Both processes are ensured with equal magnitude and are heightened by the thermal terms compared to the Boulware case.

Now let us present the radiation-reaction contributions. The calculations follow similar steps as above, and the result is

$$\left\langle \frac{dH_A}{dt} \right\rangle_{RR} = -\frac{1}{16\pi^2} \sum_{\nu'} \sum_{k,j=1}^{2} \mathcal{U}_{kj}^{rr}(\nu,\nu') \sqrt{g_{00}(r_k) g_{00}(r_j)} \exp[i(\widetilde{\Delta\nu}_k - \widetilde{\Delta\nu}_j)t]\Delta\nu$$
$$\times \int_{-\infty}^{\infty} d\omega\omega \left\{ C_H^-(\omega, r_k, r_j) \left[\frac{e^{i\Delta t/2(\widetilde{\Delta\nu}_j-\omega)} \sin(\Delta t/2(\widetilde{\Delta\nu}_j - \omega))}{\widetilde{\Delta\nu}_j - \omega} + \frac{e^{-i\Delta t/2(\widetilde{\Delta\nu}_k-\omega)} \sin(\Delta t/2(\widetilde{\Delta\nu}_k - \omega))}{\widetilde{\Delta\nu}_k - \omega} \right] \right.$$
$$\left. - C_H^-(\omega, r_j, r_k) \left[\frac{e^{i\Delta t/2(\widetilde{\Delta\nu}_j+\omega)} (\sin(\Delta t/2(\widetilde{\Delta\nu}_j + \omega)))}{\widetilde{\Delta\nu}_j + \omega} + \frac{e^{-i\Delta t/2(\widetilde{\Delta\nu}_k+\omega)} \sin(\Delta t/2(\widetilde{\Delta\nu}_k + \omega))}{\widetilde{\Delta\nu}_k + \omega} \right] \right\}. \tag{40}$$

Note that the contribution from the radiation reaction is also altered by the appearance of the thermal terms encoded in C_H^-. This is in sharp contrast to the situation of uniformly accelerated atoms coupled with a quantum field prepared in the Minkowski vacuum [40] (for a related result, see Ref. [50]). Nevertheless, as in the Boulware case, the radiation reaction does not contribute to the generation of entanglement between the atoms through an absorption process, leading always to disentanglement via spontaneous emission processes.

The total rate of change of the atomic energy is obtained by adding the contributions of vacuum fluctuations and the radiation reaction. One gets, for sufficiently large Δt,

$$\left\langle \frac{dH_A}{dt} \right\rangle_{\rm tot} = -\frac{1}{8\pi} \sum_{\nu'} \sum_{k,j=1}^{2} \mathcal{U}_{kj}^{rr}(\nu,\nu') \sqrt{g_{00}(r_k) g_{00}(r_j)} \exp[i(\widetilde{\Delta\nu}_k - \widetilde{\Delta\nu}_j)t]\Delta\nu$$
$$\times [\widetilde{\Delta\nu}_j \vec{C}_H(\widetilde{\Delta\nu}_j, r_k, r_j) + \widetilde{\Delta\nu}_k \vec{C}_H(\widetilde{\Delta\nu}_k, r_k, r_j) - \widetilde{\Delta\nu}_j \overleftarrow{C}_H(-\widetilde{\Delta\nu}_j, r_j, r_k) - \widetilde{\Delta\nu}_k \overleftarrow{C}_H(-\widetilde{\Delta\nu}_k, r_j, r_k)]. \tag{41}$$

Observe from Eq. (41) that, as the atoms approach each other, the entangled state $|\Omega^-\rangle$ with $c_1 = c_2 = 1/\sqrt{2}$ is again stable with respect to radiative processes.

The balance between vacuum fluctuations and the radiation reaction which existed in the Boulware vacuum is disturbed, and entanglement can be created via an absorption process in the exterior region of the black hole even for large asymptotic Δt. In other words, both possibilities $\Delta\nu > 0$ and $\Delta\nu < 0$ are allowed. The Planckian factors which appear in Eq. (41) through the functions $\vec{C}_H$ and $\overleftarrow{C}_H$ uncover the thermal nature of the Hartle-Hawking vacuum. For large enough Δt, one easily sees that the temperature of the thermal radiation is given by the Hawking temperature:

$$T_i^H = \frac{\kappa}{2\pi\sqrt{g_{00}(r_i)}} = \frac{1}{8\pi M\sqrt{1-2M/r_i}}, \tag{42}$$

which is just the Tolman relation which gives the temperature felt by a local observer at the fixed position r_i [66]. The emergence of two different temperatures is a feature of the atoms being at different fixed positions. Even though we find different temperatures, the thermal equilibrium is warranted by invoking the Tolman relation $(g_{00}(x))^{1/2}T(x) = \text{const}$.

Further inquiries must be answered by inspecting the result in the asymptotic regions. Consider a large enough Δt as above. For the atoms fixed at spatial infinity, i.e., $r_1, r_2 \to \infty$, the results discussed in Appendix B reveal that

$$C_H^{\pm}(\omega, r_i, r_j) \approx H_+(\omega, r_i, r_j)\left(1 + \frac{1}{e^{2\pi\omega/\kappa} - 1}\right) \pm \frac{F(\omega, \mathbf{x}_i, \mathbf{x}_j)}{e^{2\pi\omega/\kappa} - 1}, \tag{43}$$

where the function $F(\omega, \mathbf{x}, \mathbf{x}')$ is given by Eq. (B11) and $H_{\pm}(\omega, r, r')$ is given by expression (28) [or Eqs. (30) and (31) within the geometrical optics approximation discussed above]. At infinity and for a large distance between the atoms, $H_{\pm}(\omega, r, r')$ and $F(\omega, \mathbf{x}, \mathbf{x}')$ give vanishingly small contributions, and we are left with a summation of terms related with isolated atoms, each one following its own world line. Recalling the thermalization theorem [51,67,68], one is led to the conclusion that in the situation with a sufficiently high relative asymptotic distance we have two atoms coupled individually to two spatially separated cavities at different Hawking temperatures in a flat space-time. Hence, our results indicate a close resemblance to the outcomes of Ref. [11].

At the vicinity of the event horizon, i.e., $r_1, r_2 \to 2M$, one has that

$$\begin{aligned} C_H^{\pm}(\omega, r_i, r_j) \approx{}& \frac{16a^2(r_i)\sinh(\pi\xi(\omega))}{g_{00}(r_i)\pi\xi(\omega)} \\ &\times [A_{i\xi(\omega)}(\gamma, g_{00}(r_j), g_{00}(r_i)) \\ &+ A_{-i\xi(\omega)}(\gamma, g_{00}(r_j), g_{00}(r_i))] \\ &\times \left(1 + \frac{1}{e^{2\pi\omega/\kappa} - 1}\right) \\ &\pm \frac{H_-(\omega, r_i, r_j)}{e^{2\pi\omega/\kappa} - 1}, \end{aligned} \tag{44}$$

which clearly shows a divergent result for $a(r) \to \infty$. We also observe two kinds of contributions, namely, one related to the outgoing thermal radiation from the event horizon and the other one associated with the thermal term multiplied by H_-, which can be interpreted as a consequence of the existence of incoming thermal radiation from infinity. It is precisely this thermal nature that enables the atoms to get entangled even if they were initially prepared in the separable ground state.

Finally, consider that $r_2 \to 2M$, whereas r_1 is kept arbitrary. One gets

$$\begin{aligned} C_H^{\pm}(\omega, r_1, r_2) \approx \sum_{l=1}^{\infty} \frac{l(l+1)(2l+1)P_l(\hat{r}_1 \cdot \hat{r}_2)}{(r_2)^2(r_1)^2\omega^2} &\left[\frac{2e^{-i\omega/2\kappa}}{\Gamma(i\omega/\kappa)} \vec{\mathcal{R}}_{\omega l}^{(1)}(r_1) e^{i\omega \ln l/\kappa} K_{i\omega/\kappa}(2l\sqrt{g_{00}(r_2)})\left(1 + \frac{1}{e^{2\pi\omega/\kappa} - 1}\right)\right. \\ &\left.\pm \frac{\overleftarrow{\mathcal{R}}_{\omega l}^{(1)}(r_1)\mathcal{B}_l^*(\omega)e^{i\omega r_{*2}}}{e^{2\pi\omega/\kappa} - 1}\right] \end{aligned} \tag{45}$$

and a similar result for $\mathcal{C}_H^{\pm}(\omega, r_2, r_1)$; see Eq. (B27). From the results found in Appendix B, as $r_1 \to \infty$ the cross terms vanish, and again we are left only with terms corresponding to isolated atoms. On the other hand, as r_1 approaches $2M$, $\langle dH_A/dt \rangle$ diverges, and we recover the previous results discussed. Again, we have obtained the general result aforementioned: As the atoms approach the event horizon, the rate of variation of atomic energy grows rapidly and violently. For large Δt, this implies a greatly enhanced generation of entanglement between the atoms initially prepared in the ground state.

C. The Unruh vacuum

The Unruh vacuum state is the adequate choice of vacuum state which is most relevant to the gravitational collapse of a massive body. At spatial infinity, this vacuum is tantamount to an outgoing flux of blackbody radiation at the black-hole temperature. The renormalized expectation value of the stress tensor, in a freely falling frame, is well behaved on the future horizon but not on the past horizon [51].

We now proceed to calculate the rate of variation of atomic energy in the Unruh vacuum state. From the results derived in Appendix A, one may compute all the relevant correlation functions of the electric field which appears in Eqs. (9) and (12). The associated Hadamard's elementary functions reads

$$D^U_{rr}(x_i(t),x_j(t')) = \frac{1}{16\pi^2}\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}_i\cdot\hat{r}_j)\int_{-\infty}^{\infty}d\omega\omega\Bigg\{e^{-i\omega(t-t')}\left[\frac{\vec{R}^{(1)}_{\omega l}(r_i)\vec{R}^{(1*)}_{\omega l}(r_j)}{1-e^{-2\pi\omega/\kappa}}+\theta(\omega)\overleftarrow{R}^{(1)}_{\omega l}(r_i)\overleftarrow{R}^{(1*)}_{\omega l}(r_j)\right] + e^{i\omega(t-t')}\left[\frac{\vec{R}^{(1)}_{\omega l}(r_j)\vec{R}^{(1*)}_{\omega l}(r_i)}{1-e^{-2\pi\omega/\kappa}}+\theta(\omega)\overleftarrow{R}^{(1)}_{\omega l}(r_j)\overleftarrow{R}^{(1*)}_{\omega l}(r_i)\right]\Bigg\}, \tag{46}$$

i, $j = 1$, 2, where use was made of the foregoing addition theorem for the spherical harmonics. In turn, the Pauli-Jordan functions are given by

$$\Delta^U_{rr}(x_i(t),x_j(t')) = \frac{1}{16\pi^2}\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}_i\cdot\hat{r}_j)\int_{-\infty}^{\infty}d\omega\omega\Bigg\{e^{-i\omega(t-t')}\left[\frac{\vec{R}^{(1)}_{\omega l}(r_i)\vec{R}^{(1*)}_{\omega l}(r_j)}{1-e^{-2\pi\omega/\kappa}}+\theta(\omega)\overleftarrow{R}^{(1)}_{\omega l}(r_i)\overleftarrow{R}^{(1*)}_{\omega l}(r_j)\right] - e^{i\omega(t-t')}\left[\frac{\vec{R}^{(1)}_{\omega l}(r_j)\vec{R}^{(1*)}_{\omega l}(r_i)}{1-e^{-2\pi\omega/\kappa}}+\theta(\omega)\overleftarrow{R}^{(1)}_{\omega l}(r_j)\overleftarrow{R}^{(1*)}_{\omega l}(r_i)\right]\Bigg\}. \tag{47}$$

The contributions (9) and (12) to the rate of variation of atomic energy can be evaluated by inserting in such expressions the statistical functions of the two-atom system, given by Eqs. (15) and (16), and the electromagnetic-field statistical functions given by (46) and (47). Initially, let us focus on the contributions coming from the vacuum fluctuations. Performing a simple change of variable $u = t - t'$, these can be expressed as, with $\Delta t = t - t_0$,

$$\left\langle\frac{dH_A}{dt}\right\rangle_{VF} = -\frac{1}{32\pi^2}\sum_{\nu'}\sum_{k,j=1}^{2}\mathcal{U}^{rr}_{kj}(\nu,\nu')\sqrt{g_{00}(r_k)g_{00}(r_j)}\exp[i(\widetilde{\Delta\nu}_k-\widetilde{\Delta\nu}_j)t]\Delta\nu \times\int_{-\infty}^{\infty}d\omega\omega\Bigg\{C_U(\omega,r_k,r_j)\int_0^{\Delta t}du[e^{i(\widetilde{\Delta\nu}_j-\omega)u}+e^{-i(\widetilde{\Delta\nu}_k-\omega)u}] + C_U(\omega,r_j,r_k)\int_0^{\Delta t}du[e^{i(\widetilde{\Delta\nu}_j+\omega)u}+e^{-i(\widetilde{\Delta\nu}_k+\omega)u}]\Bigg\}, \tag{48}$$

where we have defined

$$C_U(\omega,r,r') = \vec{C}_U(\omega,r,r')+\overleftarrow{C}_U(\omega,r,r'), \tag{49}$$

with

$$\vec{C}_U(\omega,r,r') = \sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\vec{R}^{(1)}_{\omega l}(r)\vec{R}^{(1*)}_{\omega l}(r')\left(1+\frac{1}{e^{2\pi\omega/\kappa}-1}\right),$$
$$\overleftarrow{C}_U(\omega,r,r') = \theta(\omega)\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\overleftarrow{R}^{(1)}_{\omega l}(r)\overleftarrow{R}^{(1*)}_{\omega l}(r'). \tag{50}$$

The above integrals can be easily solved, and we find that

$$\left\langle\frac{dH_A}{dt}\right\rangle_{VF} = -\frac{1}{16\pi^2}\sum_{\nu'}\sum_{k,j=1}^{2}\mathcal{U}^{rr}_{kj}(\nu,\nu')\sqrt{g_{00}(r_k)g_{00}(r_j)}\exp[i(\widetilde{\Delta\nu}_k-\widetilde{\Delta\nu}_j)t]\Delta\nu \times\int_{-\infty}^{\infty}d\omega\omega\Bigg\{C_U(\omega,r_k,r_j)\left[\frac{e^{i\Delta t/2(\widetilde{\Delta\nu}_j-\omega)}\sin(\Delta t/2(\widetilde{\Delta\nu}_j-\omega))}{\widetilde{\Delta\nu}_j-\omega}+\frac{e^{-i\Delta t/2(\widetilde{\Delta\nu}_k-\omega)}\sin(\Delta t/2(\widetilde{\Delta\nu}_k-\omega))}{\widetilde{\Delta\nu}_k-\omega}\right] + C_U(\omega,r_j,r_k)\left[\frac{e^{i\Delta t/2(\sqrt{g_{00}(r_j)}\Delta E+\omega)}(\sin(\Delta t/2(\widetilde{\Delta\nu}_j+\omega)))}{\widetilde{\Delta\nu}_j+\omega}+\frac{e^{-i\Delta t/2(\widetilde{\Delta\nu}_k+\omega)}\sin(\Delta t/2(\widetilde{\Delta\nu}_k+\omega))}{\widetilde{\Delta\nu}_k+\omega}\right]\Bigg\}. \tag{51}$$

As in the cases above studied, the atoms disentangle and can also entangle in a finite observation time due to vacuum fluctuations of the electromagnetic field. The creation of entanglement due to vacuum fluctuations also persists at late times.

Now let us present the radiation-reaction contributions. Performing similar calculations as above, one gets

$$\left\langle \frac{dH_A}{dt} \right\rangle_{RR} = -\frac{1}{16\pi^2} \sum_{\nu'} \sum_{k,j=1}^{2} \mathcal{U}_{kj}^{rr}(\nu,\nu') \sqrt{g_{00}(r_k) g_{00}(r_j)} \exp[i(\widetilde{\Delta\nu}_k - \widetilde{\Delta\nu}_j)t]\Delta\nu$$
$$\times \int_{-\infty}^{\infty} d\omega\omega \Bigg\{ C_U(\omega, r_k, r_j) \left[\frac{e^{i\Delta t/2(\widetilde{\Delta\nu}_j - \omega)} \sin(\Delta t/2(\widetilde{\Delta\nu}_j - \omega))}{\widetilde{\Delta\nu}_j - \omega} + \frac{e^{-i\Delta t/2(\widetilde{\Delta\nu}_k - \omega)} \sin(\Delta t/2(\widetilde{\Delta\nu}_k - \omega))}{\widetilde{\Delta\nu}_k - \omega} \right]$$
$$- C_U(\omega, r_j, r_k) \left[\frac{e^{i\Delta t/2(\sqrt{g_{00}(r_j)}\Delta E + \omega)} (\sin(\Delta t/2(\widetilde{\Delta\nu}_j + \omega)))}{\widetilde{\Delta\nu}_j + \omega} + \frac{e^{-i\Delta t/2(\widetilde{\Delta\nu}_k + \omega)} \sin(\Delta t/2(\widetilde{\Delta\nu}_k + \omega))}{\widetilde{\Delta\nu}_k + \omega} \right] \Bigg\}. \quad (52)$$

Observe that the effect of the radiation reaction, with respect to absorption processes, does not contribute to the generation of entanglement between the atoms as in the cases investigated above; it always tends to disentangle an entangled state via spontaneous emission processes. As in the Hartle-Hawking case, such a contribution is also altered by the appearance of a thermal contribution.

For completeness, let us present the total rate of change of the atomic energy. This is obtained by adding the contributions of vacuum fluctuations and the radiation reaction. One gets, for sufficiently large Δt,

$$\left\langle \frac{dH_A}{dt} \right\rangle_{\text{tot}} = -\frac{1}{8\pi} \sum_{\nu'} \sum_{k,j=1}^{2} \mathcal{U}_{kj}^{rr}(\nu,\nu') \sqrt{g_{00}(r_k) g_{00}(r_j)}$$
$$\times \exp[i(\widetilde{\Delta\nu}_k - \widetilde{\Delta\nu}_j)t]\Delta\nu$$
$$\times [\widetilde{\Delta\nu}_j C_U(\widetilde{\Delta\nu}_j, r_k, r_j) + \widetilde{\Delta\nu}_k C_U(\widetilde{\Delta\nu}_k, r_k, r_j)]. \quad (53)$$

As in the Hartle-Hawking case, the balance between vacuum fluctuations and the radiation reaction which existed in the Boulware vacuum is disturbed, and entanglement can be created via an absorption process in the exterior region of the black hole even for large asymptotic Δt. The structure of the rate of variation of atomic energy implies the existence of thermal radiation from the black hole which is backscattered by space-time curvature. This thermal radiation is responsible for the possibility of the creation of entanglement between atoms initially prepared in the ground state. The temperature of the thermal radiation as felt by each of the atoms is again given by Eq. (42). Moreover, Eq. (53) shows us that the entangled state $|\Omega^-\rangle$ with the choice $c_1 = c_2 = 1/\sqrt{2}$ is stable with respect to radiative processes for atoms located at the same position.

Let us briefly digress on the rate of change of the atomic energy for the asymptotic regions of interest. Again, we are assuming a large enough Δt so that one could approximate $\sin x/x$ as delta functions. Consider the asymptotic region $r_1, r_2 \to \infty$. From the results derived in Appendix B, one gets

$$C_U(\omega, r_i, r_j) \approx H_+(\omega, r_i, r_j) \left(1 + \frac{1}{e^{2\pi\omega/\kappa} - 1} \right)$$
$$+ \theta(\omega) F(\omega, \mathbf{x}_i, \mathbf{x}_j). \quad (54)$$

Note that thermal terms are multiplied by the gray-body factor $H_\pm$ which vanishes at spatial infinity. Hence, as the atoms approach spatial infinity, the flux felt by them becomes more pale, which means that the creation of entanglement by absorption processes becomes rarer.

The other important region is when $r_1, r_2 \to 2M$. One has that

$$C_U(\omega, r_i, r_j) \approx \frac{16a^2(r_i)\sinh(\pi\xi(\omega))}{g_{00}(r_i)\pi\xi(\omega)}$$
$$\times [A_{i\xi(\omega)}(\gamma, g_{00}(r_j), g_{00}(r_i))$$
$$+ A_{-i\xi(\omega)}(\gamma, g_{00}(r_j), g_{00}(r_i))]$$
$$\times \left(1 + \frac{1}{e^{2\pi\omega/\kappa} - 1} \right)$$
$$+ \theta(\omega) H_-(\omega, r_i, r_j). \quad (55)$$

Again note that $g_{00}(r)$ vanishes as the event horizon is approached; hence, the rate of change of the atomic energy diverges.

As a last analysis concerning the Unruh vacuum we take, say, $r_2 \to 2M$, whereas r_1 is kept arbitrary. One gets

$$C_U(\omega, r_1, r_2) \approx \sum_{l=1}^{\infty} \frac{l(l+1)(2l+1)P_l(\hat{r}_1 \cdot \hat{r}_2)}{(r_2)^2 (r_1)^2 \omega^2}$$
$$\times \Bigg[\theta(\omega) \overleftarrow{\mathcal{R}}_{\omega l}^{(1)}(r_1) \mathcal{B}_l^*(\omega) e^{i\omega r_{*2}}$$
$$+ \frac{2e^{-i\omega/2\kappa}}{\Gamma(i\omega/\kappa)} \vec{\mathcal{R}}_{\omega l}^{(1)}(r_1) e^{i\omega \ln l/\kappa} K_{i\omega/\kappa}(2l\sqrt{g_{00}(r_2)})$$
$$\times \left(1 + \frac{1}{e^{2\pi\omega/\kappa} - 1} \right) \Bigg]. \quad (56)$$

From the results found in Appendix B, as $r_1 \to \infty$ the cross terms vanish, and again we are left only with terms corresponding to isolated atoms. On the other hand, as r_1

approaches $2M$, $\langle dH_A/dt\rangle$ diverges. This last case confirms again the general result stated above: As the atoms approach the event horizon, the rate of variation of atomic energy grows quickly. For large Δt, this implies a greatly enhanced generation of entanglement between the atoms initially prepared in the ground state.

IV. CONCLUSIONS AND PERSPECTIVES

Many works in the recent literature of quantum information theory have been devoted to investigations of entanglement in quantum field theory and quantum field theory in curved space-time. Throughout the text, some of these were already cited. Among several investigations in the field, we would also like to mention the analysis regarding quantum teleportation between noninertial observers [69–71] and relativistic approaches to the Einstein-Podolsky-Rosen framework and also to Bell's inequality [72–75]. References [76–83] provide more intriguing discussions on the subject of relativistic quantum entanglement for the interested reader. The point is that most of these studies were implemented in a framework of open quantum systems. Employing the formalism developed by Dalibard, Dupont-Roc, and Cohen-Tannoudji, we have uncovered the distinct effects of vacuum fluctuations and the radiation reaction on the quantum entanglement between two identical atoms in Schwarzschild space-time. Within such a formalism, the interplay between vacuum fluctuations and the radiation reaction can be considered to maintain the stability of the entangled state in some particular situations. We assume that both atoms are coupled to a quantum electromagnetic field. The overall picture is the following. The rate of change of the two-atom system energy is very small when the atoms are far away of the horizon. As they get closer, this rate increases in a oscillatory regime in such a way that, when the atoms approach the horizon, most contributions to the rate oscillates violently. This suggests that the generation of entanglement is highly magnified when the atoms are near the horizon and also largely suppressed when they get to spatial infinity. In turn, we have also obtained evidences that the degradation of entanglement follows the same response; i.e., it is highly enhanced when the atoms approach the horizon and also largely suppressed when they get to spatial infinity. The present analysis taken in connection with the results of Refs. [29,30] allows us to state the following assertion: Even though the thermal terms contribute decisively to the creation of entanglement between the atoms, the degree of entanglement thus generated is suppressed for atoms approaching the event horizon. In this way, we note that here the Hawking effect is a key ingredient in the discussion of creation of entanglement. We stress that one must not refrain from observing that the entanglement features of the system under consideration depend crucially on the distance of the atoms to the event horizon and also on the balance between vacuum fluctuations and the radiation reaction. This is manifest in the framework studied here.

In this work, since we are interested in mean lives, we choose an alternative perspective to understand quantum entanglement. We have carefully demonstrated that, when considering the resonant interaction between two-level atoms, the machinery underlying entanglement can be understood as an interplay between classical concepts (represented by the radiation-reaction effect) and quantum-mechanical phenomena (vacuum fluctuations). In this scenario, the usage of the DDC formalism has been proved to be essential in order to unfold this interpretation in a clear way. Nevertheless, we mention that the standard formalism for the evaluations of time evolution and correlation properties of entangled atomic systems is the traditional master equation approach. An important concept that is commonly addressed with the master equation approach is entanglement swapping. Within this approach, one derives an equation of motion for the reduced density operator of a certain subsystem A which interacts with another subsystem B. Commonly, one describes the general solution of the master equation in terms of the so-called Kraus representation. In possession of a density matrix of the pair of atoms, it is possible to quantify the degree of entanglement by employing the usual techniques, such as Wootters' concurrence or negativity. Concerning such entanglement measures, one could also consider the calculation of the entanglement entropy, which characterizes the correlations between subsystems belonging to a quantum-mechanical system. A systematic study of entanglement entropy in quantum field theory and of black holes is given in many important works; see, for instance, Refs. [84,85], and references cited therein. We reserve future studies to all the important subjects raised above.

We believe that the results presented in this paper may have an impact in the studies of radiative processes of atoms in the presence of an event horizon. A framework in which vacuum fluctuations and the radiation-reaction effect have been clearly uncovered may contribute to a deeper understanding of such results. For instance, recently the method was employed to investigate the Casimir-Polder forces between two uniformly accelerated atoms [86]. In such a work, the authors exhibit a transition from the short-distance thermal behavior dictated by the Unruh effect to a long-distance nonthermal behavior. In addition, studies of quantum entanglement in Schwarzschild space-time are attracting much attention due to their obvious applications to the problem of black-hole information loss [87–92]. One expects that the present investigation will impact the discussion on black-hole complementarity [93] or even on the possible firewall scenarios [94–97]. Indeed, the relationship between particle detectors in different vacua in Rindler and Schwarzschild space-time was undertaken in recent studies [98]. All these investigations suggest that the attempts to ascertain possible connections between the equivalence principle and quantum entanglement could unveil a different and important aspect on the black-hole information paradox. Such subjects are under investigation and results will be reported elsewhere.

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APPENDIX A: CORRELATION FUNCTIONS OF THE ELECTROMAGNETIC FIELD IN SCHWARZSCHILD SPACE-TIME

In this Appendix, we present the correlation functions for each one of the vacuum states discussed above. A detailed analysis of the quantization of the electromagnetic field in Schwarzschild space-time can be found in Ref. [99]. For a different but related method, see Ref. [100]. The associated correlation functions are also evaluated in Ref. [50]. Fundamentally, the concept is to employ the modified Feynman gauge and then use the Gupta-Bleuler quantization in this gauge employing Schwarzschild coordinates.

The action for the free electromagnetic fields in the modified Feynman gauge is given by

$$S_F = -\int d^4x\sqrt{-g}\left[\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}G^2\right], \tag{A1}$$

where $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$ and $G = \nabla^\mu A_\mu + K^\mu A_\mu$, with ∇^μ being the usual covariant derivative in curved space-time. Choosing

$$K^\mu = \left(0, \frac{2M}{r^2}, 0, 0\right),$$

the equation for A_0 decouples from the other ones. A complete set of solutions of the field equations is denoted by $A_\mu^{(\lambda n;\omega lm)}$. The label n distinguishes between modes incoming from the past null infinity $\mathcal{J}^-$ ($n =\leftarrow$) and those going out from the past horizon $\mathcal{H}^-$ ($n =\rightarrow$). There are four classes of modes which form this basis. The modes with $\lambda = 0$ do not obey the gauge condition $G = 0$ which is satisfied by all other modes with $\lambda \neq 0$. In turn, modes with $\lambda = 3$ are so-called pure gauge modes. Finally, the modes with $\lambda = 1, 2$ correspond to the physical modes. We choose them to have $A_0 = 0$. These are given by

$$A_\mu^{(1n;\omega lm)} = \left(0, R_{\omega l}^{(1n)}(r) Y_{lm} e^{-i\omega t},\right.$$
$$\left. \times \frac{(1-2M/r)}{l(l+1)}\frac{d}{dr}(r^2 R_{\omega l}^{(1n)}(r))\partial_i Y_{lm} e^{-i\omega t}\right) \tag{A2}$$

for $\lambda = 1$, with $i = \theta, \phi$ and $l \geq 1$. The functions $Y_{lm} = Y_{lm}(\theta, \phi)$ are the usual spherical harmonics and $l = 0, 1, 2, \ldots$, with $l \geq m$, m being an integer number. As for $\lambda = 2$, they can be expressed as

$$A_\mu^{(2n;\omega lm)} = (0, 0, R_{\omega l}^{(2n)}(r) Y_i^{lm} e^{-i\omega t}), \tag{A3}$$

where the functions $Y_i^{lm} = Y_i^{lm}(\theta, \phi)$ are divergence-free vector spherical harmonics defined on the unit 2-sphere with angular coordinates (θ, ϕ). The associated normalization of the modes $A_\mu^{(\lambda n;\omega lm)}$ can be fixed from the usual inner product. Expanding the field operator in terms of the complete set of basic modes as

$$\hat{A}_\mu(x) = \sum_{\lambda nlm}\int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}}[A_\mu^{(\lambda n;\omega lm)}(x)\hat{a}_{\omega lm}^{(\lambda n)}$$
$$+ (A_\mu^{(\lambda n;\omega lm)})^*(x)\hat{a}_{\omega lm}^{\dagger(\lambda n)}], \tag{A4}$$

the commutation relations between the annihilation and creation operators are given by

$$[\hat{a}_{\omega lm}^{(3n)}, \hat{a}_{\omega' l' m'}^{\dagger(3n')}] = -[\hat{a}_{\omega lm}^{(0n)}, \hat{a}_{\omega' l' m'}^{\dagger(3n')}]$$
$$= \delta^{nn'}\delta_{ll'}\delta_{mm'}\delta(\omega - \omega') \tag{A5}$$

and

$$[\hat{a}_{\omega lm}^{(1n)}, \hat{a}_{\omega' l' m'}^{\dagger(1n')}] = [\hat{a}_{\omega lm}^{(2n)}, \hat{a}_{\omega' l' m'}^{\dagger(2n')}]$$
$$= \delta^{nn'}\delta_{ll'}\delta_{mm'}\delta(\omega - \omega'). \tag{A6}$$

All other commutators vanish. From the Gupta-Bleuler condition, one gets $\hat{G}^+|\phi\rangle = 0$, for any physical state $|\phi\rangle$, where $\hat{G}^+$ is the positive-frequency part of the operator $\hat{G} = \nabla^\mu \hat{A}_\mu + K^\mu \hat{A}_\mu$. This condition means that any state with $\lambda = 3$ is unphysical and the states with $\lambda = 0$ have zero norm and are orthogonal to any physical states. The Boulware vacuum $|0_B\rangle$ [101] is defined by requiring that it be annihilated by all annihilation operators $\hat{a}_{\omega lm}^{\dagger(\lambda n)}$. One can take as the representative elements the states obtained by applying the creation operators $\hat{a}_{\omega lm}^{\dagger(3n)}$, $\lambda = 1$, 2 on the vacuum $|0_B\rangle$. As discussed in Ref. [99], unphysical particles created by $\hat{a}_{\omega lm}^{\dagger(3n)}$ will be in thermal equilibrium in the so-called Hartle-Hawking vacuum [102] for a static black hole if one demands that the gauge-fixed two-point function be nonsingular on the horizons similar to the procedure taken in the scalar-field case [103]. There will also be a flux of unphysical particles in the so-called Unruh vacuum [104].

Observe that the positive-frequency states defined as above are related to the timelike Killing vector field $\partial/\partial t$ with respect to which the exterior region of the black hole is static [51]. However, as argued in Refs. [51,104], there are other possible prescriptions when considering the metric in

Kruskal-Szekeres coordinates instead of the usual Schwarzschild coordinates. In this respect, one easily notes that the null coordinates on the past horizon $\mathcal{H}^-$ ($U = v - u$) and the null coordinate on the future horizon $\mathcal{H}^+$ ($V = v + u$) also act as Killing vector fields $\partial/\partial U$ and $\partial/\partial V$ on the respective horizons. Therefore, one can define basis modes in terms of such Kruskal null coordinates U, V. This set is regular on the entire manifold. The associated vacuum $|0_H\rangle$ is known as the Hartle-Hawking vacuum. On the other hand, it is also known that one may take the incoming modes to be positive frequency with respect to $\partial/\partial t$ and the outgoing modes to be positive frequency with respect to $\partial/\partial U$—one can show that such a prescription leads to a definition of a set of modes which oscillate infinitely rapidly on the past event horizon [54]. The associated vacuum $|0_U\rangle$ is known as the Unruh vacuum [104]. This last prescription is the one required in order to mock up the geometrical effects associated with the gravitational collapse of a spherically symmetric electrically neutral star.

Let us present the expansion of the field operator in terms of the complete set of modes associated with the Hartle-Hawking vacuum and the Unruh vacuum. Our discussion has grounds in Ref. [104]. As discussed in such a reference, the field operators can also be expanded as

$$\hat{A}_\mu(x) = \sum_{\lambda lm}\bigg\{\int_{-\infty}^{\infty} d\omega \frac{1}{\sqrt{8\pi\omega\sinh(4\pi M\omega)}}[\bar{A}_\mu^{(\lambda\leftarrow;\omega lm)}(x)\hat{h}_{\omega lm}^{(\lambda\leftarrow)} + (\bar{A}_\mu^{(\lambda\leftarrow;\omega lm)})^*(x)\hat{h}_{\omega lm}^{\dagger(\lambda\leftarrow)}] + \int_{-\infty}^{\infty} d\omega \frac{1}{\sqrt{8\pi\omega\sinh(4\pi M\omega)}}[\bar{A}_\mu^{(\lambda\rightarrow;\omega lm)}(x)\hat{h}_{\omega lm}^{(\lambda\rightarrow)} + (\bar{A}_\mu^{(\lambda\rightarrow;\omega lm)})^*(x)\hat{h}_{\omega lm}^{\dagger(\lambda\rightarrow)}]\bigg\}, \tag{A7}$$

where

$$\bar{A}_\mu^{(\lambda\rightarrow;\omega lm)}(x) = e^{2\pi M\omega}A_{\mu\mathrm{I}}^{(\lambda\rightarrow;\omega lm)}(x) + e^{-2\pi M\omega}(A_{\mu\mathrm{III}}^{(\lambda\rightarrow;\omega lm)})^*(x),$$
$$\bar{A}_\mu^{(\lambda\leftarrow;\omega lm)}(x) = e^{-2\pi M\omega}(A_{\mu\mathrm{I}}^{(\lambda\leftarrow;\omega lm)})^*(x) + e^{2\pi M\omega}A_{\mu\mathrm{III}}^{(\lambda\leftarrow;\omega lm)}(x), \tag{A8}$$

with $\bar{A}_{\mu\mathrm{I}}^{(\lambda n;\omega lm)}(x) = A_\mu^{(\lambda n;\omega lm)}(x)$ for $x \in$ I and zero for $x \in$ III, I and III different regions of the Kruskal-Szekeres diagram as commented above (see Fig. 31.3 of Ref. [53]) and similarly for $\bar{A}_{\mu\mathrm{III}}^{(\lambda n;\omega lm)}(x)$ which is zero for $x \in$ I. One has

$$[\hat{h}_{\omega lm}^{(3n)}, \hat{h}_{\omega' l' m'}^{\dagger(3n')}] = -[\hat{h}_{\omega lm}^{(0n)}, \hat{h}_{\omega' l' m'}^{\dagger(3n')}] = \delta^{nn'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'),$$
$$[\hat{h}_{\omega lm}^{(1n)}, \hat{h}_{\omega' l' m'}^{\dagger(1n')}] = [\hat{h}_{\omega lm}^{(2n)}, \hat{h}_{\omega' l' m'}^{\dagger(2n')}] = \delta^{nn'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'), \tag{A9}$$

with all other commutators vanishing, and also $\hat{h}_{\omega lm}^{(\lambda n)}|0_H\rangle = 0$. In turn, one may also expand the field operators as

$$\hat{A}_\mu(x) = \sum_{\lambda lm}\bigg\{\int_0^{\infty} \frac{d\omega}{\sqrt{4\pi\omega}}[A_\mu^{(\lambda\leftarrow;\omega lm)}(x)\hat{u}_{\omega lm}^{(\lambda\leftarrow)} + (A_\mu^{(\lambda\leftarrow;\omega lm)})^*(x)\hat{u}_{\omega lm}^{\dagger(\lambda\leftarrow)}] + \int_{-\infty}^{\infty} d\omega \frac{1}{\sqrt{8\pi\omega\sinh(4\pi M\omega)}}[\bar{A}_\mu^{(\lambda\rightarrow;\omega lm)}(x)\hat{u}_{\omega lm}^{(\lambda\rightarrow)} + (\bar{A}_\mu^{(\lambda\rightarrow;\omega lm)})^*(x)\hat{u}_{\omega lm}^{\dagger(\lambda\rightarrow)}]\bigg\}, \tag{A10}$$

with

$$[\hat{u}_{\omega lm}^{(3n)}, \hat{u}_{\omega' l' m'}^{\dagger(3n')}] = -[\hat{u}_{\omega lm}^{(0n)}, \hat{u}_{\omega' l' m'}^{\dagger(3n')}] = \delta^{nn'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'),$$
$$[\hat{u}_{\omega lm}^{(1n)}, \hat{u}_{\omega' l' m'}^{\dagger(1n')}] = [\hat{u}_{\omega lm}^{(2n)}, \hat{u}_{\omega' l' m'}^{\dagger(2n')}] = \delta^{nn'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'), \tag{A11}$$

with all other commutators vanishing, and $\hat{u}_{\omega lm}^{(\lambda n)}|0_U\rangle = 0$. For simplicity, we assume that the atoms are polarized along the radial direction defined by their positions relative to the black-hole space-time rotational Killing vector fields. This assumption significantly simplifies the calculations in that the contributions associated with the polarizations in angular directions do not need to be considered. Therefore, with the usual relationships $E_i = F_{0i}$, one can calculate the various correlation functions which will be important in our calculations. The important object to be considered is

$$E_r = F_{0r} = \nabla_0 A_r - \nabla_r A_0 = \partial_0 A_r - \partial_r A_0$$

(the connection terms cancel). Hence,

$$\langle 0|\hat{E}_r(x)\hat{E}_r(x')|0\rangle = \langle 0|(\partial_0\hat{A}_r - \partial_r\hat{A}_0)(\partial'_0\hat{A}_r - \partial'_r\hat{A}_0)|0\rangle. \tag{A12}$$

Let us present the correlation functions for each one of the vacuum states for $x, x' \in \mathrm{I}$.

(1) *The Boulware vacuum.*—One has

$$\langle 0_B|\hat{E}_r(x)\hat{E}_r(x')|0_B\rangle = \frac{1}{4\pi}\sum_{lm}\int_0^\infty d\omega\omega e^{-i\omega(t-t')}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') \times [\vec{R}^{(1)}_{\omega l}(r)\vec{R}^{(1*)}_{\omega l}(r') + \overleftarrow{R}^{(1)}_{\omega l}(r)\overleftarrow{R}^{(1*)}_{\omega l}(r')], \quad \text{(A13)}$$

where we have used that

$$\langle 0_B|\hat{a}^{(1n)}_{\omega lm}\hat{a}^{\dagger(1n')}_{\omega' l' m'}|0_B\rangle = \delta^{nn'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'), \quad \text{(A14)}$$

and all other possible combinations coming from the product $\hat{E}_r(x)\hat{E}_r(x')$ vanish.

(2) *The Hartle-Hawking vacuum.*—One has

$$\langle 0_H|\hat{E}_r(x)\hat{E}_r(x')|0_H\rangle = \frac{1}{4\pi}\sum_{lm}\int_{-\infty}^\infty d\omega\omega\left[e^{-i\omega(t-t')}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi')\frac{\vec{R}^{(1)}_{\omega l}(r)\vec{R}^{(1*)}_{\omega l}(r')}{1-e^{-2\pi\omega/\kappa}} + e^{i\omega(t-t')}Y^*_{lm}(\theta,\phi)Y_{lm}(\theta',\phi')\frac{\overleftarrow{R}^{(1*)}_{\omega l}(r)\overleftarrow{R}^{(1)}_{\omega l}(r')}{e^{2\pi\omega/\kappa}-1}\right], \quad \text{(A15)}$$

where $\kappa = 1/4M$ is the surface gravity of the black hole. A relation similar to (A15) holds for the operators $\hat{h}^{(1n)}_{\omega lm}$ and $\hat{h}^{\dagger(1n)}_{\omega lm}$.

(3) *The Unruh vacuum.*—One has

$$\langle 0_U|\hat{E}_r(x)\hat{E}_r(x')|0_U\rangle = \frac{1}{4\pi}\sum_{lm}\int_{-\infty}^\infty d\omega\omega e^{-i\omega(t-t')}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') \times \left[\frac{\vec{R}^{(1)}_{\omega l}(r)\vec{R}^{(1*)}_{\omega l}(r')}{1-e^{-2\pi\omega/\kappa}} + \theta(\omega)\overleftarrow{R}^{(1)}_{\omega l}(r)\overleftarrow{R}^{(1*)}_{\omega l}(r')\right], \quad \text{(A16)}$$

where $\theta(z)$ is the usual Heaviside theta function. A relation similar to (A15) also holds for the operators $\hat{u}^{(1n)}_{\omega lm}$ and $\hat{u}^{\dagger(1n)}_{\omega lm}$.

We stress that each of the vacua discussed above represent different physical scenarios. The Boulware state, empty at infinity, is the ground state of quantum fields around a cold neutron star. The Unruh state, empty at past null infinity and regular on the future horizon, is customarily taken as the ground state for an evaporating black hole; i.e., such a vacuum state reproduces the effects of a gravitational collapsing body. In turn, the Hartle-Hawking vacuum corresponds to a black hole in equilibrium with an infinite sea of blackbody radiation.

APPENDIX B: EVALUATION OF MODE SUMS

In order to evaluate the correlation functions, one needs to present explicit expressions for the radial functions. Even though it is a remarkable task, fortunately, one is usually interested in two asymptotic regions, namely, $r \to 2M$ (near the event horizon) and $r \to \infty$ (away from the event horizon). In this case, the behavior of the radial functions is well known. We shall briefly discuss such limits in the present Appendix. We extend the results of Refs. [50,105] for correlation functions calculated at different points of the space-time. From standard considerations, one has that

$$\begin{aligned}
&\vec{\mathcal{R}}^{(1)}_{\omega l}(r) \sim e^{i\omega r_*} + \vec{\mathcal{A}}_l(\omega)e^{-i\omega r_*}, \qquad r \to 2M,\\
&\vec{\mathcal{R}}^{(1)}_{\omega l}(r) \sim \vec{\mathcal{B}}_l(\omega)e^{i\omega r_*}, \qquad r \to \infty,\\
&\overleftarrow{\mathcal{R}}^{(1)}_{\omega l}(r) \sim \overleftarrow{\mathcal{B}}_l(\omega)e^{-i\omega r_*}, \qquad r \to 2M,\\
&\overleftarrow{\mathcal{R}}^{(1)}_{\omega l}(r) \sim e^{-i\omega r_*} + \overleftarrow{\mathcal{A}}_l(\omega)e^{i\omega r_*}, \qquad r \to \infty,
\end{aligned} \quad \text{(B1)}$$

where $r_* = r + 2M\ln(r/2M - 1)$ is the Regge-Wheeler tortoise coordinate and $\mathcal{R}^{(1n)}_{\omega l}(r)$ is defined through the equation

$$R^{(1n)}_{\omega l}(r) = \frac{\sqrt{l(l+1)}}{\omega}\frac{\mathcal{R}^{(1n)}_{\omega l}(r)}{r^2}.$$

In the above, $\mathcal{A}$ and $\mathcal{B}$ are the usual reflection and transmission coefficients, respectively, with the following properties:

$$\begin{aligned}
&\vec{\mathcal{B}}_l(\omega) = \overleftarrow{\mathcal{B}}_l(\omega) = \mathcal{B}_l(\omega),\\
&|\vec{\mathcal{A}}_l(\omega)| = |\overleftarrow{\mathcal{A}}_l(\omega)|,\\
&1 - |\vec{\mathcal{A}}_l(\omega)|^2 = 1 - |\overleftarrow{\mathcal{A}}_l(\omega)|^2 = |\mathcal{B}_l(\omega)|^2,\\
&\vec{\mathcal{A}}^*_l(\omega)\mathcal{B}_l(\omega) = -\mathcal{B}^*_l(\omega)\overleftarrow{\mathcal{A}}_l(\omega).
\end{aligned} \quad \text{(B2)}$$

Key results involving the mode summations in the asymptotic regions $r \to 2M$ and $r \to \infty$ will be now considered. At fixed radial distances r and r', the radial correlation function of the field in the Boulware vacuum is given by

$$\langle 0_B|\hat{E}_r(x)\hat{E}_r(x')|0_B\rangle = \frac{1}{16\pi^2}\sum_{l=1}^{\infty}\int_0^{\infty} d\omega\omega e^{-i\omega(t-t')}(2l+1)P_l(\hat{r}\cdot\hat{r}') \times [\vec{R}^{(1)}_{\omega l}(r)\vec{R}^{(1*)}_{\omega l}(r') + \overleftarrow{R}^{(1)}_{\omega l}(r)\overleftarrow{R}^{(1*)}_{\omega l}(r')], \quad \text{(B3)}$$

where we have used the addition theorem for the spherical harmonics [60]

$$\frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') = P_l(\hat{r}\cdot\hat{r}'),$$

where $\hat{r}$ and $\hat{r}'$ are two unit vectors with spherical coordinates (θ,ϕ) and (θ',ϕ'), respectively, and P_l is the Legendre polynomial of degree l [61]. From Eq. (B1), one has that

$$\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\vec{R}^{(1)}_{\omega l}(r)\vec{R}^{(1*)}_{\omega l}(r') \approx \sum_{l=1}^{\infty}\frac{l(l+1)(2l+1)P_l(\hat{r}\cdot\hat{r}')|\mathcal{B}_l(\omega)|^2 e^{i\omega(r_*-r'_*)}}{\omega^2 r^2 r'^2}, \quad r,r'\to\infty. \quad \text{(B4)}$$

For $\mathbf{x}=\mathbf{x}'$ one gets $[P_l(1)=1]$

$$\sum_{l=1}^{\infty}(2l+1)|\vec{R}_{\omega l}|^2 \approx \sum_{l=1}^{\infty}\frac{l(l+1)(2l+1)|\mathcal{B}_l(\omega)|^2}{r^4\omega^2}, \quad r\to\infty. \quad \text{(B5)}$$

In turn, in order to estimate the remaining sum, one should note that the above correlation function at large radii should agree with the correlation function of the electric field in the Minkowski vacuum (a similar consideration was undertaken in Ref. [50]). The latter is given by [106]

$$\langle 0|\hat{E}^i(x)\hat{E}^j(x')|0\rangle = \left(\frac{\partial}{\partial t}\frac{\partial}{\partial t'}\delta^{ij} - \frac{\partial}{\partial x_i}\frac{\partial}{\partial x'_j}\right)D(t-t',\mathbf{x}-\mathbf{x}') = -\left(\frac{\partial^2}{\partial\eta^2}\delta^{ij} - \frac{\partial}{\partial\rho_i}\frac{\partial}{\partial\rho_j}\right)D(\eta,\boldsymbol{\rho}), \quad \text{(B6)}$$

where $\eta = t-t'$, $\rho_i = (\mathbf{x}-\mathbf{x}')_i$ and

$$D(t-t',\mathbf{x}-\mathbf{x}') = \frac{1}{(2\pi)^3}\int\frac{d^3k}{2\omega_{\mathbf{k}}}e^{i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega_{\mathbf{k}}(t-t')]} = \frac{1}{(2\pi)^2}\int_0^{\infty}d\omega e^{-i\omega(t-t')}\frac{\sin(\omega|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}, \quad \text{(B7)}$$

with $\omega_{\mathbf{k}} = \omega = |\mathbf{k}|$. Performing the derivatives, one gets, with $\Delta t = t-t'$ and $\Delta\mathbf{x} = \mathbf{x}-\mathbf{x}'$,

$$\langle 0|\hat{E}^i(x)\hat{E}^j(x')|0\rangle = \frac{1}{16\pi^2}\int_0^{\infty}d\omega\omega e^{-i\omega\Delta t}D^{ij}(\omega,\mathbf{x},\mathbf{x}'), \quad \text{(B8)}$$

where

$$D_{ij}(\omega,\mathbf{x},\mathbf{x}') = -\frac{4}{\omega|\Delta\mathbf{x}|^3}\left[\delta_{ij}\mathcal{S}_1(\omega,|\Delta\mathbf{x}|) - \frac{(\Delta\mathbf{x})_i(\Delta\mathbf{x})_j}{|\Delta\mathbf{x}|^2}\mathcal{S}_3(\omega,|\Delta\mathbf{x}|)\right], \quad \text{(B9)}$$

with

$$\mathcal{S}_n(\omega,|\Delta\mathbf{x}|) = (n-\omega^2|\Delta\mathbf{x}|^2)\sin(\omega|\Delta\mathbf{x}|) - n\omega|\Delta\mathbf{x}|\cos(\omega|\Delta\mathbf{x}|),$$

and we have used that

$$\frac{\partial}{\partial\rho_i}f(\rho) = \frac{\rho^i}{\rho}\frac{d}{d\rho}f(\rho),$$
$$\frac{\partial}{\partial\rho_i}\frac{\partial}{\partial\rho_j}f(\rho) = \frac{\delta^{ij}}{\rho}\frac{d}{d\rho}f(\rho) + \frac{\rho^i\rho^j}{\rho}\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}f(\rho)\right].$$

In order to compare Eqs. (B3) and (B8), the latter should be expressed in spherical coordinates. This can be achieved using the usual transformation formula between the Cartesian unit vectors and the spherical unit vectors, which leads us to

$$\langle 0|\hat{E}_i(y)\hat{E}_j(y')|0\rangle = \frac{\partial x^a}{\partial y^i}\frac{\partial x'^b}{\partial y'^j}\langle 0|\hat{E}_a(x)\hat{E}_b(x')|0\rangle,$$

where the y_j are the usual spherical coordinates r, θ, ϕ. Hence,

$$\langle 0|\hat{E}_r(y)\hat{E}_r(y')|0\rangle = \frac{1}{16\pi^2}\int_0^{\infty}d\omega\omega e^{-i\omega\Delta t}F(\omega,\mathbf{x},\mathbf{x}'), \quad \text{(B10)}$$

where

$$\begin{aligned}F(\omega,\mathbf{x},\mathbf{x}') &= \sin\theta\sin\theta'[\cos\phi\cos\phi' D_{11}(\omega,\mathbf{x},\mathbf{x}') \\ &\quad + \sin\phi\sin\phi' D_{22}(\omega,\mathbf{x},\mathbf{x}')] \\ &\quad + \cos\theta\cos\theta' D_{33}(\omega,\mathbf{x},\mathbf{x}') \\ &\quad + (\cos\theta\sin\theta'\cos\phi' \\ &\quad + \cos\theta'\sin\theta\cos\phi)D_{13}(\omega,\mathbf{x},\mathbf{x}') \\ &\quad + (\cos\theta\sin\theta'\sin\phi' \\ &\quad + \cos\theta'\sin\theta\sin\phi)D_{23}(\omega,\mathbf{x},\mathbf{x}') \\ &\quad + \sin\theta\sin\theta'\sin(\phi+\phi')D_{12}(\omega,\mathbf{x},\mathbf{x}'),\end{aligned} \quad \text{(B11)}$$

with $\mathbf{x}$, $\mathbf{x}'$ expressed in spherical coordinates. Therefore, comparing Eqs. (B3) and (B10), one gets, for $r, r' \to \infty$,

$$\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\overleftarrow{R}^{(1)}_{\omega l}(r)\overleftarrow{R}^{(1*)}_{\omega l}(r') \approx F(\omega,\mathbf{x},\mathbf{x}'), \qquad r, r' \to \infty. \tag{B12}$$

For $\mathbf{x} = \mathbf{x}'$,

$$\sum_{l=1}^{\infty}(2l+1)|\overleftarrow{R}_{\omega l}|^2 \approx \frac{8\omega^2}{3}, \qquad r \to \infty. \tag{B13}$$

In order to evaluate the mode sums in the region $r \sim 2M$, a certain amount of caution is mandatory. We begin by defining

$$\zeta^2 = \frac{r}{2M} - 1$$

and

$$\xi = 4M\omega.$$

With these definitions, and using that $l(l+1)\zeta^2 \approx (l\zeta)^2$ (since $\zeta \sim 0$), one can easily prove that $\vec{\mathcal{R}}^{(1)}_{\omega l}$, taken as a function of ζ, obeys the following differential equation:

$$\left[\zeta^2\frac{d^2}{d\zeta^2} + \zeta\frac{d}{d\zeta} + (\xi^2 - (2l\zeta)^2)\right]\vec{\mathcal{R}}^{(1)}_{\omega l}(\zeta) = 0, \tag{B14}$$

whose solutions are the modified Bessel functions $K_{i\xi}(2l\zeta)$ and $I_{i\xi}(2l\zeta)$. Hence, the general solution can be conveniently expressed as

$$\vec{\mathcal{R}}^{(1)}_{\omega l}|_{r\to 2M} \approx c_l K_{i\xi}(2l\zeta) + d_l I_{-i\xi}(2l\zeta). \tag{B15}$$

As $l \to \infty$ for fixed ζ, the function $\vec{\mathcal{R}}^{(1)}_{\omega l} \to 0$, since r lies then in the region for which the effective potential for the radial function is large. One deduces from this that d_l is an exponentially small function of l for large l, since [61]

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}},$$

which is valid for large z and fixed ν. The second term in Eq. (B15) will therefore make a contribution to the sum in Eq. (B3) which remains bounded as $\zeta \to 0$ and which may be neglected in comparison with that of the first term in (B15), which will be of the order of $(\zeta\zeta')^{-2}$. The coefficient c_l may be determined by comparing the result [61]

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}$$

[$\Gamma(z)$ is the usual gamma function], which is valid for Re[ν] > 0 fixed and $z \to 0$ with the asymptotic solution

$$\vec{\mathcal{R}}^{(1)}_{\omega l}(r) \sim e^{i\omega r_*} + \vec{\mathcal{A}}_l(\omega)e^{-i\omega r_*}, \qquad r \to 2M.$$

One finds that

$$c_l \sim \frac{2e^{i\xi/2}l^{-i\xi}}{\Gamma(-i\xi)}. \tag{B16}$$

Hence, to leading order

$$\begin{aligned}\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\vec{R}^{(1)}_{\omega l}(r)\vec{R}^{(1*)}_{\omega l}(r') &\approx \frac{4}{\Gamma(i\xi)\Gamma(-i\xi)}\sum_{l=1}^{\infty}\frac{l(l+1)(2l+1)P_l(\cos\gamma)}{\omega^2 r^2 r'^2}K_{i\xi}(2l\zeta)K_{i\xi}(2l\zeta')\\ &\approx \frac{2\sinh(4\pi M\omega)}{\pi M^3\omega}\int_0^\infty dl l^3 J_0(l\gamma)K_{i\xi}(2l\sqrt{g_{00}(r)})K_{i\xi}(2l\sqrt{g_{00}(r')}), \qquad r, r' \to 2M,\end{aligned} \tag{B17}$$

where $g_{00} = (1 - 2M/r)$, $\cos\gamma = \hat{r}\cdot\hat{r}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$, and we have used that [61]

$$\Gamma(i\xi)\Gamma(-i\xi) = \frac{\pi}{\xi\sinh(\pi\xi)},$$

together with the asymptotic result

$$P_\nu\left(\cos\frac{x}{\nu}\right) \approx J_0(x) + \mathcal{O}(\nu^{-1}),$$

in which $J_\mu(x)$ is a Bessel function of the first kind. Employing the result [65]

$$\int_0^\infty dx x^{\alpha-1}J_\lambda(ax)K_\mu(2\sqrt{b}x)K_\nu(2\sqrt{c}x) = \frac{2^{\alpha-3}a^\lambda}{(2\sqrt{c})^{\alpha+\lambda}\Gamma(\lambda+1)}[A^\nu_\mu(a,b,c) + A^\nu_{-\mu}(a,b,c)], \tag{B18}$$

where ($A^\mu_\mu = A_\mu$)

$$A_{\mu}^{\nu}(a,b,c)=\left(\frac{b}{c}\right)^{\mu/2}\Gamma\left[-\mu,\frac{\alpha+\lambda+\mu-\nu}{2},\frac{\alpha+\lambda+\mu+\nu}{2}\right]\times F_4\left(\frac{\alpha+\lambda+\mu-\nu}{2},\frac{\alpha+\lambda+\mu+\nu}{2};\times\lambda+1,\mu+1;-\frac{a^2}{4c},\frac{b}{c}\right), \tag{B19}$$

$F_4(a,b;c,c';x,y)$ being the Appell hypergeometric function F_4

$$F_4(a,b;c,c';x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n m!n!}x^m y^n$$

$[(q)_m$ is the Pochhammer symbol representing the rising factorial] and also

$$\Gamma[a_1,\ldots,a_m]=\prod_{k=1}^{m}\Gamma(a_k),$$

one gets

$$\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\vec{R}_{\omega l}^{(1)}(r)\vec{R}_{\omega l}^{(1*)}(r')\approx\frac{\sinh(4\pi M\omega)}{4\pi g_{00}^2(r)M^3\omega}\times[A_{4M\omega i}(\gamma,g_{00}(r'),g_{00}(r))+A_{-4M\omega i}(\gamma,g_{00}(r'),g_{00}(r))],\quad r,r'\to 2M. \tag{B20}$$

One may derive a much simpler result by considering that $r\approx r'$ but $\hat{\mathbf{r}}\neq\hat{\mathbf{r}}'$. With [65]

$$\int_0^\infty dx x^3 J_0(ax)[K_{iq}(2bx)]^2=\frac{4\pi\mathrm{csch}(\pi q)}{a^4(a^2+16b^2)^2\sqrt{\frac{16b^2}{a^2}+1}}\times\left\{2a^2q\sqrt{\frac{16b^2}{a^2}+1}(a^2+4b^2)\cos\left[2q\mathrm{csch}^{-1}\left(\frac{4b}{a}\right)\right]+(a^4(q^2-1)+8a^2b^2(2q^2-1)-64b^4)\sin\left[2q\mathrm{csch}^{-1}\left(\frac{4b}{a}\right)\right]\right\}, \tag{B21}$$

where we assume a small positive imaginary part for a so that the integral converges, one gets

$$\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\vec{R}_{\omega l}^{(1)}(r)\vec{R}_{\omega l}^{(1*)}(r')\approx\frac{4\pi\mathrm{csch}(\pi\xi)}{\gamma^3(16g_{00}(r)+\gamma^2)^{5/2}}\times\left\{2\gamma\xi[16g_{00}(r)+\gamma^2]^{1/2}[4g_{00}(r)+\gamma^2]\cos\left[2\xi\mathrm{csch}^{-1}\left(\frac{4\sqrt{g_{00}(r)}}{\gamma}\right)\right]+[\gamma^4(\xi^2-1)+8\gamma^2g_{00}(r)(2\xi^2-1)-64(g_{00}(r))^2]\sin\left[2\xi\mathrm{csch}^{-1}\left(\frac{4\sqrt{g_{00}(r)}}{\gamma}\right)\right]\right\},\quad r,r'\to 2M. \tag{B22}$$

For $\mathbf{x}=\mathbf{x}'$,

$$\sum_{l=1}^{\infty}(2l+1)|\vec{R}_{\omega l}^{(1)}|^2\approx\frac{8\omega^2}{3g_{00}^2}+\frac{1}{6M^2g_{00}^2},\qquad r\to 2M, \tag{B23}$$

where we have used that [65]

$$\frac{8}{\Gamma(i\xi)\Gamma(-i\xi)}\int_0^\infty dt t^3[K_{i\xi}(2tx)]^2=\frac{\xi^2(\xi^2+1)}{6x^4}.$$

The other mode sum in the region $r\sim 2M$ appearing in Eq. (B3) can be easily estimated using Eq. (B1). One finds

$$\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\overleftarrow{R}_{\omega l}^{(1)}(r)\overleftarrow{R}_{\omega l}^{(1*)}(r')\approx\sum_{l=1}^{\infty}\frac{l(l+1)(2l+1)P_l(\hat{r}\cdot\hat{r}')|\mathcal{B}_l(\omega)|^2e^{-i\omega(r_*-r'_*)}}{(2M)^4\omega^2},\qquad r,r'\to 2M. \tag{B24}$$

For $\mathbf{x}=\mathbf{x}'$,

$$\sum_{l=1}^{\infty}(2l+1)|\overleftarrow{R}_{\omega l}|^2 \approx \sum_{l=1}^{\infty}\frac{l(l+1)(2l+1)|\mathcal{B}_l(\omega)|^2}{(2M)^4\omega^2}, \qquad r\to 2M. \tag{B25}$$

Other important estimate that one may evaluate is the one in which, say, $r\to\infty$ but $r'\to 2M$. One finds that

$$\begin{aligned}&\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\overleftarrow{R}^{(1)}_{\omega l}(r)\overleftarrow{R}^{(1*)}_{\omega l}(r')\\ &\quad\approx\sum_{l=1}^{\infty}\frac{l(l+1)(2l+1)P_l(\hat{r}\cdot\hat{r}')\mathcal{B}^*_l(\omega)}{(2M)^2r^2\omega^2}\left(e^{-i\omega(r_*-r'_*)}+\overleftarrow{\mathcal{A}}_l(\omega)e^{i\omega(r_*+r'_*)}\right), \qquad r\to\infty, r'\to 2M.\end{aligned} \tag{B26}$$

The estimate for the other mode sum yields

$$\begin{aligned}&\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\vec{R}^{(1)}_{\omega l}(r)\vec{R}^{(1*)}_{\omega l}(r')\\ &\quad\approx\frac{2e^{-i\xi/2}}{\Gamma(i\xi)}\sum_{l=1}^{\infty}\frac{l(l+1)(2l+1)P_l(\hat{r}\cdot\hat{r}')\mathcal{B}_l(\omega)e^{i\omega r_*}e^{i\xi\ln l}}{(2M)^2r^2\omega^2}K_{4M\omega i}(2l\sqrt{g_{00}(r')}), \qquad r\to\infty, r'\to 2M.\end{aligned} \tag{B27}$$

Observe that such expressions yield a vanishingly small contribution to Eq. (B3) as $r\to\infty$ and hence can be neglected. For a fixed r and $r'\to 2M$, one gets

$$\begin{aligned}&\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\vec{R}^{(1)}_{\omega l}(r)\vec{R}^{(1*)}_{\omega l}(r')\\ &\quad\approx\frac{2e^{-i\xi/2}}{\Gamma(i\xi)}\sum_{l=1}^{\infty}\frac{l(l+1)(2l+1)P_l(\hat{r}\cdot\hat{r}')\vec{\mathcal{R}}^{(1)}_{\omega l}(r)e^{i\xi\ln l}}{(2M)^2r^2\omega^2}K_{4M\omega i}(2l\sqrt{g_{00}(r')})\end{aligned} \tag{B28}$$

and

$$\sum_{l=1}^{\infty}(2l+1)P_l(\hat{r}\cdot\hat{r}')\overleftarrow{R}^{(1)}_{\omega l}(r)\overleftarrow{R}^{(1*)}_{\omega l}(r')\approx\sum_{l=1}^{\infty}\frac{l(l+1)(2l+1)P_l(\hat{r}\cdot\hat{r}')\overleftarrow{\mathcal{R}}^{(1)}_{\omega l}(r)\mathcal{B}^*_l(\omega)e^{i\omega r'_*}}{(2M)^2r^2\omega^2}. \tag{B29}$$

For other details concerning the evaluation of asymptotic correlation functions at equal space-time points, see Refs. [50,105].

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