Thermodynamics of $(2+1)$-dimensional charged black holes with power-law Maxwell field

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In this work, the three-dimensional nonlinearly charged black holes have been considered with a power-law modified electromagnetic theory. The black hole solutions to Einstein's three-dimensional field equations with a negative cosmological constant have been constructed in the presence of power-law nonlinear electrodynamics. Through the physical and mathematical interpretation of the solutions, a new class of asymptotically anti–de Sitter (AdS) black hole solutions has been introduced. The area law, surface gravity, and Gauss's law are utilized to obtain the entropy, temperature, and electric charge of the new AdS black holes, respectively. The quasilocal mass of the solutions has been calculated based on the counterterm method. A Smarr-type formula for the mass as a function of entropy and charge has been obtained. It has been shown that the thermodynamical quantities satisfy the first law of thermodynamics for the new AdS black holes. Also, it has been found that in order for the Smarr mass formula to be compatible with the first law of black hole thermodynamics, the cosmological parameter Λ should be treated as a thermodynamical variable and the generalized first law of thermodynamics has been introduced. Through the canonical ensemble method, the black hole remnant or phase transitions have been investigated regarding the black hole heat capacity. It has been found that the AdS black hole solutions we just obtained are thermodynamically stable.

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I. INTRODUCTION

Black holes, as one of the interesting predictions of Einstein's theory of relativity, have been investigated by theoretical physicists in four and higher dimensional spacetimes for a long time. From 1992, when the first $(2+1)$-dimensional black hole solutions were discovered by Banados, Teitelboim, and Zanelli (BTZ) [1], this subject area has been considered extensively [2], and it still has many unknown parts to be studied. A large number of studies on charged and uncharged black holes, dilatonic black holes, hairy black holes, rotating black holes, etc., in the presence or absence of a cosmological constant in $(2+1)$-dimensional spacetimes have been done by many authors [3]. Among the reasons it should be interesting to study the physics of lower dimensional spacetimes, we mention that black holes in lower dimensions are easier to study and can essentially lead to a deeper insight into the fundamental ideas in comparison to higher dimensional black holes. In addition, according to (A)dS/CFT correspondence, there is a dual between quantum gravity on anti (–de Sitter) space and a Euclidean conformal field theory on the lower dimensional spacetimes [4,5]. Therefore, it is useful for understanding quantum field theory on A(dS) spacetimes.

Recently, the study of black hole physics, and especially charged three-dimensional black holes, has been generalized making use of the nonlinear theory of electrodynamics. The initial idea to modify Maxwell's theory of electromagnetics was apparently outlined for the first time by Born and Infeld [6]. The modification itself originates from the quest of finding a new electromagnetic theory that is able to produce a finite amount of self-energy for pointlike charges. From 1934 up to now, a large number of modified theories of electromagnetics (or nonlinear electromagnetic theory) have been introduced, which are constructed by nonlinear combinations of the Maxwell invariant [7–10]. All of the proposed nonlinear electromagnetic theories reduce to the usual theory of electromagnetics in the special cases [8]. Models of nonlinear electrodynamics can be considered as the effective models with the quantum corrections are taken into account. Maxwell's theory of electrodynamics is a special case of nonlinear electromagnetic theory for weak fields. If the electromagnetic fields are high strength, the self-interaction of the photons cannot be forgotten and the electromagnetic theory has to be generalized to nonlinear models. Also the properties and their applications to the geometrical physics and especially physics of charged black holes have been studied extensively [7–11]. Even if the nonlinear electrodynamics originally was proposed as an instrument for removing the divergences from the classical electrodynamics, it is now a helpful theory for studying the properties of the charged black holes.

On the other hand, after the works of Bekenstein [12], of Bardeen, Carter, and Hawking [13], and of Hawking [14], it is well known that black holes can be considered as the thermodynamical systems with a temperature proportional

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to the surface gravity and with entropy equal to one-fourth of the horizon area. The main object here is to provide a detailed analysis of the thermodynamical properties of three-dimensional electrically charged black holes in the presence of the power-law Maxwell field. The power-law Maxwell field is one of the nonlinear models that has been considered in the context of black hole physical and thermodynamical properties by many authors [9,10].

This paper is outlined based on the following order. In Sec. II, we obtained the general form of the nonlinear electromagnetic and gravitational field equations by varying the proper three-dimensional action. By introducing the power-law model of nonlinear electrodynamics we solved the field equations in a spherically symmetric three-dimensional geometry. The physical and mathematical properties of the solutions are analyzed, and through consideration of the mathematical constraints on the parameters in the proposed model, we introduced a new class of asymptotically AdS black hole solutions. Section III is devoted to thermodynamical analysis of the new black hole solutions we just obtained. We have calculated the temperature, entropy, electric potential, conserved mass, and charge of the black holes. We showed that the black hole thermodynamical first law is satisfied for the BTZ and the new black hole solution we just obtained. We found that, in order to overcome the incompatibility between the Smarr mass formula and the first law of thermodynamics, the cosmological parameter Λ must be treated as a thermodynamical variable. Also the proper form of the Smarr mass formula, as the integral form of the thermodynamical first law, has been obtained. Finally, making use of the canonical ensemble method and regarding the black hole heat capacity, the thermodynamical stability and phase transitions of the black holes are studied. The results are summarized and discussed in Sec. IV.

II. FIELD EQUATIONS AND SOLUTIONS

The action for three-dimensional Einstein-Λ-nonlinear electrodynamics theory can be written in the following general form:

$$I = -\frac{1}{16\pi}\int \sqrt{-g}d^3x[R - 2\Lambda + \mathcal{L}(\mathcal{F})], \qquad (2.1)$$

where R is the Ricci scalar, $\Lambda = -1/\ell^2$ is the AdS cosmological constant, and $\mathcal{L}(\mathcal{F})$ denotes the Lagrangian of nonlinear electrodynamics as a function of Maxwell's invariant $\mathcal{F} = F^{\mu\nu}F_{\mu\nu}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is the electromagnetic potential. Varying action (2.1) with respect to the gravitational field we get Einstein's field equations as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{1}{\ell^2}g_{\mu\nu} = T_{\mu\nu}, \qquad (2.2)$$

and the corresponding stress-energy tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \frac{1}{2}\mathcal{L}(\mathcal{F})g_{\mu\nu} - 2\mathcal{L}'(\mathcal{F})F_{\mu\alpha}F_\nu{}^\alpha, \qquad (2.3)$$

where $\mathcal{L}'(\mathcal{F})$ means the derivative of $\mathcal{L}(\mathcal{F})$ with respect to the argument. Also varying action (2.1) with respect to the electromagnetic field yields

$$\nabla_\mu[\mathcal{L}'(\mathcal{F})F^{\mu\nu}] = 0$$
$$\text{or equivalently} \quad \partial_\mu[\sqrt{-g}\mathcal{L}'(\mathcal{F})F^{\mu\nu}] = 0. \qquad (2.4)$$

The only nonvanishing component of the electromagnetic field is that of F^{tr}. Assuming as a function of r, that is, $F^{\mathrm{tr}} = E(r) = -h'(r)$, we have

$$\mathcal{F} = -2E^2(r) = -2(h'(r))^2. \qquad (2.5)$$

We consider the following ansutz as the three-dimensional spherically symmetric solution to Einstein's field equations (2.2):

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\theta^2. \qquad (2.6)$$

It leads to the following independent differential equations:

$$f' - \frac{2}{\ell^2}r = r[\mathcal{L}(\mathcal{F}) + 4(h'(r))^2\mathcal{L}'(\mathcal{F})], \qquad (2.7)$$

$$f'' - \frac{2}{\ell^2} = \mathcal{L}(\mathcal{F}). \qquad (2.8)$$

Also, we obtain the Ricci scalar, Ricci, and Riemann invariants as

$$R = g^{\mu\nu}R_{\mu\nu} = -f'' - \frac{2f'}{r}, \qquad (2.9)$$

$$R^{\mu\nu}R_{\mu\nu} = \frac{1}{2}\left(f'' + \frac{f'}{r}\right)^2 + \left(\frac{f'}{r}\right)^2, \qquad (2.10)$$

$$R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda} = (f'')^2 + 2\left(\frac{f'}{r}\right)^2. \qquad (2.11)$$

In the following subsection, by introducing a nonlinear model of electrodynamics we try to solve the above differential equations for obtaining f, h, and F_{tr} as functions of radial component r. Also, we consider the singularities of the spacetime described by metric (2.6).

A. Power-law nonlinear electrodynamics

We begin with the following Lagrangian for the power-law nonlinear electrodynamics [10]

$$\mathcal{L}(\mathcal{F}) = (\alpha_p \mathcal{F})^p, \tag{2.12}$$

where α_p is a coupling constant. Since the power-law nonlinear electrodynamics is considered as the extension of Maxwell's electromagnetic theory, it must be reduced to Maxwell's electromagnetic theory as a special case. For this purpose we set $\alpha_p = -1$. In the case of $p = 1$ the power-law nonlinear theory of electromagnetics recovers the standard Maxwell's electromagnetic theory.

From Eq. (2.12) with $\alpha_p = -1$ we can write

$$\mathcal{L}'(\mathcal{F}) = -p(-\mathcal{F})^{p-1}. \tag{2.13}$$

Now making use of Eq. (2.13) together with Eqs. (2.4) and (2.5) we have

$$(h'(r))^{2p-2}[h'(r) + (2p-1)rh''(r)] = 0. \tag{2.14}$$

The solution to the differential equation (2.14) can be obtained as

$$h(r) = \begin{cases} q\ln(\frac{r}{\ell}) & \text{for } p = 1, \\ q(\frac{r}{\ell})^{\frac{2p-2}{2p-1}} & \text{for } p \neq 1, \end{cases} \tag{2.15}$$

where q is the integration constant related to the black hole charge. Also the nonzero component of the electromagnetic field is given by

$$F_{\rm tr} = \begin{cases} \frac{q}{r} & \text{for } p = 1, \\ \frac{q}{\ell}\left(\frac{2p-2}{2p-1}\right)(\frac{r}{\ell})^{\frac{-1}{2p-1}} & \text{for } p \neq 1. \end{cases} \tag{2.16}$$

Now, making use of Eqs. (2.12) and (2.13) in Eq. (2.7), the metric function $f(r)$ can be calculated as

$$f(r) = \begin{cases} \frac{r^2}{\ell^2} - 2q^2\ln(\frac{r}{\ell}) - m & \text{for } p = 1, \\ \frac{r^2}{\ell^2} + \ell^2(2)^p(1-2p)q^{2p}\left(\frac{2p-2}{2p-1}\right)^{2p-1}(\frac{r}{\ell})^{\frac{2p-2}{2p-1}} - m & \text{for } p \neq 1, \end{cases} \tag{2.17}$$

where m is an integration constant related to the black hole mass. Noting Eq. (2.12), one can easily show that the metric function $f(r)$ satisfies the last independent differential equation (2.8).

Let us determine the range of parameter p for which our obtained solutions have reasonable behavior and physically are more interesting for us. There is a restriction on p based on the fact that the electric potential $h(r)$ should have a finite value at infinity. This leads to

$$\frac{2p-2}{2p-1} < 0 \quad \text{or equivalently} \quad \frac{1}{2} < p < 1. \tag{2.18}$$

Since $(\frac{2p-2}{2p-1})^{2p-1}$ appears in the metric function as a coefficient, it must be noted that for the metric function to be a real one, $2p$ cannot be a rational number with an even number in the denominator. Figure 1 shows the plot of $f(r)$ versus r for alternative allowed p values. It is clear from Fig. 1 that, if one fixes the black hole mass and charge properly, it is possible to produce different kinds of black holes (i.e., naked singularity, extreme, and two-horizon black holes) that correspond to suitable choices of allowed p values.

To investigate the asymptotic behavior of the solutions, we notice the metric function $f(r)$ for the limit of $r \to \infty$. One can show that the p dependent power of r (i.e., $\frac{2p-2}{2p-1}$) is negative for $\frac{1}{2} < p < 1$, positive for $p < \frac{1}{2}$ or $p > 1$, and equal to 2 for $p = 0$. Thus it can be obtained from (2.17) that

$$\lim_{r\to\infty} f(r) = \frac{r^2}{\ell^2} - m \quad \text{for } \frac{1}{2} < p < 1, \tag{2.19}$$

which confirms that the metric function $f(r)$ describes an asymptotically AdS spacetime for the allowed p values. Also the spacetime is pure AdS for $p = 0$ with the following effective cosmological constant:

$$\frac{1}{\ell_{\rm eff}^2} = \frac{1}{\ell^2} + \frac{1}{2}. \tag{2.20}$$

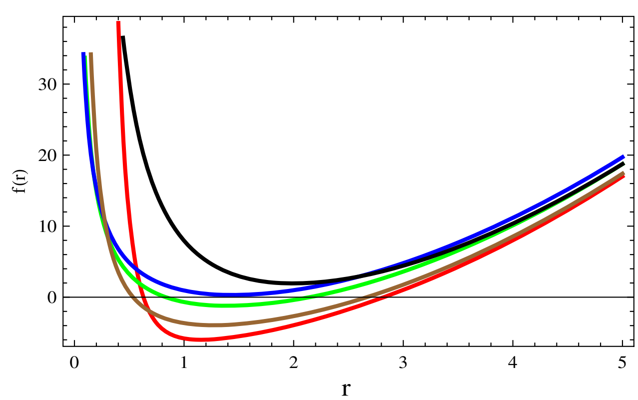

FIG. 1. $f(r)$ versus r for $M = 1$, $Q = 1$, and $\ell = 1$. Red, brown, and green lines correspond to $p = \frac{3}{5}$, $p = \frac{5}{7}$, and $p = \frac{9}{7}$, respectively. They show black holes with two horizons. The blue line ($p = \frac{4}{5}$) shows an extreme black hole, and the black line ($p = \frac{6}{7}$) shows the naked singularity. Note that Eqs. (3.16) and (3.24) have been used.

Making use of Eqs. (2.9)–(2.11) and (2.17), one can rewrite the Ricci scalar, Ricci, and Riemann invariants in the following forms:

$$R = -\frac{6}{\ell^2} + \begin{cases} \frac{2q^2}{r^2} & \text{for } p = 1, \\ (2)^p q^{2p}(4p-3)\left(\frac{2p-2}{2p-1}\right)^{2p}\left(\frac{r}{\ell}\right)^{\frac{-2p}{2p-1}} & \text{for } \frac{1}{2} < p < 1, \end{cases} \tag{2.21}$$

$$R^{\mu\nu}R_{\mu\nu} = \frac{12}{\ell^4} + \begin{cases} \frac{4q^4}{r^4} - \frac{8q^2}{\ell^2 r^2} & \text{for } p = 1, \\ \frac{(2)^{p+2}q^{2p}}{\ell^2}\left(\frac{2p-2}{2p-1}\right)^{2p}(3-4p)\left(\frac{r}{\ell}\right)^{\frac{-2p}{2p-1}} + (4)^p q^{4p}(6p^2-8p+3)\left(\frac{2p-2}{2p-1}\right)^{4p}\left(\frac{r}{\ell}\right)^{\frac{-4p}{2p-1}} & \text{for } \frac{1}{2} < p < 1, \end{cases} \tag{2.22}$$

$$R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda} = \frac{12}{\ell^4} + \begin{cases} \frac{12q^4}{r^4} - \frac{8q^2}{\ell^2 r^2} & \text{for } p = 1, \\ \frac{(2)^{p+2}q^{2p}}{\ell^2}\left(\frac{2p-2}{2p-1}\right)^{2p}(4p-3)\left(\frac{r}{\ell}\right)^{\frac{-2p}{2p-1}} + (4)^p q^{4p}(8p^2-8p+3)\left(\frac{2p-2}{2p-1}\right)^{4p}\left(\frac{r}{\ell}\right)^{\frac{-4p}{2p-1}} & \text{for } \frac{1}{2} < p < 1. \end{cases} \tag{2.23}$$

Note that the Ricci scalar, Ricci, and Riemann invariants diverge for the allowed p values. There is singularity at $r = 0$ (i.e., $r = 0$ is an essential singularity) for both the charged BTZ black holes ($p = 1$) and the asymptotically anti–de Sitter ($\frac{1}{2} < p < 1$) black holes.

III. THERMODYNAMICS

In this section we explore thermodynamical properties of three-dimensional nonlinearly charged black hole solutions we have obtained. Also the black hole stability and phase transitions are considered regarding the black hole heat capacity. At first we are interested in the charged BTZ black holes.

A. The BTZ black hole with Maxwell electrodynamics ($p = 1$)

The BTZ black hole is a solution to the Einstein-Maxwell theory in AdS space. It is identified with the metric function (2.17) with $p = 1$, that is,

$$f(r) = \frac{r^2}{\ell^2} - 2q^2 \ln\left(\frac{r}{l}\right) - m, \tag{3.1}$$

where q and m are the integration constants and are related to the black hole charge Q and mass M through the following relations [15]:

$$m = 8M, \qquad q = 2Q. \tag{3.2}$$

One can obtain the Hawking temperature associated with the black hole horizon $r = r_+$, which is the root of $f(r_+) = 0$, in terms of the surface gravity κ as

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\frac{d}{dr}f(r)|_{r=r_+} = \frac{1}{2\pi}\left(\frac{r_+}{\ell^2} - \frac{q^2}{r_+}\right). \tag{3.3}$$

Next, we calculate the entropy of the black hole. It can be considered as the three-dimensional generalization of the Hawking-Bekenstein entropy-area law, that is,

$$S = \frac{A}{4} = \frac{\pi r_+}{2}, \tag{3.4}$$

and the electric potential can obtained as [10]

$$\Phi = A_\mu \chi^\mu|_{\text{reference}} - A_\mu \chi^\mu|_{r=r_+} = -2Q \ln\left(\frac{r_+}{\ell}\right), \tag{3.5}$$

where $\chi = \partial_t$ is the null generator of the horizon. Note that Eq. (3.2) has been used.

To investigate the consistency of these quantities with the thermodynamical first law, from Eqs. (3.1)–(3.4), we can obtain the black hole mass as the function of S and Q that is

$$M(S, Q) = \frac{S^2}{2\pi^2\ell^2} - Q^2 \ln\left(\frac{2S}{\pi\ell}\right). \tag{3.6}$$

One can regard the parameters S and Q as a complete set of extensive parameters for the mass $M(S, Q)$ and define T and Φ as the intensive parameters conjugate to S and Q. These quantities can be obtained as

$$T = \left(\frac{\partial M}{\partial S}\right)_Q = \frac{S}{\pi^2\ell^2} - \frac{Q^2}{S},$$
$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_S = -2Q \ln\left(\frac{2S}{\pi\ell}\right), \tag{3.7}$$

which are compatible with the temperature and electric potential given in Eqs. (3.3) and (3.5). It means that the thermodynamics quantities we obtained in this section satisfy the first law of black hole thermodynamics,

$$dM = TdS + \Phi dQ. \qquad (3.8)$$

Finally, we would like to study the local stability and phase transition of the BTZ black hole in the canonical ensemble. It is well known that a black hole, as a thermodynamical system, is locally stable if its heat capacity is positive. A nonstable black hole may undergo phase transitions to be stabilized. The phase transition points are where the heat capacity vanishes or diverges. In the vanishing points (roots of heat capacity) the phase transition is named conventionally as the type one phase transition. The points where the heat capacity diverges are known as the type two phase transition points. With these issues in mind we proceed with the heat capacity $C_Q = T(\partial S/\partial T)_Q$. Making use of the chine rule it can be rewritten as

$$C_Q = T\left(\frac{\partial S}{\partial r_+}\right)_Q\left(\frac{\partial r_+}{\partial T}\right)_Q = \frac{\pi r_+}{2}\left(\frac{\frac{r_+^2}{\ell^2} - 4Q^2}{\frac{r_+^2}{\ell^2} + 4Q^2}\right). \qquad (3.9)$$

From Eq. (3.9), it is obvious that the BTZ black hole is stable if $r_+ > 2\ell Q$. It undergoes a type one phase transition if $r_+ = 2\ell Q$. The heat capacity does not diverge, and there is no type two phase transition. If we notice the Hawking temperature (3.3) and if we accept that it must be positive to have physical meaning, we have $r_+ > 2\ell Q$ and as the final result the positivity of the heat capacity and stability of the BTZ black holes is guaranteed. It is notable that the positivity of the black hole temperature restricts the black hole charge as $2Q < r_+/\ell$. It means that it is necessary for the black hole to be thermodynamically stable to have an electric charge in this region. Most of the results obtained in this subsection have been covered by Frassino *et al.* [16].

At this stage we return to the Smarr mass formula as the integral form of the first law of thermodynamics, that is,

$$\frac{D-3}{D-2}M = TS + \frac{D-3}{D-2}\Phi Q + \Omega J, \qquad (3.10)$$

for arbitrary D-dimensional black holes. For the present case (i.e., $D = 3$ and $\Omega = 0$), the Smarr formula results in $ST = 0$, which is in contradiction to Eqs. (3.3) and (3.4). As in Refs. [17–20], to overcome this problem, the cosmological parameter Λ must be treated as a thermodynamical variable. This idea has been proposed originally by Kastor *et al.* [21]. It has recently been used by Hendi *et al.* [8] for overcoming the problem of ensemble dependency of the BTZ-like black hole stability. Now the thermodynamical first law must be generalized to

$$dM = TdS + \Phi dQ + \Omega dJ + \Xi d\Lambda, \qquad (3.11)$$

with

$$\Xi = \left(\frac{\partial M}{\partial \Lambda}\right)_{S,Q,J} = -\frac{1}{2}\left(\frac{S^2}{\pi^2} + \frac{Q^2}{\Lambda^2}\right). \qquad (3.12)$$

Also the Smarr formula can be generalized as [22]

$$\frac{D-3}{D-2}M = TS + \frac{D-3}{D-2}\Phi Q + \Omega J - \frac{2}{D-2}\Xi\Lambda. \qquad (3.13)$$

It removes the above-mentioned problem. The generalized nonlinearly charged BTZ will be considered in the following subsection.

B. Asymptotically AdS black holes with nonlinear electrodynamics ($\frac{1}{2} < p < 1$)

In this section, we seek satisfaction of the first law of thermodynamics for our three-dimensional AdS black hole solutions. As it has been emphasized before, in order for the metric function $f(r)$ as well as the electromagnetic strength to be reasonable, physically, only the numbers in the range $\frac{1}{2} < p < 1$ are allowed.

We start with the calculation of the black hole electric charge Q, as a conserved quantity, by calculating the flux of the electromagnetic field at infinity (i.e., $r \to \infty$), that is,

$$Q = \frac{1}{4\pi}\int \sqrt{-g}\mathcal{L}'(\mathcal{F})F_{\mu\nu}n^\mu u^\nu d\Omega, \qquad (3.14)$$

where n^μ and u^ν are the unit spacelike and timelike normals to the hypersurface of radius r defined through the following relations:

$$n^\mu = \frac{1}{\sqrt{-g_{tt}}} = \frac{dt}{\sqrt{f(r)}}, \qquad u^\nu = \frac{1}{\sqrt{g_{rr}}} = \sqrt{f(r)}dr.$$

Making use of Eq. (2.16) after some simple calculations we arrived at

$$Q = -p(2)^{(p-2)}\left[\frac{q(2p-2)}{2p-1}\right]^{2p-1}, \qquad (3.15)$$

which can be rewritten as

$$q = \frac{2p-1}{2p-2}\left[\frac{-Q}{p(2)^{(p-2)}}\right]^{1/(2p-1)}. \qquad (3.16)$$

The electric potential Φ, measured at infinity with respect to the horizon, is defined by Eq. (3.5). In terms of the nonzero component of electromagnetic potential $A_t = h(r)$ given in Eq. (2.15), it can be written as

$$\Phi = -q\left(\frac{r_+}{\ell}\right)^{\frac{2p-2}{2p-1}}. \qquad (3.17)$$

The black hole entropy is given by Eq. (3.4), and the black hole horizons are the roots of lapse function

$$f(r_+) = \frac{r_+^2}{\ell^2} + (2)^p q^{2p} \left(\frac{2p-2}{2p-1}\right)^{2p-1} (1-2p) \left(\frac{r_+}{\ell}\right)^{\frac{2p-2}{2p-1}} - m = 0. \quad (3.18)$$

Next, the Hawking temperature at the black hole horizon can be calculated through the use of the definition of surface gravity, as

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \frac{d}{dr} f(r)|_{r=r_+} = \frac{1}{4\pi\ell} \left[\frac{2r_+}{\ell} + (2)^p q^{2p} (1-2p) \left(\frac{2p-2}{2p-1}\right)^{2p} \left(\frac{r_+}{\ell}\right)^{\frac{-1}{2p-1}}\right]. \quad (3.19)$$

Since the spacetime under consideration is an asymptotically AdS one, we can use the counterterm method [18] to obtain the conserved mass. In the counterterm method the metric of asymptotically AdS spacetime must be written in the following form:

$$ds^2 = \frac{\ell^2}{r^2} dr^2 + \frac{r^2}{\ell^2} (-dt^2 + dx^2) + \delta g_{ab} dx^a dx^b. \quad (3.20)$$

Making use of the first order δg_{ab} the tt component of the divergence free stress tensor is written as

$$8\pi T_{tt} = \frac{r^4}{2\ell^5} \delta g_{rr} + \frac{\delta g_{xx}}{\ell} - \frac{r}{2\ell} \partial_r \delta g_{xx}, \quad (3.21)$$

and the black hole mass M in terms of the mass parameter m may be calculated as

$$M = \int dx T_{tt}, \qquad \text{for large } r. \quad (3.22)$$

In the present case, $x = \ell\theta$, $\delta g_{xx} = 0$, and δg_{rr} can be obtained as

$$\delta g_{rr} = \frac{1}{f(r)} - \frac{\ell^2}{r^2}. \quad (3.23)$$

Now, substituting Eqs. (3.21) and (3.23) in Eq. (3.22), after integration it yields

$$m = 8M \quad (3.24)$$

once again, but this time m is obtained from Eq. (3.18).

Here, we check the first law of thermodynamics for the quantities obtained in this subsection. At first we obtain the mass as a function of the extensive quantities S and Q as

$$M(Q,S) = \frac{S^2}{2\pi^2\ell^2} - (2)^{\frac{3-3p}{2p-1}} (p)^{\frac{-2p}{2p-1}} \frac{(2p-1)^2}{2p-2} (Q)^{\frac{2p}{2p-1}} \left(\frac{2S}{\pi\ell}\right)^{\frac{2p-2}{2p-1}}, \quad (3.25)$$

where Eqs. (3.4), (3.16), and (3.18) have been used. By treating Q and S, as the thermodynamical extensive variables, one can calculate

$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_S, \qquad T = \left(\frac{\partial M}{\partial S}\right)_Q, \quad (3.26)$$

and show that the results are compatible with those of Eqs. (3.17) and (3.19). It confirms the validity of the thermodynamical first law in the form of Eq. (3.8).

The positivity of heat capacity $C_Q = T(\partial S/\partial T)_Q = T/(\partial^2 M/\partial S^2)_Q$ or equivalently the positivity of $(\partial S/\partial T)_Q$ or $(\partial^2 M/\partial S^2)_Q$ with $T > 0$ is sufficient to ensure the local stability of the black hole. The unstable black holes undergo phase transitions to be stabilized. Making use of Eq. (3.19) together with the relation $S = \pi r_+/2$, the black hole heat capacity is

$$C_Q = \frac{\pi T}{2} \left(\frac{\partial T}{\partial r_+}\right)_Q^{-1} = \pi^2 T \ell^2 \left[1 + (2)^{p-1} q^{2p} \left(\frac{2p-2}{2p-1}\right)^{2p} \left(\frac{r_+}{\ell}\right)^{-\frac{2p}{2p-1}}\right]^{-1}. \quad (3.27)$$

On the other hand, Eq. (3.19) can be rewritten as

$$T = \frac{r_+}{2\pi\ell^2} \left[1 - (2)^{p-1} (2p-1) \left(\frac{q(2p-2)}{2p-1}\right)^{2p} \left(\frac{r_+}{\ell}\right)^{-\frac{2p}{2p-1}}\right]. \quad (3.28)$$

It is notable that for $\frac{1}{2} < p < 1$ the p-dependent term in the brackets is positive; it results in a positive temperature, if

$$(2)^{p-1} (2p-1) \left(\frac{q(2p-2)}{2p-1}\right)^{2p} \left(\frac{r_+}{\ell}\right)^{-\frac{2p}{2p-1}} < 1, \quad (3.29)$$

as it must be for a physically acceptable black hole. Thus the heat capacity does not vanish. Indeed, the inequality (3.29) is a restriction on the black hole charge. Making use of Eq. (3.16), it can be rewritten as

$$Q < p \frac{r_+}{\ell} 2^{\frac{3p-1}{2p}} (2p-1)^{-\frac{2p-1}{2p}}. \quad (3.30)$$

For a stability interpretation of the black hole we examine the roots of the denominator in Eq. (3.27). It yields

$$\left(\frac{r_+}{\ell}\right)^{\frac{2p}{2p-1}} + (2)^{p-1}\left[\frac{q(2p-2)}{2p-1}\right]^{2p} = 0. \tag{3.31}$$

Since both terms in this equation are positive, it does not have any real roots. In summary, the heat capacity neither diverges nor vanishes, and as a result the black hole is thermodynamically stable and does not undergo any types of phase transitions.

Here, we return to the Smarr mass formula once again. In the case of $D = 3$ it results in $ST = 0$, which is not compatible with the T and S values we have obtained in this subsection. Noting Eq. (3.19) we have

$$ST = \frac{r_+^2}{8}\left[-2\Lambda - 2^p(2p-1)\left(\frac{q(2p-2)}{2p-1}\right)^{2p} \times (r_+)^{-\frac{2p}{2p-1}}(-\Lambda)^{\frac{p-1}{2p-1}}\right]. \tag{3.32}$$

To overcome this problem we consider the cosmological parameter Λ as a thermodynamical variable with the same definition in Eqs. (3.11) and (3.12). But this time M is given by Eq. (3.25) with $\Lambda = -1/\ell^2$, which yields

$$\Xi = \left(\frac{\partial M}{\partial \Lambda}\right)_{S,Q,J} = \frac{r_+^2}{8}\left[-1 + 2^{p-1}(2p-1)\left(\frac{q(2p-2)}{2p-1}\right)^{2p} \times (r_+)^{-\frac{2p}{2p-1}}(-\Lambda)^{\frac{-p}{2p-1}}\right], \tag{3.33}$$

and the generalized Smarr mass formula is none other than the statement given in Eq. (3.13).

IV. CONCLUSION

This work considers an extension of the Einstein-Maxwell three-dimensional black hole solutions in which the usual Maxwell theory has been generalized to a power-law nonlinear one. We obtained the gravitational and electromagnetic field equations by varying the action of the Einstein–nonlinear Maxwell with respect to the metric tensor and electromagnetic potential, respectively. Through considering a static spherically symmetric geometry, we obtained the solutions of the gravitational and electromagnetic field equations and found that different black hole solutions can be distinguished according to different choices of parameters in the nonlinear theory of electrodynamics. For the obtained solutions having physical meaning, we found that the value of p (i.e., the power of the Maxwell invariant in the nonlinear theory) must be restricted to the range $\frac{1}{2} < p < 1$ except the values for which $2p$ is a rational value with an even number in the denominator. The black solutions with these constraints are a new class of asymptotically AdS black hole solutions we just obtained. Also in the case of $p = 1$ the solution coincides with the charged BTZ black hole.

In the next part of the paper, we considered the thermodynamics of the obtained black hole solutions and discussed the thermal stability and phase transitions through the canonical ensemble approach. In the case of BTZ solutions, at first we confirmed the validity of the first law of black hole thermodynamics. Next we showed that the black hole is thermally stable and no phase transitions take place. Also we found that, for the BTZ black hole to be thermodynamically stable, the black hole charge must be restricted to $2Q < r_+/\ell$. For the asymptotically AdS black hole solutions, as the new black hole solutions obtained here, at first we calculated the entropy and temperature of the black hole, making use of the concepts of the horizon area and surface gravity. Also, we obtained the charge and mass of the black hole, as the conserved quantities, by using Gauss's law and counterterm method, respectively. By considering the black hole mass as a function of both the charge and the entropy, we confirmed the validity of the black hole thermodynamical first low. Finally, through the canonical ensemble method by calculating the black hole heat capacity, we analyzed the black hole remnant or phase transitions and found that the new AdS black holes are thermally stable and do not undergo any types of phase transitions. The charge of the physically reasonable AdS black holes has been restricted through Eq. (3.30). In either of the black hole solutions (i.e., the BTZ and the new AdS black holes) we found that, in order for the Smarr mass formula to be consistent with the first law of black hole thermodynamics, the cosmological parameter Λ must be treated as a thermodynamical variable.

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