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Notes on nonsingular models of black holes

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We discuss static spherically symmetric metrics which represent nonsingular black holes in four- and higher-dimensional spacetime. We impose a set of restrictions, such as a regularity of the metric at the center $r = 0$ and Schwarzschild asymptotic behavior at large r. We assume that the metric besides mass M contains an additional parameter ℓ, which determines the scale where modification of the solution of the Einstein equations becomes significant. We require that the modified metric obeys the limiting curvature condition; that is, its curvature is uniformly restricted by the value $\sim\ell^{-2}$. We also make a "more technical" assumption that the metric coefficients are rational functions of r. In particular, the invariant $(\nabla r)^2$ has the form $P_n(r)/\tilde{P}_n(r)$, where P_n and $\tilde{P}_n$ are polynomials of the order of n. We discuss first the case of four dimensions. We show that when $n \leq 2$ such a metric cannot describe a nonsingular black hole. For $n = 3$ we find a suitable metric, which besides M and ℓ contains a dimensionless numerical parameter. When this parameter vanishes, the obtained metric coincides with Hayward's one. The characteristic property of such spacetimes is $-\xi^2 = (\nabla r)^2$, where ξ^2 is a timelike at infinity Killing vector. We describe a possible generalization of a nonsingular black-hole metric to the case when this equality is violated. We also obtain a metric for a charged nonsingular black hole obeying similar restrictions as the neutral one and construct higher dimensional models of neutral and charged black holes.

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I. INTRODUCTION

The general relativity is ultraviolet incomplete, in both the classical and the quantum domains. A well-known problem of the classical Einstein's theory of gravity is the inevitable existence of singularities. For example, solutions of the Einstein equations, describing stationary isolated black holes, such as Schwarzschild, Reisner-Nordström, and Kerr metric, have curvature singularity in their interior. It is generally believed that the general relativity should be modified in the regions where the spacetime curvature becomes high. Such a modification is also required if one searches for a theory which is ultraviolet (UV) complete. There exist several different proposals how such a modification can be achieved. For example, quite a long time ago it has been demonstrated that the addition of the higher order in curvature terms, as well as the terms containing higher derivatives, can improve the UV properties of the Einstein gravity [1–4]. However, such theories usually have nonphysical degrees of freedom (ghosts). Recently a new version of the UV complete modification of the general relativity was proposed which is free from this problem [5–7]. It was named a ghost-free gravity [5–14]. Such a theory contains an infinite number of derivatives and, in fact, is nonlocal [10,13]. A similar theory appears naturally also in the context of noncommutative geometry deformation of the Einstein gravity [15,16] (see a review [17] and references therein). The application of the ghost-free theory of gravity to the problem of singularities in cosmology and black holes can be found in [18–25].

In the absence of the adopted theory it is instructive to discuss what kind of modifications one could expect when gravity is UV complete. Such an analysis can be fruitful only if some "natural" assumptions concerning the properties of such a "full" theory are imposed. In the present paper we present some results on so-called nonsingular models of black holes.

One of the main assumptions is that there exists a critical energy μ and the corresponding length-scale parameter $\ell = \mu^{-1}$. The metric should be modified when the spacetime curvature $\mathcal{R}$ becomes comparable with ℓ^{-2}. At the same time we assume that one can use the classical metric $g_{\mu\nu}$, which is a solution of the effective action of the modified gravity. In other words, the length scale λ, where quantum gravity effects become important, is much smaller than ℓ. We are looking for black-hole metrics which do not have curvature singularities. The first model of a nonsingular black hole was proposed by Bardeen [26], who considered a collapse of a charged matter with a charged matter core inside the black hole instead of its singularity. Different models of neutral, charged, and rotating nonsingular black holes were proposed and discussed later [27–39]. A general review of different models of nonsingular black holes and additional references on this subject can be found in [40].

In a general case a regular solution besides some critical scale parameter ℓ, which is a parameter of the

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corresponding UV complete theory, contains also such parameters as mass and charge, which specify a concrete solution. The regularity of the solution means that for a fixed value of these parameters the curvature of the spacetime is finite. However, in a general case there is no guarantee that the maximal value of the curvature would not infinitely grow when, say, the mass (and/or charge) becomes infinitely large. It is reasonable to assume that for a viable fundamental theory the absolute maximal value of the curvature is restricted by some fundamental value. In other words, the curvature invariants are uniformly restricted by some universal value

$$|\mathcal{R}| \leq \frac{c}{\ell^2}, \tag{1.1}$$

where c is a dimensionless constant which depends only on the type of the curvature invariant. This assumption, called the limiting curvature conjecture, was proposed in [41–43]. In the present paper we assume that the limiting curvature condition is satisfied. To make this condition more concrete, we shall check restriction (1.1) for the Ricci scalar, as well as for the square roots of the quadratic in the curvature invariants. We shall demonstrate that the limiting curvature condition imposes a significant restriction on regular metrics. Many of the proposed earlier nonsingular black-hole models violate this property.

In the present paper we consider spherically symmetric nonsingular black holes. This simplifying assumption allows one to make many results more concrete. For example, simple analysis of the spherically symmetric metrics shows that if such a spacetime has an apparent horizon, it cannot cross the center $r = 0$ without the creation of the curvature singularity (see next section). This means that besides an outer horizon, close to the classical gravitational radius, such a regular metric has also an inner horizon, close to $r = 0$. Either these two horizons never meet in the future or the apparent horizon is closed [44]. New (baby) universe creation inside a black hole, discussed in [45,46], is an example of the former case. (See also [47–52].) A model, describing a complete quantum evaporation of a nonsingular black hole, described in [44], is an example of the latter case. Such models were later intensively discussed in the literature [20,53–60]. Nonsingular black hole models in the dilaton two-dimensional (2D) gravity were discussed in [61–65].

A special interest to a nonsingular model of a completely evaporating black hole is connected with a long-standing problem of the information loss in black holes. In a case when the apparent horizon is closed, the event horizon is absent, and all the information, accumulated inside such a "black hole," can return to the spacetime, visible by an external observer, after the evaporation is completed (see e.g. [58] and references therein).

A model of a nonsingular black hole, which is convenient for the analysis, was proposed by Hayward [54] (see also [58]). It describes an isolated four-dimensional spherically symmetric regular spacetime and, besides the fundamental length ℓ, contains only one parameter, mass M. At a large distance it reproduces the Schwarzschild metric, while at the origin it is regular and has de Sitter form. For $M \geq 3\sqrt{3}\ell/4$ the metric has two branches (outer and inner) of the apparent horizon. The outer horizon is located near $2M$, while the inner one is close to ℓ. The property which makes this metric simple for the analysis is its scaling behavior. Namely, there exists a scaling transformation of the coordinates, metric, and its parameters, which preserves the form of the metric. We discuss this property in Sec. II.

In the present paper we propose and discuss useful generalizations of the Hayward metric. A first important generalization is a wide class of metrics, with a nontrivial redshift factor. We also present a higher dimension generalization of a nonsingular metric, as well as metrics for charged nonsingular black hole spacetimes.

II. METRIC OF A NONSINGULAR BLACK HOLE

A. A nonsingular black-hole model

A general static metric in a four-dimensional spacetime can be written in the form

$$dS^2 = -FA^2 dV^2 + 2A dV dr + r^2 d\omega^2, \tag{2.1}$$

where $F = F(r)$ and $A = A(r)$ are two arbitrary functions. This metric has the Killing vector $\xi^\mu \partial_\mu = \partial_V$. It is easy to see that

$$F = (\nabla r)^2, \qquad FA^2 = -\xi^2. \tag{2.2}$$

In a spacetime with a horizon, $F(r)$ vanishes at the position r_0 of the apparent horizon. For a regular static metric such a horizon is at the same time the Killing horizon, so that $A(r_0)$ is finite there.

If the metric has a horizon where $F(r_0) = 0$, then

$$\xi^\nu \xi^\mu_{;\nu} \overset{H}{=} \kappa \xi^\mu, \qquad \kappa = \frac{1}{2}(AF')|_{r=r_0}. \tag{2.3}$$

By definition, κ is the surface gravity. The value of κ depends on the choice of the normalization of the Killing vector. In an asymptotically flat spacetime one usually puts $\xi^2|_{r=\infty} = -1$. A condition that there is no solid angle deficit implies $F|_\infty = 1$. Hence one also has $A|_\infty = 1$.

Let R be the Ricci scalar, $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R$, and $C_{\mu\nu\alpha\beta}$ be the Weyl tensor. Let us define the following quadratic in the curvature invariants:

$$\mathcal{S}^2 = S_{\mu\nu} S^{\mu\nu}, \qquad \mathcal{C}^2 = C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta}. \tag{2.4}$$

Then one has

$$R = F'' + \frac{4}{r}F' - 2\frac{F-1}{r^2} + \frac{1}{A}\left(2FA'' + 3F'A' + \frac{4}{r}FA'\right), \qquad (2.5)$$

$$\mathcal{C} = \frac{1}{\sqrt{3}}\left[F'' - \frac{2}{r}F' + 2\frac{F-1}{r^2} + \frac{1}{A}\left(2FA'' + 3F'A' - \frac{2}{r}FA'\right)\right]. \qquad (2.6)$$

An expression for $\mathcal{S}$ is similar but quite long, and we do not present it here.

We assume that the metric (2.1) is finite at the origin $r = 0$, so that

$$F = F_0 + F_1 r + F_2 r^2 + O(r^3),$$
$$A = A_0 + A_1 r + A_2 r^2 + O(r^3). \qquad (2.7)$$

In a general case the curvature invariants R and $\mathcal{C}$ are singular at the origin and have divergences $\sim r^{-2}$ and $\sim r^{-1}$. Conditions that these divergences are absent and the metric is regular are

$$F_0 = 1, \qquad F_1 = A_1 = 0. \qquad (2.8)$$

By substituting (2.7) in the expression for $\mathcal{S}^2$ and using relations (2.8) one can check that the conditions of regularity of $\mathcal{S}^2$ at $r = 0$ are identically satisfied. One also has

$$R = -12\left(F_2 + \frac{A_2}{A_0}\right) + O(r^2), \qquad (2.9)$$

$$\mathcal{S} = 2\sqrt{3}\frac{A_2}{A_0} + O(r^2), \qquad \mathcal{C} = O(r^2). \qquad (2.10)$$

Let us notice that A_0 is an arbitrary constant. Its meaning is connected with a time delay between infinity and $r = 0$. For a fixed value of r at far distance one has $\Delta\tau_\infty \equiv \Delta t = \Delta V$, where τ_∞ is the proper time measured by the clocks at the infinity. For the same interval ΔV the proper time measured at the center $r = 0$ is $\Delta\tau_0 = A_0\Delta V$. Hence

$$\Delta\tau_0 = A_0\Delta\tau_\infty. \qquad (2.11)$$

For $A_0 < 1$ ($A_0 > 1$) the time at the center "goes" slower (faster) than at the infinity. For a monochromatic wave φ propagating in a static spacetime one can write $\varphi \sim \exp(i\Phi)$, where $\Phi = \omega V$. If $\omega_\infty = d\Phi/d\tau_\infty$ and $\omega_0 = d\Phi/d\tau_0$, then one has

$$\omega_0 = A_0^{-1}\omega_\infty. \qquad (2.12)$$

For $A_0 > 1$ ($A_0 < 1$) the frequency of a signal, registered at the center $r = 0$, is redshifted (blueshifted) with respect to the frequency of the signal emitted at the infinity. In what follows we refer to $A(r)$ as a redshift function.

We are interested in a metric which describes a black hole. For this reason we assume that the function $F(r)$ vanishes at some value $r = r_+$, where the event horizon is located. In order for the metric to be regular at $r = 0$ it must have at least one more zero at $r = r_- > 0$. For simplicity we assume that the function $F(r)$ has exactly two zeros at $r_+ > r_- > 0$. Our final assumption is that the curvature invariants $|R|$, $|\mathcal{S}|$, and $|\mathcal{C}|$ are uniformly restricted by some values proportional to ℓ^{-2}. We call this parameter ℓ the fundamental length. The latter requirement means that our metric satisfies the Markov's limiting curvature conjecture [41–43].

To fix the scale of the parameter ℓ one can put $|F_2| = \ell^{-2}$, so that the metric function F at the origin $r = 0$ has the form

$$F = 1 + \varepsilon r^2/\ell^2 + O(r^4), \qquad \varepsilon = \pm 1. \qquad (2.13)$$

We assume that the spacetime is asymptotically flat, so that

$$F = 1 - \frac{r_g}{r} + O(r^{-2}), \qquad r_g = 2M. \qquad (2.14)$$

We call a spacetime (2.1) satisfying the above described properties (including the limiting curvature condition) a nonsingular black hole. Certainly one cannot require that this metric is a solution of the Einstein equations. One should assume that the Einstein equations should be modified in the UV domain. The curvature of the Schwarzschild spacetime, $\sim r_g/r^3$, reaches the critical value ℓ^{-2} at the radius $r_\ell = (r_g\ell^2)^{1/3}$. At this radius the modified solution becomes essentially different from the Einstein's solution.

B. Uncharged nonsingular black-hole metric

1. Scaling property

In the absence of a "final" UV complete theory of gravity there is a wide ambiguity in the choice of metric functions F and A for the metric describing a modified black hole. This ambiguity is reduced by adopting constraints described in the previous section, but it is still quite wide. We impose additional "natural" restrictions.

Let us consider the Schwarzschild metric which is a vacuum spherically symmetric solution of the Einstein equations. For this metric

$$F = 1 - \frac{r_g}{r} = \frac{r - r_g}{r}, \qquad A = 1. \qquad (2.15)$$

The form of the metric is fixed by the Einstein equations, and it contains one parameter, $r_g = 2M$, which is the integration constant. Moreover, F has the form of the rational function, which is the ratio of two first order in r polynomials. For a special value of the parameter $r_g = 0$ the spacetime is flat. An additional property of the Schwarzschild metric is its scale invariance. Namely, its form does not change under the following scale transformations

$$r \to \beta r, \qquad r_g \to \beta r_g, \qquad v \to \beta v, \qquad ds^2 \to \beta^2 ds^2. \tag{2.16}$$

This symmetry property allows one to write

$$ds^2 = r_g^2 (ds^2)|_{r_g=1}. \tag{2.17}$$

In other words, by using the dimensional quantity r_g as a general scale parameter, one reduces the original metric with one parameter (mass), to the metric $(ds^2)|_{r_g=1}$, which does not contain any arbitrary parameters at all.

2. Case $n \le 2$

Let us consider a generalization of this metric obeying the condition $A = 1$. We assume that F is a rational function of r

$$F(r) = \frac{P_n(r)}{\tilde{P}_n(r)}, \tag{2.18}$$

where P_n and $\tilde{P}_n$ are polynomials of the order $n > 1$. For example, one may try the following form of F:

$$F = 1 - \frac{r_g r}{r^2 + \ell^2}. \tag{2.19}$$

Here the fundamental scale ℓ plays the role of the regularizer. At a far distance the metric correctly reproduces the Schwarzschild solution, and deflection from it is of the order of ℓ^2. At the origin the metric is finite. However, it is not regular. Moreover, its curvature invariants have the form

$$\mathcal{R} = \frac{1}{\ell^2}\frac{r_g}{\ell} f(\rho), \tag{2.20}$$

where $\rho = r/\ell$. For any fixed ρ the corresponding curvature invariant can be made arbitrarily large by simply increasing the mass parameter r_g. This means that the metric (2.1) with (2.19) for a black hole does not satisfy the limiting curvature condition [66].

Can we reach desired properties of the metric when $n = 2$? One can write (2.18) for this case as follows:

$$F = \frac{r^2 + a_1 r + a_0}{r^2 + b_1 r + b_0}. \tag{2.21}$$

Condition $F_0 = 1$ implies $b_0 = a_0$. It is also easy to check that the condition $F_1 = 0$ requires that $b_1 = a_1$. But in this case F is identically 1 and the spacetime is flat. To summarize, metrics (2.18) with $n \le 2$ cannot be used as a consistent model of a nonsingular black hole.

3. Case $n = 3$

Let us analyze metrics (2.18) with $n = 3$,

$$F = \frac{r^3 + a_2 r^2 + a_1 r + a_0}{r^3 + b_2 r^2 + b_1 r + b_0}. \tag{2.22}$$

Regularity of the spacetime at the origin $r = 0$ implies

$$b_0 = a_0, \qquad b_1 = a_1, \tag{2.23}$$

so that one has

$$F = 1 - \frac{(b_2 - a_2) r^2}{r^3 + b_2 r^2 + b_1 r + a_0}. \tag{2.24}$$

In order to have a proper Schwarzschild asymptotic form one must put $b_2 - a_2 = r_g$. To satisfy the condition (2.13) one must choose $a_0 = r_g \ell^2$. Hence the metric function F takes the form

$$F = 1 - \frac{r_g r^2}{r^3 + b_2 r^2 + b_1 r + r_g \ell^2}. \tag{2.25}$$

This function, besides the fundamental length ℓ and the gravitational radius $r_g = 2M$, contains two arbitrary parameter b_1 and b_2 with the dimensionality $[\text{length}]^2$ and [length], respectively. We assume that these parameters have the form of the product of non-negative integer powers of r_g and ℓ. Then $b_2 \sim \ell$ and $b_1 \sim r_g \ell$ or $b_1 \sim \ell^2$. The cases when $b_2 \sim r_g$ and $b_1 \sim r_g^2$ are excluded by the condition that in the limit $\ell \to 0$ the metric must coincide with the Schwarzschild one.

We write the metric function (2.25) as

$$F = 1 - \frac{r_g r^2}{r^3 + c_2 \ell r^2 + (c\ell^2 + c_1 \ell r_g) r + r_g \ell^2}. \tag{2.26}$$

One has the following series expansion

$$F = 1 - \frac{r_g}{r} + \frac{c_2 \ell r_g}{r^2} + \frac{\ell r_g [(c - c_2)\ell + c_1 r_g]}{r^3} + O(r^{-4}). \tag{2.27}$$

In the quantum gravity the fundamental length is $\ell = \sqrt{\hbar c / G}$. Loop expansions contain integer powers of $\hbar$, or, what is equivalent, integer powers of ℓ^2. In such a case the terms linear in ℓ in (2.27) should vanish, and one has

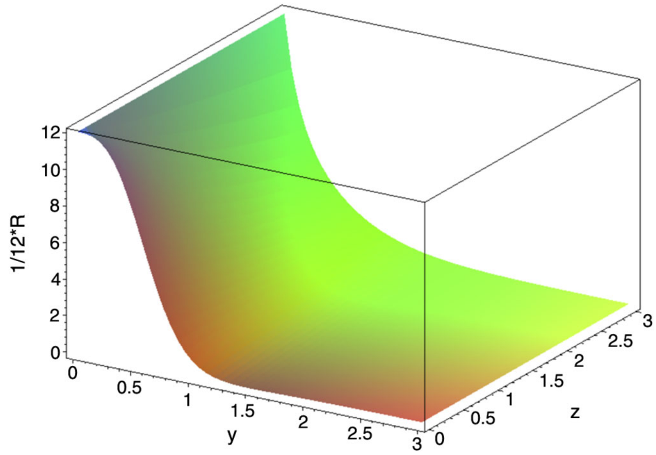

FIG. 1. Plot of $\ell^2 R/12$ as a function of y and z.

$$F = 1 - \frac{r_g r^2}{r^3 + c\ell^2 r + r_g \ell^2}, \tag{2.28}$$

where c is a dimensionless numerical parameter. Let us denote

$$\beta = (\ell/r_g)^{1/3}, \qquad q = (2M\ell^2)^{1/3}, \qquad r = qy. \tag{2.29}$$

The curvature invariants for metric (2.28) are

$$R = \frac{2}{\ell^2}\frac{-y^4 z + 3y^2 z^2 - 3y^3 + 8yz + 6}{(y^3 + yz + 1)^3},$$
$$\mathcal{C} = \frac{1}{\sqrt{3}\ell^2}\frac{2y(-3y^5 + y^3 z + 6y^2 + z)}{(y^3 + yz + 1)^3},$$
$$\mathcal{S} = \frac{1}{\ell^2}\frac{y(5y^3 z + yz^2 + 9y^2 + 2z)}{(y^3 + yz + 1)^3}. \tag{2.30}$$

Here $z = c\beta^2$. We assume that $z \geq 0$, so that the denominators in (2.30) are strictly positive.

Figures 1, 2, and 3 show plots of the invariants $\ell^2 R/12$, $\ell^2\mathcal{C}$, and $\ell^2\mathcal{S}$, respectively. These plots demonstrate that these invariants are uniformly bounded. Hence, the metric (2.28) satisfies the limiting curvature condition, and it can be used as a nonsingular model of a black hole.

C. Hayward model

The metric (2.28) contains a free dimensionless parameter c. It takes a simpler form when this parameter vanishes,

$$F = 1 - \frac{2Mr^2}{r^3 + 2M\ell^2}. \tag{2.31}$$

This form of metric for a nonsingular black hole was proposed and discussed in [54].

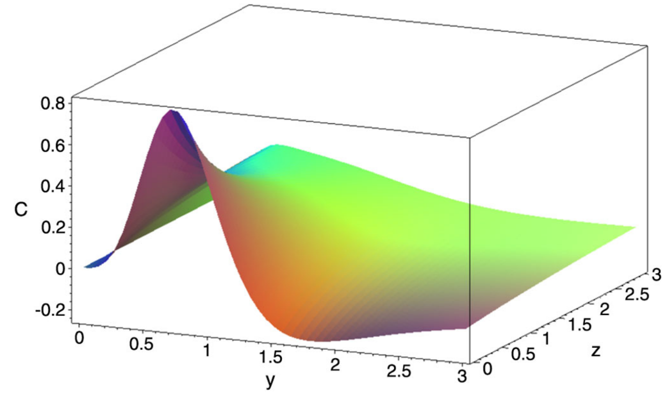

FIG. 2. Plot of $\ell^2\mathcal{C}$ as a function of y and z.

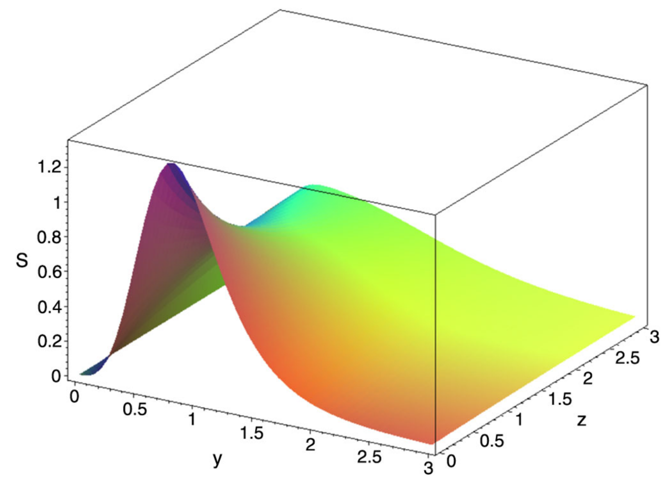

FIG. 3. Plots of $\ell^2\mathcal{S}$ as a function of y and z.

At large r one has

$$F = 1 - \frac{2M}{r} + O(r^{-4}). \tag{2.32}$$

Let us denote by y the dimensionless coordinate

$$r = (2M\ell^2)^{1/3} y. \tag{2.33}$$

Then one has

$$F = 1 - \mathcal{B}\frac{y^2}{y^3 + 1}, \qquad R = -\frac{6}{\ell^2}\frac{y^3 - 2}{(y^3 + 1)^3}, \tag{2.34}$$

$$\mathcal{S} = \frac{9}{\ell^2}\frac{y^3}{(y^3 + 1)^3}, \qquad \mathcal{C} = \frac{\sqrt{12}}{\ell^2}\frac{y^3(y^3 - 2)}{(y^3 + 1)^3}. \tag{2.35}$$

Here $\mathcal{B} = (2M/\ell)^{2/3}$.

The rational functions of y, which enter expressions for R, $\mathcal{S}$, and $\mathcal{C}$, are regular, are finite, and have their absolute value restricted by some numerical factor. This means that

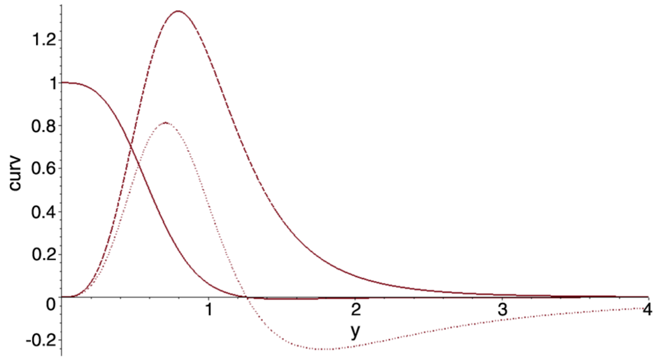

FIG. 4. Plots of $\ell^2 R/12$ (solid line), $\ell^2 \mathcal{C}$ (dotted line), and $\ell^2 \mathcal{S}$ (dashed line) as functions of y.

the metric (2.1), (2.31) satisfies the limiting curvature condition. The plots of these invariants as functions of y are presented in Fig. 4.

The metric (2.1), (2.31) is invariant under the following scaling transformation:

$$\begin{aligned} &r \to \alpha r, \qquad V \to \alpha V, \qquad M \to \alpha M, \\ &\ell \to \alpha \ell, \qquad dS^2 \to \alpha^2 dS^2. \end{aligned} \tag{2.36}$$

Using this property one can choose the scale parameter as it is convenient as well as to work after this with the dimensionless form of the metric. The original metric (2.1), (2.31) contains two parameters, the mass M and the fundamental scale ℓ. However, only their (dimensionless) ratio is really important.

The metric function (2.31) can be written in the form

$$F = \frac{(r - r_-)(r - r_+)(r - r_0)}{r^3 + B}. \tag{2.37}$$

This form of F contains 4 parameters: r_-, r_+, r_0, and B. However, they are not independent. The condition $F'(0) = 0$ gives

$$r_0 = -\frac{r_+ r_-}{r_+ + r_-}, \tag{2.38}$$

while the condition $F(0) = 1$ implies

$$B = \frac{r_+^2 r_-^2}{r_+ + r_-}. \tag{2.39}$$

Using relations (2.13) and (2.32) one finds

$$\ell = \frac{r_+ r_-}{\sqrt{r_+^2 + r_+ r_- + r_-^2}}, \tag{2.40}$$

$$M = \frac{r_+^2 + r_+ r_- + r_-^2}{2(r_+ + r_-)}. \tag{2.41}$$

For given ℓ and M one can solve Eqs. (2.40) and (2.41) and find the radii of the outer and inner horizons, r_- and r_+. Solutions exist only if $M \geq M_* = 3\sqrt{3}\ell/4$. For this minimal value of mass one has $r_+ = r_- = \sqrt{3}\ell$.

There exists another convenient parametrization of the metric. One can choose r_- as a scale factor and define new dimensional coordinates and parameters as follows:

$$\begin{aligned} &x = r/r_-, \qquad p = x_+/r_-, \qquad m = M/r_-, \\ &b = \ell/r_-, \qquad v = V/r_-. \end{aligned} \tag{2.42}$$

One has

$$\begin{aligned} &dS^2 = r_-^2 ds^2, \\ &ds^2 = -f dv^2 + 2dvdx + x^2 d\omega^2, \\ &f = \frac{(x-p)(x-1)(x + \frac{p}{p+1})}{x^3 + \frac{p^2}{p+1}}. \end{aligned} \tag{2.43}$$

One also has

$$r_- = \frac{\ell}{b}, \qquad b = \frac{p}{\sqrt{p^2 + p + 1}}, \qquad m = \frac{p^2 + p + 1}{2(p+1)}. \tag{2.44}$$

The metric (2.43) has two horizons located at p and 1. The corresponding dimensionless surface gravity at these horizons is

$$\kappa_+ = \frac{(p-1)(p+2)}{2p(p^2 + p + 1)}, \qquad \kappa_- = -\frac{(p-1)(2p+1)}{2(p^2 + p + 1)}. \tag{2.45}$$

We denoted by p a position of the outer horizon, so that one has $p \geq 1$. In the limit of the large mass, $p \to \infty$, one has

$$\begin{aligned} &r_-/\ell \to 1, \qquad r_+/\ell \to 2m, \qquad \kappa_+ \to \frac{1}{2p}, \\ &\kappa_- \to -1. \end{aligned} \tag{2.46}$$

D. Modified Hayward metric

In the previous sections we have assumed that the redshift function $A(r)$ is trivial, $A = 1$. This means that there is no frequency shift for the radiation propagating from infinity to the center $r = 0$ of the regular black hole. This assumption is rather restrictive. We describe now nonsingular black hole models with a nontrivial frequency-shift property. We show that there exist such smooth functions $A(r)$ which produce an arbitrary red- or blueshift effect in the center of the black hole without violating its regularity [67].

It is convenient to start with the form (2.43) of the Hayward metric and modify it as follows:

$$dS^2 = r_-^2 ds^2,$$
$$ds^2 = -fA^2dv^2 + 2Advdx + x^2d\omega^2,$$
$$f = \frac{(x-p)(x-1)(x+\frac{p}{p+1})}{x^3+\frac{p^2}{p+1}},$$
$$A = \frac{x^n+1}{x^n+p^k}. \tag{2.47}$$

Here n and k are properly chosen positive integer numbers. At a large distance one has

$$A \sim 1 + \frac{1-p^k}{x^n} + O(x^{-2n}). \tag{2.48}$$

In order to preserve the correct Schwarzschild asymptotic form one must put $n \geq 2$. In the presence of the function A the surface gravity (2.45) is modified,

$$\kappa_- \to \frac{2}{p^k+1}\kappa_-,$$
$$\kappa_+ \to \frac{p^n+1}{p^n+p^k}\kappa_+ = 1 - \frac{1-p^k}{1+p^{n-k}}\kappa_+. \tag{2.49}$$

These relations show that for a large mass, $p \to \infty$, the surface gravity at the inner horizon is reduced by the factor p^k, while κ_+ remains practically the same.

To illustrate properties of the modified metric we consider a special case $n = 6$ and $k = 4$,

$$A = \frac{x^6+1}{x^6+p^4}. \tag{2.50}$$

For this form of the redshift function, in the limit of large mass ($p \to \infty$), the surface gravity of the inner horizon becomes small, $\kappa_- \sim p^{-4}$, while the surface gravity κ_+ remains practically unchanged.

Let us check that the curvature invariants (2.5) for such a metric satisfy the limiting curvature condition. Figures 5–7 show plots R, $\mathcal{C}$, and $\mathcal{S}$ as functions of x and p. They have similar behavior. Namely, they are uniformly bounded by numerical factor, independent of x and p. In the limit $p \to \infty$ they have the following asymptotic form:

$$R \sim \frac{12(11x^6 - 7x^4 + 1)}{x^6+1} + O(p^{-1}),$$
$$\mathcal{C} \sim \frac{4\sqrt{3}x^4(7x^2-4)}{x^6+1} + O(p^{-1}),$$
$$\mathcal{S} \sim \frac{6\sqrt{3}x^4\sqrt{22x^4-28x^2+9}}{x^6+1} + O(p^{-1}). \tag{2.51}$$

Hence, the metric (2.47) with nontrivial redshift property satisfies the limiting curvature condition.

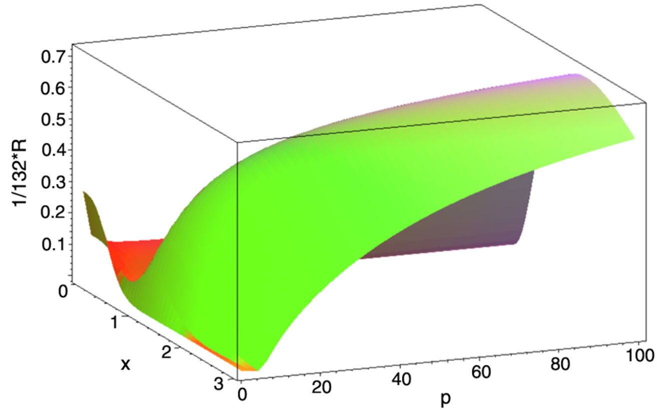

FIG. 5. Plot of $r_-^2 R/132$ as a function of x and p.

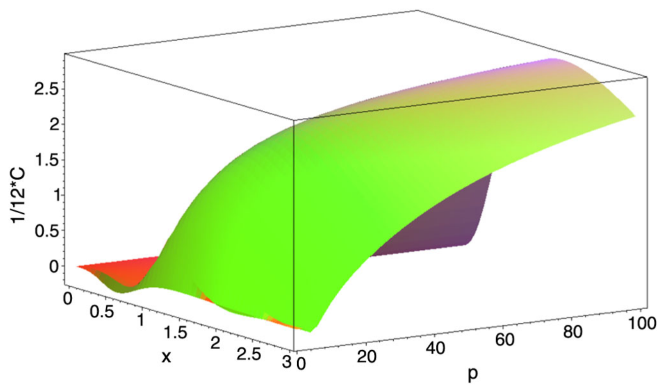

FIG. 6. Plot of $r_-^2\mathcal{C}/12$ as a function of x and p.

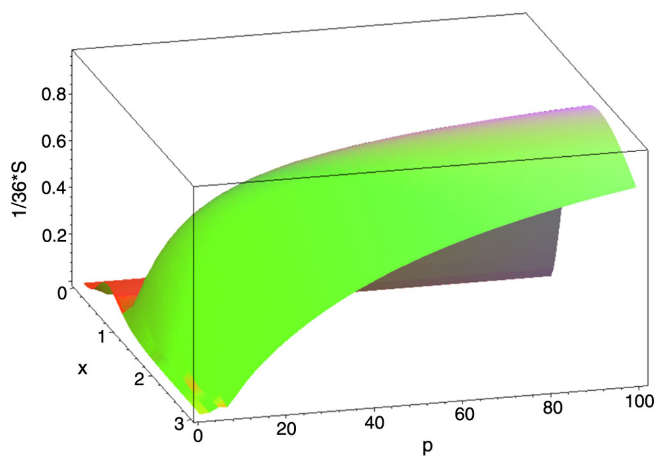

FIG. 7. Plots of $r_-^2\mathcal{S}/36$ as a function of x and p.

III. HIGHER-DIMENSIONAL NONSINGULAR BLACK HOLES

The metric (2.1), (2.31) allows a higher-dimensional generalization. Let us consider static, spherically symmetric $D = n + 1$-dimensional spacetime

$$dS^2 = -FdV^2 + 2dVdr + r^2d\omega_{n-1}^2, \tag{3.1}$$

where $F = F(r)$ is of the form

$$F = 1 - \frac{r_g^{n-2} r^2}{r^n + r_g^{n-2}\ell^2}. \tag{3.2}$$

For $\ell = 0$ this metric reproduces the Tangherlini solution of the Einstein equations. In the four-dimensional case ($n = 3$) this metric reduces to (2.1), (2.31) with $r_g = 2M$. At $r \to \infty$ and $r = 0$ one has

$$F = 1 - \left(\frac{r_g}{r}\right)^{n-2} + O(r^{-2(n-1)}), \tag{3.3}$$

$$F = 1 - \left(\frac{r}{\ell}\right)^2 + O(r^{n+2}). \tag{3.4}$$

Conditions $F(r*) = F'(r*) = 0$ determine the critical value of the gravitational radius

$$r_g^* = \left(\frac{n}{n-2}\right)^{1/2}\left(\frac{n}{2}\right)^{1/(n-2)}\ell. \tag{3.5}$$

For $r_g > r_g^*$ the metric (3.1)–(3.2) has two horizons, while for $r_g < r_g^*$ the horizons do not exist. For the four-dimensional spacetime, where $n = 3$, one reproduces the result of Sec. II. 2.

IV. CHARGED NONSINGULAR BLACK HOLE

A. Four-dimensional spacetime

Let us consider the metric (2.1) with

$$F = 1 - \frac{(2Mr - Q^2)r^2}{r^4 + (2Mr + Q^2)\ell^2}. \tag{4.1}$$

In the limit $\ell \to 0$ this metric reproduces the Reissner-Nordström metric. At $r \to \infty$ and $r = 0$ one has

$$F = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \ell^2 O(r^{-4}), \tag{4.2}$$

$$F = 1 + \frac{r^2}{\ell^2} + O(r^6). \tag{4.3}$$

The latter relation means that the corresponding metric is regular at the origin, where its curvature is of the order of ℓ^{-2}. Deflection of this metric from the Reissner-Nordström metric at a far distance is small, and it is controlled by the parameter ℓ^2.

Let us demonstrate that the metric (4.1) satisfies the limiting curvature property. For this purpose we rewrite F in the form

$$F = 1 - \frac{1}{\ell^2}\frac{(ar - z)r^2}{r^4 + ar + z}, \qquad a = 2M\ell^2, \qquad z = Q^2\ell^2. \tag{4.4}$$

The calculations give

$$R = \frac{\mathcal{A}_R}{\ell^2 N}, \qquad \mathcal{S} = \frac{\mathcal{A}_S}{\ell^2 N}, \qquad \mathcal{C} = \frac{\mathcal{A}_C}{\ell^2 N}, \tag{4.5}$$

$$\mathcal{A}_R = -6a^2r^6 + 12a^3r^3 + 20r^4z^2 + 24a^2r^2z + 4arz^2 - 12z^3, \tag{4.6}$$

$$\mathcal{A}_S = r(2r^7z + 9a^2r^5 + 6ar^4z - 14r^3z^2 - 2a^2rz - 4az^2), \tag{4.7}$$

$$\mathcal{A}_C = 2r(-3ar^8 + 6r^7z + 6a^2r^5 + 9ar^4z - 10r^3z^2 - 2az^2), \tag{4.8}$$

$$N = (r^4 + ar + z)^3. \tag{4.9}$$

We use a notation $\mathcal{A}_\bullet$ to denote any of the quantities $\mathcal{A}$ which enter the expressions for the curvature invariants (4.5)–(4.9). Each of $\mathcal{A}_\bullet$ contains three types of the terms: (i) Terms independent of charge; (ii) Terms independent of mass; and (iii) Terms which depend on both mass and charge. We show now that the contribution of each of these types to the curvature invariants obeys an inequality

$$\frac{|\mathcal{A}_\bullet|}{N} < c_\bullet, \tag{4.10}$$

where the constants $c_\bullet$ are independent of mass and charge.

For the case (i) one has

$$\frac{|\mathcal{A}_\bullet(z=0)|}{N} \le \frac{|\mathcal{A}_\bullet(z=0)|}{N(z=0)}. \tag{4.11}$$

But the expressions in the right-hand side coincide with similar expressions for the uncharged nonsingular black hole, (2.34), and for this reason they obey the property (4.10).

For the case (ii) we denote $r = z^{1/4}y$. Then one has

$$\frac{|\mathcal{A}_\bullet(a=0)|}{N} \le \frac{|\mathcal{A}_\bullet(a=0)|}{N(a=0)} = \frac{|P(y)|}{(y^4+1)^3}. \tag{4.12}$$

Here $P(y)$ is a polynomial of y of the power less than or equal to 8. Thus the inequality (4.10) is also valid for this contribution.

Let us focus now on the case (iii). Simple analysis shows that there are three types of contributions,

$$Q_1 = \frac{az^2r}{N}, \qquad Q_2 = \frac{a^2zr^2}{N}, \qquad Q_3 = \frac{azr^5}{N}. \tag{4.13}$$

Since $a \geq 0$ and $z \geq 0$, these functions have similar behavior. They are non-negative and vanish at $r = 0$ and $r \to \infty$. Putting equal to zero the derivatives of these objects with respect to r and solving the obtained relations with respect to z, one finds

$$z_1 = r(11r^3 + 2a), \qquad z_2 = r(5r^3 + a/2), \\ z_3 = \frac{r}{5}(7r^3 - 2a). \tag{4.14}$$

In the last case one should have $7r^3 - 2a > 0$. Under these conditions the second derivatives of Q_i with respect to r at the critical points are negative. Thus the functions Q_i have a maximum. Putting $r = a^{1/3}u$ one obtains that at the points of their maximum the values of Q_i are

$$\max(Q_1) = \frac{1}{27}\frac{(11u^3 + 2)^2}{(4u^3 + 1)^3}, \tag{4.15}$$

$$\max(Q_2) = \frac{4}{27}\frac{10u^3 + 1}{(4u^3 + 1)^3}, \tag{4.16}$$

$$\max(Q_3) = \frac{25}{27}\frac{(7u^3 - 2)u^3}{(4u^3 + 1)^3}. \tag{4.17}$$

This implies that these contributions satisfy (4.10). Thus we proved that the invariants $|R|$, $|\mathcal{S}|$, and $|\mathcal{C}|$ satisfy the limiting curvature condition.

To summarize, the metric (2.1) with the metric function (4.1) describes a nonsingular black hole. Its asymptotic at large r correctly reproduces the Reissner-Nordström metric, so that one can interpret this metric as a nonsingular version of the charged black hole. Certainly, one should assume that besides the gravitational field a system contains also the electromagnetic field, so that the metric (4.1) is a solution of a coupled system of Maxwell and modified gravity equations.

Let us find a relation between charge Q and mass M for which the outer and inner horizons coincide. It happens when

$$F = F' = 0, \tag{4.18}$$

where F is given by (4.1) and F' is

$$F' = -\frac{r(-ar^5 + 2a^2r^2 + 2arz + 2zr^4 - 2z^2)}{(r^4 + ar + z)^2}. \tag{4.19}$$

Solving equation $F = 0$ one finds

$$a = \frac{(\ell^2 r^4 + \ell^2 z + r^2 z)}{r(-\ell^2 + r^2)}. \tag{4.20}$$

Substituting (4.20) into the equation $F' = 0$ and solving it, one gets

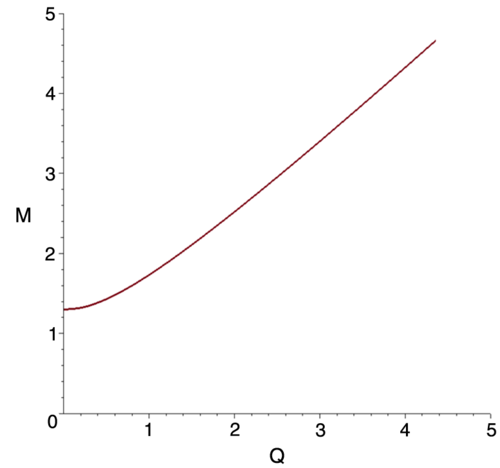

FIG. 8. Critical mass M/ℓ for the charged nonsingular black hole as a function of Q/ℓ.

$$z = \frac{\ell^2 r^4 (r^2 - 3\ell^2)}{-\ell^4 + r^4 + 4\ell^2 r^2}. \tag{4.21}$$

And finally substituting (4.21), one finds

$$a = \frac{2\ell^2(r^2 + 2\ell^2)r^3}{-\ell^4 + r^4 + 4\ell^2 r^2}. \tag{4.22}$$

Relations (4.21) and (4.22) determined the relation between mass and charge, written in the parametric form. Since both of the quantities a and z should be positive, one has $r \geq \sqrt{3}$.

The plot presented in Fig. 8 shows critical mass M as a function of charge Q. For small Q one has

$$M = r_g/2 \sim \frac{3\sqrt{3}}{4}\ell + \frac{1}{\sqrt{3}}\frac{Q^2}{\ell} + O(Q^4). \tag{4.23}$$

B. Higher dimensional generalization

By comparing (3.2) with (4.1) one can obtain the following higher dimensional version of the charged nonsingular black hole. Namely, one uses the form of the metric (3.1) with the metric function of the form

$$F = 1 - \frac{(r_g^{n-2}r^{n-2} - Q^{2(n-2)})r^2}{r^{2(n-2)} + \ell^2(r_g^{n-1}r^{n-2} + Q^{2(n-2)})}. \tag{4.24}$$

This metric in the limit $\ell \to 0$ correctly reproduces the higher dimensional Reissner-Nordström metric

$$F = 1 - \left(\frac{r_g}{r}\right)^{n-2} + \left(\frac{Q}{r}\right)^{2(n-2)} + \ell^2 O(r^{2(n-1)}). \tag{4.25}$$

It is regular at the origin $r = 0$

$$F = 1 + \frac{r^2}{\ell^2} + \cdots. \quad (4.26)$$

V. DISCUSSION

In the present paper we discussed nonsingular black hole metrics. We restrict ourselves by metrics which are spherically symmetric and static. Besides natural assumptions of regularity at the center $r = 0$ and proper asymptotic behavior at infinity, we required that the corresponding metric also satisfies the limiting curvature condition. The latter condition is rather restrictive and narrows the class of feasible models. The metric proposed by Hayward [54] is an important example of the metric for a neutral black hole satisfying these conditions. However, this metric has a property which makes it problematic for a self-consistent description of the evaporating black hole. Outgoing field modes propagating near the inner horizon are accumulating near it and experience huge blueshift [58]. One can expect that this effect for a quantum field results in the quantum emission of energy [68]

$$\Delta E \sim \exp(-\kappa_- T_{\mathrm{BH}}). \quad (5.1)$$

Here κ_- is the (negative) surface gravity at the inner horizon, which for the Hayward model is of the order of ℓ^{-1}, and $T_{\mathrm{BH}} \sim M^3$ is the lifetime of the evaporating black hole. For a self-consistent model of an evaporating black hole one should expect $\Delta E < M$. The expression (5.1) for $M \gg \ell$ does not satisfy this restriction. This indicates that there exists a severe self-consistency problem when one tries to apply the Hayward model to "realistic" quantum black holes.

In the present paper we proposed a class of metrics, which may help to solve this problem. Namely, we used a modification of the metrics with a nontrivial redshift function $A(r)$. We demonstrate that this function can be chosen so that the surface gravity $|\kappa_-|$ becomes sufficiently small, so that ΔE, estimated as in (5.1), can be made rather small.

We also presented a nonsingular model for a charged black hole, which obeys the limiting curvature condition. We briefly discussed higher dimensional versions of such nonsingular black holes. It would be interesting to discuss the application of the presented nonsingular metrics for study of the self-consistent models of evaporating black holes. They also might be interesting for a discussion of the information loss paradox. We hope to address these problems in our future work.

ACKNOWLEDGMENTS

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