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Hamiltonian formulation of teleparallel gravity

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The Hamiltonian formulation of the teleparallel equivalent of general relativity is developed from an ordinary second-order Lagrangian, which is written as a quadratic form of the coefficients of anholonomy of the orthonormal frames (vielbeins). We analyze the structure of eigenvalues of the multi-index matrix entering the (linear) relation between canonical velocities and momenta to obtain the set of primary constraints. The canonical Hamiltonian is then built with the Moore-Penrose pseudoinverse of that matrix. The set of constraints, including the subsequent secondary constraints, completes a first-class algebra. This means that all of them generate gauge transformations. The gauge freedoms are basically the diffeomorphisms and the (local) Lorentz transformations of the vielbein. In particular, the Arnowitt, Deser, and Misner algebra of general relativity is recovered as a subalgebra.

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I. INTRODUCTION

The determination of the independent dynamical degrees of freedom is of the utmost importance in any field theory, since it allows us to exhibit the internal consistency of the theory and tackle the issue of the well-posedness of the Cauchy problem. It also puts the theory into a different perspective, because it helps us to find the minimal number of variables specifying the state of the system, is thus vital for the quantization of the theory. According to Dirac's procedure [1], the number of genuine degrees of freedom can be determined from the algebra of the constraints among the canonical variables of the theory. The constraints first appear when the canonical momenta are computed. These *primary* constraints have to be consistent with the Hamiltonian evolution of the system, which leads to *secondary* constraints, and so on. Finally, the set of all the constraints is reclassified as first-class and second-class constraints, depending whether their Poisson brackets are or are not zero on the constraint surface in the phase space. First-class constraints generate gauge transformations; thus, each of them is related to a spurious degree of freedom. On the other hand, second-class constraints can be reorganized as pairs of spurious conjugated variables. Thus, the number of genuine degrees of freedom can be computed as

$$\begin{aligned}\text{Number of d.o.f.} = {} & \text{Number of pairs of canonical variables} \\ & - \text{Number of first class constraints} \\ & - \frac{1}{2}\text{Number of second class constraints.}\end{aligned} \tag{1}$$

A nice example is the Maxwell potential, described by four dynamical variables A_μ that are governed by the Lagrangian $L[A_\mu] \propto F_{\lambda\rho}F^{\lambda\rho}$ (the field tensor $F_{\lambda\rho}$ is $F_{\lambda\rho} = \partial_\lambda A_\rho - \partial_\rho A_\lambda$). Since $F_{\lambda\rho}$ is antisymmetric, $\partial_0 A_0$ is not present in the Lagrangian. Thus the canonical momentum $\pi^0 = \partial L/\partial(\partial_0 A_0)$ identically vanishes; it is a primary constraint. The consistency of the constraint $\pi^0 = 0$ with the evolution requires the vanishing of the Poisson bracket between π^0 and the Hamiltonian; this leads to the secondary constraint $\nabla_i \pi^i \propto \nabla_i F^{0i} = 0$ (Gauss's law). Both constraints are first class, since the Poisson brackets between canonical momenta are identically zero. Therefore, according to Eq. (1), one realizes that the electromagnetic field does not have four degrees of freedom A_μ at each event, but only two (electromagnetic waves are transversal). At the level of the initial data, the existence of constraints imply a restriction on the spectrum of allowed initial configurations. Besides, the absence of kinetic term for A_0 in the Lagrangian implies that the evolution of this dynamical variable, conjugate to the first-class constraint π^0, remains completely undetermined. The same happens to the evolution of the longitudinal component of the potential $A_\parallel$, which also remains undetermined as a consequence of the existence of the first-class constraint $\nabla_i \pi^i$. Thus, A_0 and $A_\parallel$ are gauge freedoms. The former conclusions can also be derived from a slightly modified Lagrangian. The integration by parts of one of the terms containing $\partial_i A_0$ leads to a surface term, which can be eliminated, plus the term $A_0 \nabla_i F^{0i}$. In such a way, the spurious degree of freedom A_0 becomes a Lagrange multiplier whose variation leads to the Gauss's law constraint (any other presence of A_0 is captured in the canonical momenta π^i) [2].

The canonical formulation of general relativity (GR) relies on the widely spread formalism by Arnowitt, Deser,

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and Misner (ADM) [3], in which the spacetime is foliated into a family of spacelike hypersurfaces that induces a proper decomposition of the metric tensor $g_{\mu\nu}$. The Einstein-Hilbert Lagrangian can be integrated by parts to realize that the temporal sector of the metric (the lapse N and the shift vector N_i) is thrown into the role of Lagrange multipliers associated to four first-class constraints (the super-Hamiltonian and supermomenta constraints). So written, the Lagrangian gives dynamics only to the six components of the three-dimensional metric g_{ij} on the spacelike hypersurfaces of the foliation; however, the canonical variables (g_{ij}, π^{ij}) are still constrained by the four first-class constraints. Thus the gravitational field contains only two genuine degrees of freedom. In fact, apart from the undetermined evolutions of the four Lagrange multipliers (N, N_i), there are also four gauge freedoms among the six components of g_{ij} (gravitational waves are transversal and traceless). As a feature that distinguishes GR from electromagnetism, the GR Hamiltonian vanishes because of the constraints. This feature is typical of systems having the time hidden among their canonical variables [4].

In 1918, Weyl's unsuccessful attempt to unify gravitation and electromagnetism introduced for the first time the notion of gauge theories [5]. Einstein tried the same unification idea ten years later, taking advantage of the sixteen components of the tetrad field in order to include the electromagnetic field [6]. Later he realized that the arbitrariness in the choice of the tetrad comes from the set of local Lorentz transformations that leave the metric unchanged; therefore, the extra degrees of freedom could not account for the electromagnetism. However, he introduced the concept of teleparallelism that remains important today, presenting for the first time the teleparallel equivalent of general relativity (TEGR), an equivalent formulation of general relativity. In fact, although both theories have different Lagrangian formulations, they are equivalent at the level of the equations of motion. Nonetheless, they are based on completely different Lagrangian constructions. This is so because TEGR describes gravity as the effect of torsion in the curvatureless Weitzenböck geometry; the dynamical variables are not the components of the metric $g_{\mu\nu}$ but those of the field of orthonormal frames—tetrads or vierbeins—e^a_μ [a and μ are $SO(3,1)$ and coordinate indices, respectively] [6,7]. As a consequence, the Hamiltonian formalisms of GR and TEGR are different too. Among the works treating the Hamiltonian formulation of TEGR we specially mention Ref. [8], which introduces a set of auxiliary variables in a first-order approach that lowers the order of the Euler-Lagrange equations (cf. [9–12]), and Ref. [13], which deals with an enlarged set of variables and constraints to enforce the vanishing of the curvature. The canonical formulation of TEGR has been also stated in the geometric language of differential forms [14,15].

In this work we will put forward the Hamiltonian formalism for TEGR in a way as close as possible to the second-order formalism of electrodynamics that was sketched above. This work is organized as follows: in Sec. II we introduce the standard TEGR dynamics, which is governed by a Lagrangian quadratic in the torsion. In Sec. III we show that the TEGR Lagrangian can be reformulated as the quadratic inner product of the anholonomy coefficients with respect to a supermetric that is defined in the tangent space. In Sec. IV we obtain the set of primary and secondary constraints that are equivalent to those of electrodynamics and GR geometrodynamics. In Sec. V we study the gauge transformations generated by these constraints (they will prove to be first class). Compared with geometrodynamics, TEGR has an additional gauge symmetry associated to local Lorentz transformations of frames, which is the source of the constraints analyzed in Sec. VI. In Sec. VII the (constrained) linear relations between canonical momenta and velocities is inverted to build the canonical TEGR Hamiltonian $\mathcal{H}$; the procedure implies a careful analysis of the eigenvector structure involved in these linear relations in order to build the respective pseudoinverse matrix. The entire set of $n(n+3)/2$ constraints (n is the spacetime dimension) is consistent with the evolution governed by $\mathcal{H}$; they are first class as proven by the algebra of constraints computed in Sec. VIII. In Sec. IX we summarize the main steps and the achievements of the paper. The Appendix shows some useful computations that are needed throughout the work.

II. TEGR AND STANDARD LAGRANGIAN FORMULATION

TEGR is a theory of gravity where the field of orthonormal frames plays the role of dynamical variable. Let M be a manifold, $\{\mathbf{e}_a\}$ a basis in the tangent space $T_p(M)$, and $\{\mathbf{E}^a\}$ its dual basis in the cotangent space $T^*_p(M)$ [i.e., if the 1-forms $\mathbf{E}^a$ are applied to the vectors $\mathbf{e}_b$ one obtains $\mathbf{E}^a(\mathbf{e}_b) = \delta^a_b$]. They can be expanded in a coordinate basis as $\mathbf{e}_a = e^\mu_a \partial_\mu$ and $\mathbf{E}^a = E^a_\mu dx^\mu$; thus, duality means that

$$E^a_\mu e^\mu_b = \delta^a_b, \qquad e^\mu_a E^a_\nu = \delta^\mu_\nu. \tag{2}$$

Here and from now on, we will use Greek letters $\mu, \nu, \ldots = 0, \ldots, n-1$ for spacetime coordinate indices and Latin letters $a, b, \ldots, g, h = 0, \ldots, n-1$ for Lorentzian tangent space indices. A vielbein (vierbein or tetrad in $n = 4$ dimensions) is a basis encoding the metric structure of the spacetime,

$$\mathbf{g} = \eta_{ab} \mathbf{E}^a \otimes \mathbf{E}^b, \tag{3}$$

therefore,

$$\mathbf{E}^a \cdot \mathbf{E}^b = \mathbf{g}(\mathbf{E}^a, \mathbf{E}^b) = \eta_{ab}, \tag{4}$$

which means that the vielbein is an orthonormal basis. In component notation, the former expressions look like

$$g_{\mu\nu} = \eta_{ab} E^a_\mu E^b_\nu, \qquad \eta_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b, \tag{5}$$

which implies that the relation between the metric volume and the determinant of the matrix E^a_μ is

$$\sqrt{|g|} = \det[E^a_\mu] \doteq E. \tag{6}$$

Since the vielbein encodes the metric structure of the spacetime, one can formulate a dynamical theory of the spacetime geometry by defining a Lagrangian for the vielbein field. In particular, there is a Lagrangian which leads to dynamical equations for the vielbeins that are equivalent to Einstein equations for the metric [16]. The so-called teleparallel equivalent of general relativity is governed by the Lagrangian density

$$L = ET, \tag{7}$$

where T is the torsion scalar

$$T \doteq T^\rho{}_{\mu\nu} S_\rho{}^{\mu\nu}, \tag{8}$$

which is made up of

$$T^\mu{}_{\nu\rho} \doteq e_a{}^\mu (\partial_\nu E^a_\rho - \partial_\rho E^a_\nu), \tag{9}$$

and

$$S_\rho{}^{\mu\nu} \doteq \frac{1}{2}(K^{\mu\nu}{}_\rho + T_\lambda{}^{\lambda\mu}\delta^\nu_\rho - T_\lambda{}^{\lambda\nu}\delta^\mu_\rho), \tag{10}$$

where

$$K^{\mu\nu}{}_\rho \doteq \frac{1}{2}(T_\rho{}^{\mu\nu} - T^{\mu\nu}{}_\rho + T^{\nu\mu}{}_\rho). \tag{11}$$

In Lagrangian (7), the strength field $T^\mu{}_{\nu\rho}$ is the torsion associated with the Weitzenböck connection $\Gamma^\mu_{\nu\rho} \doteq e_a{}^\mu \partial_\nu E^a_\rho$, and $K^{\mu\nu}{}_\rho$ is the contorsion [17,18]. In geometric language, torsion is the 2-form $\mathbf{T}^a \doteq d\mathbf{E}^a + \omega^a{}_b \wedge \mathbf{E}^b$, where the 1-form $\omega^a{}_b$ is the spin connection. The Weitzenböck connection is the choice $\omega^a{}_b = 0$, because it leads to $(\mathbf{T}^a)_{\nu\rho} = (d\mathbf{E}^a)_{\nu\rho} = \partial_\nu E^a_\rho - \partial_\rho E^a_\nu = E^a_\mu T^\mu{}_{\nu\rho}$. The Weitzenböck connection is metric compatible, since $\nabla_\nu E^a_\mu = \partial_\nu E^a_\mu - \Gamma^\lambda_{\nu\mu} E^a_\lambda = 0$. Besides, from Eq. (2) we also get that $\nabla_\nu e^\mu_a = 0$. This means that the vielbein is automatically parallel transported along any curve. Furthermore, the parallel transport of any vector does not depend on the path (it is *absolute*), since Weitzenböck connection has the remarkable feature that the curvature $\mathbf{R}^a{}_b \doteq d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ is identically zero. The (Weitzenböck) covariant derivative of a vector is $\nabla_\nu \mathbf{U} = \nabla_\nu (U^a \mathbf{e}_a) = \mathbf{e}_a \partial_\nu U^a$; thus, vector $\mathbf{U}$ will be parallel transported if and only if its components U^a are constant.

Although TEGR Lagrangian can be understood in terms of the Weitzenböck connection and its respective torsion, it should be emphasized that the TEGR Lagrangian fixes neither the connection nor the vielbein; it only determines the metric, as is well known. Furthermore, whenever matter couples minimally to the metric, as usual, the free particles will follow geodesics of the (torsionless) Levi-Civita connection $\bar{\Gamma}^\mu_{\nu\rho}$.[1] Setting aside this point, we used to say that TEGR is a theory where the gravitational effects are fully encoded in the torsion. On the contrary, GR associates gravity to curvature; it assumes that the spacetime is endowed with the torsionless Levi-Civita connection, whose curvature enters the Einstein-Hilbert Lagrangian $L = E\bar{R}$. The reason why TEGR is indeed equivalent to GR is traced to the fact that their respective Lagrangian densities differ in a surface term,

$$-E\bar{R} = ET - 2\partial_\rho (ET_\mu{}^{\mu\rho}). \tag{12}$$

Even so, the vielbein field contains n^2 components, while the metric tensor has only $n(n+1)/2$. However, TEGR dynamical equations are invariant under local Lorentz transformations of the vielbein, which involve $\binom{n}{2}$ generators. Such a gauge invariance means that $\binom{n}{2} = n(n-1)/2$ degrees of freedom cancels out, which allows for the theories to turn out to be equivalent at the level of the equations of motion.

III. TEGR LAGRANGIAN IN TERMS OF THE VIELBEIN FIELD

With the aim of preparing the TEGR Lagrangian for the study of its canonical structure, we will rewrite it completely in terms of e^μ_a, E^a_ν and the derivatives $\partial_\mu E^a_\nu$. This imply the removing of any presence of the metric field, since such contributions hide a dependence on the vielbein. We transform the scalar torsion into

$$T = \frac{1}{4} T_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu} - \frac{1}{2} T^\rho{}_{\mu\nu} T^{\mu\nu}{}_\rho - T^\rho{}_{\mu\rho} T^{\nu\mu}{}_\nu. \tag{13}$$

We note that all terms in T are quadratic in the antisymmetrized derivatives of the vielbein; writing term by term one gets

$$\frac{1}{4} T_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu} = \frac{1}{4} g_{\rho\alpha} g^{\beta\mu} g^{\gamma\nu} T^\alpha{}_{\beta\gamma} T^\rho{}_{\mu\nu}. \tag{14}$$

Then, one substitutes the expression for the torsion tensor [Eq. (9)] and the metric in terms of the vielbein field and its inverse [Eq. (5)],

[1] However, Levi-Civita and Weitzenböck connections are related through the contorsion $\bar{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - K^\mu{}_{\nu\rho}$.

$$\frac{1}{4}T_{\rho}{}^{\mu\nu}T^{\rho}{}_{\mu\nu} = \eta_{ab}\eta^{c[d}\eta^{f]e}E\partial_{\mu}E^{a}{}_{\nu}\partial_{\rho}E^{b}{}_{\lambda}e^{\mu}_{c}e^{\nu}_{e}e^{\rho}_{d}e^{\lambda}_{f}. \quad (15)$$

After this procedure has been performed in all the terms, the TEGR Lagrangian becomes

$$L = ET = \frac{1}{2}E\partial_{\mu}E^{a}{}_{\nu}\partial_{\rho}E^{b}{}_{\lambda}e^{\mu}{}_{c}e^{\nu}_{e}e^{\rho}_{d}e^{\lambda}_{f}M_{ab}{}^{cedf}, \quad (16)$$

where we call supermetric $M_{ab}{}^{cedf}$ the emerging Lorentz invariant tensor given by

$$M_{ab}{}^{cedf} \doteq 2\eta_{ab}\eta^{c[d}\eta^{f]e} - 4\delta_{a}^{[d}\eta^{f][c}\delta_{b}^{e]} + 8\delta_{a}^{[c}\eta^{e][d}\delta_{b}^{f]}. \quad (17)$$

The supermetric is antisymmetric in the pairs of indices $c-e$ and $d-f$, which implies that only the antisymmetric parts of $\partial_{\mu}E^{\alpha}{}_{\nu}$ and $\partial_{\lambda}E^{b}{}_{\rho}$ take part in the Lagrangian (16). Other properties of the supermetric are summarized in the Appendix.

We remark that the index structure of the supermetric is natural when we recognize in Eq. (16) the anholonomy coefficients f^{c}_{ab}, which are defined by the commutator $[\mathbf{e}_a, \mathbf{e}_b] = f^{c}_{ab}\mathbf{e}_c$. In fact, by using Eqs. (2) the coefficients f^{a}_{bc} can be rewritten as

$$f^{a}_{bc} = -e^{\mu}_{b}e^{\nu}_{c}(\partial_{\mu}E^{a}_{\nu} - \partial_{\nu}E^{a}_{\mu}) = -2e^{\mu}_{b}e^{\nu}_{c}\partial_{[\mu}E^{a}_{\nu]}, \quad (18)$$

which can be related to other geometrical magnitudes, such as the Weitzenböck torsion and the Lie derivative of the vielbein,

$$f^{a}_{bc} = \mathbf{T}^{a}(\mathbf{e}_c, \mathbf{e}_b) = (\mathcal{L}_{\mathbf{e}_c}\mathbf{E}^{a})(\mathbf{e}_b). \quad (19)$$

In terms of these coefficients, the Lagrangian density looks in a very elegant form,

$$L = \frac{1}{8}Ef^{a}_{ce}f^{b}_{df}M_{ab}{}^{cedf}. \quad (20)$$

A similar expression for the Lagrangian can be found in Ref. [19], where the anholonomy coefficients are identified with a Yang-Mills-like field strength; however, that Lagrangian still mixed tangent space and coordinate indices. Instead, Lagrangian (20) does not involve coordinate indices; it shows that supermetric $M_{ab}{}^{cedf}$ is a relevant geometric object in the (co)tangent space structure of the spacetime. We intend to analyze the Hamiltonian structure of TEGR by starting from Lagrangians (16) and (20), and then following a canonical second-order procedure.

IV. SUPER-HAMILTONIAN AND SUPERMOMENTA CONSTRAINTS

We compute the canonical momenta by differentiating the Lagrangian (16) with respect to the time derivative of the canonical variable E^{a}_{μ},

$$\Pi^{\mu}_{a} = \frac{\partial L}{\partial(\partial_0 E^{a}_{\mu})} = E\partial_{\rho}E^{b}{}_{\lambda}e^{0}_{c}e^{\mu}_{e}e^{\rho}_{d}e^{\lambda}_{f}M_{ab}{}^{cedf}$$
$$= -\frac{1}{2}Ee^{0}_{c}e^{\mu}_{e}f^{b}_{df}M_{ab}{}^{cedf}. \quad (21)$$

Thus, the Poisson brackets in TEGR are defined as

$$\{A(t,\mathbf{x}), B(t,\mathbf{y})\} \doteq \int d\mathbf{z}\left(\frac{\delta A(t,\mathbf{x})}{\delta E^{a}_{\lambda}(\mathbf{z})}\frac{\delta B(t,\mathbf{y})}{\delta\Pi^{\lambda}_{a}(\mathbf{z})} - \frac{\delta A(t,\mathbf{x})}{\delta\Pi^{\lambda}_{a}(\mathbf{z})}\frac{\delta B(t,\mathbf{y})}{\delta E^{a}_{\lambda}(\mathbf{z})}\right). \quad (22)$$

The brackets between fundamental canonical variables are

$$\{E^{a}_{\mu}(t,\mathbf{x}), \Pi^{\nu}_{b}(t,\mathbf{y})\} = \delta^{a}_{b}\delta^{\nu}_{\mu}\delta(\mathbf{x}-\mathbf{y}). \quad (23)$$

Additional fundamental Poisson brackets, including E, e^{μ}_{a}, etc., are summarized in the Appendix.

From Eq. (21) we immediately get n trivial primary constraints

$$G^{(1)}_{a} \doteq \Pi^{0}_{a} \equiv 0, \quad (24)$$

which are derived by noticing that $e^{0}_{c}e^{0}_{e}$ is symmetric in $c-e$ but $M_{ab}{}^{cedf}$ is antisymmetric. Although we cannot prove yet that they are first class (i.e., we do not know yet whether they generate gauge transformations), the electromagnetic analogue tells us that they mean the E^{a}_{0}'s are spurious gauge-dependent variables, that would become Lagrange multipliers if an integration by parts were performed in the action. This is in line with the spurious character of the temporal sector of the metric tensor we have commented on in Sec. I.

The primary constraints must be satisfied at any time. In other words, if the system is on the constraint surface at the initial time, it must remain there along the evolution. If this consistency requirement were not fulfilled, then it could be enforced by resorting to new (secondary) constraints [20]. From a Hamiltonian perspective, the consistency of the primary constraints is controlled by means of the primary Hamiltonian [2]

$$H_p = H + \int d\mathbf{x}\, u^{a}(t,\mathbf{x})\phi^{(1)}_{a}(t,\mathbf{x}), \quad (25)$$

where H is the canonical Hamiltonian, $u^{a}(t,\mathbf{x})$ are arbitrary functions, and $\phi^{(1)}_{a}$ are *all* the primary constraints. The consistency will be fulfilled if the Poisson brackets $\{\phi^{(1)}_{a}, H_p\}$ are zero on the constraint surface. This requirement could be satisfied by properly choosing the functions $u^{a}(t,\mathbf{x})$; if not, new (secondary) constraints will be needed to enforce it, and so on. Actually, in TEGR we will find that all the Poisson brackets between constraints are zero on the constraint surface. This means that primary and secondary constraints are all first class; they generate gauge transformations. Thus the constraints will be consistent with the evolution if their Poisson brackets with H vanish on the constraint surface (i.e., if H is gauge invariant, as it should be expected).

Though the entire set of primary constraints has not been obtained yet, the evolution of constraints (24) can be analyzed at the level of the Euler-Lagrange evolution equations,

$$\partial_\mu \frac{\partial L}{\partial(\partial_\mu E^a_\nu)} - \frac{\partial L}{\partial E^a_\nu} = 0. \tag{26}$$

By splitting the first term, one gets

$$\partial_0 \Pi^\nu_a + \partial_i \frac{\partial L}{\partial(\partial_i E^a_\nu)} - \frac{\partial L}{\partial E^a_\nu} = 0. \tag{27}$$

Therefore, if the constraints (24) must be fulfilled at any time, we obtain n equations—those having $\nu = 0$—which do not contain second-order temporal derivatives,

$$\partial_i \frac{\partial L}{\partial(\partial_i E^a_0)} - \frac{\partial L}{\partial E^a_0} = 0. \tag{28}$$

Like the Gauss's law in electromagnetism these equations do not contain dynamics, but they constrain the dynamics. Since the derivatives of the vielbein enter the Lagrangian only in antisymmetric combinations, then it is

$$\partial_i \frac{\partial L}{\partial(\partial_i E^a_0)} = -\partial_i \frac{\partial L}{\partial(\partial_0 E^a_i)} = -\partial_i \Pi^i_a. \tag{29}$$

Thus, we have found n secondary constraints,

$$\partial_i \Pi^i_a + \frac{\partial L}{\partial E^a_0} = 0. \tag{30}$$

We will prove that these constraints are consistent with the evolution; thus, they do not generate new constraints. For this, we will apply the derivative ∂_0 to the constraints (30), and use Eq. (27) to substitute $\partial_0 \Pi^i_a$,

$$\begin{aligned}\partial_0\left(\partial_i \Pi^i_a + \frac{\partial L}{\partial E^a_0}\right) &= -\partial_i\partial_j \frac{\partial L}{\partial(\partial_j E^a_i)} + \partial_\mu \frac{\partial L}{\partial E^a_\mu} \\ &= -\partial_i\partial_j \frac{\partial L}{\partial(\partial_j E^a_i)} + \partial_\nu\partial_\mu \frac{\partial L}{\partial(\partial_\mu E^a_\nu)} \equiv 0\end{aligned} \tag{31}$$

[we also use Eq. (26) in the last step]. The zero result comes from the fact that $\partial_\mu E^a_\nu$ enters the Lagrangian in antisymmetric combinations but the operators $\partial_i\partial_j$ and $\partial_\nu\partial_\mu$ are symmetric.

So far, we have obtained a set of constraints which is consistent with the evolution. To write them in a fully canonical way, we have to compute the derivative $\partial L/\partial E^a_0$ and express it as a function of the momenta, the vielbein and its spatial derivatives. This computation is made in the Appendix, where we obtain that the canonical Hamiltonian density $\mathcal{H}$ (i.e., $H = \int d\mathbf{x}\mathcal{H}$) takes part in the results. These results are better understood when projected on E^a_0 and E^a_k. Thus, we get the secondary constraints written in canonical form,

$$G^{(2)}_0 \doteq \mathcal{H} - \partial_i(E^c_0 \Pi^i_c) \approx 0, \tag{32}$$

$$G^{(2)}_k \doteq \partial_k E^c_i \Pi^i_c - \partial_i(E^c_k \Pi^i_c) \approx 0 \tag{33}$$

(the symbol $\approx$ stands for equalities that are valid on the constraint surface). The constraints (32) and (33) are equivalent to the super-Hamiltonian and supermomenta constraints of the ADM formalism. While the ADM Hamiltonian vanishes on the constraint surface, the TEGR Hamiltonian does not. The reason can be traced to the surface term in Eq. (12); in fact, according to Eq. (32), $\mathcal{H}$ is not zero but a divergence [which became a spatial divergence thanks to the constraints (24)].

V. GAUGE TRANSFORMATIONS

We have already anticipated—although have not yet proven—that all the constraints will be first class. Let us now consider the gauge transformations generated by $G^{(1)}_a$ and $G^{(2)}_\mu$. In general, the infinitesimal gauge transformation generated by a first-class constraint G is [2]

$$\delta E^a_\mu(t,\mathbf{x}) = \int d\mathbf{y}\epsilon(t,\mathbf{y})\{E^a_\mu(t,\mathbf{x}), G(t,\mathbf{y})\}. \tag{34}$$

Any transformation of the vielbein has to be accompanied by a transformation of the basis $\{\mathbf{e}_a\}$, in order to respect the duality relations $\mathbf{E}^a(\mathbf{e}_b) = \delta^a_b$ of Eq. (2). Therefore

$$\mathbf{E}^a(\delta\mathbf{e}_b) + \delta\mathbf{E}^a(\mathbf{e}_b) = 0, \tag{35}$$

or

$$\delta e^\nu_b = -e^\nu_a e^\mu_b \delta E^a_\mu. \tag{36}$$

According to Eq. (34), any linear combination of primary constraints $\epsilon^b(t,\mathbf{x})G^{(1)}_b$ generates a transformation that only affects the temporal component of the 1-forms $\mathbf{E}^a$

$$\delta E^a_0(t,\mathbf{x}) = \epsilon^a(t,\mathbf{x}), \tag{37}$$

(or $\delta\mathbf{E}^a = \epsilon^a dt$), which also implies[2]

$$\delta e^\nu_b = -\epsilon^a e^\nu_a e^0_b. \tag{38}$$

Instead, the transformations generated by $G^{(2)}_0$, $G^{(2)}_k$ only affect the spatial components of the forms $\mathbf{E}^a$ (the canonical Hamiltonian density $\mathcal{H}$ does not contain Π^0_a). Then, the infinitesimal gauge transformations generated by $G^{(2)}_0$ and any arbitrary combination $\xi^k G^{(2)}_k$ are respectively

$$\begin{aligned}\delta E^a_i(t,\mathbf{x}) &= \xi \dot{E}^a_i(t,\mathbf{x}) + E^a_0 \partial_i \xi \\ &= \partial_i(E^a_0 \xi) + \xi 2\partial_{[0} E^a_{i]},\end{aligned} \tag{39}$$

$$\begin{aligned}\delta E^a_i(t,\mathbf{x}) &= \xi^k \partial_k E^a_i + E^a_k \partial_i \xi^k \\ &= \partial_i(E^a_k \xi^k) + \xi^k 2\partial_{[k} E^a_{i]}.\end{aligned} \tag{40}$$

In these results there is a term resembling the gauge transformation of the electromagnetic potential. However, they come together with a term related to the Weitzenböck torsion $\mathbf{T}^a = d\mathbf{E}^a$. Both terms are needed because, differing from the electromagnetic Lagrangian, TEGR Lagrangian depends not only on the exterior derivative of the field $\mathbf{E}^a$ but on the field itself. Even so, the whole result exhibits a clear geometric content, which can be evidenced by means of the Lie derivative of a p-form α along a vector ξ,

$$\mathcal{L}_\xi \alpha = d[\alpha(\xi)] + d\alpha(\xi). \tag{41}$$

In fact, the rhs of Eqs. (39) and (40) constitute the spatial components of $\mathcal{L}_\xi \mathbf{E}^a$, where ξ is the arbitrary vector field formed by the infinitesimal parameters $\xi(t,\mathbf{x})$, $\xi^k(t,\mathbf{x})$. We notice that Eqs. (39) and (40) can be extended to the temporal component of the 1-forms $\mathbf{E}^a$, since any change of E^a_0 is a gauge transformation. Therefore, we have obtained that TEGR is insensitive to $2n$ independent gauge transformations of the vielbein on the constraint surface, which are given by Eq. (37) and

$$\delta\mathbf{E}^a = \mathcal{L}_\xi \mathbf{E}^a. \tag{42}$$

The derivative character of transformation (42) together with Eq. (35) imply that

$$\delta\mathbf{e}_b = \mathcal{L}_\xi \mathbf{e}_b = [\xi, \mathbf{e}_b]. \tag{43}$$

In turn, this last transformation leads to a change of the anholonomy coefficients,

$$\delta f^c_{ab} = \mathcal{L}_\xi f^c_{ab} = \xi(f^c_{ab}), \tag{44}$$

as can be easily verified by using the Jacobi identity to compute $\delta[\mathbf{e}_a, \mathbf{e}_b] = [\delta\mathbf{e}_a, \mathbf{e}_b] + [\mathbf{e}_a, \delta\mathbf{e}_b]$.

We remark that the Lie derivative of any Lagrangian—understood as the n-form $\mathbf{L} = L dx^0 \wedge \ldots \wedge dx^{n-1}$, where L is the Lagrangian density—is always a boundary term. In fact, if α is a n-form in Eq. (41), then its Lie derivative $\mathcal{L}_\xi \alpha$ is the exact form $d[\alpha(\xi)]$. But in a theory of gravity, like TEGR, this kind of (quasi-)invariance of the Lagrangian comes from a symmetry of its dynamical variables generated by a proper combination of the trivial primary constraints and the secondary ones. In fact, the change of the TEGR Lagrangian n-form,

$$\begin{aligned}\mathbf{L} &= \frac{1}{8} E f^a_{ce} f^b_{df} M_{ab}{}^{cedf} dx^0 \wedge \ldots \wedge dx^{n-1} \\ &= \frac{1}{8} f^a_{ce} f^b_{df} M_{ab}{}^{cedf} \mathbf{E}^0 \wedge \ldots \wedge \mathbf{E}^{n-1},\end{aligned} \tag{45}$$

(we used that the vielbein is orthonormal to rewrite the volume) under the gauge transformation (42) is equal to its Lie derivative by virtue of Eqs. (42) and (44),

$$\begin{aligned}\delta\mathbf{L} &= \frac{1}{4} \delta f^a_{ce} f^b_{df} M_{ab}{}^{cedf} \mathbf{E}^0 \wedge \ldots \wedge \mathbf{E}^{n-1} \\ &\quad + \frac{1}{8} f^a_{ce} f^b_{df} M_{ab}{}^{cedf} \delta\mathbf{E}^0 \wedge \ldots \wedge \mathbf{E}^{n-1} + \cdots \\ &= \mathcal{L}_\xi \mathbf{L} = d[\mathbf{L}(\xi)].\end{aligned} \tag{46}$$

VI. MORE PRIMARY CONSTRAINTS: THE LORENTZ GAUGE GROUP

So far we have found the $2n$ constraints that reflect the constraint structure of the ADM formulation of general relativity. However, TEGR describes the $n(n+1)$ components of the metric tensor through a $n \times n$ matrix E^a_μ. The relation between both sets of dynamical variables is given by the Eq. (5), which is invariant under local Lorentz transformations of the vielbein. Since we know that TEGR has dynamics only for the metric, as is clear from the equivalence between TEGR and Einstein-Hilbert Lagrangians expressed in Eq. (12), the local Lorentz symmetry has to be a property not only of the relation (5) but also the set of dynamical equations. Then, we should find that Lorentz transformations in the tangent space constitute a gauge group in TEGR. Therefore, we will search for more primary constraints in Eq. (21).

Equation (21) is a system of n^2 equations that are not linearly independent. In the previous section we have already shown that they contain a set of n constraints that trivially emerge for $\mu = 0$. The existence of constraints associated to the temporal coordinate index is a consequence of the privileged role the temporal coordinate plays in the canonical formalism. We expect that the rest of the

[2]Since $E = \varepsilon_{ab\ldots g} E^a_0 E^b_1 \ldots E^g_{n-1}$, where $\varepsilon_{ab\ldots g}$ is the Levi-Civita symbol, we also obtain $e^0_h \delta E = e^0_h \epsilon^a \varepsilon_{ab\ldots g} E^b_1 \ldots E^g_{n-1} = -E^a_\nu \delta e^\nu_h \varepsilon_{ab\ldots g} E^b_1 \ldots E^g_{n-1} = -E \delta^0_h$. Therefore $e^0_h E$ is invariant under the transformation (37).

primary constraints are exclusively related to tangent space indices. Therefore, we will look for constraints among the coordinate invariant combinations $\Pi^\mu_a E^e_\mu$; according to Eq. (21) they are

$$\Pi^\mu_a E^e_\mu = E C_{ab}{}^{ef} e^\lambda_f \partial_0 E^b_\lambda + E\partial_i E^b_\lambda e^0_c e^i_d e^\lambda_f M_{ab}{}^{cedf}, \tag{47}$$

where $C_{ab}{}^{ef}$ is defined as

$$C_{ab}{}^{ef} \doteq e^0_c e^0_d M_{ab}{}^{cedf}. \tag{48}$$

To find constraints (relations among the canonical variables) in Eq. (47), we should find (vielbein-depending) coefficients v^a_e such that $v^a_e \Pi^\mu_a E^e_\mu$ does not contain canonical velocities. In other words, since the square matrix e^λ_f is not singular, it should be

$$v^a_e C_{ab}{}^{ef} = 0. \tag{49}$$

Notice that even the n trivial primary constraints $G^{(1)}_g \doteq \Pi^0_g$ can be recovered in this way. In fact $G^{(1)}_g$ requires coefficients $v^a_{|g|e} \doteq e^0_e \delta^a_g$ (the index between vertical bars is a label for each independent set of coefficients), since $e^0_e \delta^a_g \Pi^\mu_a E^e_\mu = \Pi^0_g$. On the other hand, these coefficients satisfy Eq. (49), because $M_{gb}{}^{cedf}$ is antisymmetric in $c-e$,

$$v^a_{|g|e} C_{ab}{}^{ef} = e^0_e e^0_c e^0_d M_{gb}{}^{cedf} \equiv 0. \tag{50}$$

We will introduce an independent set of coefficients $v^a{}_e$ leading to the primary constraints associated with the Lorentz group. Let the following be the set of coefficients $v^a{}_e$ labeled by gh

$$v_{|gh|e}{}^a \doteq 2\delta^a_{[g}\eta_{h]e}. \tag{51}$$

Taking into account the form (17) of the supermetric, we obtain

$$v_{|gh|e}{}^a C_{ab}{}^{ef} = 2e^0_c e^0_d \eta_{e[h} M_{g]b}{}^{cedf} = 4e^0_c e^0_d \delta^{cdf}_{hgb} \equiv 0, \tag{52}$$

since δ^{cdf}_{hgb} is completely antisymmetric [see Eq. (A5) for details of the calculation]. The antisymmetric labels gh classify $n(n-1)/2$ new constraints. By combining Eqs. (47), (52) and (A5), one gets

$$\begin{aligned} 0 &\equiv v_{|gh|e}{}^a (\Pi^\mu_a E^e_\mu - E\partial_i E^b_\lambda e^0_c e^i_d e^\lambda_f M_{ab}{}^{cedf}) \\ &= 2\eta_{e[h}\Pi^\mu_{g]} E^e_\mu + 4E\partial_i E^b_\lambda e^0_{[h} e^i_g e^\lambda_{b]}. \end{aligned} \tag{53}$$

In the last line, λ can be substituted with j due to the antisymmetrization of the pair $h-b$. Besides, on the constraint surface it is $\Pi^0_g = 0$. So, we define the primary constraints

$$G^{(1)}_{gh} \doteq 2\eta_{e[h}\Pi^i_{g]} E^e_i + 4E\partial_i E^b_j e^0_{[h} e^i_g e^j_{b]} \approx 0. \tag{54}$$

In Sec. VIII we will prove that these $n(n-1)/2$ constraints fulfill the Lorentz algebra. In addition, they will be consistent with the evolution. The entire set of constraints will prove to be first class. According to Eq. (34), the gauge transformation of the vielbein generated by a combination $\epsilon^{gh} G^{(1)}_{gh}$ is

$$\delta E^a_j(t,\mathbf{x}) = \int d\mathbf{y}\epsilon^{gh}(t,\mathbf{y})\{E^a_j(t,\mathbf{x}), 2\eta_{e[h}\Pi^i_{g]}(t,\mathbf{y})E^e_i\}, \tag{55}$$

which can be extended to the component E^a_0 by virtue of the gauge transformation (37), thus leading to the local Lorentz transformation

$$\delta\mathbf{E}^a = \epsilon^{gh}(t,\mathbf{x})(\eta_{eh}\delta^a_g - \eta_{eg}\delta^a_h)\mathbf{E}^e. \tag{56}$$

At this point, one could ask whether we have exhausted the solutions to Eq. (49). We remark that $C_{ab}{}^{ef}$ can be rephrased as a symmetric $n^2 \times n^2$ matrix by using a notation that take pairs of flat indices $a, b, \ldots$ to define a multi-index $A = ()^a{}_e$ such that the Eq. (49) becomes

$$v^A C_{AB} = 0. \tag{57}$$

For this, we use the indexation formulas for $A = ()^a{}_e$, $B = ()^b{}_f$ as follows[3]:

$$A = (a-1)n + e, \qquad B = (b-1)n + f, \tag{58}$$

so, $A, B, \ldots = 1, \ldots, n^2$. Equation (A1) implies the symmetry of C_{AB},

$$C_{AB} = C_{BA}. \tag{59}$$

Equation (57) means that there are as many linear constraints as there are zero eigenvalues in the symmetric $n^2 \times n^2$ matrix C_{AB}. The coefficients $v^A = v^a_e$ of the constrained combinations $v^a_e \Pi^\mu_a E^e_\mu$ are the components of the respective eigenvectors. So far we have found $n + n(n-1)/2 = n(n+1)/2$ zero eigenvalues. As we will see in the forthcoming sections, the other $n(n-1)/2$ eigenvalues are different from zero.

VII. TEGR CANONICAL HAMILTONIAN

We will fully exploit the multi-index notation introduced at the end of the previous section. For this, we define a set of objects of n^2 components,

[3]The formula can be inverted by taking $a = [A/n]$, so $e = A - n[A/n] - 1$, where [] means the integer part.

$$\dot{E}^B \doteq e^\lambda_f \dot{E}^b_\lambda, \qquad E^B_0 \doteq e^i_f \partial_i E^b_0,$$
$$\Pi_A \doteq \Pi^\mu_a E^e_\mu, \qquad P_A \doteq E\partial_i E^b_k e^0_c e^i_d e^k_f M_{ab}{}^{cedf}. \tag{60}$$

Thus the Lagrangian density (16) reads

$$L = \frac{1}{2}(\Pi_A + P_A)(\dot{E}^A - E^A_0) - U, \tag{61}$$

where

$$U \doteq -\frac{1}{2} E\partial_i E^a_j \partial_k E^b_l e^i_c e^j_e e^k_d e^l_f M_{ab}{}^{cedf}. \tag{62}$$

Therefore, the canonical Hamiltonian density turns out to be

$$\begin{aligned}\mathcal{H} &\doteq \Pi^\mu_a \dot{E}^a_\mu - L = \Pi_A \dot{E}^A - L \\ &= \frac{1}{2}(\Pi_A - P_A)\dot{E}^A + \frac{1}{2}(\Pi_A + P_A)E^A_0 + U. \end{aligned}\tag{63}$$

To write $\mathcal{H}$ in a canonical way, the velocities $\dot{E}^B$ must be solved in terms of the momenta. Equation (47) displays the linear relation among velocities and momenta; this equation now reads

$$\Pi_A - P_A = EC_{AB}(\dot{E}^B - E^B_0). \tag{64}$$

In Eq. (64) $\dot{E}^B$ cannot be straightforwardly solved because the matrix C_{AB} is singular. Matrix C_{AB} has $n(n+1)/2$ zero eigenvalues, since there are $n(n+1)/2$ primary constraints linear in the momenta. Despite the fact that C_{AB} is not invertible, we can still solve the subspace of velocities that is orthogonal to the subspace of zero eigenvalues. In fact, by using a proper basis for splitting the subspace of zero eigenvalues, C_{AB} would look like

$$C'_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{C} \end{pmatrix}. \tag{65}$$

In such a basis we would find $n(n+1)/2$ constraints $\Pi_A - P_A = 0$; in addition, we would trivially solve $n(n-1)/2$ relevant velocities,

$$\dot{E}^A - E^A_0 = E^{-1} D'^{AB}(\Pi_B - P_B), \tag{66}$$

where the matrix D' is

$$D' = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{D} \end{pmatrix}, \tag{67}$$

and satisfies

$$D'C' = C'D' = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{68}$$

Equation (66) declares zero the $n(n+1)/2$ first velocities. This causes no harm, since these velocities enter the Hamiltonian (63) as the coefficients of the primary constraints $\Pi_A - P_A = 0$. So, the values of the $n(n+1)/2$ first velocities are irrelevant, because different choices modify the Hamiltonian by terms proportional to the constraints. Anyway this kind of terms are reintroduced in the primary Hamiltonian (25).

Let us use the matrix N of change of basis to return to the original basis: $C' = NCN^{-1}$. Then, the previous equation becomes

$$N^{-1}D'NC = CN^{-1}D'N = N^{-1}\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}N. \tag{69}$$

The rhs is not the identity, but is a symmetric matrix. The matrix

$$D \doteq N^{-1}D'N \tag{70}$$

satisfies that $CDC = C$ and $DCD = D$. Therefore D is the Moore-Penrose pseudoinverse of C. We will use the Eq. (66) in the original basis, so we must substitute D' with D. Thus, we substitute Eq. (66) in Eq. (63) to obtain the canonical form of the Hamiltonian density,

$$\mathcal{H} = \frac{1}{2}e(\Pi_A - P_A)D^{AB}(\Pi_B - P_B) + \Pi_A E^A_0 + U, \tag{71}$$

where $e = E^{-1} = \det(e^\mu_a)$. The canonical Hamiltonian is the integral of $\mathcal{H}$. We can remind the form (32) of the constraint $G^{(2)}_0$ to write

$$H = \int d\mathbf{x}\mathcal{H} = \int d\mathbf{x} G^{(2)}_0 + \int E^c_0 \Pi^i_c dS_i. \tag{72}$$

Then, the canonical Hamiltonian is a constraint plus a boundary term. As a consequence, the set of first-class constraints will be automatically consistent with the evolution.

A. Dimension $n = 3$

Let us work with the matrix $C^A{}_B$,

$$C^A{}_B = C^a{}_{eb}{}^f = e^0_c e^0_d M^a{}_b{}^g{}_e{}^{hf}, \tag{73}$$

where $M^a{}_b{}^g{}_e{}^{hf} = \eta^{ac}\eta_{de}M_{cb}{}^{gdhf}$. C_{AB} and $C^A{}_B$ share the eigenvectors of zero eigenvalue (see the Appendix for the forms of these matrices). The nonzero eigenvalues of $C^A{}_B$ are

$$\lambda_1 = \lambda_2 = 2[(e^0_0)^2 - (e^0_1)^2 - (e^0_2)^2] = 2g^{00} \doteq \lambda,$$
$$\lambda_3 = -\lambda. \tag{74}$$

The case $n=3$ is very simple because the matrix $C^A{}_B$ satisfies

$$C^A{}_B C^B{}_C C^C{}_D = \lambda^2 C^A{}_D. \tag{75}$$

This is a consequence of the fact that the nonzero eigenvalues have the same absolute value. This means that the pseudoinverse of $C^A{}_B$ is $D^A{}_B = \lambda^{-2} C^A{}_B$. Therefore, the matrix D^{AB} in Eq. (71) is

$$D^{AB} = \lambda^{-2} C^{AB} = \lambda^{-2} e^0_g e^0_h M^{abg}{}_e{}^h{}_f. \tag{76}$$

B. Dimension $n > 3$

In $n=4$ dimensions, the matrix $C^A{}_B$ has six nonzero eigenvalues; they are

$$\begin{aligned}\lambda_1 &= \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 \\ &= 2[(e^0_0)^2 - (e^0_1)^2 - (e^0_2)^2 - (e^0_3)^2] = 2g^{00} \doteq \lambda, \\ \lambda_6 &= -2\lambda.\end{aligned} \tag{77}$$

Since their absolute values are not equal, the pseudoinverse matrix D^{AB} cannot be inferred in a straightforward way as we did in $n=3$ dimensions. In fact, the matrix $C^A{}_B$ does not satisfy Eq. (75) when $n>3$. The eigenvector related to the odd eigenvalue is

$$w^B = w^b{}_f = -\frac{\lambda}{2}\delta^b_f + e^0_f \eta^{bh} e^0_h. \tag{78}$$

In fact, in any dimension n, vector w^B satisfies the eigenvalue equation

$$\begin{aligned}C^A{}_B w^B &= e^0_g e^0_h M^a{}_b{}^g{}_e{}^{hf} w^b{}_f \\ &= -(n-2)\lambda w^a{}_e = -(n-2)\lambda w^A.\end{aligned} \tag{79}$$

We will show that the pseudoinverse of $C^A{}_B$ can be formulated as the matrix

$$D^A{}_B = \lambda^{-2}(C^A{}_B + \alpha w^A w_B), \tag{80}$$

where α is a factor to be determined. The idea is to use the projector associated to the odd eigenvalue to "improve" the matrix $C^A{}_B$ and get the desired result. In order for $D^A{}_B$ to be the pseudoinverse of $C^A{}_B$, the rhs of the equation

$$C^A{}_C D^C{}_D C^D{}_B = \lambda^{-2} C^A{}_C C^D{}_C C^D{}_B + \alpha(n-2)^2 w^A w_B \tag{81}$$

should be $C^A{}_B$. To find α, we will introduce the auxiliary matrix

$$\tilde{C}^A{}_B = C^A{}_B + 4\lambda^{-1} w^A w_B, \tag{82}$$

which satisfies

$$\begin{aligned}\tilde{C}^A{}_B w^B &= C^A{}_B w^B + 4\lambda^{-1} w^A w_B w^B \\ &= -(n-2)\lambda w^A + (n-1)\lambda w^A = \lambda w^A.\end{aligned} \tag{83}$$

Besides, for any vector ℓ^B orthogonal to w^B it is $\tilde{C}^A{}_B \ell^B = C^A{}_B \ell^B = \lambda \ell^A$. Then, $\tilde{C}^A{}_B$ is isotropic in the subspace of nonzero eigenvalues.[4] Since all the nonzero eigenvalues of $\tilde{C}^A{}_B$ are equal to λ, then $\tilde{C}^A{}_B$ fulfills Eq. (75). Therefore

$$\begin{aligned}\lambda^2(C^A{}_B + 4\lambda^{-1} w^A w_B) &= \lambda^2 \tilde{C}^A{}_B = \tilde{C}^A{}_C \tilde{C}^C{}_D \tilde{C}^D{}_B \\ &= C^A{}_C C^D{}_C C^D{}_B + 4\lambda(n^2 - 5n + 7) w^A w_B,\end{aligned} \tag{84}$$

i.e.,

$$\lambda^{-2} C^A{}_C C^D{}_C C^D{}_B = C^A{}_B - 4\lambda^{-1}(n-3)(n-2) w^A w_B. \tag{85}$$

Substituting this result in Eq. (81), we obtain that $D^A{}_B$ is the pseudoinverse of $C^A{}_B$ if α has the value

$$\alpha = \lambda^{-1}\frac{4(n-3)}{(n-2)}. \tag{86}$$

In $n=4$ dimensions, α is equal to $2\lambda^{-1}$. Thus the contravariant pseudoinverse matrix $D^{AB} = \lambda^{-2}(C^{AB} + \alpha w^A w^B)$ in four dimensions is

$$\begin{aligned}D^{AB} = D^{ab}{}_{ef} &= \lambda^{-1}\left(\delta^{[a}_f \delta^{b]}_e + \frac{1}{2}\eta^{ab}\eta_{ef}\right) \\ &\quad - \lambda^{-2}(e^0_e e^0_f \eta^{ab} + 4 e^0_g e^0_{[e} \delta^{[a}_{f]} \eta^{b]g} \\ &\quad + e^0_g e^0_h \eta^{ag}\eta^{bh}\eta_{ef}) \\ &\quad + 2\lambda^{-3}\eta^{ag}\eta^{bh} e^0_g e^0_h e^0_e e^0_f.\end{aligned} \tag{87}$$

VIII. ALGEBRA OF CONSTRAINTS

The Hamiltonian formalism for TEGR is not complete without checking that the set of constraints is first class. For this, we have to compute the entire set of Poisson brackets between the constraints. The pseudoinverse matrix D^{AB} will enter the algebra of those Poisson brackets involving the constraint $G^{(2)}_0$. It is worth mentioning that Eq. (78) can be substituted in the lhs of Eq. (79) to obtain

$$\frac{1}{2(n-2)} C^a{}_{be}{}^b = w^a{}_e = w^A. \tag{88}$$

Therefore, matrix D in Eq. (80) can be written entirely in terms of the matrix C as

[4] This is true not only for $n=4$; it has also been checked for arbitrary n through the computer algebra program CADABRA [21].

$$D^{ab}{}_{ef} = \lambda^{-2} C^{ab}{}_{ef} + \frac{\lambda^{-3}(n-3)}{(n-2)^3} C^{a}{}_{ce}{}^{c} C^{b}{}_{df}{}^{d}, \quad (89)$$

which will be useful to compute those brackets involving $G_0^{(2)}$.

The simplest brackets are those related to $G_a^{(1)}$,

$$\{G_a^{(1)}(t,\mathbf{x}), G_b^{(1)}(t,\mathbf{y})\} = 0, \quad (90)$$

$$\{G_i^{(2)}(t,\mathbf{x}), G_a^{(1)}(t,\mathbf{y})\} = 0, \quad (91)$$

$$\{G_{ab}^{(1)}(t,\mathbf{x}), G_c^{(1)}(t,\mathbf{y})\} = 0, \quad (92)$$

$$\{G_0^{(2)}(t,\mathbf{x}), G_a^{(1)}(t,\mathbf{y})\} = (e_a^0 G_0^{(2)} + e_a^i G_i^{(2)})\delta(\mathbf{x}-\mathbf{y}). \quad (93)$$

However, the last one requires the knowledge of the brackets between the momenta Π_a^0 and the matrix D^{AB}. In the Appendix we summarize useful hints in order to simplify this calculation.

The Poisson brackets between secondary constraints $G_\mu^{(2)}$ reproduce the algebra of constraints of the ADM formulation of general relativity,

$$\{G_i^{(2)}(t,\mathbf{x}), G_j^{(2)}(t,\mathbf{y})\} = -G_i^{(2)}(\mathbf{x})\partial_j^{\mathbf{y}}\delta(\mathbf{x}-\mathbf{y}) + G_j^{(2)}(\mathbf{y})\partial_i^{\mathbf{x}}\delta(\mathbf{x}-\mathbf{y}), \quad (94)$$

$$\{G_0^{(2)}(t,\mathbf{x}), G_0^{(2)}(t,\mathbf{y})\} = g^{ij}(\mathbf{x})G_i^{(2)}(\mathbf{x})\partial_j^{\mathbf{y}}\delta(\mathbf{x}-\mathbf{y}) - g^{ij}(\mathbf{y})G_i^{(2)}(\mathbf{y})\partial_j^{\mathbf{x}}\delta(\mathbf{x}-\mathbf{y}), \quad (95)$$

$$\{G_0^{(2)}(t,\mathbf{x}), G_i^{(2)}(t,\mathbf{y})\} = G_0^{(2)}(\mathbf{x})\partial_i^{\mathbf{y}}\delta(\mathbf{x}-\mathbf{y}). \quad (96)$$

We have also verified that the Poisson brackets for the constraints $G_{ab}^{(1)}$ reproduces the Lorentz algebra,

$$\{G_{ac}^{(1)}(t,\mathbf{x}), G_{fe}^{(1)}(t,\mathbf{y})\} = (\eta_{ec}G_{af}^{(1)} + \eta_{af}G_{ce}^{(1)} - \eta_{cf}G_{ae}^{(1)} - \eta_{ae}G_{cf}^{(1)})\delta(\mathbf{x}-\mathbf{y}). \quad (97)$$

It is

$$\{G_{ab}^{(1)}(t,\mathbf{x}), G_i^{(2)}(t,\mathbf{y})\} = 0. \quad (98)$$

Finally the most intricate calculation is required by the bracket

$$\{G_0^{(2)}(t,\mathbf{x}), G_{ab}^{(1)}(t,\mathbf{y})\} = E_0^c \eta_{c[a} e_{b]}^0 G_0^{(2)}\delta(\mathbf{x}-\mathbf{y}). \quad (99)$$

In order to alleviate some difficult parts of it, some useful computations are summarized in the Appendix.

As a result we have obtained n trivial primary constraints $G_a^{(1)}$, together with $n(n-1)/2$ primary constraints that come from the Lorentz algebra. We have also obtained n secondary constraints $G_\mu^{(2)}$ that are equivalent to the super-Hamiltonian and supermomenta constraints of the ADM formalism. Since we just proved that all constraints are first class, then the counting of degrees of freedom goes as

$$\text{Number of d.o.f.} = \text{Number of } (\mathrm{p},\mathrm{q}) - \text{Number of fcc} = n^2 - \frac{n(n+3)}{2} = \frac{n(n-3)}{2} \quad (100)$$

which is the number of degrees of freedom of general relativity in n dimensions.

IX. SUMMARY

The essence of a Hamiltonian constrained system lies in the impossibility of solving all the canonical velocities in terms of canonical momenta. This is because the momenta are not independent, but satisfy constraint equations, which in turn means that some dynamical variables are spurious degrees of freedom. In the case of the teleparallel equivalent of general relativity (TEGR), such obstruction is expressed in Eq. (47), since $C_{ab}{}^{ef}$ cannot be inverted. $C_{ab}{}^{ef}$ is an object intimately linked to the Lorentz invariant supermetric $M_{ab}{}^{cedf}$ entering the TEGR Lagrangian (20). In order to analyze how many constraints are involved in the Eq. (47), and how many canonical velocities can be solved, we have arranged the components of $C_{ab}{}^{ef}$ in a $n^2 \times n^2$ symmetric matrix C_{AB} [the relation between the superindex A and the tangent space indices is given in Eq. (58)]. We have shown that the eigenvalues of $C^A{}_B$ follow a very simple pattern: $n(n+1)/2$ eigenvalues are zero, $n(n-1)/2-1$ of them are equal to $2g^{00} \doteq \lambda$, and the remaining one is equal to $(2-n)\lambda$. The primary constraints result from the contraction of the Eq. (64) with each eigenvector of zero eigenvalue; they include the n trivial constraints $G_a^{(1)}$ [see Eq. (24)] and the $n(n-1)/2$ Lorentz constraints $G_{ab}^{(1)}$ [see Eq. (54)]. To build the canonical Hamiltonian we must identify the subset of canonical velocities that can be still solved in terms of the momenta. For this, we employed the Moore-Penrose pseudoinverse of matrix C, which can be sought in the form proposed in Eq. (80) thanks to the simple pattern of eigenvalues exhibited by the matrix C. The so obtained matrix D^{AB} is the piece we need to write the canonical Hamiltonian density $\mathcal{H}$ [see Eq. (71)]. Those terms associated with the unsolved velocities are absorbed into the terms added to the primary Hamiltonian H_p (25). Besides the primary constraints $G_a^{(1)}$, $G_{ab}^{(1)}$, we have also obtained n secondary constraints $G^{(2)}_\mu$—the diffeomorphism constraints—that guarantee that the primary constraints remain valid along the evolution dictated by H_p (we have examined this requirement at the level of the Euler-Lagrange equations). The consistency under the evolution of the system must be checked for the secondary constraints too. Not surprisingly, the canonical Hamiltonian density $\mathcal{H}$ is equal to $G^{(2)}_0$ except for a boundary term. Thus, the consistency of the entire set of constraints is guaranteed by the first-class constraint algebra [Eqs. (90)–(99)]. Since the constraints are first class, they generate gauge transformations. Therefore, there are $n(n+3)/2$ spurious variables, which reduces the number of degrees of freedom to $n(n-3)/2$. The independent gauge transformations are those displayed in Eqs. (37), (42), and (56).

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APPENDIX

1. Properties of the supermetric

There are many properties of the supermetric that were used throughout this work, and that can be deduced from its definition. Some of them are

$$M_{ab}{}^{cedf} = M_{ba}{}^{dfce} = -M_{ab}{}^{ecdf} = -M_{ab}{}^{cedf}. \tag{A1}$$

We can calculate "traces" of the supermetric, which depend on the dimension n. Some of them are

$$M_{ab}{}^{aedf} = M_{ba}{}^{dfae} = 4(n-2)\eta^{e[d}\delta^{f]}_b, \tag{A2}$$

$$M_{ab}{}^{dfae} = M_{ba}{}^{aedf} = 2(n-2)\eta^{e[f}\delta^{d]}_b, \tag{A3}$$

$$M_{ab}{}^{aebf} = -2(n-1)(n-2)\eta^{ef}. \tag{A4}$$

The totally antisymmetric Kronecker delta δ^{ghf}_{cab} appears in the antisymmetrized product

$$\begin{aligned}\eta_{e[c}M_{a]b}{}^{gehf} &= 2(\delta^h_{[a}\delta^f_{c]}\delta^g_b + \delta^g_{[a}\delta^h_{c]}\delta^f_b + \delta^f_{[a}\delta^g_{c]}\delta^h_b)\\ &= -2\delta^{ghf}_{cab}.\end{aligned} \tag{A5}$$

We also obtain

$$M^{abg}{}_{ef}{}^h\eta_{a[q}\delta^e_{p]} = \eta^{bg}\eta_{f[q}\delta^h_{q]} + \eta^{bh}\eta_{f[q}\delta^g_{p]} + \delta^b_f\delta^g_{[q}\delta^h_{p]}, \tag{A6}$$

$$\eta^{e[c}C^{d]b}{}_{ef} = 4e^0_f e^{0[c}\eta^{d]b}. \tag{A7}$$

Some other combinations quadratic in M appear in the calculations, and it is useful to have them on hand,

$$\begin{aligned}C^{ac}{}_{ec}M_{ag}{}^{cedf} &= 4(n-3)(n-2)g^{00}\eta^{c[d}\delta^{f]}_g\\ &\quad + 8(n-2)e^0_g e^{0[f}\eta^{d]c} + e^{0c}e^{0[d}\delta^{f]}_g,\end{aligned} \tag{A8}$$

$$\begin{aligned}C_{ab}{}^{ef}M^{abc}{}_e{}^d{}_f &= 6(n-3)(n-2)\eta^{cd}g^{00}\\ &\quad + 12(n-2)e^{0c}e^{0d}.\end{aligned} \tag{A9}$$

2. Calculation of $\partial L/\partial E^a_0$

For computing $\partial L/\partial E^a_0$, it is important to notice that, in contrast to electromagnetism, E^a_0 appears in the Lagrangian not just in the spatial derivatives $\partial_i E^a_0$ but also as a part of e^μ_a and E. First of all we need the quotient $\partial e^\mu_c/\partial E^a_\lambda$, which is obtained from the duality relation,

$$\delta^\mu_\nu = e^\mu_b E^b_\nu \to 0 = \frac{\partial e^\mu_b}{\partial E^a_\lambda}E^b_\nu + e^\mu_a\delta^\lambda_\nu. \tag{A10}$$

This implies that

$$\frac{\partial e^\mu_c}{\partial E^a_\lambda} = -e^\mu_a e^\lambda_c. \tag{A11}$$

We will need also the expression $\partial E/\partial E^a_0$, which is obtained from the explicit formula for the determinant,

$$E = \epsilon_{abcd\ldots g}E^a_0 E^b_1 E^c_2 E^d_3 \ldots E^g_n. \tag{A12}$$

Then we obtain

$$E^a_\lambda \frac{\partial E}{\partial E^a_0} = \delta^0_\lambda E \to \frac{\partial E}{\partial E^a_0} = E e^0_a. \tag{A13}$$

In this way,

$$\begin{aligned}\frac{\partial L}{\partial E^a_0} &= \frac{1}{2}E(e^0_a e^\mu_g e^\nu_e e^\rho_h e^\lambda_f - e^\mu_a e^0_g e^\nu_e e^\rho_h e^\lambda_f - e^\nu_a e^\mu_g e^0_e e^\rho_h e^\lambda_f\\ &\quad - e^\rho_a e^\mu_g e^\nu_e e^0_h e^\lambda_f - e^\lambda_a e^\mu_g e^\nu_e e^\rho_h e^0_f)\partial_\mu E^c_\nu \partial_\rho E^d_\lambda M_{cd}{}^{gehf}.\end{aligned} \tag{A14}$$

In the last expression we identify the Lagrangian in the first term, and different index combinations of the momenta. We rewrite it and continue with the algebraic manipulation

$$\begin{aligned}\frac{\partial L}{\partial E^a_0} &= e^0_a L - \frac{1}{2}e^\mu_a\partial_\mu E^c_\nu \Pi^\nu_c + \frac{1}{2}e^\nu_a\partial_\mu E^c_\nu\Pi^\mu_c\\ &\quad - \frac{1}{2}e^\rho_a\partial_\rho E^d_\lambda\Pi^\lambda_d + \frac{1}{2}e^\lambda_a\partial_\rho E^d_\lambda \Pi^\rho_d\\ &= e^0_a L + 2e^\nu_a\partial_{[\mu}E^c_{\nu]}\Pi^\mu_c\\ &= e^0_a L + 2e^0_a\partial_{[i}E^c_{0]}\Pi^i_c + 2e^j_a\partial_{[i}E^c_{j]}\Pi^i_c.\end{aligned} \tag{A15}$$

The Hamiltonian density can be extracted from the first terms, to obtain

$$\begin{aligned}\frac{\partial L}{\partial E_0^a} &= -e_a^0\mathcal{H} + e_a^0\partial_i E_0^c\Pi_c^i + 2e_a^j\partial_{[i}E_{j]}^c\Pi_c^i \\ &= e_a^0(\partial_i(E_0^c\Pi_c^i) - \mathcal{H}) - E_0^c e_a^0\partial_i\Pi_c^i + 2e_a^j\partial_{[i}E_{j]}^c\Pi_c^i.\end{aligned} \tag{A16}$$

This result is substituted in Eq. (30) to obtain n secondary constraints,

$$E_j^c e_a^j\partial_i\Pi_c^i + e_a^0(\partial_i(E_0^c\Pi_c^i) - \mathcal{H}) + 2e_a^j\partial_{[i}E_{j]}^c\Pi_c^i \approx 0. \tag{A17}$$

We note that only spatial derivatives are present, and the canonical Hamiltonian takes part in the secondary constraints. We can isolate the contribution of the Hamiltonian by doing the contraction with E_0^a; thus, we get

$$G_0^{(2)} = \mathcal{H} - \partial_i(E_0^c\Pi_c^i) \approx 0. \tag{A18}$$

We also perform the contraction with E_k^a, yielding

$$G_k^{(2)} = \partial_k E_i^c\Pi_c^i - \partial_i(E_k^c\Pi_c^i) \approx 0. \tag{A19}$$

3. Matrix $C^A{}_B$

We present the full expression for the matrix $C^A{}_B$ in $n = 4$, which appears in the definition of the canonical momenta. It is

$$C^A{}_B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 2c_{23} & -2d_{12} & -2d_{13} & 0 & -2d_{12} & 2c_{13} & -2d_{23} & 0 & -2d_{13} & -2d_{23} & 2c_{12} \\
0 & -c_{23} & d_{12} & d_{13} & c_{23} & 0 & -d_{02} & -d_{03} & -d_{12} & -d_{02} & 2d_{01} & 0 & -d_{13} & -d_{03} & 0 & 2d_{01} \\
0 & d_{12} & -c_{13} & d_{23} & -d_{12} & 2d_{02} & -d_{01} & 0 & c_{13} & -d_{01} & 0 & -d_{03} & -d_{23} & 0 & -d_{03} & 2d_{02} \\
0 & d_{13} & d_{23} & -c_{12} & -d_{13} & 2d_{03} & 0 & -d_{01} & -d_{23} & 0 & 2d_{03} & -d_{02} & c_{12} & -d_{01} & -d_{02} & 0 \\
0 & c_{23} & -d_{12} & -d_{13} & -c_{23} & 0 & d_{02} & d_{03} & d_{12} & d_{02} & -2d_{01} & 0 & d_{13} & d_{03} & 0 & -2d_{01} \\
2c_{23} & 0 & -2d_{02} & -2d_{03} & 0 & 0 & 0 & 0 & 2d_{02} & 0 & -2c_{03} & -2d_{23} & 2d_{03} & 0 & -2d_{23} & -2c_{02} \\
-2d_{12} & d_{02} & d_{01} & 0 & -d_{02} & 0 & c_{03} & d_{23} & -d_{01} & c_{03} & 0 & d_{13} & 0 & d_{23} & d_{13} & -2d_{12} \\
-2d_{13} & d_{03} & 0 & d_{01} & -d_{03} & 0 & d_{23} & c_{02} & 0 & d_{23} & -2d_{13} & d_{12} & -d_{01} & c_{02} & d_{12} & 0 \\
0 & -d_{12} & c_{13} & -d_{23} & d_{12} & -2d_{02} & d_{01} & 0 & -c_{13} & d_{01} & 0 & d_{03} & d_{23} & 0 & d_{03} & -2d_{02} \\
-2d_{12} & d_{02} & d_{01} & 0 & -d_{02} & 0 & c_{03} & d_{23} & -d_{01} & c_{02} & 0 & d_{13} & 0 & d_{23} & d_{13} & -2d_{12} \\
2c_{13} & -2d_{01} & 0 & -2d_{03} & 2d_{01} & -2c_{03} & 0 & -2d_{13} & 0 & 0 & 0 & 0 & 2d_{03} & -2d_{13} & 0 & -2c_{01} \\
-2d_{23} & 0 & d_{03} & d_{02} & 0 & -2d_{23} & d_{13} & d_{12} & -d_{03} & d_{13} & 0 & c_{01} & -d_{02} & d_{12} & c_{01} & 0 \\
0 & -d_{13} & -d_{23} & c_{12} & d_{13} & -2d_{03} & 0 & d_{01} & d_{23} & 0 & -2d_{03} & d_{02} & -c_{12} & d_{01} & d_{02} & 0 \\
-2d_{13} & d_{03} & 0 & d_{01} & -d_{03} & 0 & d_{23} & c_{02} & 0 & d_{23} & -2d_{13} & d_{12} & -d_{01} & c_{02} & d_{12} & 0 \\
-2d_{23} & 0 & d_{03} & d_{02} & 0 & -2d_{23} & d_{13} & d_{12} & -d_{03} & d_{13} & 0 & c_{01} & -d_{02} & d_{12} & c_{01} & 0 \\
2c_{12} & -2d_{01} & -2d_{02} & 0 & 2d_{01} & -2c_{02} & -2d_{12} & 0 & 2d_{02} & -2d_{12} & -2c_{01} & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{A20}$$

where

$$\begin{aligned}
&c_{01} = (e_0^0)^2 - (e_1^0)^2, \qquad c_{02} = (e_0^0)^2 - (e_2^0)^2, \\
&c_{03} = (e_0^0)^2 - (e_3^0)^2, \qquad c_{12} = (e_1^0)^2 + (e_2^0)^2, \\
&c_{13} = (e_1^0)^2 + (e_3^0)^2, \qquad c_{23} = (e_2^0)^2 + (e_3^0)^2, \\
&d_{01} = e_0^0 e_1^0, \qquad d_{02} = e_0^0 e_2^0, \qquad d_{03} = e_0^0 e_3^0, \\
&d_{12} = e_1^0 e_2^0, \qquad d_{13} = e_1^0 e_3^0, \qquad d_{23} = e_2^0 e_3^0.
\end{aligned} \tag{A21}$$

The matrices C_{AB} and C^{AB} are obtained by raising and lowering indices with the corresponding η tensors. The matrix $D^A{}_B$ is obtained starting from (80).

4. Poisson brackets

Some useful fundamental Poisson brackets between the canonical variables and their derivatives are given below,

$$\{E(t,\mathbf{x}), \Pi_a^\mu(t,\mathbf{y})\} = E e_a^\mu \delta(\mathbf{x}-\mathbf{y}), \tag{A22}$$

$$\{e(t,\mathbf{x}), \Pi_a^\mu(t,\mathbf{y})\} = -e e_a^\mu \delta(\mathbf{x}-\mathbf{y}), \tag{A23}$$

$$\{e_a^\mu(t,\mathbf{x}), \Pi_b^\nu(t,\mathbf{y})\} = -e_b^\mu e_a^\nu \delta(\mathbf{x}-\mathbf{y}), \tag{A24}$$

$$\begin{aligned}\{\partial_\lambda E_\mu^a(t,\mathbf{x}), \Pi_b^\nu(t,\mathbf{y})\} &= -\{E_\mu^a(t,\mathbf{x}), \partial_\lambda \Pi_b^\nu(t,\mathbf{y})\} \\ &= \delta_b^a \delta_\mu^\nu \partial_\lambda^{\mathbf{x}} \delta(\mathbf{x}-\mathbf{y}),\end{aligned} \tag{A25}$$

$$\{\partial_\mu E^b_\nu(t,\mathbf{x}),\partial_\lambda\Pi^\lambda_c(t,\mathbf{y})\} = \int d\mathbf{z}\delta^b_c\partial^{\mathbf{x}}_\mu\delta(\mathbf{x}-\mathbf{z})\partial^{\mathbf{y}}_\nu\delta(\mathbf{y}-\mathbf{z}). \tag{A26}$$

These expressions are enough (together with patience and a lot of calculations) to calculate those Poisson brackets that do not involve $G^{(2)}_0$. For the remaining Poisson brackets, we provide some easy-to-derive expressions

$$\{G^{(2)}_0(t,\mathbf{x}),E^c_i(t,\mathbf{y})\} = (eD^{AB}(\Pi_B-P_B)E^e_i\delta^c_a+\partial_i E^c_0)\delta(\mathbf{x}-\mathbf{y}), \tag{A27}$$

$$\{G^{(2)}_0(t,\mathbf{x}),\partial_\lambda E^c_\mu(t,\mathbf{y})\} = (eD^{AB}(\Pi_B-P_B)E^e_\mu\delta^c_a+\partial^{\mathbf{x}}_\mu E^c_0(\mathbf{x}))\partial^{\mathbf{y}}_\lambda\delta(\mathbf{x}-\mathbf{y}). \tag{A28}$$

Other combinations of brackets between canonical momenta and some basic building blocks of the secondary constraint $G^{(2)}_0$ that recurrently appear in the calculations are the following:

$$\{\lambda^{-\gamma},\Pi^0_c\} = 2\gamma\lambda^{-\gamma}e^0_c, \tag{A29}$$

$$\{\lambda^{-\gamma},\Pi^i_c\} = 4\gamma\lambda^{-(\gamma+1)}e^0_c g^{0i}, \tag{A30}$$

$$\{w^A,\Pi^0_c\} = -2e^0_c w^A, \tag{A31}$$

$$\{w^A,\Pi^i_c\} = -\frac{1}{2(n-2)}e^0_c(e^i_g e^0_h+e^i_h e^0_g)M^a{}_d{}^g{}_e{}^{hd}. \tag{A32}$$

Finally, we give some help to calculate the brackets of the momenta and the matrix D^{AB}. It is very simple to get the brackets

$$\{D^{AB},\Pi^0_c\} = 2e^0_c D^{AB}. \tag{A33}$$

However, for the spatial part of the momenta Π^i_a, the brackets with the matrix D do not simplify so easily. After using all the developed tools, we get

$$\begin{aligned}\{D^{AB},\Pi^i_c\} &= 8e^0_c g^{0i}\lambda^{-3}(C^{AB}+\alpha w^A w^B)-\lambda^{-2}e^0_c(e^i_g e^0_h+e^i_h e^0_g)M^{abg}{}_{ef}{}^h\\ &\quad-\frac{\alpha\lambda^{-2}e^0_c}{2(n-2)}(e^i_g e^0_h+e^i_h e^0_g)(M^a{}_d{}^g{}_e{}^{hd}w^B+M^b{}_d{}^g{}_f{}^{hd}w^A)\\ &\quad+4\alpha\lambda^{-3}e^0_c g^{0i}w^A w^B.\end{aligned} \tag{A34}$$

In Eqs. (A29)–(A34) a factor $\delta(\mathbf{x}-\mathbf{y})$ is understood. As general advice, the raising and lowering of indices in the supermetric $M_{ab}{}^{cedf}$ must be carefully done, in order to keep the original order of the indices and protect the symmetries of the object.

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