Hypersurface-deformation algebroids and effective spacetime models

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In canonical gravity, covariance is implemented by brackets of hypersurface-deformation generators forming a Lie algebroid. Lie-algebroid morphisms, therefore, allow one to relate different versions of the brackets that correspond to the same spacetime structure. An application to examples of modified brackets found mainly in models of loop quantum gravity can, in some cases, map the spacetime structure back to the classical Riemannian form after a field redefinition. For one type of quantum corrections (holonomies), signature change appears to be a generic feature of effective spacetime, and it is shown here to be a new quantum spacetime phenomenon which cannot be mapped to an equivalent classical structure. In low-curvature regimes, our constructions not only prove the existence of classical spacetime structures assumed elsewhere in models of loop quantum cosmology, they also show the existence of additional quantum corrections that have not always been included.

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I. INTRODUCTION

Several independent examples of modified gauge transformations have been found in different models of canonical quantum gravity, using effective [1–7] and operator calculations [8–12]. In classical canonical formulations, spacetime structure is encoded not in the usual form of general covariance of tensors but by the equivalent version of gauge covariance under hypersurface deformations in spacetime [13,14]. The new structures found as a direct consequence of key ingredients of the quantization process using holonomies instead of connections, therefore, confirm a general expectation: quantum geometry may lead to modified spacetime structures [15,16].

Although these modified gauge structures have been found within a variety of models of loop quantum gravity and by virtue of different computational methods, they all share some important properties. There is a phase-space function β modifying only the Poisson bracket of two smeared Hamiltonian constraints (or normal deformations of hypersurfaces). Denoting the constraints by $H[N]$ with the lapse function N that specifies the magnitude of the normal deformation at every point on a spatial hypersurface, we have

$$\{H[N], H[M]\} = -H_a[\beta q^{ab}(N\partial_b M - M\partial_b N)]. \quad (1)$$

On the right-hand side, H_a represents the components of the diffeomorphism constraint (generating tangential deformations) and q^{ab} is the inverse metric on a spatial hypersurface. Brackets involving $H_a[M^a]$ retain the classical form

$$\{H_a[M_1^a], H_b[M_2^b]\} = -H_c[\mathcal{L}_{M_2} M_1^c] \quad (2)$$

$$\{H[N], H_a[M^a]\} = -H[\mathcal{L}_M N]. \quad (3)$$

There have been attempts to modify the brackets involving not only the Hamiltonian constraint as in (1) but also the diffeomorphism constraint [17,18]. Other such examples are given by fractional spacetime models, in which the modification functions can, however, be absorbed [19]. A discrete version of the brackets has been defined in [20], which differs from (2) and (3). In this paper, we focus on continuum effective theories in which space (but not necessarily spacetime) has the classical structure. Accordingly, (2) will not be modified. We will derive a new form of brackets in which (3) is modified, but (2) is not. Nevertheless, our main focus will be on brackets with modifications, as in (1).

The correction function $\beta \neq 1$ depends on the phase-space variables and transforms as a spatial scalar. In the classical case, the hypersurface-deformation brackets are (on shell) related to the Lie algebra of spacetime diffeomorphisms, reflecting the coordinate invariance of general relativity. Brackets with $\beta \neq 1$ modify the general covariance of the effective theory, but in such a way that no gauge transformations are violated. (Obeying the condition of anomaly freedom, gauge transformations are allowed to be modified by quantum corrections, but not to be destroyed.)

With modified brackets, the effective metric q_{ab} appearing in (1) cannot be part of a spacetime line element of classical form: modified gauge transformations of q_{ab}, generated by $H[\epsilon]$ and $H_a[\epsilon^a]$, do not complement coordinate transformations of $\mathrm{d}x^a$ to form an invariant spacetime line element,

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + q_{ab}(\mathrm{d}x^a + N^a\mathrm{d}t)(\mathrm{d}x^b + N^b\mathrm{d}t), \quad (4)$$

in canonical form. Nevertheless, there may be field redefinitions of different kinds which allow one to find a classical spacetime picture for some function β of the phase-space variables. For instance, in some cases, β can be absorbed in the lapse function by $N' \coloneqq \sqrt{|\beta|}N$, with classical brackets in terms of N', or a combination of the original spatial metric and extrinsic curvature could determine the spatial geometry of an effective spacetime of the classical type. The question was investigated in certain spherically symmetric models in [21], with some encouraging results: gauge transformations of the original canonical fields of the effective theory (including q_{ab}) are deformed, but by applying canonical transformations it is possible, in some cases, to recover the classical hypersurface-deformation brackets, and hence to restore general covariance. For example, a canonical transformation with this effect was found in [21] when β depended only on metric components. Absorbing β in q^{ab} then provides a simple canonical transformation. If β depends on the momentum (extrinsic curvature) as well, it is more difficult to see whether it can be removed from the brackets.

In this paper, we analyze the same question from a different perspective which is insensitive to the availability of canonical transformations. Our discussion makes use of the general setting of Lie algebroids, of which a suitable fiber-bundle formulation of (1) provides an example [22]. More generally, the language of Lie algebroids is a well-defined mathematical structure that allows one to formalize theories with structure functions. Our results are independent of details of any specific form of quantum gravity, in the sense that we will not use equations or methods characteristic of a specific approach. Instead, we use the general form (1) of the modified bracket of two normal deformations as a guiding principle and study possible Lie-algebroid realizations. Modifications of the classical brackets can be understood as a generic form of quantum corrections, introduced by some effective quantum-gravity theory.

We will be able to classify different inequivalent spacetime structures corresponding to modified brackets of the type (1) that cannot be related by morphisms. While there appears to be an arbitrary modification function β in (1) with virtually unrestricted quantum corrections, only $\mathrm{sgn}\beta$ remains as the single choice left after equivalence classes of brackets up to morphisms are considered. This result helps to clarify the implications of modified brackets (1) for spacetime structures. Specifically, they can be related to the classical brackets by Lie-algebroid morphisms as long as β has a definite sign and is nonzero. The existence of effective Riemannian spacetime structures is confirmed in this case, which so far has only been assumed (for instance, in [23–26]). Such modifications, therefore, do not imply radical changes of the spacetime structure, even though they may still lead to a modified dynamics on and of the effective spacetime. If β does not have a fixed sign, a new version of quantum spacetime is obtained which exhibits signature change as a new physical effect.

In some cases, concrete morphisms can be formulated with simple interpretations of their implications on canonical variables and the dynamics. For instance, with spatially constant $\beta \neq 1$, as in cosmological models with first-order perturbative inhomogeneity, a suitable morphism is obtained by changing the usual conventions in setting up the canonical formulation based on spacetime foliations into spatial slices. Somewhat akin to absorbing β in the lapse function, one can make use of a generalized canonical formulation which is a hybrid version of, on one side, the Dirac [13] and the Arnowitt-Deser-Misner (ADM) [27] formulation, with variables adapted to directions normal and tangent to a spatial hypersurface, and, on the other, Rosenfeld's [28] earlier derivation of canonical gravity without reference to a foliation or preferred directions. We will use a foliation but do not require the timelike vector n^μ to be normalized or orthogonal to the spatial tangent plane. The normalization function $n^\mu n_\mu$ can be related to β. Therefore, nonstandard normalizations present a more-general way of relating modified brackets to classical spacetime structures than absorbing β in the lapse function would do. The angles between n^μ and the spatial tangent plane give rise to new modifications of the brackets that have not yet been encountered elsewhere. At the same time, we make use of a concise derivation of the hypersurface-deformation brackets and use the example to introduce Lie algebroids in this context. Morphisms of Lie algebroids will lead to further transformations that can be used to relate modified brackets of different types, still with the classical signature as the only parameter that characterizes inequivalent spacetime structures of brackets of the form (1) via $\mathrm{sgn}\beta$. This result allows us to draw rather general conclusions about implications of the modified dynamics according to (1).

II. CANONICAL GRAVITY AND LIE ALGEBROIDS

In order to set up the canonical formalism, we assume, as usual, spacetime $\mathcal{M}$ to be globally hyperbolic and introduce a foliation by constant-level surfaces of a parameter $t \in \mathbb{R}$, such that the hypersurfaces are all spacelike. Each spatial slice is homeomorphic to a 3-manifold σ, on which we may choose local coordinates x^a, $a \in \{1,2,3\}$. We realize σ as a spatial hypersurface $\Sigma_t := X_t(\sigma)$ at constant t by an embedding $X: \mathbb{R} \times \sigma \hookrightarrow \mathcal{M}$, with $(t,x) \mapsto X(t,x) =: X_t(x)$.

We choose a foliation $X_t = X(t,\cdot)$ and define a time-evolution vector field τ^μ by

$$\tau(X) := \partial_t X^\mu(t,x)\partial_\mu. \tag{5}$$

This vector field is, in general, not normal to Σ_t. Following ADM [27], it is convenient to introduce vector fields tangential to Σ_t, given by

$$X_a(X) := \partial_a X^\mu(t,x)\partial_\mu, \tag{6}$$

and to define a timelike vector field normal to the time slice Σ_t by

$$g_{\mu\nu} n^\mu X^\nu_a = 0, \qquad g_{\mu\nu} n^\mu n^\nu = -1. \tag{7}$$

If we further require that n^μ point toward the future, that is, $n^\mu \partial_\mu t > 0$, it is uniquely defined. By introducing the lapse function $N(X)$ and the shift vector field $M^a(X)$, the time-evolution vector field τ^μ is decomposed into its components normal and tangential to Σ_t:

$$\tau^\mu(X) = N(X) n^\mu(X) + N^a(X) X^\mu_a(X). \tag{8}$$

Since the choice of the embedding X is arbitrary, the components of lapse and shift are free functions as long as they give rise to a timelike τ^μ.

So far, we have used only well-known and basic ingredients of the canonical formulation. (See [29] for further details.) The decomposition (8) and the normalization condition of n^μ in (7) play a key role in our considerations of modified spacetime structures. In order to exhibit the full freedom of the formalism, we will not follow the common convention of normalizing n^μ by $g_{\mu\nu} n^\mu n^\nu = -1$. We may fix any other negative constant, or even a phase-space function, for the Lorentzian spacetime signature, or a positive constant (or phase-space function) for the Euclidean signature. We may, therefore, require that $g_{\mu\nu} n^\mu n^\nu = \epsilon\beta$, where $\epsilon = -1$ in the Lorentzian case and $\epsilon = +1$ in the Euclidean case. If the signature is constant, $\beta > 0$ is a positive phase-space function. However, in anticipation of applying these methods to some of the models found in the context of loop quantum gravity, we allow for β to change its sign, so that $\mathrm{sgn}\beta =: \epsilon_\beta$ may not be constant. The overall signature is then locally given by the product $\epsilon\epsilon_\beta$.

In order to compare dynamical results obtained with different normalizations, we should demand that $\tau^\mu(X)$ remain the same and be independent of β:

$$\begin{aligned}\tau^\mu(X) &= \frac{1}{\sqrt{|\beta|}} N(X) n^\mu(X) + M^a(X) X^\mu_a(X) \\ &=: N_\beta(X) n^\mu(X) + M^a(X) X^\mu_a(X),\end{aligned} \tag{9}$$

where $n^\mu/\sqrt{|\beta|}$ is now normalized to $\pm 1 = \epsilon\epsilon_\beta$. This condition ensures that equations of motion for evolution along τ^μ exist independently of the canonical decomposition in terms of hypersurfaces. At this stage, we see the simple result that the lapse function has to absorb any nonstandard normalization factor β, but later on we will be able to draw more benefit from these simple-looking considerations. The only requirement for (9) to be used is that n^μ and $M^\mu = M^a X^\mu_a$ form a basis of the tangent space to $\mathcal{M}$ at each point. We may, therefore, drop the normalization conditions as well as the orthogonality of n^μ and M^μ.

A. A concise derivation of the hypersurface-deformation brackets

We derive the brackets of hypersurface deformations with nonstandard normalization by repurposing a derivation of the usual result given in [22]. The main aim of this paper is to analyze the Lie-algebroid structure of the brackets, which we will describe in the following subsection. Some part of the mathematical analysis of [22] amounts to a brief derivation of the brackets, which we formulate here in abstract index notation, and, at the same time, use it to derive the brackets with nonstandard normalization. As a further generalization, we will also assume a nonorthogonality relation between n^μ and X^μ_a. More traditional derivations using ADM-style evolution equations or geometrodynamics are given in the Appendix for the case of a nonunit normal n^μ, with equivalent results.

The explicit derivation of hypersurface deformations depends on choices of coordinates or embedding functions, but the brackets must be covariant under changes of these auxiliary structures. As in [22], one can exploit the coordinate freedom by working with embeddings such that the spacetime metric, from which the spatial metric q_{ab} in the structure functions is induced, is Gaussian with respect to the hypersurfaces:

$$\mathrm{d}s^2 = \epsilon \mathrm{d}t^2 + q_{ab}\mathrm{d}x^a \mathrm{d}x^b. \tag{10}$$

In this way, one fixes a representative in each equivalence class of hypersurface embeddings. The remaining coordinate freedom is given by diffeomorphisms generated by so-called g-Gaussian vector fields v^μ, which preserve the Gaussian form of the metric and therefore satisfy

$$n^\mu \mathcal{L}_v g_{\mu\nu} = 0, \tag{11}$$

with some vector field n^μ normal to $t = \text{constant}$, but not necessarily normalized. This condition ensures that an infinitesimal diffeomorphism along v^μ, changing $g_{\mu\nu}$ to $g'_{\mu\nu} \coloneqq g_{\mu\nu} + \mathcal{L}_v g_{\mu\nu}$, respects the relations $n^\mu n^\nu g'_{\mu\nu} = n^\mu n^\nu g_{\mu\nu} = \epsilon$ and $n^\mu w^\nu g'_{\mu\nu} = 0$ if $n^\mu w^\nu g_{\mu\nu} = 0$ of the Gaussian system. Because they generate diffeomorphisms preserving the Gaussian form of the metric, g-Gaussian vector fields form a subalgebra of the Lie algebra of all vector fields with a bracket that is the usual Lie bracket. As found in [22], one can derive the hypersurface-deformation brackets by rewriting the Lie bracket using properties of vector fields v^μ satisfying (11).

Some restriction on the form of vector fields is necessary because the hypersurface deformations as gauge transformations are known to be equal to infinitesimal spacetime diffeomorphisms only on shell [14], that is, when some of the generators H and H_a and the equations of motion they generate are set to zero as phase-space functions. The restriction is implemented here by using g-Gaussian vector fields, which turn out to have Lie brackets directly related to the hypersurface-deformation brackets. Such a restriction cannot be chosen arbitrarily but must fulfill three conditions. (i) The vector fields considered must provide a unique extension from spatial (lapse) functions N and spatial (shift) vector fields M^a to a spacetime vector field v^μ which equals $Nn^\mu + M^a X_a^\mu$ on the spatial slice. If this condition is fulfilled, it is possible to compute spacetime Lie brackets. (ii) The vector fields considered must form a subalgebra of the Lie algebra of all spacetime vector fields. (iii) The Lie bracket of spacetime extensions of two pairs, (N_1, M_1^a) and (N_2, M_2^a), should depend not on the extensions but only on the spatial derivatives of N_i and M_i^a, in addition to the functions and vector fields themselves. With conditions (ii) and (iii) fulfilled, it is then possible to interpret the Lie bracket of extensions of the two pairs (N_1, M_1^a) and (N_2, M_2^a) as the unique extension of a third pair, (N_3, M_3^a), and to define a new bracket, $[(N_1, M_1^a), (N_2, M_2^a)] \coloneqq (N_3, M_3^a)$. All three conditions can be shown to be true for g-Gaussian vector fields [22], recovered as a special case ($\beta = 1$ and $\alpha^a = 0$) of the following calculations. To the best of our knowledge, it is not known whether g-Gaussian vector fields are the only choice fulfilling all three conditions, but having one such choice is sufficient for a derivation of the brackets.

We first derive properties (i), (ii), and (iii), found in [22], using abstract index notation. We write (11) as

$$0 = n^\mu \mathcal{L}_v g_{\mu\nu} = n^\mu v^\rho \partial_\rho g_{\mu\nu} + n^\mu g_{\nu\rho}\partial_\mu v^\rho + n^\mu g_{\mu\rho}\partial_\nu v^\rho. \tag{12}$$

The first two terms can be expressed by the Lie bracket of n^μ and v^ν if we write $n^\mu v^\rho \partial_\rho g_{\mu\nu} = v^\rho \partial_\rho n_\nu - g_{\mu\nu} v^\rho \partial_\rho n^\mu$. The last term in $n^\mu \mathcal{L}_v g_{\mu\nu}$ can be replaced by a total derivative using $n^\mu g_{\mu\rho}\partial_\nu v^\rho = \partial_\nu (n^\mu v^\rho g_{\mu\rho}) - v^\rho \partial_\nu n_\rho$. In addition to the Lie bracket and the total derivative, there remain two extra terms related to the 2-form $\mathrm{d}n$:

$$0 = n^\mu \mathcal{L}_v g_{\mu\nu} = [n, v]^\mu g_{\mu\nu} + \partial_\nu (n^\mu v^\rho g_{\mu\rho}) + v^\rho (\mathrm{d}n)_{\rho\nu}. \tag{13}$$

If n^μ is hypersurface orthogonal, by the Frobenius theorem we have $\mathrm{d}n = n \wedge w$ with some 1-form w which can, without loss of generality, be assumed to be orthogonal to n^μ. For $n^\mu n_\mu = \epsilon$ and the metric in Gaussian form, $w = 0$ because $n = \epsilon \mathrm{d}t$ is closed. In this case, n^μ is hypersurface orthogonal in a neighborhood of the initial slice by construction of the Gaussian system. If $n^\mu n_\mu = \epsilon\beta$, the analog of the Gaussian system has an n^μ hypersurface orthogonal only if β is spatially constant. In order to allow for spatially nonconstant β, we use a Gaussian system constructed from a unit normal, which would be $\tilde{n}^\mu \coloneqq n^\mu/\sqrt{|\beta|}$ if $n^\mu n_\mu = \epsilon\beta$. This rescaled normal is extended to a closed 1-form in its Gaussian system. We can compute $\mathrm{d}n = n \wedge w$ from the equation $\mathrm{d}\tilde{n} = 0$, resulting in $w = -\frac{1}{2}\beta^{-1}(\mathrm{d}\beta - \epsilon|\beta|^{-1/2}\dot{\beta} n)$. The second term, with $\dot{\beta} = \partial\beta/\partial t$, is chosen such that $n^\mu w_\mu = 0$.

We include one further generalization by relaxing the usual orthogonality relation to $g_{\mu\nu} n^\mu X_a^\nu = \alpha_a$, with fixed phase-space functions α_a allowed to be nonzero. The components of α_a are related to the direction cosines (hyperbolicus) of n^μ with respect to the spatial basis X_a^ν. The new condition can equivalently be written as an orthogonality relation $g_{\mu\nu} n'^\mu X_a^\nu = 0$ with a redefined $n'^\mu \coloneqq n^\mu - \alpha^a X_a^\mu$. With the nonstandard normalization of n^μ, the redefined vector satisfies $n'^\mu n'_\mu = \epsilon\beta - \alpha_a \alpha^a \eqqcolon \epsilon\gamma$. In the Euclidean case, $\epsilon = 1$, we must have $\gamma > 0$ and therefore $\alpha^a \alpha_a < \beta$. The same condition ensures that n^μ and X_a^μ form a basis because the angle between the direction n^μ and the spatial tangent plane spanned by X_a^μ is less than 90°. In the Lorentzian case, $\alpha^a \alpha_a$ is unrestricted.

We construct a Gaussian system as before. The hypersurface orthogonal vector is now given by $\tilde{n}'^\mu \coloneqq n'^\mu/\sqrt{|\beta - \epsilon\alpha^a\alpha_a|} = n'^\mu/\sqrt{|\gamma|}$. Computing $\mathrm{d}n' = n' \wedge w$ from the equation $\mathrm{d}\tilde{n}' = 0$ now results in $w = -\frac{1}{2}\gamma^{-1}(\mathrm{d}\gamma - \epsilon|\gamma|^{-1/2}\dot{\gamma} n')$. With the redefined normal, (13) takes the form

$$0 = n'^\mu \mathcal{L}_v g_{\mu\nu} = [n', v]^\mu g_{\mu\nu} + \partial_\nu (n'^\mu v^\rho g_{\mu\rho}) + v^\rho (\mathrm{d}n')_{\rho\nu}. \tag{14}$$

We use n'^μ because we need a normal vector for the condition of a g-Gaussian vector field. However, we may decompose a g-Gaussian vector field v^μ according to our original basis (n^μ, X_a^ν) or according to the redefined basis using n'^μ instead of n^μ:

$$v^\mu = Nn^\mu + M^\mu = Nn'^\mu + M'^\mu, \tag{15}$$

with $M^\mu = M^a X_a^\mu$ and $M'^\mu = M^\mu + N\alpha^a X_a^\mu$, or $M'^a = M^a + N\alpha^a$. The latter choice simplifies some derivations and is, therefore, employed below, but for full generality we will transform the final result to a decomposition with respect to (n^μ, X_a^ν).

We will need the following ingredients in order to rewrite (14) with a decomposed vector field v^μ. In contrast to the standard case, $n'^\mu n'_\mu = \epsilon\gamma$ is not a constant because α^a and β may depend on space and time via phase-space variables. Therefore, for spatial M^μ (or M'^μ), $[n', M]^\mu$ has a normal component given by

$$\begin{aligned}\frac{n'^\mu n'_\nu [n', M]^\nu}{n'^\kappa n'_\kappa} &= \frac{1}{\epsilon\gamma} n'^\mu (n'_\nu n'^\rho \nabla_\rho M^\nu - n'_\nu M^\rho \nabla_\rho n'^\nu)\\ &= -\frac{1}{\epsilon\gamma} n'^\mu (M^\nu n'^\rho \nabla_\rho n'_\nu + n'_\nu M^\rho \nabla_\rho n'^\nu)\\ &= -\frac{1}{\epsilon\gamma} n'^\mu (2M^\nu n'^\rho \nabla_\rho n'_\nu + n'^\nu M^\rho (\mathrm{d}n')_{\rho\nu})\\ &= -\frac{1}{\epsilon\gamma} n'^\mu \Big(2M^\nu \sqrt{|\gamma|}\tilde{n}'^\rho \nabla_\rho \Big(\sqrt{|\gamma|}\tilde{n}'_\nu\Big)\\ &\quad + 2n'^\nu M^\rho n'_{[\rho} w_{\nu]}\Big)\\ &= \frac{1}{\epsilon\gamma} n'^\mu n'^\nu n'_\nu M^\rho w_\rho = n'^\mu M^\rho w_\rho,\end{aligned} \tag{16}$$

using $M^\nu \tilde{n}'_\nu = 0$ and the geodesic property $\tilde{n}'^\rho \nabla_\rho \tilde{n}'_\nu = 0$ of the normal in a Gaussian system. With

$$v^\rho (\mathrm{d}n')_{\rho\nu} = 2(Nn'^\rho + M'^\rho) n'_{[\rho} w_{\nu]} = \epsilon\gamma N w_\nu - M'^\rho w_\rho n'_\nu, \tag{17}$$

we can write (14) as

$$\begin{aligned}0 &= n'^\mu g_{\mu\nu} n'^\rho \partial_\rho N + [n', M']^\mu g_{\mu\nu} + \partial_\nu (N n'^\mu n'^\rho g_{\mu\rho})\\ &\quad + \epsilon\gamma N w_\nu - M'^\rho w_\rho n'_\nu\end{aligned} \tag{18}$$

or

$$\begin{aligned}0 &= [n', M']^\mu + n'^\mu n'^\rho \partial_\rho N + \epsilon \partial^\mu (\gamma N)\\ &\quad + \epsilon\gamma N w^\mu - M'^\rho w_\rho n'^\mu.\end{aligned} \tag{19}$$

The equation can now be split into components parallel and orthogonal to n'^μ. The normal component implies

$$n'^\rho \partial_\rho N = -\frac{1}{2}\frac{N}{\gamma} n'^\nu \partial_\nu \gamma \tag{20}$$

(the contribution from $\mathrm{d}n'$ canceling out with the normal contribution from $[n', M']$), while the spatial component gives

$$\begin{aligned}[n', M']^a &= q^a_\mu [n', M']^\mu\\ &= -\epsilon q^{ab} \partial_b (\gamma N) - \epsilon\gamma N w^a\\ &= -\epsilon (\mathrm{grad}_q (\gamma N))^a - \epsilon\gamma N w^a.\end{aligned} \tag{21}$$

The full spacetime commutator is

$$[n', M']^\mu = q^\mu_a [n', M']^a + M'^\rho w_\rho n'^\mu, \tag{22}$$

combining (21) with (16).

With these relations, the hypersurface-deformation brackets follow immediately from the Lie brackets of g-Gaussian vector fields. First, in the Gaussian system, (20) and (22) provide first-order partial differential equations for N and M^μ or M'^μ to be extended into a neighborhood of the initial slice. [Importantly, all M^μ-dependent terms cancel out in (20), even with nonstandard normalization. The equation for N is therefore decoupled from the equation for M^μ.] We can then compute spacetime Lie brackets of two g-Gaussian vector fields,

$$\begin{aligned}[v_1, v_2] &= [N_1 n' + M'_1, N_2 n' + M'_2]\\ &= [N_1 n', N_2 n'] + [N_1 n', M'_2]\\ &\quad + [M'_1, N_2 n'] + [M'_1, M'_2]\\ &= (N_1 \mathcal{L}_{n'} N_2 - N_2 \mathcal{L}_{n'} N_1) n' + (\mathcal{L}_{M'_1} N_2 - \mathcal{L}_{M'_2} N_1) n'\\ &\quad + N_1 [n', M'_2] - N_2 [n', M'_1] + [M'_1, M'_2].\end{aligned} \tag{23}$$

The first term, $N_1 \mathcal{L}_{n'} N_2 - N_2 \mathcal{L}_{n'} N_1$, is zero even with the new contributions in (20) for a nonconstant γ. Similarly, the w^a term in (21) does not contribute to $N_1 [n', M'_2] - N_2 [n', M'_1]$. However, the normal contribution $M'^\rho w_\rho n'^\mu = -\frac{1}{2}\gamma^{-1} n'^\mu M'^\rho \partial_\rho \gamma$ in (22) does not cancel out and provides a new normal term in

$$\begin{aligned}[v_1, v_2] &= \left(\mathcal{L}_{M'_1} N_2 - \mathcal{L}_{M'_2} N_1 - \frac{1}{2\gamma}(N_1 \mathcal{L}_{M'_2}\gamma - N_2 \mathcal{L}_{M'_1}\gamma)\right) n'\\ &\quad - \epsilon N_1 \mathrm{grad}_q (N_2 \gamma) + \epsilon N_2 \mathrm{grad}_q (N\gamma) + [M'_1, M'_2]\\ &= \frac{1}{\sqrt{|\gamma|}}\Big(\mathcal{L}_{M'_1}\Big(\sqrt{|\gamma|} N_2\Big) - \mathcal{L}_{M'_2}\Big(\sqrt{|\gamma|} N_1\Big)\Big) n'\\ &\quad - \epsilon\gamma (N_1 \mathrm{grad}_q N_2 - N_2 \mathrm{grad}_q N_1) + [M'_1, M'_2].\end{aligned} \tag{24}$$

The last line can now be transformed from $n'^\mu = n^\mu - \alpha^\mu$ and $M'^\mu = M^\mu + N\alpha^\mu$ to n^μ and M^μ. Inserting the expressions for the primed vectors leads to several extra terms, most of which cancel out. However, two new contributions remain:

$$[v_1, v_2] = \left(\frac{1}{\sqrt{|\gamma|}}\left(\mathcal{L}_{M_1}\left(\sqrt{|\gamma|}N_2\right) - \mathcal{L}_{M_2}\left(\sqrt{|\gamma|}N_1\right)\right) + N_1\mathcal{L}_\alpha N_2 - N_2\mathcal{L}_\alpha N_1\right)n - \epsilon\gamma(N_1 \mathrm{grad}_q N_2 - N_2 \mathrm{grad}_q N_1)$$
$$- \sqrt{|\gamma|}\left(N_1\mathcal{L}_{M_2}\frac{\alpha}{\sqrt{|\gamma|}} - N_2\mathcal{L}_{M_1}\frac{\alpha}{\sqrt{|\gamma|}}\right) + [M_1, M_2]. \quad (25)$$

By extracting terms parallel to n or the tangent plane, we write this Lie bracket as bracket relationships between pairs (N, M^a):

$$[(0, M_1^a), (0, M_2^b)] = (0, [M_1, M_2]^c) \quad (26)$$

$$[(N, 0), (0, M^a)] = (-|\gamma^{-1/2}|\mathcal{L}_M(|\gamma|^{1/2}N), - |\gamma|^{1/2}N\mathcal{L}_M(|\gamma|^{-1/2}\alpha^a)) \quad (27)$$

$$[(N_1, 0), (N_2, 0)] = (N_1\mathcal{L}_\alpha N_2 - N_2\mathcal{L}_\alpha N_1, - \epsilon\gamma(N_1 \mathrm{grad}_q^a N_2 - N_2 \mathrm{grad}_q^a N_1)). \quad (28)$$

The following special cases are of interest:

(i) If $\alpha^a \neq 0$, there is a new class of modified brackets which have not been derived explicitly in models of loop quantum gravity. New features are a transversal deformation (along a non-normal n^μ) contributing to the bracket of two transversal deformations, and a spatial diffeomorphism contributing to the bracket of a transversal deformation and a spatial diffeomorphism. If this example is realized by quantum-gravity effects, it would require the existence of a preferred spatial direction α^a.

(ii) If $\alpha^a = 0$, the bracket of two normal deformations is a spatial diffeomorphism, as in the classical version, but with a multiplicative correction function $\gamma = \beta$. One can obtain the modified brackets (28) by replacing N_i with $\sqrt{|\gamma|}N_i$ and n' with $n'/\sqrt{|\gamma|}$ in the standard brackets, in accordance with the rescaling transformations of the normal keeping Nn' invariant for (9) to be preserved. However, our calculation shows more than this because it ensures that the three conditions required for a meaningful relation between hypersurface-deformation brackets and spacetime Lie brackets are still satisfied for g-Gaussian vector fields with a nonstandard normal.

(iii) If $\alpha^a = 0$ and $\gamma = \beta$ is spatially constant, all derivatives of γ cancel out and the bracket of a normal deformation and a spatial diffeomorphism is unmodified. A time-dependent γ therefore leads only to a multiplicative modification of the standard brackets, and it appears only in the bracket of two normal deformations. This is the example (1) found in models of loop quantum cosmology with first-order perturbative inhomogeneity.

B. Lie algebroids

The hypersurface-deformation generators do not form a Lie algebra, owing to the appearance of structure functions. Structure functions can be elegantly described by the notion of Lie algebroids, which may be motivated as follows. Assume that we have a finite number of constraints C_I, $I = 1, \ldots, n$, on a Poisson manifold B, which satisfy an algebra $\{C_I, C_J\} = c_{IJ}^K(x)C_K$ with structure functions $c_{IJ}^K(x)$ depending on $x \in B$. We can formally rewrite brackets with structure functions in terms of structure constants by defining an extended system of infinitely many constraints:

$$C_I, C_{IJ} \coloneqq \{C_I, C_J\} = c_{IJ}^K C_K$$
$$C_{HIJ} \coloneqq \{C_H, C_{IJ}\} = (\{C_H, c_{IJ}^K\} + c_{IJ}^L c_{HL}^K)C_K \ldots \quad (29)$$

The brackets $\{C_I, C_J\} = C_{IJ}, \{C_H, C_{IJ}\} = C_{HIJ}, \ldots$ of the extended system then have structure constants.

These constraints span a certain linear subspace of the space $\Gamma(A)$ of sections $\alpha = \alpha(x)^I C_I$ of a vector bundle A over the base manifold B (phase space) with fiber $\pi^{-1}(x) \approx \mathbb{R}^n \ni \{\alpha(x)^1, \ldots, \alpha(x)^n\}$. The sections of this bundle form a Lie algebra by taking Poisson brackets $[\alpha_1, \alpha_2] = \{\alpha_1(x)^I C_I, \alpha_2(x)^J C_J\}$. Moreover, we can define a linear map $\rho\colon \Gamma(A) \to \Gamma(TB), \alpha = \alpha^I(x)C_I \mapsto \{\alpha(x)^I C_I, \cdot\}$ which appears in a Leibniz rule,

$$\begin{aligned}[\alpha, g\beta] &= \{\alpha(x)^I C_I, g(x)\beta(x)^J C_J\} \\ &= g(x)\{\alpha(x)^I C_I, \beta(x)^J C_J\} \\ &\quad + \{\alpha(x)^I C_I, g(x)\}\beta(x)^J C_J \\ &= g(x)\{\alpha(x)^I C_I, \beta(x)^J C_J\} \\ &\quad + (\rho(\alpha(x)^I C_I)g(x))\beta(x)^J C_J \\ &= g[\alpha, \beta] + (\rho(\alpha)g)\beta, \end{aligned} \quad (30)$$

and ρ is a homomorphism of Lie algebras:

$$\begin{aligned}\rho([\alpha, \beta]) &= \{\{\alpha(x)^I C_I, \beta(x)^J C_J\}, \cdot\} \\ &= \{\alpha(x)^I C_I, \{\beta(x)^J C_J, \cdot\}\} \\ &\quad - \{\beta(x)^J C_J, \{\alpha(x)^I C_I, \cdot\}\} \\ &= \rho(\alpha)\rho(\beta) - \rho(\beta)\rho(\alpha) = [\rho(\alpha), \rho(\beta)], \end{aligned} \quad (31)$$

using the Jacobi identity. The Lie bracket on sections together with a homomorphism ρ characterize A as a Lie algebroid [30].

Definition 1.—A *Lie algebroid* is a vector bundle A over a smooth base manifold B together with a Lie bracket $[\cdot,\cdot]_A$ on the set $\Gamma(A)$ of sections of A and a bundle map $\rho\colon\Gamma(A)\to\Gamma(TB)$, called the *anchor*, provided that

(i) $\rho\colon(\Gamma(A),[\cdot,\cdot]_A)\to(\Gamma(TB),[\cdot,\cdot])$ is a homomorphism of Lie algebras, that is,

$$\rho([\xi,\eta]_A)=[\rho(\xi),\rho(\eta)],$$

where $[\cdot,\cdot]$ is the commutator of vector fields in $\Gamma(TB)$.

(ii) For any ξ, $\eta\in\Gamma(A)$ and for any $f\in C^\infty(B)$, the Leibniz identity

$$[\xi,f\eta]_A=f[\xi,\eta]_A+(\rho(\xi)f)\eta$$

holds.

If the base manifold B is a point, the Lie algebroid is a Lie algebra. Another example for a Lie algebroid is the tangent bundle TB of a manifold B, with $\rho\colon\Gamma(TB)\to\Gamma(TB)$ being the identity map and the Lie bracket of vector fields as the bracket on sections. The hypersurface-deformation brackets have been shown in [22] to be captured by a certain Lie algebroid more specific than the construction based on (29). This notion can, therefore, provide useful methods in an analysis of different versions of hypersurface deformations. In order to identify classes of equivalent Lie algebroids, one may generalize the notion of a Lie algebra morphism to the Lie-algebroid case.

Definition 2.—A *base-preserving morphism* between Lie algebroids $(A,[\cdot,\cdot]_A,\rho)$ and $(A',[\cdot,\cdot]_{A'},\rho')$ is a bundle map $\Phi\colon A\to A'$ over $\mathrm{id}_B\colon B\to B'=B$, such that Φ induces a Lie algebra homomorphism $\Phi\colon(\Gamma(A),[\cdot,\cdot]_A)\to(\Gamma(A'),[\cdot,\cdot]_{A'})$ and satisfies $\rho'\circ\Phi=\rho$.

If the induced base map ϕ_0 is a diffeomorphism, the definition can still be used. In such cases, which will be of interest to us, the bundle map induces a map on sections via $\Phi(\xi)(y)=\xi(\phi_0^{-1}(y))$ for $\xi\in\Gamma(A)$ and $y\in B'$. For completeness, we mention that a Lie-algebroid morphism which does not preserve the base manifold can be defined as follows (see, for instance [31]).

Definition 3.—A Lie-algebroid morphism from $A\to B$ to $A'\to B'$ is a bundle map $\phi\colon A'^*\to A^*$ with induced base map $\phi_0\colon B'\to B$, such that

(i) The induced map $\Phi\colon\Gamma(A)\to\Gamma(A')$, defined by $\Phi(\xi)(y)=\phi^*\xi(\phi_0(y))$ for $y\in B'$, preserves the Lie bracket on sections: $[\Phi(\xi),\Phi(\eta)]=\Phi([\xi,\eta])$ for all $\xi,\eta\in\Gamma(A)$.

(ii) We have $\rho=\phi_{0*}\circ\rho'\circ\Phi$.

We will not use general morphisms in this paper, but note that an example of a morphism as in the preceding definition could be used to relate the spacetime structures underlying general relativity and higher-curvature actions, respectively. The latter are higher-derivative theories and have additional canonical degrees of freedom compared with general relativity; therefore, the base manifolds are not diffeomorphic. Nevertheless, the hypersurface-deformation brackets are the same in both settings [32] and could be used to construct a Lie-algebroid morphism.

From now on, we focus on the specific example of the algebroid underlying general relativity. We quote useful definitions and one central result from [22]:

(i) A connected Lorentzian manifold (or spacetime) $(\mathcal{M},g)$ is called Σ *adapted* if it admits an embedding of Σ as a spacelike hypersurface. Such an embedding is called a Σ *space* in $\mathcal{M}$, and a pair consisting of a spacetime and a Σ space in it is called a Σ *spacetime*. On every Σ space, we have an induced, or spatial, metric $q=i^*g$ using the embedding $i\colon(\Sigma,q)\hookrightarrow(\mathcal{M},g)$.

(ii) Coordinate independence leads to the concept of a Σ *universe*, an equivalence class $[i]$ of Σ spacetimes where $i\colon(\Sigma,q)\hookrightarrow(\mathcal{M},g)$ and $i'\colon(\Sigma,q)\hookrightarrow(\mathcal{M}',g')$ are equivalent if there is an isometry $\Psi\colon(\mathcal{M},g)\to(\mathcal{M}',g')$, which preserves the coorientation of Σ and satisfies $\Psi\circ i=i'$. The set of all Σ universes is denoted by $\mathcal{U}\Sigma$. In order to confirm that this definition is consistent, we pull back g' along i' and obtain the same result as before applying the isometry: $(i')^*g'=(\Psi\circ i)^*g'=i^*(\Psi^*g')=i^*g=q$.

(iii) At this point, the relations between a Cauchy hypersurface Σ and a spacetime $\mathcal{M}$ have been formalized. The next step is to look at the evolutions of one time slice into another time slice. A time slice is defined to be an embedding i_t for a fixed time parameter $t=\text{constant}$ within a one-parameter family. Different time slices are related by Σ *evolutions*, equivalence classes $[i_1,i_0]$ of pairs (i_1,i_0) of Σ spaces in the same spacetime, where a pair (i_1,i_0) in $\mathcal{M}$ is equivalent to (i_1',i_0') in $\mathcal{M}'$ if there is a single isometry $\Psi\colon\mathcal{M}\to\mathcal{M}'$ which is consistent with the coorientations of time slices and which satisfies both $\Psi\circ i_1=i_1'$ and $\Psi\circ i_0=i_0'$. The set of all Σ evolutions is denoted by $\mathcal{E}\Sigma$.

The set of Σ evolutions, $\mathcal{E}\Sigma$, forms a Lie groupoid [22] with elements in $\mathcal{U}\Sigma$, source map $s([i_1,i_0])=[i_0]$, and target map $t([i_1,i_0])=[i_1]$, with multiplication given by $[i_2,i_1][i_1,i_0]=[i_2,i_0]$ and inversion by $[i_1,i_0]^{-1}=[i_0,i_1]$. The definition, therefore, gives rise to an evolution picture in terms of groupoid multiplication. The Lie algebroid $A\mathcal{E}\Sigma$ belonging to the Lie groupoid $\mathcal{E}\Sigma$ provides the link between this formulation and the infinitesimal one used, for instance, in [14]. According to [22], we have the following:

Proposition 1.—The Lie algebroid $A\mathcal{E}\Sigma$ of $\mathcal{E}\Sigma$ is isomorphic as a vector bundle to the trivial bundle $\mathcal{U}\Sigma\times(\Gamma(T\Sigma)\oplus C^\infty(\Sigma))$ over the base manifold $\mathcal{U}\Sigma$.

Proposition 1 tells us that infinitesimal evolutions of an equivalence class in $\mathcal{U}\Sigma$ are described by (shift) vector fields in $\Gamma(T\Sigma)$ and (lapse) C^∞ functions on Σ. The base manifold of the Lie algebroid is the space of equivalence classes of spatial embeddings. Structure functions of the classical hypersurface-deformation brackets depend on the spatial metric, which in turn depends only on the equivalence class of embeddings $\Sigma \hookrightarrow \mathcal{M}$ for a given spacetime metric. Similarly, extrinsic curvature on Σ depends on the embedding in $(\mathcal{M}, g)$, but it is not invariant under spacetime isometries fixing (Σ, q). Since the modification function β may depend on all phase-space variables, we should refine the equivalence classes to those transformations that keep both q_{ab} and K_{cd} fixed on Σ. However, if the hypersurface-deformation brackets are modified, it is not clear whether a spacetime metric structure exists which can induce a spatial metric. It is then more appropriate to formulate the Lie algebroid directly over a base manifold of spatial metrics and extrinsic-curvature tensors on Σ (or the classical phase space). In fact, [22] indicates the way to such a formulation using Gaussian representatives.

For an explicit construction of Lie-algebroid brackets and the anchor, [22] chooses as a representative for a Σ universe a slicing which is locally of Gaussian form, as in the derivation of Sec. II A. A representative of a class in $\mathcal{U}\Sigma$ can then be fixed by specifying the induced metric q instead of the embedding. The tangent space of the resulting base manifold of spatial metrics is, at a point q, given by $T_q\mathcal{U}\Sigma = S^2T^*\Sigma$, the space of symmetric tensors identified with Lie derivatives of the spacetime metric by g-Gaussian vector fields $v^\mu = Nn^\mu + M^\mu$: since such vector fields preserve the Gaussian form, $\mathcal{L}_v g$ is equivalent to a change $\delta_v q \coloneqq \mathcal{L}_M q + N\dot{q}$ of just the spatial metric, where $\dot{q} = \mathcal{L}_n q = 2K$ is proportional to the extrinsic-curvature tensor. The latter changes by $\delta_v K = \mathcal{L}_M K + N\dot{K}(q, K)$, where $\dot{K} = \mathcal{L}_n K$ is a function of q_{ab} and K_{cd} via the field equations. (The field equations had been bypassed on [22] by working with equivalence classes of the entire neighborhood of embeddings of Σ in M.) Notice that the anchor ρ depends on the field equations of the theory, while the brackets do not.

The anchor map of the Lie algebroid with the gravitational phase space as base manifold is given by $(N, M) \mapsto (\delta_{Nn+M}q, \delta_{Nn+M}K)$. This base manifold and the anchor have been extended to the space of induced metrics and extrinsic-curvature tensors, which is necessary if one works with modified brackets where β depends on q_{ab} and K_{ab}. The same calculations as in Sec. II A imply that the Lie algebra of g-Gaussian vector fields v leads to a Lie-algebroid bracket

$$[(N_1, M_1), (N_2, M_2)] = \Bigg(\frac{1}{\sqrt{|\beta|}}\Big(\mathcal{L}_{M_1}\Big(\sqrt{|\beta|}N_2\Big) - \mathcal{L}_{M_2}\Big(\sqrt{|\beta|}N_1\Big)\Big), \epsilon\beta(N_1 \mathrm{grad}_q N_2 - N_2 \mathrm{grad}_q N_1) + [M_1, M_2]\Bigg) \tag{32}$$

(if $\alpha^a = 0$) once the decomposition $v^\mu = Nn^\mu + M^\mu$ is introduced.

III. PHYSICS FROM HYPERSURFACE-DEFORMATION ALGEBROIDS

Using the Lie-algebroid structure of hypersurface deformations, we can now look at possible modified versions and their relations to the classical brackets. In some cases, they turn out to be related by algebroid morphisms. We begin with a review of existing examples for deformed brackets.

A. Modified brackets

The classical hypersurface-deformation brackets have been derived from the usual spacetime structure, using, for instance, infinitesimal spacetime diffeomorphisms in (24). They are independent of specific solutions to Einstein's or modified field equations as long as the theory is based on Riemannian geometry. For instance, the same brackets are obtained for higher-curvature actions [32]. In several effective models of loop quantum gravity, however, modified versions of the brackets have been found, and it has not been clear what spacetime structure or what effective actions they may correspond to. In this subsection, we discuss several relevant conceptual details of such models, leaving aside technical features.

Modified brackets have been derived canonically, by including possible quantum corrections in the classical constraints and checking under which conditions they still give rise to a closed set of Poisson brackets. Generically, quantum corrections suggested by loop quantum gravity, based on real connection variables, could be implemented consistently only when the brackets were modified as in (1). For complex connections, the derivative structure of the Hamiltonian constraint is different, in that there are no second-order derivatives of the triad, unlike in real formulations which have the generic pattern responsible for signature-change-type deformations [33]. At least in spherically symmetric models, it is then possible to have undeformed brackets even in the presence of holonomy modifications [34]. Such models are less restrictive than the full theory, and therefore it is not clear whether the full brackets can be undeformed.

Two main classes of models in which deformed brackets have been derived are (i) cosmological perturbations [1,35] where, to linear order, β is a function only of time (via the background spatial metric and extrinsic curvature) and (ii) spherically symmetric models [2–4,7] where β may also depend on the radial coordinate. With so-called holonomy modifications of the classical dynamics, β depends on K_{ab} as some kind a of higher-curvature correction, but only in spatial terms, so that the modification is not necessarily spacetime covariant. Detailed calculations have shown that it is possible to have such spatial-curvature modifications and still maintain closed brackets of correspondingly modified hypersurface-deformation generators, but only when β and the way it appears in the equations of motion are restricted. This is the condition of anomaly freedom. Generically, whenever β depends on K_{ab}, it changes sign at large curvature if quantum effects lead to bounded curvature or densities (so-called bounce models). The same observations have been found in cosmological and spherically symmetric models, with agreement also in the specific functional form of β [36]. There are, however, obstructions in models with local physical degrees of freedom [37,38], in which no anomaly-free holonomy-modified versions have been found yet. (There are also obstructions in some operator versions of spherically symmetric models that implement spatial discreteness [39].)

In these two classes of models, two kinds of methods have been used to provide complementary insights. Effective calculations proceed by computing Poisson brackets of classical hypersurface-deformation generators modified by potential quantum corrections, following a systematic canonical version of effective-action techniques [40–43]. Operator methods compute commutators of quantized generators. Also here, there is full agreement between results from these two different methods: the operator calculations of [12] in spherically symmetric models provide the same restrictions on modifications and the function β as found by effective methods [2]. It is not known how to implement cosmological perturbations at the operator level, but there is a set of $(2+1)$-dimensional models which provide complementary insights. In [8], a modification function for holonomies was found that shows the same features related to the change of sign of β; see also [44].

Other operator calculations in $(2+1)$-dimensional models [9–11] are only partially off shell at this point and, therefore, are not able to show the full brackets. Particularly since they amount to factoring out spatial diffeomorphisms everywhere except at a finite number of isolated points, they cannot exhibit holonomy modifications which are spatially nonlocal. The interesting conclusion of β changing sign, therefore, cannot yet be tested in this setting. Nevertheless, these models have confirmed the presence of modified brackets for metric-dependent modifications. For instance, Eq. (9.27) in [9] gives a definition of the right-hand side of the operator equivalent of (1), which contains an inverse-metric operator with a factor of $(\det q)^{-1/4}$ modified by so-called inverse-triad corrections 45,46]]. We note that reading off modified brackets from commutators is not straightforward because, in addition to the commutator, an effective bracket contains information about semiclassical states. Defining such states and computing expectation values in them is notoriously difficult in background-independent quantum-gravity theories. Nevertheless, it is clear that the naive classical limit of the equation just cited shows a modification of the classical bracket. (In the naive classical limit, one replaces operator factors in the quantized constraints and structure functions with their expectation values in simple states, thus ignoring fluctuations and higher moments.)

Some quantization schemes of constrained gravitational systems represent hypersurface deformations in an indirect way, after reformulating the classical constraints so as to make them easier to quantize. In the present context, two examples are relevant in which one can use reformulations in order to eliminate structure functions from the constraint brackets. In [11], $(2+1)$-dimensional gravity is quantized by writing the bracket of two Hamiltonian constraints in the schematic form $\{H[N], H[M]\} = \{D[N'^a], D[M'^b]\}$, where N'^a and M'^b are shift vector fields related to N and M, respectively. There are no structure functions on the right-hand side, and it is possible to represent the bracket relation without modifications. However, this result does not imply that the hypersurface-deformation brackets are undeformed; in fact, one can check that $\{H[N], H[M]\}$, written as a single diffeomorphism constraint, has quantum-corrected structure functions. (The vector fields N'^a and M'^b mentioned above depend on the spatial metric and give rise to new terms in structure functions when $\{D[N'^a], D[M'^b]\}$ is expressed as a term linear in D.)

Similarly, spherically symmetric systems can be reformulated in a way that partially Abelianizes the constraint algebra [47,48]. The Hamiltonian constraint is here replaced by a linear combination $C[L] \coloneqq H[L'] + D[L'']$, with L' and L'' suitably related to L, such that $\{C[L_1], C[L_2]\} = 0$. Structure functions are thus eliminated from the constrained system (C, D), and the brackets can be represented without quantum corrections in their coefficients. However, if one tries to find hypersurface-deformation generators of quantum constraints with the correct classical limit, it turns out that this is possible only if the hypersurface-deformation brackets are deformed [37,38].

Since all of these examples are obtained after quantizing generators of normal deformations with respect to n^μ such that $g_{\mu\nu} n^\mu n^\nu = \epsilon$ and the vector field n^μ is not subject to quantum corrections, the deformed algebra refers to a unit normal vector. With such modified brackets—but standard normalization—the spacetime considerations of [14] no longer apply and, therefore, a nonclassical spacetime structure seems to be realized.

The new brackets, in general, cannot be viewed as describing deformations of hypersurfaces in a Riemannian spacetime with metric $g_{\mu\nu}$. They do, however, determine a well-defined canonical theory in which one can, in principle, solve the constraints and compute gauge-invariant observables, which is all that is needed for physical predictions. Importantly, the brackets are still closed, which is the challenging part of their constructions. If the brackets were not closed, the models would be anomalous and inconsistent because gauge transformations would be violated and results would depend on the choice of coordinates.

Modified brackets can be formulated as a Lie algebroid over the space of pairs of symmetric tensor fields (q_{ab}, K_{cd}) with a positive-definite q_{ab}. The inverse of q_{ab}, as well as K_{ab} through possible modifications in β, appears in the structure functions of the constraint brackets, but they together play the role only of phase-space functions, which need not have a geometrical interpretation as spatial metric and extrinsic curvature associated with a slice Σ in spacetime $(\mathcal{M}, g)$. Instead of defining these spatial tensors in terms of the embedding functions $X(x)$ and a spacetime metric $g_{\mu\nu}$, the only option is to view q_{ab} and K_{ab} as independent phase-space degrees of freedom on which the constraints and the structure functions depend. The modification function must be covariant under transformations with brackets (2), (3), and (1). Particularly since these brackets contain infinitesimal spatial diffeomorphisms as a subalgebra, β must be a spatial scalar. In the modified case, the theory is not necessarily standard spacetime covariant, but, if the brackets close, β and the resulting theory are covariant under transformations generated by Poisson brackets with the modified constraints. In the absence of a spacetime picture, the physical meaning of q_{ab} and K_{cd} is supplied by how they appear in canonical observables. The latter have a known interpretation in the classical limit of $\beta \to 1$ (low curvature), which is extended to nonclassical regimes in an anomaly-free deformed theory. Alternatively, one may employ field redefinitions such that a relation of Lie-algebroid elements to spacetime metrics becomes possible. We discuss two possible types in the following subsections.

B. Base transformations

In (1), β always appears in combination with the inverse of q^{ab}, whose components can be used as coordinates on the base manifold along with the components of K_{ab}. We can define a transformation of the base manifold by mapping (q_{ab}, K_{cd}) to $(|\beta|^{-1} q_{ab}, K_{cd})$ and can extend it to a fiber map $(q_{ab}, K_{cd}, N, M^e) \mapsto (|\beta|^{-1} q_{ab}, K_{cd}, N, M^e)$. Here, the fiber coordinates N and M^e as well as K_{cd} are unchanged, while q_{ab} absorbs $|\beta|$. As long as $\beta \neq 0$, the base map is a diffeomorphism and a well-defined Lie-algebroid morphism is obtained, eliminating $|\beta|$ from the brackets. The only parameter that cannot be absorbed is $\mathrm{sgn}\beta$ because q_{ab} is required to be positive definite and, especially, invertible.

We may then consider $|\beta|^{-1} q_{ab}$ as the spatial metric on a spatial slice in a spacetime with the line element

$$\mathrm{d}s^2 = \epsilon\epsilon_\beta N^2 \mathrm{d}t^2 + |\beta|^{-1} q_{ab}(\mathrm{d}x^a + M^a \mathrm{d}t)(\mathrm{d}x^b + M^b \mathrm{d}t), \tag{33}$$

which cannot be obtained generically by a coordinate transformation from (4). (If this were possible, one could eliminate the scale factor $a = |\beta|^{-1/2}$ of a Friedmann-Robertson-Walker metric by a coordinate transformation.) The extrinsic curvature of a $t =$ constant slice in (33) is not equal to K_{ab}. However, we can use the field equations of the modified theory in order to relate K_{ab} to $\dot{q}_{ab} = \mathcal{L}_n q_{ab}$. Using the standard equation for extrinsic curvature computed from (33), a relationship between K_{ab} and extrinsic curvature is obtained, which may not be the identity.

The new variables $(|\beta|^{-1} q_{ab}, K_{cd})$ are no longer canonical coordinates on the base manifold. Noncanonical base coordinates do not make a difference for a Lie algebroid, which, in general, does not even have a Poisson structure on its base. However, we need a Poisson structure on the base manifold in order to derive the dynamics generated by the constraints, and for this it is useful to have a canonical set of variables. Modifying the map $(q_{ab}, K_{cd}) \mapsto (|\beta|^{-1} q_{ab}, K_{cd})$ such that it becomes canonical is possible in some models [21], but it may be complicated in general.

While base transformations can map modified brackets to the classical version—as long as β does not change sign—it is not easy to derive general, theory-independent effects because the interpretation of K_{ab} depends on the dynamics, and there may be no simple canonical sets of variables. It turns out that general aspects of physical implications of the absorption are easier to discern if one uses morphisms that originate from fiber maps. We will be able to do so by absorbing $|\beta|$ in the normalization condition, at least partially, allowing us to discuss the possible physical implications in general terms.

C. Change of normalization as algebroid morphism

One usually expects that the classical theory can be recovered when β approaches one in some regime, such as low curvature. However, as already mentioned, the classical theory can be described with a more-general β if one uses nonstandard normalizations $g_{\mu\nu} n^\mu n^\nu = \epsilon\beta$ of normal vectors to hypersurfaces. Even the classical brackets can, therefore, be modified without changing the implied physics. Although it is customary to assume the normal vector n^μ to be normalized to $\epsilon = \pm 1$, depending on the signature, this choice is a mere convention and one may as well introduce a different normalization. Thus, the requirement of having the correct classical limit does not restrict β much, except that β should not be identically zero.

Since we know from Sec. II A that, for a spatially constant β, the hypersurface-deformation brackets belong to a Lie algebroid, irrespective of how the normal is normalized, there are no further conditions on β from the Jacobi identity. As in our explicit derivation of the brackets, we may obtain a deformation by using a nonstandard normalization of the normal vector field in classical general relativity.

We introduce a bundle map Φ with fiber map $(N, M^a) \mapsto (\sqrt{|\beta|} N, M^a)$ and the identity as base map. It obeys

$$\begin{aligned}[\Phi((N_1,0)),\Phi((N_2,0))] &= \left[\left(\sqrt{|\beta|}N_1,0\right),\left(\sqrt{|\beta|}N_2,0\right)\right] \\ &= (0,|\beta|M_{12}^a) = \Phi((0,|\beta|M_{12}^a)) \\ &= \Phi([(N_1,0),(N_2,0)]_\beta), \end{aligned} \tag{34}$$

where $M_{12}^a = q^{ab}(N_1\partial_b N_2 - N_2\partial_b N_1)$ and we have more specifically denoted the modified bracket by $[\cdot,\cdot]_\beta$, while $[\cdot,\cdot]$ is the classical bracket. The anchor is preserved because $Nn^\mu = \sqrt{|\beta|}\tilde{n}^\mu$ with a nonstandard normal $\tilde{n}^\mu$, such that $g_{\mu\nu}\tilde{n}^\mu\tilde{n}^\nu = 1/|\beta|$. If β is spatially constant, as in models of first-order cosmological perturbations, modified brackets of sections in the Lie algebroid A are mapped to the classical brackets on A', with the required anchor because $Nn^\mu \mapsto (N/\sqrt{|\beta|})n^\mu = N\tilde{n}^\mu$. With a spatially dependent β, the existence of a morphism is less clear because $\{H[N], H_a[M^a]\}$ is not modified in effective models of loop quantum gravity, while it would change in (24). Fiber transformations are, therefore, less general than base transformations in mapping modified brackets to the classical ones.

The fiber map just introduced is valid only if β has a constant sign. When β is of an indefinite sign, no β-absorbing morphism can exist: for opposite signs of β, the corresponding groupoids are inequivalent because their compositions are concatenations of slices in Lorentzian spacetime and the four-dimensional space of the Euclidean signature, respectively.

For a spatially constant $\beta > 0$, we have a Lie-algebroid morphism between modified and unmodified brackets, irrespective of where the deformation function β originates. In the modified case, we then have the classical spacetime structure after applying the morphism that absorbs β in the normalization. But the classical structure is obtained after a field redefinition: the spacetime metric obtained from q_{ab} is not of the standard canonical form but reads

$$\mathrm{d}s^2 = \epsilon\beta N^2\mathrm{d}t^2 + q_{ab}(\mathrm{d}x^a + N^a\mathrm{d}t)(\mathrm{d}x^b + N^b\mathrm{d}t), \tag{35}$$

depending, in general, on q_{ab} and K_{ab}. This line element is conformally related to (33).

D. Equations of motion

When we interpret hypersurface deformations as actual moves in spacetime, we refer to time-evolution vector fields and, therefore, to coordinate structures. Spacetime coordinates are not quantized in canonical quantum gravity, and, therefore, the vector field should not receive quantum corrections if there is a classical manifold picture for the effective theory. Deformed brackets with $\beta > 0$ can sometimes be mapped to the classical spacetime structure in terms of hypersurface deformations, but this does not necessarily lead to the same physics in terms of time evolution.

For a classical deformation with standard normalization, we use

$$\tau^\mu = \delta X^\mu = \delta N\tilde{n}^\mu + \delta N^a X_a^\mu \tag{36}$$

in order to identify time deformations, while, in the classical case with nonstandard normalization, we have

$$\delta X^\mu = \delta N_\beta n^\mu + \delta N^a X_a^\mu, \tag{37}$$

with $n^\mu = \sqrt{|\beta|}\tilde{n}^\mu$. These vector fields must be the same: changing the normalization of the normal vector should not affect the relative position of two hypersurfaces X^μ and $X^\mu + \delta X^\mu$ embedded in spacetime. Thus, the two time-evolution vector fields have to be the same, and it follows that the infinitesimal lapse function δN_β of the modified theory must be given by

$$\delta N_\beta = \frac{1}{\sqrt{|\beta|}}\delta N. \tag{38}$$

1. Classical theory with nonstandard normalization

Classically, we have standard hypersurface-deformation brackets with the normalization condition $g_{\mu\nu}n^\mu n^\nu = \epsilon$, and we know, from [14], that second-order equations of motion for the metric are the classical field equations of general relativity. However, we may change the normalization condition to $g_{\mu\nu}n^\mu n^\nu = \epsilon|\beta|$. The theory is still classical, but the generator of normal deformations is rescaled. Accordingly, the hypersurface-deformation brackets are modified. Since the physics is insensitive to our choice of normalization, we should be able to recover Einstein's field equations from the new brackets.

In [14,49] the Lie derivative with respect to the normal vector field plays an important role in the derivation of possible Hamiltonian constraints consistent with the brackets, and hence in the derivation of the equations of motion. One obtains a partial differential equation which the Hamiltonian constraint, as the generator of normal deformations, must obey [14], and, similarly, there is a related partial differential equation for the Lagrangian [49]. If the brackets are modified, the differential equation is changed by a new coefficient β. For instance, a metric-dependent Lagrangian $L[q_{ab}(x), K_{ab}(x)]$ consistent with constraints satisfying (1) must satisfy the functional equation [15]

$$\begin{aligned}&\frac{\delta L(x)}{\delta q_{ab}(x')}K_{ab}(x') + 2(\partial_b\beta)(x)\frac{\partial L(x)}{K_{ab}(x)}\partial_a\delta(x,x') \\ &\quad + 2\beta(x)\frac{\partial L(x)}{\partial K_{ab}(x)}\partial_a\partial_b\delta(x,x') - (x\leftrightarrow x') = 0,\end{aligned} \tag{39}$$

where $K_{ab} = \frac{1}{2}\mathcal{L}_n q_{ab}$ is taken with a nonstandard normal n^μ. The normal derivative is subsequently written as a Lie derivative along τ^μ in order to arrive at equations of motion with respect to the time-evolution vector field. For the classical equations to result in this second case, in which the algebroid and the normalization are modified in such a way that we are still dealing with the classical theory, the function β appearing in n^μ with nonstandard normalization (and therefore in the Lie derivative $\mathcal{L}_n$ as well) must cancel the function β appearing in the modified brackets. We will make use of the presence of such cancellations in our discussion of the modified case.

2. Modified theory

In models of loop quantum gravity, the hypersurface-deformation brackets are modified. However, since one sets up the models in the standard canonical formulation, the normalization $g_{\mu\nu}n^\mu n^\nu = \epsilon$ is preserved. Since the normal does not depend on phase-space variables and is not quantized, the normalization convention does not change. Yet the brackets are modified. This case is, therefore, different from simply rescaling the normal vector. Nevertheless, one can understand the resulting structures by rescaling the normal after new brackets have been obtained from quantum effects. For a spatially constant β, a morphism to the classical brackets is obtained. By applying the preceding arguments, we nevertheless expect non-classical equations of motion: there is a function β from the modified brackets appearing in the Hamiltonian constraint or the Lagrangian regained from the brackets, but now there is no compensating for β in the normal Lie derivative in relation to the τ^μ derivative because it is defined with respect to the standard normal vector n^μ.

The dynamics is, therefore, modified, which is consistent with the results of several detailed investigations of cosmological [6,50–57] and black-hole consequences [3,4,58,59] in terms of physical, coordinate-independent effects. An open question has been whether one can introduce a modified effective spacetime metric which is generally covariant in the standard sense, or whether the deformed algebroid modifies this symmetry and leads to an entirely new spacetime structure.

For a spatially constant β, we know that deformed brackets can be mapped to classical brackets by a Lie-algebroid morphism so long as β does not change sign. In terms of spacetime geometries, rescaling the normal vector n^μ to $\tilde{n}^\mu = |\beta|^{-1/2} n^\mu$ then leads us back to the unmodified brackets. We already know that this algebroid implements standard spacetime covariance in the canonical formalism. Therefore, we see, in qualitative agreement with [21], that a field redefinition allows us to restore the undeformed brackets, and consequently general covariance in the classical form. The equations of motion are, nevertheless, different from the classical ones because we moved the β appearing in the modified brackets into the new normal vector, which is not canceled out when we finally switch to equations of motion with respect to τ^μ.

IV. CONSEQUENCES

Hypersurface-deformation brackets can be modified by replacing the usual normalization of the normal vector by $g_{\mu\nu}n^\mu n^\nu = \epsilon\beta$, while the time-evolution vector field must be the same for the modified as well as the unmodified theory. These two facts raise the question of whether it is possible to distinguish between classical modifications from non-standard normalizations and modifications induced by quantum-gravity theories. We have answered this question in the affirmative because equations of motion with respect to a fixed time-evolution vector field do change.

A. Field equations and matter couplings

If β has a definite sign and is spatially constant, one can absorb the bracket modifications in a nonstandard normalization. Gauge transformations generated by the algebroid then amount to the standard symmetries of covariance. Accordingly, regained constraints or Lagrangians must belong to the canonical theory of some higher-curvature action, assuming that a local effective action exists.

We expect higher-curvature effective actions when a local derivative expansion exists. In canonical terms, a nonlocal quantum effective action is obtained by coupling expectation values to independent quantum moments [40,41], which formally play the role of auxiliary fields in a nonlocal theory. Only when moments behave adiabatically can they be eliminated from the equations of motion, and a local effective action results. As shown in [42], moments do not appear in structure functions such as β here, but they lead to higher-order constraints which restrict the moments as independent variables. For a local, higher-curvature version of the effective theory, one would have to solve for almost all of the higher-order constraints, which may not always be possible. A canonical effective theory still exists.

However, even if we have a standard higher-curvature effective action after a field redefinition, there are additional effects from modified brackets. The Hamiltonian constraint in such a system generates deformations along a nonstandard normal vector. Therefore, when equations of motion are written with respect to a time coordinate, they belong to an effective action in which time derivatives are multiplied by a factor of β. The main consequence of modified algebroids is, therefore, a nonclassical propagation speed, which is in agreement with the specific results obtained in [5,6,35,50,51,54] for cosmological scalar and tensor modes. From (35), we have the kinetic term $\ddot{\phi}/\beta - \Delta\phi$ in an equation of motion for a scalar field on the effective Riemannian spacetime. This result is in agreement with a related one derived in [15] for a metric-dependent β, following [15,49]. At the same time, we have generalized the result of [15] by extending it to β functions that may depend on extrinsic curvature, as in the cases of interest for signature change.

One can turn these arguments around and try to generate explicit consistent models with modified brackets by introducing nonstandard normalizations in different classical actions or constraints. More generally, we could relax the orthogonality condition between n^μ and X^μ_a in order to find models with the new modified brackets (27) and (28), with $\alpha^a \neq 0$. The recent analysis of [60] suggests that such modified versions of constraints will have to be of higher than second order in extrinsic curvature.

B. (Non)existence of an effective Riemannian structure

Sometimes, the classical spacetime structure is *assumed* in toy models of quantum gravity, without checking closure of modified constraints. In fact, one should consider such constructions not as models of quantum gravity but of quantum-field theory on (modified) curved spacetimes because quantum gravity is usually understood as including a derivation of nonclassical spacetime structures, in addition to a modified dynamics. For instance, some constructions [24–26] use perturbation equations on a modified background $\bar{q}_{ab}$ subject to evolution equations with quantum corrections. Perturbations are gauge fixed or combined into gauge-invariant expressions before quantization, and therefore one assumes the classical spacetime structure. As confirmed here, an effective formulation with the classical spacetime structure does exist, as long as $\beta > 0$, but only after a field redefinition using either base transformations or, in the case of a spatially constant β as it is realized in first-order cosmological perturbation theory, fiber transformations of the hypersurface-deformation algebroid.

There are, therefore, two important caveats regarding assumptions such as those made in [24–26]. First, if the evolution of $\bar{q}_{ab}$ is modified, a consistent description of spacetime transformations for inhomogeneous modes requires a modified N, which can only be computed if one knows a consistent set of β-modified brackets. (The lapse function of the postulated spacetime metrics in [24–26] do have quantum corrections, but in an incomplete way that ignores the field redefinition required for a consistent spacetime structure.) The modified N, as opposed to the classical N, then implies further quantum corrections not directly present in the evolving $\bar{q}_{ab}$. One can, of course, partially absorb $\sqrt{|\beta|}$ in $N' = \sqrt{|\beta|}N$ by introducing a new time coordinate t' with $\mathrm{d}t' = \sqrt{|\beta|}\mathrm{d}t$. However, the dependence of $\bar{q}_{ab}$ on this new t' is different from the original dependence on t, so additional quantum corrections are present.

C. Signature change

Specifically, as the second caveat, the signature of the effective spacetime metric can be determined only if one knows the sign $\epsilon\epsilon_\beta$ by which $\beta N^2 \mathrm{d}t^2$ enters the metric (35) in the equivalent Riemannian spacetime structure, which can differ from the classical value if β does not have definite sign. The sign, in turn, affects the form of well-posed partial differential equations on the background; see, for instance, [33,61]. In the presence of a signature change, there is no deterministic evolution through large curvature. Also, even if one tries to ignore this conclusion for a formal analysis of the resulting phenomenology, no viable results are obtained [62].

If β is of indefinite sign, it can no longer be absorbed globally. The classical spacetime structure can be used only to model disjoint pieces of a solution in which β has a definite sign, corresponding to Lorentzian spacetime patches when β is positive and Euclidean spatial patches when it is negative. We then have nonisomorphic Lie algebroids. A nonconstant sign of β therefore triggers signature change [15,61,63], with the effective signature locally given by $\epsilon\epsilon_\beta$. Globally, such a solution of an effective quantum-gravity model can be described consistently only with a modified algebroid, in which all structure functions are continuous and well defined even when β goes through zero. It is no longer possible to absorb β globally and, therefore, a new version of quantum spacetime is obtained.

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APPENDIX: ADM AND GEOMETRODYNAMICS DERIVATION OF NONSTANDARD CLASSICAL CONSTRAINTS

We derive the results of Sec. II A for $\alpha^a = 0$ using more familiar methods.

1. ADM

Given a spacetime metric $g_{\mu\nu}$ and a time-evolution vector field of the form (9) with respect to a foliation, we obtain the canonical form of the metric by expanding $g_{\mu\nu}\mathrm{d}X^\mu \mathrm{d}X^\nu$ using

$$\mathrm{d}X^\mu = \partial_t X^\mu \mathrm{d}t + \partial_a X^\mu \mathrm{d}x^a = (N_\beta n^\mu + N^a X^\mu_a)\mathrm{d}t + X^\mu_a \mathrm{d}x^a, \tag{A1}$$

with $N_\beta = N/\sqrt{|\beta|}$. If n^μ has the nonstandard normalization $g_{\mu\nu}n^\mu n^\nu = \epsilon\beta$, the metric components are

$$g_{tt} = N^a N_a + \epsilon\beta N_\beta^2 = N^a N_a + \epsilon\epsilon_\beta N^2,$$
$$g_{at} = N_a, \qquad g_{tb} = N_b, \qquad g_{ab} = q_{ab}. \tag{A2}$$

With respect to a nonstandard normal, we define the tensor

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n q_{\mu\nu}. \tag{A3}$$

It differs from the extrinsic-curvature tensor by a factor of $\sqrt{|\beta|}$, as can be seen from the alternative version

$$K_{\mu\nu} = \frac{1}{2N_\beta}\mathcal{L}_{\tau-\vec{N}} q_{\mu\nu} \tag{A4}$$

derived from (A3) using (9). The relationship between $K_{ab} = K_{\mu\nu}X^\mu_a X^\nu_b$ and the τ derivative $\dot{q}_{ab} = \mathcal{L}_\tau q_{ab}$ is, therefore,

$$K_{ab} = \frac{1}{2N_\beta}(\dot{q}_{ab} - \mathcal{L}_{\vec{N}} q_{ab}). \tag{A5}$$

In order to relate K_{ab} to the momentum of q_{ab}, we need the gravitational action $S = \int \mathrm{d}y^4 \sqrt{|\det g|}R$, in new variables defined with respect to a nonstandard normal. (We set $16\pi G = 1$.) The standard derivation from Gauss-Codazzi equations gives us the spacetime Ricci scalar

$$R = \mathcal{R} - \frac{\epsilon}{\beta}(K_{ab}K^{ab} - K^2), \tag{A6}$$

expressed as a combination of the spatial Ricci scalar $\mathcal{R}$ and K_{ab}. (See also [64], where a time-dependent β has been assumed to study classical signature change.) Together with

$$\sqrt{|\det(X^* g)|} = N\sqrt{\det(g_{ab})} = N_\beta \sqrt{|\beta|}\sqrt{\det(q_{ab})}, \tag{A7}$$

all contributions to the Einstein-Hilbert action that appear are written in terms of new variables. The momentum of q_{ab} is

$$P^{ab}(t,x) = \frac{\delta \mathcal{S}}{\delta \dot{q}_{ab}} = -\frac{\epsilon\epsilon_\beta}{\sqrt{|\beta|}}\sqrt{\det(q_{ab})}(K^{ab} - q^{ab}K^c_c), \tag{A8}$$

while the momenta P of N and P_a of N^a vanish as usual. [The factor of $\epsilon\epsilon_\beta/\sqrt{|\beta|} = (\epsilon/\beta)(N/N_\beta)$ in (A8) is a result of combining ϵ/β in (A6) with N in (A7) and one of the N_β's obtained after converting K_{ab} to $\dot{q}_{ab}$ using (A5).] For the primary constraints $P = 0$ and $P_a = 0$ to be preserved in time, we obtain as secondary constraints the diffeomorphism and Hamiltonian constraints

$$\mathcal{H}_a \coloneqq -2q_{ab}\nabla_b P^{bc} \tag{A9}$$

$$\mathcal{H} \coloneqq -\frac{\epsilon\epsilon_\beta\sqrt{|\beta|}}{\sqrt{\det(q_{ab})}}\left(q_{ac}q_{bd} - \frac{1}{2}q_{ab}q_{cd}\right)P^{ab}P^{cd} - \sqrt{|\beta|}\sqrt{\det(q_{ab})}\mathcal{R}. \tag{A10}$$

These constraints have closed Poisson brackets corresponding to (24). In terms of the extrinsic curvature instead of the momentum, the first term of (A10) has a factor of $\epsilon_\beta/\sqrt{|\beta|}$, in agreement with expressions regained from modified brackets [15], following the methods of [14,49].

2. Geometrodynamics

Using the formalism of hyperspace [65–67], the hypersurface-deformation brackets can be derived from infinitesimal deformations, irrespective of the dynamics. An infinitesimal deformation δX^μ may be decomposed as

$$\delta X^\mu = \delta N_\beta n^\mu + \delta N^a X^\mu_a. \tag{A11}$$

The (nonstandard) normalization and orthogonality relations $g_{\mu\nu}n^\mu n^\nu = \epsilon\beta$ and $g_{\mu\nu}n^\mu X^\nu_a = 0$ allow us to compute δN_β and δN^a from δX^μ:

$$\delta N_\beta = \frac{\epsilon}{\beta}n_\mu \delta X^\mu, \qquad \delta N^a = X^a_\mu \delta X^\mu. \tag{A12}$$

Here, we do not refer to τ^μ or δN because the present geometrical considerations refer to what is considered the normal vector with a nonstandard normalization.

An arbitrary functional $F = F[X^\mu(x^a)]$ on hyperspace changes if we deform the hypersurface by $\delta N_\beta(x)$ along a normal geodesic and stretch it by $\delta N^a(x)$. Using (A11), we write the infinitesimal change of F as

$$\delta F = \int_\sigma \mathrm{d}^3x \delta X^\mu(x)\frac{\delta}{\delta X^\mu(x)}F = \int_\sigma \mathrm{d}^3x(\delta N_\beta(x)\rho_0(x) + \delta N^a(x)\rho_a(x))F, \tag{A13}$$

with the generators of pure deformations and pure stretchings given by

$$\rho_0(x) \coloneqq n^\mu(X(x))\frac{\delta}{\delta X^\mu(x)}, \qquad \rho_a(x) \coloneqq X^\mu_a(x)\frac{\delta}{\delta X^\mu(x)}. \tag{A14}$$

These generators can be interpreted as the Lie-algebroid anchor $\rho:\Gamma(A) \to \Gamma(TB)$, with base manifold B being the space of embeddings $X:\sigma \to \mathcal{M}$, expressed in a local basis: in a neighborhood $U \subset B$, we introduce a smooth chart $(U, \{x^a\})$ of the manifold B and a local frame $\{e_i\}$ for sections of the Lie algebroid $\pi^{-1}(U) \subset A$. Then there exist smooth functions $c^k_{ij}, \rho^a_i : B \to \mathbb{R}$, such that

$$[e_i, e_j]_A = c^k_{ij} e_k, \qquad \rho(e_i) = \rho^a_i \frac{\partial}{\partial x^a}. \tag{A15}$$

These functions are called the structure functions of the Lie algebroid with respect to the local frame $\{e_i\}$ and the local coordinates $\{x^a\}$. For the hypersurface-deformation algebroid, $\rho_0 = \rho(e_0)$ and $\rho_a = \rho(e_a)$.

There are infinitely many generators $\rho_0(x)$ and $\rho_a(x)$ which span the tangent space to hyperspace at each hypersurface. Compared to the coordinate basis $\delta/\delta X^\mu$, an important advantage of this basis is its independence of the choice of spacetime coordinates X^μ. We can, therefore, describe the kinematics in terms intrinsic to the hypersurfaces. However, the basis is nonholonomic: commutators of the generators $\rho_0(x)$ and $\rho_a(x)$ do not, in general, vanish.

In order to establish the commutators of deformation generators (A14), we have to know how the normal vector changes under an infinitesimal deformation. To this end, the formula

$$\delta n^\mu = -\epsilon X^{\mu a} \delta N_{,a} + K_{ab} X^{\mu a} \delta N^b - \Gamma^\mu_{\rho\sigma} X^\rho_c n^\sigma \delta N^c - \Gamma^\mu_{\rho\sigma} n^\rho n^\sigma \delta N \tag{A16}$$

was used in [14,65] in order to compute the commutator of normal deformations $\rho_0(x)$ in which $\delta n^\mu(x)/\delta X^\nu(x')$ appear. Only the first term in (A16) contributes to this commutator, while all other terms are irrelevant for this purpose because they present variations proportional to delta functions. Since delta functions are symmetric in their arguments, they will cancel out thanks to the antisymmetry of a commutator. The variation given by the first term in (A16), on the other hand, is proportional to $\delta_{,a}(x,x') = -\delta_{,a'}(x',x)$, which is antisymmetric and does contribute.

The first term in (A16) follows from a simple consideration that can easily be extended to nonstandard normalizations of n^μ. One can compute the full (A16) in terms of its normal and tangential components by varying $g_{\mu\nu} n^\mu n^\nu = \epsilon$ and $g_{\mu\nu} X^\mu_a n^\nu = 0$. Since the first term in (A16) does not contribute to the normal component $n_\mu \delta n^\mu$, it must result from $\delta(g_{\mu\nu} X^\mu_a n^\nu) = 0$. This variation has three terms, so the equation can be solved for

$$\begin{aligned} X_{a\mu} \delta n^\mu &= -n^\mu \delta X_{\mu,a} - X^\mu_a n^\nu \delta g_{\mu\nu} \\ &= -(n^\mu \delta X_\mu)_{,a} - n^\mu_{,a} \delta X_\mu - X^\mu_a n^\nu \delta g_{\mu\nu}. \end{aligned} \tag{A17}$$

The metric variations in the last term, as well as $n^\mu_{,a}$ in the second term, can be written in terms of the extrinsic curvature and the Christoffel symbol, while the first term provides the first part of (A16) upon using (A12) with $\beta = 1$. For $\beta \neq 1$, the first term in (A16) is replaced by $-\epsilon(\beta \delta N_\beta)_{,a}$, or $-\epsilon\beta(\delta N_\beta)_{,a}$ if the derivative of β is combined with the last term in (A16) which drops out of commutators. As a result, there is a factor of β in the commutator

$$[\rho_0(x), \rho_0(x')] = \epsilon\beta(q^{ab}(x)\delta_{,a}(x,x')\rho_b(x) - q^{ab}(x')\delta_{,a}(x',x)\rho_b(x')). \tag{A18}$$

This result agrees with (24).

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